Hyperbolic Embeddings in Toric Geometry: Effectivity and Deformation Stability

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Abstract. We study the deformation behavior of Kobayashi hyperbolic embeddings for complements of divisors in projective toric varieties. In the toric setting, the Zariski closure of an entire curve in a torus complement is a translate of an algebraic subtorus contained in that complement. This translates hyperbolicity questions into combinatorial conditions on lattice-point configurations of Newton polytopes. Building on a theorem of Tiba, which guarantees hyperbolic embedding for a general divisor under suitable facewise lattice conditions, we develop an effective refinement of his argument. We construct an explicit constructible bad locus in the coefficient parameter space, characterized by the presence of translated subtori in the divisor or its complement on a torus stratum, and use its proper Zariski closure as an exceptional locus. This description makes the exceptional set amenable to computation. Using this effectivity, we prove a deformation stability result: along any algebraic one-parameter family of divisors whose initial member avoids the exceptional locus, hyperbolic embedding fails for at most finitely many parameters. Under strengthened lattice-point hypotheses, we further exhibit distinguished one-parameter families for which hyperbolic embedding persists without any exceptional parameters. We also analyze deformations arising from diagonal torus reparametrizations. If the initial divisor lies outside $\mathcal{Y}$, the inverse image of $\mathcal{Y}$ under the similarity family is empty on the nondegenerate parameter domain, while the smallest possible degeneracy set of a rational similarity presentation is controlled by a multiplicative rank invariant of the scaling vector. Finally, we illustrate the theory through examples in projective spaces and Hirzebruch surfaces. For cubic surfaces in $\mathbb{P}^3$, a sparse-resultant calculation shows that the reduced divisorial part of the global exceptional locus has degree 240.

Key words: Kobayashi metric; Kobayashi hyperbolicity; hyperbolic embeddings in toric varieties

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1 Introduction and results

1.1 Deformation and (non)hyperbolicity in the noncompact world

Deformation theory has long played a decisive role in the study of (non)hyperbolicity in complex geometry. A precise propagation result for the Kobayashi pseudodistance was proved by Kamenova–Lu–Verbitsky: if this pseudodistance vanishes identically for one complex structure, then it vanishes for every ergodic complex structure in the same deformation class [9, Theorem 2.1]. They deduced, in particular, that all complex K3 surfaces have identically vanishing Kobayashi pseudodistance [9, Corollary 2.2]. More recently, one of the authors together with

D.R. Morrison proved non-measure-hyperbolicity for every complex K3 surface S and every Hilbert scheme of points $\mathrm{Hilb}^n(S)$, $n \geq 1$ [2, Theorem 2 and Corollary 3]. These results illustrate the role of deformation methods in proving vanishing statements.

The situation is markedly different for hyperbolicity. For compact complex manifolds, Kobayashi hyperbolicity is open under deformation. For general deformations, upper semi-continuity of the infinitesimal Kobayashi–Royden pseudometric is available [12]; see also [8] for the compact hyperbolic case. For divisor complements, however, hyperbolicity need not persist under small deformations. Zaidenberg gives an explicit pencil deforming five lines in general position in $\mathbb{P}^2$ to smooth quintics whose complements contain an affine line [14, Proposition 3.2 and Example 3.2]. Thus, deformation theory in the noncompact world is a genuinely delicate operation, and stability of hyperbolicity cannot be taken for granted.

This leads naturally to the central question addressed in this paper: *Under what conditions does Kobayashi hyperbolicity of a noncompact complement persist under algebraic deformation of divisors?*

1.2 Toric geometry and combinatorial control of entire curves

We study this question in the toric setting, where analytic properties of divisor complements admit a precise translation into combinatorial data. Throughout, let $X = X_\Sigma$ be a projective toric variety of dimension $r \geq 1$ with dense torus $T_N \cong (\mathbb{C}^*)^r$ and character lattice $M \cong \mathbb{Z}^r$, and let D be a torus-invariant ample Cartier divisor. Its polytope $P_D \subset M_\mathbb{R}$ is a full-dimensional lattice polytope with normal fan Σ. We consider divisors in $|D|$ represented by Laurent polynomials supported in $P_D \cap M$; the actual Newton polytope of a member may be smaller when some coefficients vanish.

In this framework, the Zariski closure of an entire curve in a torus complement is a translate of an algebraic subtorus contained in that complement; see Theorem 4.5. As a consequence, the exclusion of entire curves can be detected via lattice-point configurations along faces of the Newton polytope. Combined with Tiba's boundary criterion on all torus strata, this yields Kobayashi hyperbolic embeddings. This philosophy was made precise by Tiba, who proved that if a finite support $S \subset P_D \cap M$ satisfies suitable facewise combinatorial conditions, then the complement of a *general* divisor in the corresponding linear system is Kobayashi hyperbolically embedded into X (see Theorem 3.1).

Tiba's proof already uses layer polynomials, incidence dimension estimates, and a finite collection of hyperplane directions to obtain a proper exceptional subset. We use this construction as our starting point. Our emphasis is on distinguishing its constructible incidence images from their Zariski closures, making their coefficient conditions usable in specific families, and carrying out explicit elimination calculations.

1.3 Main contributions and structure of the paper

We give a coefficient-wise reformulation of Tiba's argument and apply it to special deformations and explicit examples. The generic hyperbolic embedding criterion remains Tiba's theorem; the finite-exception statement below is an algebraic consequence of a proper closed exceptional locus.

Our first contribution is an effective reformulation of Tiba's argument. Under the same facewise lattice hypotheses, we construct a constructible bad locus and a proper Zariski closed exceptional locus $\mathcal{Y} \subset |D|$ in the coefficient parameter space. The bad locus is defined by concrete facewise degeneration conditions that detect the presence of translated subtori, and $\mathcal{Y}$ is its reduced Zariski closure (together with the boundary of the relevant coefficient chart). This description makes the exceptional set amenable to explicit computation, a feature illustrated by several concrete examples.

The construction has the following deformation consequence. Along an algebraic one-parameter family whose image is not contained in $\mathcal{Y}$, hyperbolic embedding holds outside a finite set of parameters. The hypothesis on the initial member in Theorem 1.1 ensures this noncontainment.

Theorem 1.1 (deformation stability with finite exceptions)**.** *Assume that, for every positive-dimensional face τ of P_D and every $H \in \mathcal{H}_{\tau\cap M}$,*

$$|\phi_H((\tau \cap M) - x)| \geq \dim \tau + 1$$

for some $x \in \tau \cap M$. (Equivalently, the complete support $P_D \cap M$ satisfies the facewise hypotheses of Theorem 3.1.) Let $D_0 \in \mathfrak{d}_D \setminus \mathcal{Y}$, where $\mathcal{Y} \subset |D|$ is the proper Zariski closed exceptional locus constructed in Section 4. Let $\gamma\colon \mathbb{P}^1 \to |D|$ be a morphism with $\gamma(0) = D_0$. Then for all but finitely many $t \in \mathbb{P}^1$, the complement

$$T_N \setminus \operatorname{Supp}(D_t \cap T_N), \qquad D_t := \gamma(t),$$

is Kobayashi hyperbolically embedded in X.

The same conclusion holds for a morphism $\gamma\colon C \to |D|$ from an irreducible complex algebraic curve C, provided some member lies outside $\mathcal{Y}$. Here and below algebraic curves are of finite type over $\mathbb{C}$. If C is reducible, the hypothesis is required on each irreducible component to which the conclusion is applied; see Section 5.4 for the proof.

Finally, under strengthened lattice-point conditions, we show that exceptional parameters can be eliminated altogether for certain distinguished affine one-parameter families.

Theorem 1.2 (exception-free deformations)**.** *Let $S \subset P_D \cap M$ be finite and meet every facet of P_D (equivalently, $\mathfrak{d}_S \neq \varnothing$). Assume that for every positive-dimensional face τ of P_D, the set $S_\tau := \tau \cap S$ satisfies $\operatorname{aff}(S_\tau) = \operatorname{aff}(\tau)$ and $|\phi_H(S_\tau - x)| \geq \dim \tau + 2$ for every $H \in \mathcal{H}_{S_\tau}$ and $x \in S_\tau$. Choose a representative $a = (a_I)_{I\in S}$ whose projective class $[a] \in \mathbb{P}^{|S|-1}$ lies outside the enlarged facewise exceptional locus $\mathcal{Y}'_S$ of Definition 5.3, and fix $I_0 \in S$. Then*

$$D_t = \operatorname{div}\Big(tz^{I_0} + \sum_{I\in S\setminus\{I_0\}} a_I z^I\Big), \qquad t \in \mathbb{C},$$

where D_t denotes the divisor of the corresponding global section of $\mathcal{O}_X(D)$. For every $t \in \mathbb{C}$, the inclusion $T_N \setminus \operatorname{Supp}(D_t \cap T_N) \hookrightarrow X$ is a Kobayashi hyperbolic embedding.

The original divisor occurs at $t = a_{I_0}$, whereas $t = 0$ deletes that coefficient. The assertion does not include the member at infinity of the projective closure of this family. A key feature of our approach is that the bad loci are described by explicit incidence conditions. We give examples in $\mathbb{P}^2$, $\mathbb{P}^3$, and Hirzebruch surfaces; in the cubic-surface case we compute the reduced divisorial part of the global exceptional locus by sparse resultants and obtain degree 240.

Structure of the paper. Section 3 recalls basic toric notation and fixes the Newton polytopes and combinatorial setup used throughout. In Section 4, we give an effective proof of Tiba's theorem by constructing the exceptional locus $\mathcal{Y}$ in coefficient space and relating its defining conditions to translated subtori via the theorem of Dethloff–Lu recalled below. Section 5 is devoted to deformation stability and contains the proofs of Theorems 1.1 and 1.2, including the refined enlarged exceptional locus tailored to one-monomial borderline degenerations. Section 6 studies a distinguished class of one-parameter deformations arising from diagonal torus reparametrizations (*similarity*). For an initial member outside $\mathcal{Y}$, the inverse image of $\mathcal{Y}$ under the similarity family is empty on the nondegenerate domain; the section relates the minimal number of degenerate parameters in a rational presentation to the multiplicative rank of the scaling vector λ. Finally, Section 7 illustrates the theory by explicit computations of exceptional loci and deformation behavior in concrete toric models, including projective spaces and Hirzebruch surfaces; it also contains the sparse-resultant computation of the degree-240 divisorial exceptional locus for cubic surfaces in $\mathbb{P}^3$.

2 Kobayashi hyperbolicity and hyperbolic embeddings

Throughout this paper, we work over the field $\mathbb{C}$. We briefly recall the notions of the Kobayashi–Royden metric, Kobayashi hyperbolicity, and hyperbolic embedding, emphasizing their role in deformation problems on *noncompact* complex manifolds.

2.1 Kobayashi–Royden metric and Kobayashi distance

Let X be a complex manifold. The *Kobayashi–Royden (pseudo-)metric* $KR_X(p;v)$ at a point $p \in X$ in the direction $v \in T_pX$ is defined by

$$KR_X(p;v) = \inf\left\{\frac{1}{\alpha} \mid \alpha > 0,\ f \in \mathrm{Hol}(\mathbb{D}, X),\ f(0) = p,\ f'(0) = \alpha v\right\},$$

where $\mathbb{D} \subset \mathbb{C}$ denotes the unit disk. The pseudometric measures derivatives of holomorphic disks centered at p and tangent to v; these maps need not be injective (see, for example, [3]). In general, KR_X is only a pseudometric: it may vanish for nonzero tangent vectors.

The associated *Kobayashi pseudodistance* d_X on X is defined by

$$d_X(x,y) = \inf\left\{\sum_{i=1}^{n} \rho_{\mathbb{D}}(a_i, b_i)\right\},$$

where $x = p_0, \ldots, p_n = y$, and for each i there exists a holomorphic map $f_i \in \mathrm{Hol}(\mathbb{D}, X)$ such that $f_i(a_i) = p_{i-1}$ and $f_i(b_i) = p_i$. Here $\rho_{\mathbb{D}}$ denotes the Poincaré distance on $\mathbb{D}$. It is known that d_X coincides with the length distance induced by the Kobayashi–Royden metric [10, Theorem 3.1.15].

2.2 Kobayashi hyperbolicity

A complex manifold X is said to be *Kobayashi hyperbolic* if the pseudodistance d_X is a genuine distance, i.e., separates points. Kobayashi hyperbolicity implies that X admits no nonconstant entire holomorphic curve $f\colon \mathbb{C} \to X$. The converse is the Brody criterion when X is compact, but it is false for arbitrary noncompact complex manifolds. We therefore use the term *Brody hyperbolic* for the absence of nonconstant entire curves and keep it distinct from Kobayashi hyperbolicity in the noncompact setting.

For compact complex manifolds, Kobayashi hyperbolicity is open under deformation. For general deformations, the infinitesimal Kobayashi–Royden pseudometric is upper semicontinuous [12]; see also [8] in the compact hyperbolic case. This statement alone does not ensure deformation openness of hyperbolicity for noncompact fibers.

Zaidenberg's pencil of quintics gives an explicit example in which a small deformation of five lines in general position destroys hyperbolicity of the complement [14, Proposition 3.2 and Example 3.2]. Consequently, in noncompact geometry one must distinguish between intrinsic hyperbolicity and the behavior of the manifold inside a compact ambient space.

2.3 Hyperbolic embeddings

Let X be a complex manifold and $Y \subset X$ a complex submanifold, with closure $\overline{Y} \subset X$. Following [10, Section 3.3], we say that Y is *Kobayashi hyperbolically embedded* in X if for any two distinct points $p, q \in \overline{Y}$ there exist neighborhoods U_p of p and U_q of q such that

$$d_Y(U_p \cap Y, U_q \cap Y) > 0.$$

The same neighborhood criterion defines hyperbolic embedding when the ambient space X is a possibly singular complex space, as in the toric applications below.

If $Y \hookrightarrow X$ is a hyperbolic embedding, then Y itself is Kobayashi hyperbolic. The converse, however, is false in general [10, Example 3.3.10]. From the perspective of deformation theory, hyperbolic embedding is the correct notion: it controls not only entire curves inside Y, but also their possible degenerations toward the boundary $\overline{Y} \setminus Y$; see also [13].

Our guiding principle is the following. In this paper, we study deformation problems for complements $T_N \setminus D$ of divisors in toric varieties. Our central philosophy is that *hyperbolic embedding* provides a robust replacement for Kobayashi hyperbolicity in noncompact settings, allowing effective control under deformation.

3 Toric geometry and hyperbolic embedding

In this section, we fix notation and record the toric-geometric framework used throughout the paper. Our purpose is to make precise how combinatorial data of lattice polytopes control the behavior of entire curves in complements of divisors. In particular, we recall the hyperplane-projection formalism underlying Tiba's generic hyperbolicity criterion, which will serve as a baseline for the deformation-theoretic refinements developed in later sections.

Let $X = X_\Sigma$ be a projective toric variety of dimension $r \geq 1$, and let $T_N \cong (\mathbb{C}^*)^r$ denote its dense algebraic torus. Let $M \cong \mathbb{Z}^r$, $N \cong \mathbb{Z}^r$ be the character lattice and the lattice of one-parameter subgroups, respectively, equipped with the canonical pairing $\langle \cdot, \cdot \rangle \colon M \times N \to \mathbb{Z}$. Set $M_\mathbb{R} := M \otimes_\mathbb{Z} \mathbb{R}$, $N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R}$.

Polytopes and divisors. As fixed in the introduction, D is a torus-invariant ample Cartier divisor on X. Write $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$, where $\Sigma(1)$ denotes the set of rays of Σ and D_ρ is the corresponding invariant prime divisor. Let $m_\rho \in N$ be the primitive generator of ρ. The ampleness and Cartier assumptions imply that D determines a full-dimensional lattice polytope

$$P_D = \bigcap_{\rho \in \Sigma(1)} \{u \in M_\mathbb{R} \mid \langle u, m_\rho \rangle \geq -a_\rho\} \subset M_\mathbb{R},$$

whose normal fan coincides with Σ; see [7, Chapter 2].

Via the Cox construction [4, Section 5.1], we may realize

$$X \cong (\operatorname{Spec} \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)] \setminus Z(\Sigma)) /\!\!/ G(\Sigma),$$

where $Z(\Sigma)$ is the exceptional set and $G(\Sigma) = \operatorname{Hom}_\mathbb{Z}(\operatorname{Cl}(X), \mathbb{C}^*)$.

Each lattice point $I \in P_D \cap M$ corresponds to a monomial

$$x^{\langle I, P_D \rangle} = \prod_{\rho \in \Sigma(1)} x_\rho^{\langle I, m_\rho \rangle + a_\rho},$$

whose divisor is linearly equivalent to D.

Finite supports and linear systems. Let $S \subset P_D \cap M$ be a finite set of lattice points. For a nonzero vector $a = (a_I)_{I \in S}$, write

$$s_a := \sum_{I \in S} a_I x^{\langle I, P_D \rangle}, \qquad D_X(a) := \operatorname{div}_X(s_a), \qquad D_T(a) := D_X(a)|_{T_N}.$$

We denote the effective Weil-divisor closure of $D_T(a)$ in X by $D_{\mathrm{cl}}(a) := \overline{D_T(a)}^X$, retaining the multiplicities of its components. The difference $D_X(a) - D_{\mathrm{cl}}(a)$ consists precisely of the torus-invariant components of $D_X(a)$. We also write $D(a) = D_X(a)$ when referring to a member of the coefficient linear system. We define the projective linear system

$$\mathfrak{d}_S^* = \left\{ \operatorname{div}\left(\sum_{I \in S} a_I x^{\langle I, P_D \rangle} \right) \mid (a_I)_{I \in S} \in \mathbb{P}^{|S|-1} \right\}.$$

Inside $\mathfrak{d}_S^*$, let

$$\mathfrak{d}_S := \{D(a) \in \mathfrak{d}_S^* \mid (a_I)_{I\in\tau\cap S} \neq 0 \text{ for every facet } \tau \subset P_D\}.$$

Thus $\mathfrak{d}_S$ consists precisely of the members having no torus-invariant irreducible component. Equivalently, $D_X(a) = D_{\mathrm{cl}}(a)$ for these members. Restricting a member of $\mathfrak{d}_S$ to T_N yields the Laurent polynomial

$$\sum_{I\in S} a_I z^I, \qquad z^I = z_1^{i_1}\cdots z_r^{i_r},$$

whose zero divisor determines the member on the dense torus. In particular, $\mathfrak{d}_S$ is nonempty if and only if every facet τ of P_D satisfies $\tau \cap S \neq \varnothing$. We abbreviate $\mathfrak{d}_D := \mathfrak{d}_{P_D\cap M}$. For the global facewise constructions we require $\mathfrak{d}_S \neq \varnothing$. This requirement must be stated separately when $r = 1$, since facets are then zero-dimensional and are not covered by the positive-dimensional face conditions below.

Hyperplane projections. The Zariski closure of an entire curve in $T_N \setminus D$ is a translate of an algebraic subtorus contained in that complement. Such subtori correspond to quotients of the character lattice M. Combinatorially, such degenerations arise when the support S collapses under projection along rational directions.

For a finite set $A \subset M_{\mathbb{R}}$ we define

$$\mathcal{L}_A := \{a - b \mid a, b \in A\}, \qquad V_A := \mathrm{Span}_{\mathbb{R}}(\mathcal{L}_A),$$

and let

$$\mathcal{H}_A := \{H \subset V_A \mid H \text{ is a codimension-one } \mathbb{R}\text{–subspace generated by elements of } \mathcal{L}_A\}.$$

For $H \in \mathcal{H}_A$, we denote by $\phi_H \colon V_A \longrightarrow V_A/H \simeq \mathbb{R}$ the canonical projection. When $A = \tau \cap S$ for a face τ of P_D, we write $\mathcal{H}_{\tau\cap S}$ accordingly. Throughout, a face is allowed to be the whole polytope, so every facewise hypothesis below includes $\tau = P_D$. For a positive-dimensional face τ, we use the saturated character lattice $M_\tau := M \cap \mathrm{span}_{\mathbb{R}}(\tau - \tau)$ of its torus orbit; translating $\tau\cap S$ by any of its lattice points places the support in M_τ. Projection cardinalities are independent of this translation. For any nonempty finite support A and $x \in A$, we use $|\phi_H(A)|$ only as shorthand for $|\phi_H(A - x)|$. This convention applies throughout, including the lemmas, tables, and examples. Whenever an expression $|\phi_H(A - x)|$ is used without a specified base point, x denotes an arbitrary point of A.

Tiba's hyperbolicity criterion. With the above notation, Tiba's generic hyperbolicity theorem can be stated as follows.

Theorem 3.1 (Tiba [11])**.** *Let $S \subset P_D \cap M$ be finite with $\mathfrak{d}_S \neq \varnothing$. Assume that for every positive-dimensional face τ of P_D:*

(1) *$\tau \cap S \neq \varnothing$, and the convex hull of $\tau \cap S$ has the same dimension as τ;*

(2) *for each hyperplane $H \in \mathcal{H}_{\tau\cap S}$ there exists $x \in \tau \cap S$ such that*

$$|\phi_H(\tau \cap S - x)| \geq \dim \tau + 1.$$

Then, for a general divisor $D \in \mathfrak{d}_S$, the complement $T_N \setminus \mathrm{Supp}\, D$ is Kobayashi hyperbolically embedded in X.

Remark 3.2. Theorem 3.1 is the generic existence result used below. We retain Tiba's incidence method and record its coefficient conditions in a form suited to one-coefficient deformations and the calculations in Section 7.

4 Effective proof of Tiba's hyperbolic embedding theorem

4.1 Position of Tiba's theorem in this paper

We recall that the central generic hyperbolicity statement used throughout this paper is due to Tiba and has already been stated as Theorem 3.1. That theorem provides a combinatorial criterion on a finite support $S \subset P_D \cap M$ ensuring that, for a *general* divisor $D \in \mathfrak{d}_S$, the complement $T_N \setminus \operatorname{Supp} D$ is Kobayashi hyperbolically embedded in the ambient projective toric variety X.

The purpose of the present section is to give an effective and coefficient-wise realization of Theorem 3.1. More precisely, we reformulate Tiba's argument in a way that

- isolates an explicit constructible bad locus and a proper Zariski closed exceptional locus $\mathcal{Y}_S \subset \mathfrak{d}_S^*$;
- characterizes membership in the constructible bad loci via translated subtori in torus strata;
- extracts a standalone "no entire curves" criterion on T_N that will be used later in the proof of Theorem 1.2.

No new generic hyperbolicity statement is claimed here; rather, this section should be read as an effective proof and structural refinement of Theorem 3.1.

4.2 Setup and strategy

Fix (X, T_N, D, P_D) as in Section 3. Let $S \subset P_D \cap M$ be a finite support of full affine dimension with $\mathfrak{d}_S \neq \varnothing$, and use the parameter spaces $\mathfrak{d}_S \subset \mathfrak{d}_S^*$ defined in Section 3. The complete support $S = P_D \cap M$ will be used for the global locus $\mathcal{Y} \subset |D|$.

For a nonzero coefficient vector a, the chosen toric trivialization on T_N represents s_a by

$$F_a(z) := \sum_{I \in S} a_I z^I, \qquad z \in T_N.$$

Its zero divisor is $D_T(a)$, and its closure is $D_{\mathrm{cl}}(a)$, as defined in Section 3. In the local Laurent-polynomial arguments of this section, we abbreviate $F = F_a$ and $D_0 = D_T(a)$; this local notation does not denote a divisor on all of X. In the facewise argument, we use $D_X(a)$ explicitly.

Set

$$r := \dim_{\mathbb{R}} \operatorname{span}_{\mathbb{R}}(S - S) = \dim \operatorname{Conv}(S).$$

All subtori appearing in the argument have dimension at most r.

The proof proceeds in four steps:

(1) reduction to one-dimensional subtori;

(2) construction of explicit exceptional loci in coefficient space;

(3) equivalence between bad coefficients and translated subtori;

(4) exclusion of entire curves via the theorem of Dethloff–Lu.

4.3 Reduction to one-dimensional subtori

Lemma 4.1. *Let $Z \subset T_N$ be any subset. Then Z contains a translate of a positive-dimensional subtorus if and only if it contains a translate of a one-dimensional subtorus. In particular, the same equivalence holds for $Z = \operatorname{Supp} D_0$ and $Z = T_N \setminus \operatorname{Supp} D_0$.*

Proof. If Z contains a translate of a positive-dimensional subtorus, then it contains the translate of any one-dimensional subtorus inside it. Conversely, a one-dimensional subtorus is itself positive-dimensional, so its translate already has the required form. ∎

4.4 Exceptional loci in coefficient space

Fix $S \subset P_D \cap M$ of full dimension r. For each $H \in \mathcal{H}_S$, choose a splitting

$$M \cong (H \cap M) \oplus \big(M/(H \cap M)\big), \qquad M/(H \cap M) \cong \mathbb{Z}.$$

With respect to the induced coordinates $z \leftrightarrow (u_1, \dots, u_{r-1}, u_r)$ on T_N, we may rewrite

$$F(z) = \sum_{i=1}^{\ell_H} Q_{H,i}(u_1, \dots, u_{r-1}) u_r^{d_i},$$

where $\ell_H := |\phi_H(S)|$ and each $Q_{H,i}$ is linear in the coefficients $(a_I)_{I \in S}$.

Definition 4.2 (bad and exceptional loci for (S, H))**.** For $1 \le j \le \ell_H$, define $B_{S,H,j} \subset \mathbb{P}^{|S|-1}$ to be the set of projective coefficient classes $[a]$, represented by $a = (a_I)_{I \in S}$, for which there exists $c \in (\mathbb{C}^*)^{r-1}$ such that $Q_{H,k}(c) = 0$ for all $k \neq j$. Set

$$B_{S,H} := \bigcup_{j=1}^{\ell_H} B_{S,H,j}, \qquad B_S := \bigcup_{H \in \mathcal{H}_S} B_{S,H}.$$

We call B_S the *bad locus* and define the exceptional locus to be its reduced Zariski closure

$$Y_S := \big(\overline{B_S}^{\mathrm{Zar}}\big)_{\mathrm{red}} \subset \mathbb{P}^{|S|-1}.$$

Lemma 4.3. *Assume that $\ell_H = |\phi_H(S)| \ge r + 1$ for every $H \in \mathcal{H}_S$. The bad locus B_S is constructible, and Y_S is a proper Zariski closed reduced subscheme of $\mathbb{P}^{|S|-1}$ of codimension at least 1.*

Proof. Fix H and j, and let

$$Z_{S,H,j} := \{(c, a) \in (\mathbb{C}^*)^{r-1} \times \mathbb{P}^{|S|-1} \mid Q_{H,k}(c) = 0 \text{ for } k \neq j\}.$$

This is closed in $(\mathbb{C}^*)^{r-1} \times \mathbb{P}^{|S|-1}$. For a fixed c, the equations $Q_{H,k}(c) = 0$, $k \neq j$, are $\ell_H - 1$ independent linear equations in the coefficients: their coefficient supports are nonempty and pairwise disjoint, and every monomial evaluated at $c \in (\mathbb{C}^*)^{r-1}$ is nonzero. Hence

$$\dim Z_{S,H,j} \le (r-1) + (|S| - \ell_H) = |S| - \ell_H + r - 1.$$

The projection to coefficient space is not proper, so its image need not be closed. Chevalley's theorem does, however, show that $B_{S,H,j} = \mathrm{pr}_2(Z_{S,H,j})$ is constructible, and the same dimension estimate holds for its Zariski closure. Since $\ell_H \ge r+1$, that closure has dimension at most $|S| - 2$, whereas $\dim \mathbb{P}^{|S|-1} = |S| - 1$. Taking the finite union over H and j proves the assertion for $Y_S = \big(\overline{B_S}^{\mathrm{Zar}}\big)_{\mathrm{red}}$. ∎

The translated-subtorus characterization in Lemma 4.4 concerns B_S, not its closure Y_S. We make no general assertion that $B_S = Y_S$, or that membership in Y_S characterizes failure of hyperbolicity of the complement. A translated subtorus in the divisor itself is different from an entire curve in its complement.

4.5 Translated subtori and exceptional coefficients

Lemma 4.4. *Let $(a_I)_{I\in S} \in \mathbb{P}^{|S|-1}$ and $D_0 = \operatorname{div}(\sum_{I\in S} a_I z^I)$. Then $(a_I) \in B_S$ if and only if either* $\operatorname{Supp} D_0$ *or* $T_N \setminus \operatorname{Supp} D_0$ *contains a translate of a one-dimensional subtorus of T_N.*

Proof. Suppose first that $a \in B_{S,H,j}$, and choose c as in Definition 4.2. On the translated one-dimensional subtorus obtained by fixing $(u_1, \dots, u_{r-1}) = c$, the restriction of F is

$$F(c, u_r) = Q_{H,j}(c) u_r^{d_j}.$$

If $Q_{H,j}(c) = 0$, this subtorus lies in $\operatorname{Supp} D_0$; otherwise, it is disjoint from $\operatorname{Supp} D_0$.

Conversely, write a translated one-dimensional subtorus as $c\lambda_v(\mathbb{C}^*)$, where $v \in N$ is primitive. Grouping the exponents according to $m = \langle I, v\rangle$ gives

$$F(c\lambda_v(t)) = \sum_m \Big(\sum_{\langle I,v\rangle = m} a_I c^I \Big) t^m.$$

If the translate is contained in $\operatorname{Supp} D_0$, every displayed coefficient vanishes. If it is contained in the complement, this Laurent polynomial has no zero in $\mathbb{C}^*$ and is therefore a single monomial. Thus in either case at most one layer coefficient is nonzero.

For completeness, the standard finite-hyperplane reduction is as follows. Put $S_m = \{I \in S \mid \langle I, v\rangle = m\}$ and $b_m = \sum_{I\in S_m} a_I c^I$. Let $W \subset v^\perp$ be the span of the differences within the sets S_m. Since $S - S$ spans $M_{\mathbb{R}}$, a basis of W can be extended by differences from $S - S$ to a hyperplane $H \in \mathcal{H}_S$. Each H-layer L_d is a union of sets S_m, so evaluation at the fixed base point c gives

$$\sum_{I\in L_d} a_I c^I = \sum_{S_m \subset L_d} b_m.$$

At most one of these sums is nonzero. In the coordinates adapted to H, write $c = (c_H, c_r)$. The left-hand side is $Q_{H,d}(c_H) c_r^d$, where $Q_{H,d}$ denotes $Q_{H,i}$ for the unique index i satisfying $d_i = d$. Since $c_r \neq 0$, at most one specialized H-layer polynomial is nonzero. Therefore, $a \in B_{S,H,j} \subset B_S$ for a suitable j. This is the hyperplane-reduction argument used in [11, Lemma 2]. ∎

4.6 The Dethloff–Lu theorem and a "no entire curves" criterion on T_N

Theorem 4.5 (Dethloff–Lu [6, Theorem 5.1])**.** *Let G be a semi-abelian variety and let D be a reduced algebraic divisor on G. If $f\colon \mathbb{C} \to G$ is holomorphic, then the Zariski closure $X(f)$ of its image in G is a translate of an algebraic subgroup of G. If in addition $f(\mathbb{C}) \subset G \setminus D$, then $X(f) \cap D = \varnothing$.*

For an algebraic torus, the same degeneracy statement can also be obtained from the classical Borel lemma. The semi-abelian formulation above applies directly to the setting needed here.

The bad-locus construction and Lemma 4.4 depend only on a finite support of full affine dimension in the character lattice. The following local statements therefore require neither a nonempty global coefficient chart nor conditions on proper faces. They apply also in the affine lattice of any face after translating the support by a lattice point.

Lemma 4.6. *Let $S \subset M$ be finite of full affine dimension r, and let $D_0 = \operatorname{div}(\sum_{I\in S} a_I z^I) \subset T_N$. If $(a_I) \notin B_S$, then there exists no nonconstant holomorphic map from $\mathbb{C}$ to either* $\operatorname{Supp} D_0$ *or* $T_N \setminus \operatorname{Supp} D_0$.

Proof. Suppose first that a nonconstant $f\colon \mathbb{C} \to \operatorname{Supp} D_0$ exists. By Theorem 4.5, the Zariski closure of $f(\mathbb{C})$ is a translate of a positive-dimensional algebraic subtorus contained in $\operatorname{Supp} D_0$. If instead $f\colon \mathbb{C} \to T_N \setminus \operatorname{Supp} D_0$ is nonconstant, the same theorem shows that its Zariski closure is a translate of a positive-dimensional algebraic subtorus disjoint from $\operatorname{Supp} D_0$. In either case, Lemma 4.1 produces a translated one-dimensional subtorus in the corresponding set, and Lemma 4.4 gives $(a_I) \in B_S$, a contradiction. ∎

Corollary 4.7 (no entire curves in the torus complement)**.** *Let $S \subset M$ be finite of full affine dimension r. Let $D_0 = \operatorname{div}(\sum_{I\in S} a_I z^I) \subset T_N$. If $(a_I) \notin B_S$, then $T_N \setminus \operatorname{Supp} D_0$ is Brody hyperbolic, i.e., there is no nonconstant holomorphic map $\mathbb{C} \to T_N \setminus \operatorname{Supp} D_0$.*

Proof. This is the complement case of Lemma 4.6. ∎

4.7 Completion of the proof of Tiba's theorem

Lemma 4.8. *Assume the hypotheses of Theorem 3.1, and let $D_X(a) \in \mathfrak{d}_S$. For every positive-dimensional face τ, put $S_\tau = \tau \cap S$ and let $\rho_\tau(a) = (a_I)_{I\in S_\tau}$ be the restricted coefficient vector. If, for every such face, $\rho_\tau(a) \neq 0$, $[\rho_\tau(a)] \notin B_{S_\tau}$, then $T_N \setminus \operatorname{Supp} D_T(a)$ is Kobayashi hyperbolically embedded in X.*

Proof. We want to verify Tiba's boundary criterion using $D_{\mathrm{cl}}(a)$, the closure of the torus divisor. Since $D_X(a) \in \mathfrak{d}_S$, this closure equals $D_X(a)$.

Fix a positive-dimensional face τ and let $T_\tau \subset X$ be the corresponding torus orbit. Choose $I_\tau \in S_\tau$. The monomial section indexed by I_τ is nonvanishing on T_τ and trivializes $\mathcal{O}_X(D)|_{T_\tau}$. In this trivialization the restricted section is represented by

$$F_{\tau,a}(z) = \sum_{I\in S_\tau} a_I z^{I-I_\tau}, \qquad I - I_\tau \in M_\tau.$$

The hypothesis $\rho_\tau(a) \neq 0$ gives $F_{\tau,a} \not\equiv 0$. Lemma 4.6, applied in the character lattice M_τ, excludes nonconstant entire curves both in its zero set and in its complement in T_τ. The full face $\tau = P_D$ treats the dense torus. Zero-dimensional orbits are points, and therefore need no further condition. Hence all the stratum conditions of [11, Lemma 3] hold for the reduced support of $D_T(a)$ and its closure. The conclusion of [11, Lemma 3] is the required hyperbolic embedding. ∎

Proof of Theorem 3.1. For every positive-dimensional face τ, put $S_\tau = \tau \cap S$ and let $\rho_\tau\colon \mathbb{C}^S \to \mathbb{C}^{S_\tau}$ be coordinate restriction. If $\widehat{Y}_{S_\tau} \subset \mathbb{C}^{S_\tau}$ denotes the affine cone over Y_{S_τ}, including its vertex, define

$$\mathcal{Y}_S := (\mathfrak{d}_S^* \setminus \mathfrak{d}_S) \cup \bigcup_{\dim \tau > 0} \mathbb{P}(\rho_\tau^{-1}(\widehat{Y}_{S_\tau})) \subset \mathfrak{d}_S^*.$$

Each term is closed and proper. Since only finitely many faces occur, $\mathcal{Y}_S$ is a proper Zariski closed subset of $\mathfrak{d}_S^*$. For every $D_X(a) \notin \mathcal{Y}_S$, we have $D_X(a) \in \mathfrak{d}_S$. Moreover, every positive-dimensional face vector $\rho_\tau(a)$ is nonzero: otherwise it would lie at the vertex of the affine cone $\widehat{Y}_{S_\tau}$ and the coefficient class would belong to $\mathcal{Y}_S$. Its projective class lies outside Y_{S_τ} and hence outside B_{S_τ}. Lemma 4.8 therefore proves the generic assertion of Theorem 3.1. For the complete support $S = P_D \cap M$, we henceforth write $\mathcal{Y} := \mathcal{Y}_{P_D\cap M} \subset |D|$. ∎

5 Stability under one-parameter deformations

In Section 4, we gave an effective proof of Tiba's hyperbolic embedding theorem (Theorem 3.1) by constructing a proper Zariski closed exceptional locus $\mathcal{Y} \subset |D|$ in the coefficient parameter space. As a consequence, for every divisor $D_0 \in \mathfrak{d}_D \setminus \mathcal{Y}$, the natural inclusion $T_N \setminus \mathrm{Supp}(D_0 \cap T_N) \hookrightarrow X$ is a Kobayashi hyperbolic embedding.

The purpose of the present section is twofold. First, we establish a general deformation-stability statement: along an algebraic one-parameter family whose image is not contained in $\mathcal{Y}$, hyperbolic embedding holds outside a finite set of parameters (see Theorem 1.1). Second, under a stronger lattice hypothesis, we prove an exception-free statement for a distinguished class of affine one-parameter families, yielding Theorem 1.2. The latter requires a refined analysis of borderline degenerations, which motivates the introduction of an enlarged exceptional locus.

5.1 Need for a larger exceptional locus

The constructible bad locus underlying $\mathcal{Y}$ detects degenerations in which, along some hyperplane direction, the defining Laurent polynomial has *at most one nonzero layer* after specializing transverse variables. If no layer remains, the corresponding translated subtorus lies in the divisor; if one remains, it lies in the complement. The exponential parametrization of any one-dimensional subtorus then gives nonconstant entire curves. Theorem 4.5 is used in the opposite direction, to pass from an entire curve to an algebraic-subgroup translate.

For one-monomial deformations of the form

$$D_t = \mathrm{div}\Big(\sum_{I \in S \setminus \{I_0\}} a_I z^I + t z^{I_0} \Big), \qquad t \in \mathbb{C},$$

a specialization with exactly two nonzero layers need not belong to the ordinary bad incidence locus for that direction. However, changing one coefficient may eliminate one of these two layers. To guarantee avoidance of the ordinary bad locus throughout the deformation, we require at least three nonzero layers initially. The enlarged locus below records the failure of this stronger initial condition.

5.2 Definition of the enlarged exceptional locus

Fix $S \subset P_D \cap M$ satisfying the hypotheses of Theorem 1.2. For each $H \in \mathcal{H}_S$, recall the decomposition

$$\sum_{I \in S} a_I z^I = \sum_{i=1}^{\ell_H} Q_{H,i}(u_1, \dots, u_{r-1}) u_r^{d_i},$$

where $\ell_H = |\phi_H(S - x)|$ and each $Q_{H,i}$ is a Laurent polynomial in $(u_1, \dots, u_{r-1})$.

Definition 5.1 (enlarged bad and exceptional loci)**.** For each $H \in \mathcal{H}_S$ and each pair of distinct indices $1 \le j_1 < j_2 \le \ell_H$, let $B'_{S,H,j_1,j_2} \subset \mathbb{P}^{|S|-1}$ be the set of projective coefficient classes $[a]$, represented by $a = (a_I)_{I \in S}$, such that there exists $(c_1, \dots, c_{r-1}) \in (\mathbb{C}^*)^{r-1}$ satisfying

$$Q_{H,i}(c_1, \dots, c_{r-1}) = 0 \qquad \text{for all} \quad i \neq j_1, j_2.$$

Define

$$B'_{S,H} := \bigcup_{j_1 < j_2} B'_{S,H,j_1,j_2}, \qquad B'_S := \bigcup_{H \in \mathcal{H}_S} B'_{S,H}, \qquad Y'_S := (\overline{B'_S}^{\mathrm{Zar}})_{\mathrm{red}}.$$

By construction, $(a_I)_{I \in S} \in B'_S$ if and only if, for some hyperplane direction, the defining polynomial degenerates to one with at most two nonzero layers after specialization of transverse variables. In particular, $B_S \subset B'_S$ and $Y_S \subset Y'_S$.

5.3 Algebraicity and codimension of Y'_S

Lemma 5.2. *The set B'_S is constructible, and $Y'_S \subset \mathbb{P}^{|S|-1}$ is a proper Zariski closed reduced subscheme of codimension at least 1.*

Proof. Fix $H \in \mathcal{H}_S$ and indices $j_1 < j_2$. Consider the closed subscheme

$$Z'_{S,H,j_1,j_2} \subset (\mathbb{C}^*)^{r-1} \times \mathbb{P}^{|S|-1}$$

defined by $Q_{H,i}(u_1, \dots, u_{r-1}) = 0$ for all $i \neq j_1, j_2$. For fixed $(u_1, \dots, u_{r-1})$, the fiber is cut out by $\ell_H - 2$ independent linear equations in $\mathbb{P}^{|S|-1}$, hence has dimension $|S|-1-(\ell_H-2) = |S|-\ell_H+1$. Since $\ell_H \geq r+2$ under the hypotheses of Theorem 1.2, we obtain

$$\dim Z'_{S,H,j_1,j_2} \leq (r-1) + (|S| - \ell_H + 1) \leq |S| - 2.$$

The projection to $\mathbb{P}^{|S|-1}$ is not proper, so we do not claim that its image is closed. Chevalley's theorem shows that the image B'_{S,H,j_1,j_2} is constructible, while the displayed estimate gives $\dim \overline{B'_{S,H,j_1,j_2}}^{\mathrm{Zar}} \leq |S| - 2$. Taking the finite union over H, j_1, j_2 proves the assertion for its reduced Zariski closure Y'_S. ∎

Definition 5.3 (facewise enlarged exceptional locus)**.** For every positive-dimensional face τ put $S_\tau = \tau \cap S$, let $\rho_\tau \colon \mathbb{C}^S \to \mathbb{C}^{S_\tau}$ be coordinate restriction, and let $\widehat{Y}'_{S_\tau}$ be the affine cone over Y'_{S_τ}. We set

$$\mathcal{Y}'_S := (\mathfrak{d}_S^* \setminus \mathfrak{d}_S) \cup \bigcup_{\dim \tau > 0} \mathbb{P}(\rho_\tau^{-1}(\widehat{Y}'_{S_\tau})) \subset \mathfrak{d}_S^*.$$

Under the hypotheses of Theorem 1.2, this is a proper Zariski closed subset by Lemma 5.2.

5.4 Proof of Theorem 1.1

Proof of Theorem 1.1. Let $D_0 \in \mathfrak{d}_D \setminus \mathcal{Y}$ and $\gamma \colon \mathbb{P}^1 \to |D|$ be a morphism with $\gamma(0) = D_0$. Set $D_t := \gamma(t)$.

The inverse image $\gamma^{-1}(\mathcal{Y})$ is Zariski closed in $\mathbb{P}^1$. It is proper because $0 \notin \gamma^{-1}(\mathcal{Y})$, and every proper closed subset of $\mathbb{P}^1$ is finite. This argument also covers the case of a constant family, when the inverse image is empty.

For all but finitely many $t \in \mathbb{P}^1$ we therefore have $D_t \in \mathfrak{d}_D \setminus \mathcal{Y}$, and the hyperbolic embedding $T_N \setminus \mathrm{Supp}(D_t \cap T_N) \hookrightarrow X$ follows from Lemma 4.8. ∎

Extension to algebraic curves. We prove that the exceptional parameter set is finite for an irreducible algebraic curve base. Let C be an irreducible complex algebraic curve and $\gamma \colon C \to |D|$ a morphism with $\gamma(c_0) \notin \mathcal{Y}$ for some $c_0 \in C$. Then $E_C := \gamma^{-1}(\mathcal{Y}) \subsetneq C$ is Zariski closed. Since C is a finite-type curve, $\dim E_C \leq 0$ and E_C is finite. For $c \in C \setminus E_C$, Lemma 4.8 gives hyperbolic embedding. Neither completeness nor smoothness of C is needed. For a reducible curve, apply this argument separately to each irreducible component whose image is not contained in $\mathcal{Y}$. No conclusion is asserted on a component mapped entirely into $\mathcal{Y}$.

5.5 Proof of Theorem 1.2

Proof. Let $a = (a_I)_{I \in S}$ satisfy $[a] \notin \mathcal{Y}'_S$ and fix $I_0 \in S$, as in the statement. Consider

$$F_t(z) = t z^{I_0} + \sum_{I \neq I_0} a_I z^I, \qquad t \in \mathbb{C}.$$

By Definition 5.3, every original face vector $\rho_\tau(a)$ is nonzero and its projective class avoids Y'_{S_τ} and hence B'_{S_τ}. Indeed, the affine cones in that definition include their vertices. Fix a positive-dimensional face τ and write $S_\tau = \tau \cap S$. If $I_0 \notin S_\tau$, the face polynomial is independent of t and its coefficient vector lies outside B'_{S_τ}, hence outside B_{S_τ}. Suppose that $I_0 \in S_\tau$. For any $H \in \mathcal{H}_{S_\tau}$ and any specialization of the transverse variables, the assumption $a|_{S_\tau} \notin B'_{S_\tau}$ says that at least three layer polynomials are nonzero at the original coefficient vector. Varying the coefficient of z^{I_0} changes only the single H-layer containing I_0. At least two of the previously nonzero layers are therefore unchanged and remain nonzero for every $t \in \mathbb{C}$. In particular, the specialized polynomial never collapses to at most one nonzero layer, so the coefficient vector of $F_t|_{S_\tau}$ lies outside B_{S_τ} for every t.

We first prove the conclusion of Theorem 1.2 for $r \geq 2$. Every facet then has positive dimension. On every positive-dimensional face the layer argument leaves at least two nonzero layers after every specialization, so the face coefficient vector is nonzero and its projective class avoids the ordinary bad locus. In particular, every facet restriction is nonzero, which means that $D_t = D_X(a(t)) \in \mathfrak{d}_S$ for all t. Lemma 4.8 therefore gives the asserted hyperbolic embedding for all $t \in \mathbb{C}$ when $r \geq 2$.

We now prove the rank-one case of Theorem 1.2 directly. For $r = 1$, the toric curve X is $\mathbb{P}^1$. There are no transverse variables, and avoidance of B'_S means that $F_{a_{I_0}}$ has at least three nonzero terms. Changing one coefficient leaves at least two nonzero terms in F_t for every t. After factoring out its lowest Laurent monomial, write $F_t(z) = z^m P_t(z)$, where $P_t(0) \neq 0$ and $\deg P_t \geq 1$. Thus the zero set $Z_t \subset \mathbb{C}^*$ is finite and nonempty. The complement is

$$T_N \setminus \operatorname{Supp} D_T(a(t)) = \mathbb{P}^1 \setminus (\{0, \infty\} \cup Z_t).$$

A sphere punctured at at least three distinct points is hyperbolically embedded in $\mathbb{P}^1$; see, for example, [13, Introduction, Paragraph 3]. If an endpoint coefficient is deleted at $t = 0$, the global divisor $D_X(a(0))$ can acquire a torus-invariant point and leave $\mathfrak{d}_S$. The inclusion $T_N \setminus \operatorname{Supp} D_T(a(0)) \hookrightarrow X$ still depends only on the torus restriction and its closure, so the punctured-sphere argument proves the conclusion at $t = 0$. ∎

Remark 5.4 (positioning of the results)**.** Under the facewise lattice hypothesis, Theorem 1.1 establishes deformation stability along projective one-parameter families whose images are not contained in $\mathcal{Y}$: hyperbolic embedding holds outside a finite set of parameters. Theorem 1.2 exploits a stronger combinatorial hypothesis to obtain an exception-free statement for a distinguished affine one-parameter family. The enlarged bad locus B'_S is tailored to isolate the borderline two-term degenerations relevant to this refinement, and Corollary 4.7 is the local "no entire curves" criterion that makes the argument uniform along the affine family.

6 Similarity deformations and degeneracy parameters

In Theorem 1.1, we proved that for any one-parameter deformation $\gamma\colon \mathbb{P}^1 \to |D|$ with $\gamma(0) = D_0 \in \mathfrak{d}_D \setminus \mathcal{Y}$, the exceptional parameter set $E_{\mathcal{Y}}(\gamma) := \{t \in \mathbb{P}^1 \mid \gamma(t) \in \mathcal{Y}\}$ is finite. In general, $E_{\mathcal{Y}}(\gamma)$ need not be empty. If $E_{\text{fail}}(\gamma)$ denotes the parameters at which the dense torus complement is not hyperbolically embedded in X, our construction proves only $E_{\text{fail}}(\gamma) \subseteq E_{\mathcal{Y}}(\gamma)$. We do not identify $\mathcal{Y}$ with the locus of all nonhyperbolic complements, nor assert equality of these two parameter sets.

For deformations arising from *diagonal torus reparametrizations* ("similarity"), invariance will show that this intrinsic exceptional set is actually empty on the domain where the reparametrization is defined. The quantity to be minimized is therefore the degeneracy set of the rational scaling functions, and we relate its optimal size to a simple invariant of $\lambda \in (\mathbb{C}^*)^r$.

6.1 Similarity via the diagonal torus action

Let $\lambda = (\lambda_1, \dots, \lambda_r) \in (\mathbb{C}^*)^r$. The diagonal map $\Delta_\lambda \colon T_N \longrightarrow T_N$, $(z_1, \dots, z_r) \longmapsto (\lambda_1 z_1, \dots, \lambda_r z_r)$ is an automorphism of T_N.

Definition 6.1 (similar divisors and Laurent polynomials)**.** Let $D_p, D_q \in \mathfrak{d}_D$ be effective divisors. We call them *similar* if, for some $\lambda \in (\mathbb{C}^*)^r$, $D_q = \Delta_\lambda^* D_p$, where the torus action is extended to X. In particular, $\mathrm{Supp}(D_q \cap T_N) = \Delta_{\lambda^{-1}}(\mathrm{Supp}(D_p \cap T_N))$. Using the fixed toric trivialization of $\mathcal{O}_X(D)$ on T_N, the corresponding Laurent equations satisfy $q(z) = cp(\lambda z)$, $c \in \mathbb{C}^*$. Indeed, the corresponding global sections have the same divisor and hence differ by a nonzero constant. After rescaling q, we use the normalized expression $q(z) = p(\lambda z)$ throughout this section. Working with effective divisors retains multiplicities; for arbitrary equations of divisors on the torus alone, the ambiguity would instead be a Laurent unit cz^m.

If $p(z) = \sum_{I \in S} a_I z^I$, then similarity by λ gives

$$q(z) = p(\lambda z) = \sum_{I \in S} a_I \lambda^I z^I, \qquad \lambda^I := \lambda_1^{i_1} \cdots \lambda_r^{i_r}.$$

6.2 Similarity preserves membership in the exceptional locus

Recall that the exceptional locus $\mathcal{Y} \subset |D|$ constructed in the proof of Theorem 3.1 is defined by facewise conditions detecting translates of one-dimensional subtori contained either in $\mathrm{Supp}\, D_0$ or in its complement. More precisely, $\mathcal{Y}$ is the reduced Zariski closure of the facewise bad loci described by Lemma 4.4, together with the boundary of $\mathfrak{d}_D$.

Lemma 6.2 (similarity invariance of $\mathcal{Y}$)**.** *Let $D_p, D_q \in \mathfrak{d}_D$ be divisors defined on T_N by similar Laurent polynomials p and $q = p(\lambda z)$. Then $D_p \in \mathcal{Y} \iff D_q \in \mathcal{Y}$.*

Proof. A translate of a one-dimensional subtorus in T_N is carried by Δ_λ to a translate of a one-dimensional subtorus, and the same holds after restricting to any torus stratum. Hence every facewise bad locus B_{S_τ} is invariant. We verify that the torus action preserves every part of the global exceptional locus. Since the action is an algebraic automorphism, it preserves the reduced Zariski closures Y_{S_τ} and their inverse images under coefficient restriction. It multiplies each coefficient a_I by the nonzero number λ^I. Therefore, a restricted coefficient vector vanishes before the action if and only if it vanishes after the action. In particular, the coefficient-chart boundary $\mathfrak{d}_D^* \setminus \mathfrak{d}_D$ is invariant. All pieces in the construction of $\mathcal{Y}$ are consequently invariant, proving the claim. ∎

6.3 Canonical similarity paths and a "degeneracy set"

Assume p and $q = p(\lambda z)$ are similar. Consider the rational functions $\Lambda_i(s:t) := \frac{s+\lambda_i t}{s+t}$, $1 \le i \le r$, on $\mathbb{P}^1$, and define (on their common domain of definition)

$$p_{(s:t)}(z) := p(\Lambda_1(s:t) z_1, \dots, \Lambda_r(s:t) z_r).$$

Let $U \subset \mathbb{P}^1$ be the maximal open subset where all Λ_i are defined and take values in $\mathbb{C}^*$ (so that the diagonal scaling is non-degenerate). Then $p_{(s:t)}$ defines a morphism $\gamma \colon U \to \mathfrak{d}_D$ by $(s:t) \mapsto D_{(s:t)} := \mathrm{div}(p_{(s:t)})$.

We have $p_{(1:0)}(z) = p(z)$, $p_{(0:1)}(z) = p(\lambda z) = q(z)$, so γ connects D_p and D_q inside U.

On U, define the exceptional preimage by $E_{\mathcal{Y}}(\gamma) := \gamma^{-1}(\mathcal{Y})$. We reserve $E_{\mathrm{fail}}(\gamma)$ for the parameters in U where hyperbolic embedding actually fails. The degeneracy set of the chosen scaling functions $\Lambda = (\Lambda_1, \dots, \Lambda_r)$ is

$$E_{\deg}(\Lambda) := \mathbb{P}^1 \setminus U = \bigcup_{i=1}^{r} \Lambda_i^{-1}(\{0, \infty\}).$$

For the above canonical choice, one checks

$$E_{\deg}(\Lambda) \subseteq \{(1:-1)\} \cup \{(\lambda_i:-1) \mid 1 \le i \le r\}, \qquad \text{hence} \quad |E_{\deg}(\Lambda)| \le r+1.$$

Every $D_{(s:t)}$ with $(s:t) \in U$ is similar to D_p. Hence Lemma 6.2 gives the sharper dichotomy

$$D_p \notin \mathcal{Y} \Longrightarrow \gamma^{-1}(\mathcal{Y}) = \varnothing, \qquad D_p \in \mathcal{Y} \Longrightarrow \gamma(U) \subset \mathcal{Y}.$$

Thus, when $D_p \notin \mathcal{Y}$, the only unavailable parameters of this rational presentation are the points of $E_{\deg}(\Lambda)$; there are no intrinsic exceptional parameters on U.

6.4 Similar deformations and a minimality problem

The previous construction is a special case of the following.

Definition 6.3 (similar deformations)**.** Fix $\lambda = (\lambda_1, \dots, \lambda_r) \in (\mathbb{C}^*)^r$. A *similar deformation* joining p and $p(\lambda z)$ is given by nonzero rational functions $\Lambda_i \in \mathbb{C}(\mathbb{P}^1)^*$ $(1 \le i \le r)$ such that

(1) $\Lambda_i(1:0) = 1$ and $\Lambda_i(0:1) = \lambda_i$;

(2) on the common open set $U \subset \mathbb{P}^1$ where all Λ_i take values in $\mathbb{C}^*$, the family

$$p_{(s:t)}(z) = p(\Lambda_1(s:t)z_1, \dots, \Lambda_r(s:t)z_r)$$

induces a morphism $\gamma\colon U \to \mathfrak{d}_D$ connecting D_p and $D_{p(\lambda z)}$.

We define the degeneracy set

$$E_{\deg}(\Lambda) := \mathbb{P}^1 \setminus U = \bigcup_{i=1}^{r} \Lambda_i^{-1}(\{0, \infty\}).$$

Zeros and poles are taken after cancellation of common factors in each function Λ_i. This notation refers to the chosen rational scaling presentation, not just to its induced map into the coefficient space. The endpoint scaling λ is fixed in the minimality problem below.

The resulting minimality problem is algebraic: realize the fixed scaling λ by rational functions Λ_i while minimizing the cardinality of the union of their zero and pole sets, namely $|E_{\deg}(\Lambda)|$.

6.5 Multiplicative rank and the optimal degeneracy size

Definition 6.4 (multiplicative rank)**.** For $\lambda = (\lambda_1, \dots, \lambda_r) \in (\mathbb{C}^*)^r$, define the *multiplicative rank* $\mathrm{mrk}(\lambda)$ to be the smallest integer $k \ge 0$ for which there exist $\mu_1, \dots, \mu_k \in \mathbb{C}^*$ and integers $c_{ij} \in \mathbb{Z}$ such that

$$\lambda_i = \prod_{j=1}^{k} \mu_j^{c_{ij}}, \qquad 1 \le i \le r.$$

By convention, $\mathrm{mrk}(1, \dots, 1) = 0$.

This invariant is not the torsion-free rank of the subgroup generated by the coordinates. For example, if $\zeta \ne 1$ is a root of unity, then $\mathrm{mrk}(\zeta, 1, \dots, 1) = 1$.

Proposition 6.5 (optimal degeneracy size for similarity deformations)**.** *Let $\lambda \in (\mathbb{C}^*)^r$ with $\lambda \ne (1, \dots, 1)$ and set $k := \mathrm{mrk}(\lambda)$. Among all similar deformations (Definition 6.3) joining p and $p(\lambda z)$, the minimal possible value of $|E_{\deg}(\Lambda)|$ equals $k+1$.*

Proof. *Step 1: construction with $k+1$ degeneracy points.* Choose $\mu_1, \dots, \mu_k \in \mathbb{C}^*$ and integers c_{ij} such that $\lambda_i = \prod_{j=1}^{k} \mu_j^{c_{ij}}$ for all i. Define

$$\Lambda_i(s:t) := \prod_{j=1}^{k} \left(\frac{s + \mu_j t}{s+t} \right)^{c_{ij}}.$$

Then $\Lambda_i(1:0) = 1$ and $\Lambda_i(0:1) = \lambda_i$. All zeros/poles of Λ_i are contained in the finite set

$$\{(1:-1)\} \cup \{(\mu_j : -1) \mid 1 \le j \le k\},$$

so $|E_{\deg}(\Lambda)| \le k+1$ for this deformation.

Step 2: lower bound $|E_{\deg}(\Lambda)| \ge k+1$. Let a similar deformation be given, and set

$$E := E_{\deg}(\Lambda) = \bigcup_{i=1}^{r} \Lambda_i^{-1}(\{0, \infty\}).$$

Since $\lambda \neq (1, \dots, 1)$, some Λ_i is nonconstant and $E \neq \varnothing$. Choose $p_0 \in E$ and a coordinate u on $\mathbb{P}^1 \setminus \{p_0\} \cong \mathbb{A}^1$. The endpoints $A = (1:0)$ and $B = (0:1)$ lie outside E; put $a = u(A)$ and $b = u(B)$. After cancelling common factors, the zero and pole orders $m_{i,\alpha} \in \mathbb{Z}$ and the normalization $\Lambda_i(A) = 1$ give

$$\Lambda_i(u) = \prod_{\alpha \in u(E \setminus \{p_0\})} \left(\frac{u - \alpha}{a - \alpha} \right)^{m_{i,\alpha}}.$$

Consequently,

$$\lambda_i = \prod_{\alpha \in u(E \setminus \{p_0\})} \left(\frac{b - \alpha}{a - \alpha} \right)^{m_{i,\alpha}}.$$

All displayed ratios are nonzero, and there are at most $|E| - 1$ of them. Thus $\operatorname{mrk}(\lambda) \le |E| - 1$. Therefore, $|E| \ge k+1$.

Combining the two steps proves the optimality. ∎

6.6 Consequence for similarity paths in Theorem 1.1

Let p and q be similar with $q(z) = p(\lambda z)$. By Lemma 6.2, either both divisors lie in $\mathcal{Y}$ or neither does. If $D_p \notin \mathcal{Y}$, then for any (non-degenerate) similarity path $\gamma\colon U \to \mathfrak{d}_D$ connecting D_p and D_q, Lemma 6.2 implies $\gamma^{-1}(\mathcal{Y}) = \varnothing$; Theorem 1.1 is not needed for this conclusion. For a fixed endpoint scaling $\lambda \neq (1, \dots, 1)$, Proposition 6.5 shows that the minimum is

$$\min |E_{\deg}(\Lambda)| = \operatorname{mrk}(\lambda) + 1.$$

The minimum is taken over rational scaling presentations with these prescribed endpoint scalings. For $\lambda = (1, \dots, 1)$, the constant scalings $\Lambda_i \equiv 1$ instead give minimum 0.

7 Examples

This section illustrates how the exceptional locus (globally $\mathcal{Y} \subset |D|$, and in restricted supports $\mathcal{Y}_S \subset \mathfrak{d}_S^*$) and the deformation principle in Theorem 1.1 behave in concrete toric settings. We focus on projective spaces (where coefficients admit explicit descriptions) and Hirzebruch surfaces (where birational modifications reveal toric boundary phenomena).

7.1 Projective spaces

7.1.1 The case $X = \mathbb{P}^2$ and $D = 2H$

Let $X = \mathbb{P}^2$ with homogeneous coordinates $[x : y : z]$ and $D = 2H$. Then P_D is the standard degree-2 simplex in $M_{\mathbb{R}} \cong \mathbb{R}^2$

$$P_D = \{(a, b) \in \mathbb{R}^2 \mid a \geq 0,\, b \geq 0,\, a + b \leq 2\},$$

and $|P_D \cap M| = 6$. A general divisor $D_0 \sim 2H$ is a plane conic

$$\operatorname{Supp} D_0 = \{a_1x^2 + a_2y^2 + a_3z^2 + b_1yz + b_2xz + b_3xy = 0\},$$

so $\mathfrak{d}_D^* = |D| \simeq \mathbb{P}^5$ with homogeneous coordinates $(a_1, a_2, a_3, b_1, b_2, b_3)$.

The bad locus is closed and explicit in this case, so $B_{P_D\cap M} = Y_{P_D\cap M} \subset \mathbb{P}^5$ can be written as a union of six hypersurfaces

$$\begin{aligned} B_{P_D\cap M} = Y_{P_D\cap M} = \{a_1 = 0\} \cup \{a_2 = 0\} \cup \{a_3 = 0\} \cup \{a_1b_1^2 + a_2b_2^2 = b_1b_2b_3\} \\ \cup \{a_2b_2^2 + a_3b_3^2 = b_1b_2b_3\} \cup \{a_3b_3^2 + a_1b_1^2 = b_1b_2b_3\}. \end{aligned}$$

To verify this description, identify the six lattice points with $(2, 0)$, $(0, 2)$, $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$. Up to sign, the six directions in $\mathcal{H}_{P_D\cap M}$ are generated by $(1, 0)$, $(0, 1)$, $(1, -1)$, $(1, 1)$, $(1, -2)$, $(2, -1)$. The first three directions have three layers. For example, for the direction generated by $(1, 0)$, after setting $z = 1$ the three layer polynomials are

$$Q_0(x) = a_1x^2 + b_2x + a_3, \qquad Q_1(x) = b_3x + b_1, \qquad Q_2 = a_2.$$

The cases in which Q_2 is one of the vanishing layers lie in the coordinate hyperplane $\{a_2 = 0\}$. A common zero of Q_0 and Q_1 in $\mathbb{C}^*$ implies $a_1b_1^2 + a_3b_3^2 - b_1b_2b_3 = 0$. The other two three-layer directions give the remaining two coordinate components and the remaining two cubic equations. Each of the last three directions has five layers, with four singleton layers; its bad incidence locus is contained in the coordinate components already listed. Thus the bad locus is contained in the displayed closed union.

For the reverse inclusion, suppose first that $a_2 = 0$, so that $p(x, y, 1) = Q_0(x) + yQ_1(x)$. If either Q_j is identically zero or has a zero in $\mathbb{C}^*$, fixing a suitable x leaves at most one nonzero layer. Otherwise both Q_j are nonzero monomials. Then p has two monomials, and its restriction to a one-dimensional subtorus in the kernel of their character ratio is zero or a single monomial. In all cases the coefficient vector belongs to the bad locus. Permuting the variables handles the other coordinate hyperplanes. Outside these hyperplanes, consider $a_1b_1^2 + a_2b_2^2 = b_1b_2b_3$. If $b_1b_2 \neq 0$, restricting to $b_1y + b_2x = 0$ and $z = 1$ gives $p = a_3 \neq 0$. If $b_1b_2 = 0$, the equality forces $b_1 = b_2 = 0$; choose a nonzero root ρ of $a_1\rho^2 + b_3\rho + a_2 = 0$. Then $p(\rho u, u, 1) = a_3$ for $u \in \mathbb{C}^*$. These are translated subtori in the complement. Permuting variables handles the other two cubic equations. This proves the displayed equality with the reduced closed union, including all degenerate coefficient vectors.

Moreover, for any one-dimensional face τ of P_D, the set $\tau \cap M$ has three lattice points and $(a, b, c) \in B_{\tau\cap M}$ if and only if at least two among a, b, c vanish. In particular, failure of an edge condition forces membership in $Y_{P_D\cap M}$, so the global condition already controls all positive-dimensional faces. We identify the global conic exceptional locus by checking the edge contributions and the coefficient-chart boundary. An edge bad locus requires at least two of its three coefficients to vanish. At least one of these is an endpoint coefficient among a_1, a_2, a_3. The coefficient-chart boundary requires all three coefficients on some edge to vanish and is contained in the same union of vertex-coefficient hyperplanes. These hyperplanes already occur in $Y_{P_D\cap M}$. Since the full face also occurs in the global construction, we obtain $\mathcal{Y} = Y_{P_D\cap M} = B_{P_D\cap M}$ in the conic case.

Proposition 7.1 (a concrete sufficient condition for hyperbolic embedding)**.** *Let* $D_0 = \operatorname{div}(a_1x^2 + a_2y^2 + a_3z^2 + b_1yz + b_2xz + b_3xy)$*. If* $a_1 \neq 0$*,* $a_2 \neq 0$*,* $a_3 \neq 0$*, and*

$$a_1b_1^2 + a_2b_2^2 \neq b_1b_2b_3, \qquad a_2b_2^2 + a_3b_3^2 \neq b_1b_2b_3, \qquad a_3b_3^2 + a_1b_1^2 \neq b_1b_2b_3,$$

then $T_N \setminus \operatorname{Supp} D_0 \hookrightarrow \mathbb{P}^2$ *is a Kobayashi hyperbolic embedding.*

Proof. The inequalities in the statement say exactly that the full coefficient vector lies outside $B_{P_D \cap M}$. On each edge of P_D, two endpoint coefficients are among a_1, a_2, a_3 and are nonzero. Hence at least two of the three edge coefficients are nonzero, so the restricted coefficient vector lies outside the one-dimensional face bad locus as well. Lemma 4.8 now gives the asserted hyperbolic embedding. ■

Failure of one of these sufficient inequalities can genuinely produce a nonhyperbolic complement. Two typical degenerations are as follows:

(*i*) *Vanishing of a diagonal coefficient.* Assume $a_1 = 0$ and $b_2b_3 \neq 0$, and put

$$q(y, z) = a_2y^2 + b_1yz + a_3z^2.$$

If $q(-b_2, b_3) \neq 0$, then on the torus locus $b_3y + b_2z = 0$ one has $p(x, y, z) = q(y, z) \neq 0$ for every $x \in \mathbb{C}^*$. Thus the complement contains the translated one-dimensional subtorus obtained by fixing $(y, z) = (-b_2, b_3)$ and varying x, and consequently it is not Brody hyperbolic.

(*ii*) *Satisfying a "cubic" equality.* Assume $a_1, a_2, a_3 \neq 0$ and $a_1b_1^2 + a_2b_2^2 = b_1b_2b_3$. If $b_1b_2 \neq 0$, then

$$p(x, y, z) = (b_1y + b_2x)\left(\frac{a_1}{b_2}x + \frac{a_2}{b_1}y + z\right) + a_3z^2.$$

On $b_1y + b_2x = 0$ with $z = 1$, the value of p is the nonzero constant a_3, so the complement again contains a translated subtorus. The equality also permits the previously omitted branch $b_1 = b_2 = 0$. In that case choose $\rho \in \mathbb{C}^*$ satisfying $a_1\rho^2 + b_3\rho + a_2 = 0$; then $u \mapsto (\rho u, u, 1)$ is a translated subtorus on which $p = a_3 \neq 0$.

Here is a one-parameter deformation with finitely many exceptions. Fix $b_1, b_2 \in \mathbb{C}^*$ and consider $p_t(x, y, z) = x^2 + y^2 + z^2 + b_1yz + b_2xz + txy$. We determine exactly the parameters at which hyperbolic embedding fails. Define

$$g_1(t) := b_1^2 + b_2^2 - b_1b_2t, \qquad g_2(t) := b_2^2 + t^2 - b_1b_2t, \qquad g_3(t) := t^2 + b_1^2 - b_1b_2t. \tag{7.1}$$

The exact failure set is

$$E_{\text{fail}} = \{\infty\} \cup \{t \in \mathbb{C} \mid g_1(t)g_2(t)g_3(t) = 0\}. \tag{7.2}$$

We first prove hyperbolic embedding outside the set in (7.2). If $g_1(t)g_2(t)g_3(t) \neq 0$, all the hypotheses of Proposition 7.1 hold with $a_1 = a_2 = a_3 = 1$ and $b_3 = t$, giving the required embedding.

We prove failure at each finite parameter in (7.2) by constructing a translated subtorus. For $g_i(t) = 0$, use the corresponding map

$$\begin{aligned} &\iota_1(u) = [b_1u : -b_2u : 1], \qquad \iota_2(u) = [1 : b_2u : -tu], \\ &\iota_3(u) = [b_1u : 1 : -tu], \qquad u \in \mathbb{C}^*. \end{aligned} \tag{7.3}$$

Using (7.1), we compute the polynomial evaluations at the representatives in (7.3): $p_t(\iota_i(u)) = 1 + g_i(t)u^2 = 1$ if $g_i(t) = 0$. The first map has nonzero coordinates because $b_1b_2 \neq 0$. For the other maps, $g_2(0) = b_2^2 \neq 0$ and $g_3(0) = b_1^2 \neq 0$ ensure $t \neq 0$ on the corresponding zero sets. Thus each relevant ι_i lies in the torus complement and $s \mapsto \iota_i(e^s)$ is a nonconstant entire curve there.

We finally treat $t = \infty$ in (7.2). The section at infinity is xy, which has no zero on T_N. The torus complement is therefore T_N itself and is not hyperbolic. The three cases establish (7.2).

7.1.2 The case $X = \mathbb{P}^2$ and $D = 3H$

Now let $D = 3H$. Then

$$P_D = \{(m_1, m_2) \in M_{\mathbb{R}} \mid m_1 \geq 0,\, m_2 \geq 0,\, m_1 + m_2 \leq 3\}$$

and $|P_D \cap M| = 10$. A general cubic divisor $D_0 \sim 3H$ is defined by

$$\begin{aligned} p(x, y, z) = a_{03}y^3 + a_{02}y^2z + a_{12}xy^2 + a_{01}yz^2 + a_{11}xyz + a_{21}x^2y \\ + a_{00}z^3 + a_{10}z^2x + a_{20}zx^2 + a_{30}x^3. \end{aligned}$$

The calculation is guided by the following observation. The arrangement $\mathcal{H}_{P_D \cap M}$ contains finitely many linear directions $H \subset M_{\mathbb{R}}$, and each H yields a determinantal-type bad condition B_H in coefficient space. Rather than listing all components, we record usable sufficient conditions by grouping the relevant H into geometric families.

We first consider the facet-parallel lines. Define

$$\begin{aligned} &H_x = \{(m_1, m_2) \mid m_1 = 0\}, \qquad H_y = \{(m_1, m_2) \mid m_2 = 0\}, \\ &H_z = \{(m_1, m_2) \mid m_1 + m_2 = 0\}. \end{aligned}$$

These equations specify vector directions through the origin; affine translates of these lines cut out the corresponding layers in P_D. The condition "$(a_{ij}) \notin B_{H_x}$" can be phrased as: for every $(y, z) \in (\mathbb{C}^*)^2$, at least two of the following four expressions are nonzero

$$\begin{aligned} &F_0(y, z) := a_{00}z^3 + a_{01}yz^2 + a_{02}y^2z + a_{03}y^3, \qquad F_1(y, z) := a_{10}z^2 + a_{11}yz + a_{12}y^2, \\ &F_2(y, z) := a_{20}z + a_{21}y, \qquad F_3 := a_{30}. \end{aligned}$$

Similarly, one obtains the corresponding lists for H_y and H_z by symmetry.

Three additional difference directions are represented by

$$H_{xy} = \{m_1 - m_2 = 0\}, \qquad H_{yz} = \{m_1 + 2m_2 = 0\}, \qquad H_{zx} = \{2m_1 + m_2 = 0\}.$$

Proposition 7.2 (a workable sufficient criterion for hyperbolic embedding in the cubic case)**.** *Let $D_0 = \operatorname{div}\left(\sum_{i+j \leq 3} a_{ij}x^iy^jz^{3-i-j}\right)$ in $\mathbb{P}^2$. Assume the following "non-concentration" conditions:*

(1) (H_x) *For every $(y, z) \in (\mathbb{C}^*)^2$, at least two among $F_0(y, z), F_1(y, z), F_2(y, z), F_3$ are nonzero.*

(2) (H_y) *For every $(x, z) \in (\mathbb{C}^*)^2$, at least two among*

$$\begin{aligned} &a_{00}z^3 + a_{10}z^2x + a_{20}zx^2 + a_{30}x^3, \\ &a_{01}z^2 + a_{11}xz + a_{21}x^2, \qquad a_{02}z + a_{12}x, \qquad a_{03}. \end{aligned}$$

are nonzero.

(3) (H_z) *For every $(x, y) \in (\mathbb{C}^*)^2$, at least two among*

$$\begin{aligned} &a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \\ &a_{20}x^2 + a_{11}xy + a_{02}y^2, \qquad a_{10}x + a_{01}y, \qquad a_{00}. \end{aligned}$$

are nonzero.

(4) (H_{xy}) *For every $(x, y, z) \in (\mathbb{C}^*)^3$, at least two among*

$$a_{30}, a_{20}, a_{02}, a_{03}, a_{10}z^2 + a_{21}xy, a_{00}z^2 + a_{11}xy, a_{01}z^2 + a_{12}xy$$

are nonzero.

(5) (H_{yz}) *For every* $(x,y,z) \in (\mathbb{C}^*)^3$, *at least two among*

$$a_{03}, a_{12}, a_{10}, a_{00}, a_{21}x^2 + a_{02}yz, a_{30}x^2 + a_{11}yz, a_{20}x^2 + a_{01}yz$$

are nonzero.

(6) (H_{zx}) *For every* $(x,y,z) \in (\mathbb{C}^*)^3$, *at least two among*

$$a_{00}, a_{01}, a_{21}, a_{30}, a_{02}y^2 + a_{10}zx, a_{03}y^2 + a_{11}zx, a_{12}y^2 + a_{20}zx$$

are nonzero.

(7) (face conditions) *In each row of*

$$\begin{pmatrix} a_{03} & a_{12} & a_{21} & a_{30} \\ a_{03} & a_{02} & a_{01} & a_{00} \\ a_{30} & a_{20} & a_{10} & a_{00} \end{pmatrix}$$

at least two entries are nonzero.

Then $T_N \setminus \operatorname{Supp} D_0 \hookrightarrow \mathbb{P}^2$ *is a Kobayashi hyperbolic embedding.*

Proof. Every element of $\mathcal{H}_{P_D \cap M}$ is the span of a primitive difference vector. Normalize such a vector $v = (p,q)$ up to sign and count the levels of the quotient functional $\nu_v(i,j) = qi - pj$. Enumerating the primitive directions in $(P_D \cap M) - (P_D \cap M)$ gives twelve directions. They split as follows:

$\lvert\phi_H(P_D \cap M)\rvert$	primitive generators of H	conditions used
4	$(1,0),\ (0,1),\ (1,-1)$	(1)–(3)
7	$(1,1),\ (1,-2),\ (2,-1)$	(4)–(6)
9	$(1,-3),\ (1,2),\ (2,-3),\ (2,1),\ (3,-2),\ (3,-1)$	(7)

For the first six directions, the displayed expressions in (1)–(6) are precisely their layer polynomials. Thus the stated non-concentration conditions say that at least two specialized layers are nonzero.

For each of the six nine-layer directions, the ten lattice points are distributed among eight singleton layers and one two-point layer. Condition (7) forces at least three boundary coefficients to be nonzero: indeed, two lattice points cannot supply two nonzero entries on each of the three edges. If at least four coefficients in total are nonzero, at least two nonzero singleton layers remain, regardless of cancellation in the two-point layer. If exactly three coefficients in total are nonzero, condition (7) requires these three points to supply at least six incidences with the edges. They must therefore be the three vertices. The vertices lie in distinct layers because none of these six directions is parallel to an edge, and there are no other nonzero coefficients to cancel them. Thus every specialization has at least two nonzero layers in each of the twelve directions. The same edge condition excludes the bad loci of the three one-dimensional faces. Lemma 4.8 proves the conclusion. ∎

Proposition 7.3 (uniform hyperbolicity along one-monomial deformations)**.** *Fix a coefficient vector* (a_{ij}) *satisfying the conditions of Proposition 7.2 with "at least two" replaced by "at least three" in* (1)–(7). *Then, for any lattice point* $I = (i_0, j_0) \in P_D \cap M$, *let* $m_I = x^{i_0}y^{j_0}z^{3-i_0-j_0}$. *Replacing its coefficient by* t *gives the family*

$$D_t = \operatorname{div}\Bigg(tm_I + \sum_{\substack{i+j\le 3 \\ (i,j)\neq I}} a_{ij}x^i y^j z^{3-i-j}\Bigg).$$

Then $T_N \setminus \operatorname{Supp} D_t \hookrightarrow \mathbb{P}^2$ *is a Kobayashi hyperbolic embedding for all* $t \in \mathbb{C}$.

Proof. The proof of Proposition 7.2 already identifies all twelve full-face directions. Conditions (1)–(6) give at least three nonzero layers for the six directions with four or seven layers. For a nine-layer direction, condition (7) with the strengthened bound forces at least six boundary coefficients to be nonzero. Indeed, if v is the number of nonzero vertex coefficients and e the number of nonzero coefficients in the relative interiors of the edges, then counting incidences with the three edges gives $2v + e \geq 9$; hence $v + e \geq 9 - v \geq 6$. Since there is only one two-point layer, at least four nonzero singleton layers remain, regardless of cancellation in that layer. In particular, at least three specialized layers are nonzero. The strengthened face conditions give the same three-layer bound on every edge.

Varying the coefficient indexed by I changes only one layer for each direction and for each face. At least two of the previously nonzero layers therefore remain unchanged for every $t \in \mathbb{C}$. Thus no member of the family enters an ordinary facewise bad locus, and Lemma 4.8 yields the conclusion for all t. ■

Proposition 7.4 (deformation to a five-term cubic)**.** *There is a nonempty Zariski open subset of the space of plane cubics such that every divisor D_0 in this subset admits a sequence of deformations preserving hyperbolic embedding to a divisor D' defined by a five-term cubic*

$$a_{30}x^3 + a_{03}y^3 + a_{10}xz^2 + a_{02}y^2z + a_{00}z^3.$$

Proof. Starting with the ten lattice points of the degree-3 simplex, eliminate the coefficients in the order a_{11}, a_{01}, a_{12}, a_{20}, a_{21}. Equivalently, remove the lattice points $(1,1)$, $(0,1)$, $(1,2)$, $(2,0)$, $(2,1)$ in that order. The layer counts used below are obtained by the following finite enumeration. For each current support S_k, take every nonzero difference $v = (p,q) \in S_k - S_k$, replace it by its primitive representative up to sign, and count the distinct integers $qi - pj$ for $(i,j) \in S_k$. For an edge, the only quotient is the identity on that one-dimensional affine lattice, so its layer count is simply the number of remaining edge points. This gives:

coefficient to be removed	$\lvert S\rvert$ before deletion	$\min_{H\in\mathcal{H}_S} \lvert\phi_H(S)\rvert$	$\min_{\text{edges }\tau} \lvert S\cap\tau\rvert$
a_{11}	10	4	4
a_{01}	9	4	4
a_{12}	8	4	3
a_{20}	7	4	3
a_{21}	6	4	3

For example, for the initial ten-point support the twelve normalized directions are those listed in the proof of Proposition 7.2; the three edge-parallel directions realize the minimum 4. Repeating the same normalization after each listed deletion gives the five rows of the table, so no unlisted direction is being omitted. These are exactly the enlarged bounds $\dim\tau + 2$ for the two-dimensional face and its one-dimensional faces. More explicitly, let S_0 be the ten-point support and let S_k $(0 \leq k \leq 4)$ be the support just before the $(k+1)$-st deletion. The coordinate restrictions $\rho_k \colon \mathbb{C}^{S_0} \to \mathbb{C}^{S_k}$ are surjective. Each affine cone $\widehat{\mathcal{Y}'_{S_k}}$ is proper and closed, so the set

$$\mathbb{C}^{S_0} \setminus \left(\bigcup_{k=0}^{4} \rho_k^{-1}(\widehat{\mathcal{Y}'_{S_k}}) \cup \bigcup_{I\in S_0} \{a_I = 0\} \right)$$

is nonempty and Zariski open. Choose the initial coefficients once in this set. At each stage the coefficients still present are exactly $\rho_k(a)$: earlier deletions changed only coordinates outside S_k. Thus no new generic choice is made during the process, and Theorem 1.2 applies successively. Each affine coefficient line preserves hyperbolic embedding from the current coefficient to zero. The remaining support is $\{(3,0),(0,3),(1,0),(0,2),(0,0)\}$, which gives the displayed five-term cubic. Its minimum full-face layer count is 3, and its three edge counts are 3, 3, 2, so it still satisfies the ordinary bounds of Theorem 3.1. ■

7.1.3 The case $X = \mathbb{P}^3$ and $D = 3H$

Let $X = \mathbb{P}^3$ and $D = 3H$. A general cubic has 20 coefficients. We record two complementary regimes: a sparse support set S, for which we verify the combinatorial hypotheses and obtain a generic conclusion, and the full support $S = P_D \cap M$, where only the directions with $\ell_H = 4$ can contribute hypersurface components.

We first consider a sparse support set. Let

$$\begin{aligned} S = \{&(0,0,0),(3,0,0),(0,3,0),(0,0,3),\\ &(1,0,0),(0,2,0),(2,0,1),(0,1,2)\} \subset P_D \cap M. \end{aligned} \tag{7.4}$$

Then D_0 is defined by

$$\begin{aligned} p(x,y,z,w) = {}& a_{300}x^3 + a_{030}y^3 + a_{003}z^3 + a_{000}w^3\\ &+ a_{100}xw^2 + a_{020}y^2w + a_{201}x^2z + a_{012}yz^2. \end{aligned} \tag{7.5}$$

Proposition 7.5 (a facewise criterion for the sparse cubic in $\mathbb{P}^3$)**.** *Let S be as in (7.4) and let p be as in (7.5). If, for every positive-dimensional face τ of P_D, the restricted coefficient vector lies outside $B_{\tau\cap S}$, then $T_N \setminus V(p) \hookrightarrow \mathbb{P}^3$ is a Kobayashi hyperbolic embedding. In particular, this conclusion holds for a nonempty Zariski open subset of the coefficient torus where all eight coefficients are nonzero.*

Proof. The support contains all four vertices of $3\Delta_3$, so every face is spanned by its restricted support and $\mathfrak{d}_S \neq \varnothing$. Normalize the nonzero cross products of pairs of difference vectors to primitive normals up to sign. Counting their projection layers gives

support	layer count: number of directions	required minimum
S	$4:7,\ 5:24,\ 6:62$	4
$S \cap \tau$ (τ a facet)	$3:2,\ 4:4$ per facet	3
$S \cap \tau$ (τ an edge)	two or three points	2

On each facet the two-dimensional enumeration uses primitive difference directions in its affine lattice. The four facets have the same layer count distribution. The xy- and zw-edges contain two support points each, and the other four edges contain three support points each. The supplementary script described in Appendix A verifies these counts, including all 93 directions for the eight-point support. Thus the facewise combinatorial hypotheses of Theorem 3.1 hold. Lemma 4.8 proves the first assertion. The global exceptional locus $\mathcal{Y}_S \subset \mathbb{P}^7$ is proper and closed by its construction. Its complement meets the dense coefficient torus, giving the asserted nonempty Zariski open subset. ∎

Nonvanishing of the eight coefficients alone does not imply the facewise bad-locus condition, because layer polynomials can cancel on the torus. Indeed, set all eight coefficients in (7.5) equal to 1. Then the translated one-dimensional subtorus $u \longmapsto [-u : u : -1 : 1]$, $u \in \mathbb{C}^*$, is contained in $V(p)$: direct substitution gives $-u^3 + u^3 - 1 + 1 - u + u^2 - u^2 + u = 0$. Thus the coefficient vector belongs to the bad locus even though every coefficient is nonzero. This example disproves the implication from coefficient nonvanishing to bad-locus avoidance. It does not by itself prove that the complement is nonhyperbolic. Accordingly, the proposition uses explicit facewise bad-locus avoidance, and its generic conclusion does not assert the result for every nonzero coefficient vector.

For the full support, the divisorial part is reduced to a finite elimination problem. When $S = P_D \cap M$, the arrangement $\mathcal{H}_S$ is large, but the incidence estimate in Lemma 4.3 shows that

a hyperplane with $\ell_H \geq 5$ contributes a subset of codimension at least 2; the enumeration below shows that every nondivisorial direction actually has $\ell_H \geq 7$. Consequently, only the finitely many hyperplanes with $\ell_H = 4$ can contribute hypersurface components.

Proposition 7.6 (divisorial degree for cubic surfaces)**.** *The codimension-one part of the global exceptional locus $\mathcal{Y} \subset |3H|$ is reduced of degree* 240*.*

Proof. Let

$$S = 3\Delta_3 \cap \mathbb{Z}^3 = \{(i_1, i_2, i_3) \in \mathbb{Z}^3_{\geq 0} \mid i_1 + i_2 + i_3 \leq 3\}.$$

Represent a hyperplane $H \subset \mathbb{R}^3$ by a primitive normal $n \in \mathbb{Z}^3$, up to sign. Then $\ell_H = |\{\langle n, I\rangle \mid I \in S\}|$. Direct enumeration gives exactly seven directions with $\ell_H = 4$ e_1, e_2, e_3, e_1+e_2, e_1+e_3, e_2+e_3, $e_1+e_2+e_3$. For reproducibility, form all linearly independent pairs $u, v \in (S - S) \setminus \{0\}$, take $n = u \times v$, divide its coordinates by their greatest common divisor, and choose the sign so that the first nonzero coordinate is positive. Removing repetitions gives

ℓ_H	4	7	9	10	12	13	14	15	17	18
$\#\{H\}$	7	18	36	12	24	12	60	72	72	192.

Thus the literal definition of $\mathcal{H}_S$ gives 505 hyperplane directions in total, exactly the seven displayed directions have four layers, and every other direction has at least seven layers.

For one of these directions, write P_0, P_1, P_2, P_3 for the Newton polygons of the four layer polynomials in the rank-two lattice $H \cap M$. For the triples without a point layer considered below, each support has dimension at least one and every pair has a two-dimensional Minkowski sum. Thus the full triple is the essential subfamily. If its support differences generate $H \cap M$, the sparse resultant has multiplicity one and equals the reduced sparse eliminant [5, Remark 3.5 and Proposition 3.6]. In this case, the closure of $B_{S,H,j}$ is its irreducible zero locus, and the degree in each retained layer block is the mixed volume of the other two Newton polygons [5, equation (3.2)]. Here MV denotes the mixed volume with respect to the lattice $H \cap M$, normalized by $\mathrm{MV}(P, P) = 2\,\mathrm{Area}(P)$, where a fundamental lattice parallelogram has area 1; see [5, Section 2.2]. We handle triples containing a point layer separately, so that no resultant multiplicity is included in the reduced degree.

For the four face-parallel directions e_1, e_2, e_3, $e_1 + e_2 + e_3$, the polygons are, up to reversing their order, $P_m = m\Delta_2$, $0 \leq m \leq 3$. If the point layer P_0 is omitted, the other three layers have degrees 1, 2, 3. Their supports satisfy the preceding dimension conditions and generate the full lattice, since the degree-one support contains a unimodular triangle. Their resultant therefore has total degree

$$\mathrm{MV}(2\Delta_2, 3\Delta_2) + \mathrm{MV}(\Delta_2, 3\Delta_2) + \mathrm{MV}(\Delta_2, 2\Delta_2) = 6 + 3 + 2 = 11.$$

If a nonpoint layer is omitted, the point-layer polynomial must vanish. The other two retained polygons have positive mixed volume, so two general polynomials with those supports have a common torus zero by Bernshtein's theorem [1, Theorem A]. Consequently, after passing to the reduced closure, the three such cases give the same coordinate hyperplane, of degree 1. Thus each face-parallel direction contributes $11 + 1 = 12$ rather than counting that coordinate hyperplane three times.

For the other three directions, the polygons can be identified with $P_m = [0, 3 - m] \times [0, m]$, $0 \leq m \leq 3$. For rectangles $[0, a] \times [0, b]$ and $[0, c] \times [0, d]$, the normalized mixed volume is $ad + bc$. Hence

(a, b)	$(0,1)$	$(0,2)$	$(0,3)$	$(1,2)$	$(1,3)$	$(2,3)$
$\mathrm{MV}(P_a, P_b)$	3	6	9	5	6	3

and the four resultants obtained by omitting, respectively, P_0, P_1, P_2, P_3 have total degrees 14, 18, 18, 14. For every omission, each retained polygon has dimension at least one and every retained pair has a two-dimensional Minkowski sum. Moreover, at least one of P_1, P_2 is retained; its support contains differences equal to the two coordinate unit vectors. Thus each triple generates the full lattice and its resultant has multiplicity one. Each of these three directions consequently contributes $14 + 18 + 18 + 14 = 64$.

It remains to distinguish the reduced components. Each noncoordinate eliminant depends on every individual retained coefficient. Indeed, fix the other two retained polynomials generically. Their positive mixed volume gives a nonempty finite set of common torus zeros by Bernshtein's theorem [1, Theorem A]. The Poisson formula [5, Theorem 3.3] expresses the eliminant, up to a nonzero factor independent of the remaining polynomial, as the product of its evaluations at those zeros, with multiplicities. Every monomial evaluates nontrivially there, so each of its coefficients occurs with a nonzero leading coefficient in this product. Hence the coefficients absent from an eliminant are exactly those in its omitted layer.

For the four noncoordinate face-parallel components, the omitted layers are the four different singleton vertex sets. For the twelve remaining components, they are six different four-point edge sets and six different six-point sets. Each six-point set consists of the cubic monomials of degree one in a specified pair of homogeneous variables and degree two in the complementary pair. The variables that occur squared determine that complementary pair, so these six sets are distinct. Thus the sixteen noncoordinate components are pairwise distinct and are not coordinate hyperplanes. The four coordinate components are the four different vertex-coefficient hyperplanes; the repetitions within each face-parallel direction have already been counted only once.

The reduced divisorial part therefore has degree $4 \cdot 12 + 3 \cdot 64 = 240$. All remaining full-support hyperplanes have $\ell_H \geq 7$. Since $r = 3$, the incidence estimate in Lemma 4.3 gives codimension at least $\ell_H - r \geq 4$, so they cannot contribute divisors. On a two-dimensional facet, the support is $3\Delta_2 \cap \mathbb{Z}^2$ and its minimum layer count is 4; the same estimate gives codimension at least $4 - 2 = 2$. On a one-dimensional edge there are four lattice points, so its bad locus has codimension $4 - 1 = 3$. Finally, a component of the coefficient boundary $\mathfrak{d}_S^* \setminus \mathfrak{d}_S$ requires all ten coefficients on one facet to vanish and therefore has codimension 10. Thus neither a proper-face bad locus nor the coefficient-chart boundary contributes a divisor. This proves the assertion. ■

The degree 240 concerns the reduced divisorial part of the constructed exceptional locus $\mathcal{Y}$. It is not the degree of an independently identified locus of all nonhyperbolic complements. The computational checks and representative lattice coordinates are given in Appendix A and the supplementary script.

7.2 Hirzebruch surfaces

Let $\mathbb{F}_\ell$ ($\ell \geq 1$) be the Hirzebruch surface. We use the toric (GIT) presentation

$$\mathbb{F}_\ell = \frac{\mathbb{C}^4 \setminus (\mathbb{C}^2 \times \{(0,0)\} \cup \{(0,0)\} \times \mathbb{C}^2)}{\{(\lambda^\ell \mu, \mu, \lambda, \lambda) \mid \lambda, \mu \in \mathbb{C}^*\}}.$$

The coordinate order is (x, w, y, z). There is a toric contraction $\pi\colon \mathbb{F}_\ell \to \mathbb{P}(\ell, 1, 1)$ of the negative section to $[1 : 0 : 0]$. This is the unique singular point when $\ell \geq 2$; for $\ell = 1$ the target is $\mathbb{P}^2$ and the image point is smooth. On the chart $w \neq 0$, the contraction is represented in weighted homogeneous coordinates by $\pi(x, w, y, z) = [x/w : y : z]$. This algebraic chart formula involves no choice of an ℓ-th root of w. The Picard group is generated by torus-invariant divisors; writing $D_w = \{w = 0\}$ and $D_y = \{y = 0\}$, $\mathrm{Pic}(\mathbb{F}_\ell) \cong \mathbb{Z}[D_w] \oplus \mathbb{Z}[D_y]$. With the Cox grading

$$\deg(x) = (1, \ell), \qquad \deg(w) = (1, 0), \qquad \deg(y) = \deg(z) = (0, 1),$$

the pair (a, b) denotes the class $aD_w + bD_y$. Here $D_w^2 = -\ell$, $D_w \cdot D_y = 1$, and $D_y^2 = 0$; consequently $aD_w + bD_y$ is ample if and only if $a > 0$ and $b > \ell a$ (equivalently, for integral classes, $a \geq 1$ and $b \geq \ell a + 1$). We concentrate on the ample class $D := 2D_w + (2\ell + 2)D_y$, for which $b - \ell a = 2$.

We use the following six-term support set. Let

$$S = \{(0,0), (0,1), (0,2), (2\ell+2, 0), (\ell+2, 1), (2,2)\},$$

where the coordinates of a lattice point are the exponents (i, j) of the torus characters

$$u := y/z, \qquad v := x/(wz^\ell). \tag{7.6}$$

We consider the Cox-homogeneous polynomial

$$p(x,w,y,z) = a_{0,y}y^{2\ell+2}w^2 + a_{0,z}z^{2\ell+2}w^2 + a_{1,y}xy^{\ell+2}w + a_{1,z}xz^{\ell+2}w + a_{2,y}x^2y^2 + a_{2,z}x^2z^2.$$

Every displayed monomial has Cox degree $(2, 2\ell + 2)$, so the polynomial indeed defines a divisor of class D. Dividing by $w^2z^{2\ell+2}$ on the dense torus gives

$$\frac{p}{w^2z^{2\ell+2}} = a_{0,y}u^{2\ell+2} + a_{0,z} + a_{1,y}u^{\ell+2}v + a_{1,z}v + a_{2,y}u^2v^2 + a_{2,z}v^2.$$

More generally, a Cox monomial of degree $(2, 2\ell + 2)$ has the form $x^j w^{2-j} y^i z^{2\ell+2-\ell j-i}$ and corresponds to $u^i v^j$. Consequently the polytope in these coordinates is

$$P_D = \{(i,j) \in \mathbb{R}^2 \mid 0 \leq j \leq 2,\ 0 \leq i \leq 2\ell + 2 - \ell j\}. \tag{7.7}$$

Its four vertices are $(0, 0)$, $(2\ell + 2, 0)$, $(2, 2)$, $(0, 2)$, all of which belong to S. For the direction determined by the three powers of x, the layer polynomials are $a_{0,y}y^{2\ell+2} + a_{0,z}z^{2\ell+2}$, $a_{1,y}y^{\ell+2} + a_{1,z}z^{\ell+2}$, $a_{2,y}y^2 + a_{2,z}z^2$ before specialization in the transverse variable. Thus one explicit part of the facewise bad-locus avoidance condition is that, for nonzero coefficients, no point $(y, z) \in (\mathbb{C}^*)^2$ makes two of these expressions vanish simultaneously.

A toric boundary limit displays a birational phenomenon. Set $w = 1$ to view the same equation on $\mathbb{P}(\ell, 1, 1)$. The morphism π is an isomorphism on the dense tori, identifying the two dense torus complements. Their intrinsic hyperbolicity is therefore the same. What depends on the compactification is the hyperbolic embedding problem and the boundary criterion used to study it. The toric resolution retains the exceptional negative section $w = 0$, on which a limiting entire curve can appear. For $\ell = 1$ this resolution is the blow-up of a smooth point of $\mathbb{P}^2$; for $\ell \geq 2$ it resolves the singular point of $\mathbb{P}(\ell, 1, 1)$.

Concretely, in the limiting configuration where the x^2z^2-term is removed, the stability condition on the Cox quotient implies $x \neq 0$ along $w = 0$, and the equation reduces to $a_{2,y}x^2y^2 = 0$. Outside $\{y = 0\}$ the divisor does not meet this boundary locus, allowing the holomorphic map $\phi\colon \mathbb{C} \to \mathbb{F}_\ell$, $s \mapsto (x, w, y, z) = (1, 0, 1, e^s)$, which lands in the boundary stratum away from the limiting divisor. The curve ϕ violates the facewise "no entire curves" condition. The proof of Proposition 7.7 establishes failure of hyperbolic embedding of the torus complement in $\mathbb{F}_\ell$.

Proposition 7.7 (a family losing hyperbolic embedding at a toric boundary limit)**.** *Fix nonzero complex numbers $a_{0,y}$, $a_{0,z}$, $a_{1,y}$, $a_{1,z}$, $a_{2,y}$ and consider the family*

$$\begin{aligned} p_t &= a_{0,y}y^{2\ell+2}w^2 + a_{0,z}z^{2\ell+2}w^2 + a_{1,y}xy^{\ell+2}w + a_{1,z}xz^{\ell+2}w + a_{2,y}x^2y^2 + tx^2z^2, \\ D_t &:= \operatorname{div}(p_t). \end{aligned} \tag{7.8}$$

Assume that there exists $t_0 \in \mathbb{C}^*$ such that $D_{t_0} \in \mathfrak{d}_S^* \setminus \mathcal{Y}_S$, where $\mathcal{Y}_S$ is the facewise exceptional locus for the displayed six-point support. Then there is a finite set $E \subset \mathbb{C}^*$ such that

$$T_N \setminus \operatorname{Supp}(D_t \cap T_N) \hookrightarrow \mathbb{F}_\ell$$

is a Kobayashi hyperbolic embedding for every $t \in \mathbb{C}^* \setminus E$. At $t = 0$, the inclusion

$$U_0 := T_N \setminus \operatorname{Supp}(D_0 \cap T_N) \hookrightarrow \mathbb{F}_\ell$$

is not a Kobayashi hyperbolic embedding for every choice of the five nonzero coefficients; the failure at $t = 0$ does not require the assumption concerning t_0.

Proof. We first verify generic hyperbolic embedding for $t \neq 0$ in (7.8). For $\ell \geq 1$, the support S spans the polygon P_D in (7.7). For a nonzero linear functional $\nu(i, j) = \alpha i + \beta j$, if $\beta \neq 0$, the points $(0, 0)$, $(0, 1)$, $(0, 2)$ already give three distinct values. If $\beta = 0$, the four distinct first coordinates 0, 2, $\ell + 2$, $2\ell + 2$ give four values. The quotient $\nu(i, j) = j$ has exactly three layers and its kernel belongs to $\mathcal{H}_S$. Hence $\min_{H \in \mathcal{H}_S} |\phi_H(S)| = 3$ for every $\ell \geq 1$. The support S contains all four vertices of P_D in (7.7). Thus each edge support contains both endpoints and has at least two layers; in particular, $\mathfrak{d}_S \neq \varnothing$. Thus the ordinary facewise hypotheses hold and $\mathcal{Y}_S \subset \mathfrak{d}_S^*$ is proper.

The affine family extends to the morphism

$$\overline{\gamma}\colon\ \mathbb{P}^1 \longrightarrow \mathfrak{d}_S^*, \qquad [s : t] \longmapsto [sa_{0,y} : sa_{0,z} : sa_{1,y} : sa_{1,z} : sa_{2,y} : t],$$

where the six entries are the coefficients in the displayed order. On the chart $s = 1$, this is the divisor D_t. Since $D_{t_0} \notin \mathcal{Y}_S$, the image of $\overline{\gamma}$ is not contained in the closed set $\mathcal{Y}_S$. Therefore, $\overline{\gamma}^{-1}(\mathcal{Y}_S)$ is a proper closed subset of $\mathbb{P}^1$ and hence is finite. Set $E := \{t \in \mathbb{C}^* \mid D_t \in \mathcal{Y}_S\}$. For $t \in \mathbb{C}^* \setminus E$, the construction of $\mathcal{Y}_S$ and Lemma 4.8 give the asserted embedding.

We now prove failure of hyperbolic embedding at $t = 0$ directly from the neighborhood criterion of Section 2.3. Fix $R > 1$ and put $A_R := |a_{1,y}| + |a_{1,z}|\mathrm{e}^{(\ell+2)R}$, $B_R := |a_{0,y}| + |a_{0,z}|\mathrm{e}^{(2\ell+2)R}$. Choose $\varepsilon_R \neq 0$ so that

$$|\varepsilon_R| < \min\left\{1, \frac{1}{R}, \frac{|a_{2,y}|}{2(1 + A_R + B_R)}\right\}.$$

For $|s| \leq R$, direct substitution gives

$$p_0(1, \varepsilon_R, 1, e^s) = a_{2,y} + \varepsilon_R(a_{1,y} + a_{1,z}\mathrm{e}^{(\ell+2)s}) + \varepsilon_R^2(a_{0,y} + a_{0,z}\mathrm{e}^{(2\ell+2)s}).$$

The choice of ε_R gives

$$|p_0(1, \varepsilon_R, 1, e^s) - a_{2,y}| \leq |\varepsilon_R| A_R + |\varepsilon_R|^2 B_R < |a_{2,y}|/2, \qquad |s| \leq R.$$

Consequently $f_R\colon \{|s| < R\} \longrightarrow U_0$, $f_R(s) = [(1, \varepsilon_R, 1, e^s)]$ is well defined. Here brackets denote the Cox quotient, and all four coordinates of $f_R(s)$ are nonzero. As $R \to \infty$, $f_R(0) \longrightarrow [(1, 0, 1, 1)]$, $f_R(1) \longrightarrow [(1, 0, 1, e)]$. The boundary points $[(1, 0, 1, 1)]$ and $[(1, 0, 1, e)]$ are distinct because their z/y coordinates are 1 and e. The convergences place both points in $\overline{U_0}$. Distance decrease for the disk map $\zeta \mapsto f_R(R\zeta)$ yields $d_{U_0}(f_R(0), f_R(1)) \leq \rho_{\mathbb{D}}(0, 1/R) \longrightarrow 0$. For any neighborhoods V_0, V_1 of $[(1, 0, 1, 1)]$ and $[(1, 0, 1, e)]$, respectively, $f_R(0) \in V_0$ and $f_R(1) \in V_1$ for sufficiently large R. Hence $d_{U_0}(V_0 \cap U_0, V_1 \cap U_0) = 0$, contrary to the positive-distance condition in Section 2.3. ∎

The contraction sends the limiting curve ϕ to $[1 : 0 : 0]$. Write p_t for the displayed Cox polynomial and $q_t(X, Y, Z) := p_t(X, 1, Y, Z)$. Every monomial of q_t contains Y or Z, so $[1 : 0 : 0]$ belongs to the closure of the torus divisor for every t. In particular, π does not map the whole complement $\mathbb{F}_\ell \setminus \operatorname{Supp} D_0$ into the complement of the pushforward divisor. It does identify the dense torus complements. Proposition 7.8 compares the embeddings of U_0 in the two compactifications.

Proposition 7.8 (comparison after contraction)**.** *For each $\ell \geq 1$, there is a nonempty Zariski open subset $\Omega \subset (\mathbb{C}^*)^5$ of coefficient tuples $(a_{0,y}, a_{0,z}, a_{1,y}, a_{1,z}, a_{2,y})$ such that the hypothesis concerning t_0 in Proposition 7.7 holds and, at $t = 0$, $U_0 \hookrightarrow \mathbb{P}(\ell, 1, 1)$ is a Kobayashi hyperbolic embedding. For the same coefficients, $U_0 \hookrightarrow \mathbb{F}_\ell$ is not a hyperbolic embedding. Thus the dense torus complement is the same complex manifold in the two compactifications, but its hyperbolic embedding behavior differs.*

Proof. We first exclude entire curves in the dense-torus divisor and its complement for general coefficients. At $t = 0$ its support in the (u, v) characters from (7.6) is

$$S_0 = S \setminus \{(0, 2)\} = \{(0, 0), (0, 1), (2\ell + 2, 0), (\ell + 2, 1), (2, 2)\}.$$

Every nonzero linear functional $\nu(i, j) = \alpha i + \beta j$ takes at least three distinct values on S_0. If $\alpha = 0$, use the three distinct second coordinates 0, 1, 2; if $\beta = 0$, use the distinct first coordinates 0, $\ell+2$, $2\ell + 2$. If $\alpha\beta \neq 0$, the values at (0,0), (0,1), $(2\ell + 2, 0)$ are distinct unless $\beta = (2\ell + 2)\alpha$. In that case the value at $(\ell + 2, 1)$ is $(3\ell + 4)\alpha$, distinct from the first two values. The support has affine dimension two, so Lemma 4.3 gives a proper closed local exceptional locus $Y_{S_0} \subset \mathbb{P}^4$. Let Ω_2 be the nonempty Zariski open subset of the coefficient torus on which $[a] \notin Y_{S_0}$. Lemma 4.6 then excludes entire curves both in the torus divisor and in U_0.

We next verify the boundary conditions in the contracted surface. Its torus-divisor closure is defined by the weighted homogeneous equation

$$q_0(X, Y, Z) = a_{0,y}Y^{2\ell+2} + a_{0,z}Z^{2\ell+2} + a_{1,y}XY^{\ell+2} + a_{1,z}XZ^{\ell+2} + a_{2,y}X^2Y^2.$$

No coordinate divisor is a component, since none of X, Y, Z divides q_0. On the one-dimensional orbits in $X = 0$, $Y = 0$, and $Z = 0$, respectively, the equations reduce, after an invertible monomial factor, to

$$a_{0,y}u^{2\ell+2} + a_{0,z}, \qquad u = Y/Z, \qquad a_{0,z} + a_{1,z}\xi, \qquad \xi = X/Z^\ell,$$
$$a_{0,y} + a_{1,y}\eta + a_{2,y}\eta^2, \qquad \eta = X/Y^\ell.$$

All these coefficients are nonzero. Each zero set in $\mathbb{C}^*$ is consequently finite and nonempty. It contains no nonconstant entire curve, and its complement in $\mathbb{C}^*$ is a sphere with at least three punctures, hence is Brody hyperbolic. The remaining orbits are points. Tiba's boundary criterion [11, Lemma 3], applied to the reduced torus divisor and its closure in the projective toric surface $\mathbb{P}(\ell, 1, 1)$, gives the asserted embedding for coefficients in Ω_2. The hypotheses of [11, Lemma 3] concern the reduced torus divisor and its closure, and impose no Cartier assumption on the weighted-projective divisor class.

Finally, we arrange the t_0 hypothesis simultaneously. Choose finitely many homogeneous equations G_j defining the proper global locus $\mathcal{Y}_S \subset \mathbb{P}^5$ for the six-point support. Define

$$\Omega_1 := \{a \in (\mathbb{C}^*)^5 \mid G_j(a, t) \not\equiv 0 \text{ for some } j\}.$$

The set Ω_1 is Zariski open because the condition is the nonvanishing of at least one coefficient of a polynomial $G_j(a, t)$ viewed as a polynomial in t. It is nonempty because the complement of the proper closed set $\mathcal{Y}_S$ meets the dense six-coordinate coefficient torus. For $a \in \Omega_1$, only finitely many t belong to $\mathcal{Y}_S$, so some $t_0 \in \mathbb{C}^*$ avoids it. The coefficient torus is irreducible, and hence $\Omega := \Omega_1 \cap \Omega_2$ is nonempty and Zariski open. Proposition 7.7 supplies the failure of embedding in $\mathbb{F}_\ell$ at $t = 0$ for these same tuples. ∎

For the Cartier comparison, we also record the auxiliary ample class $D - D_y = 2D_w + (2\ell+1)D_y$. It is included only to compare the Cartier behavior after decreasing the fiber coefficient by one.

Every member D_t of the displayed pencil, including D_0, still has class D; the class $D - D_y$ is not produced by setting $t = 0$. In the following table the symbols $\mathcal{O}(d)$ denote the rank-one reflexive sheaves associated with the pushforward Weil classes. Since D_w is contracted and D_y pushes forward to a weight-one coordinate divisor, the two pushforward degrees are $2\ell + 2$ and $2\ell + 1$, respectively. On the well-formed weighted projective surface $\mathbb{P}(\ell, 1, 1)$, the rank-one reflexive sheaf $\mathcal{O}(d)$ is invertible precisely when $\ell \mid d$. Indeed, at $[1 : 0 : 0]$ the local cyclic quotient has character group $\mathbb{Z}/\ell\mathbb{Z}$, and the class of $\mathcal{O}(d)$ is d modulo ℓ; away from this point the surface is smooth. The pushforwards of the two classes therefore satisfy

class on $\mathbb{F}_\ell$	pushforward class	values of $\ell \geq 1$ for which the class is Cartier
$2D_w + (2\ell + 2)D_y$	$\mathcal{O}(2\ell + 2)$	$\ell = 1, 2$
$2D_w + (2\ell + 1)D_y$	$\mathcal{O}(2\ell + 1)$	$\ell = 1$

For $\ell \geq 3$, even the first pushforward class is already non-Cartier. Moreover, the ample divisor on $\mathbb{F}_\ell$ is not the pullback of the corresponding class on the contracted surface: for $E = D_w$, its intersection with E is 2 (respectively 1 for the second class), whereas a Cartier pullback has intersection zero with E. Consequently, the integral-polytope criterion for ample Cartier classes cannot be transferred directly to these non-Cartier weighted-projective classes. Proposition 7.8 instead uses the local Laurent-polynomial criterion and Tiba's boundary criterion for the closure of the torus divisor, with the boundary orbits checked explicitly.

A Reproducibility of the finite computations

The supplementary file `verify_toric_computations.py` uses only the Python standard library (Python 3.9 or later) and exact integer arithmetic. Running

```
python3 verify_toric_computations.py --output verification_results.json
```

checks the assertions below and writes the direction list, layer supports, lattice coordinates, and mixed-volume data to the indicated JSON file. The supplied `verification_results.json` is the output of this command. The program checks the finite combinatorics and the degree bookkeeping; it does not expand symbolic resultants or replace the analytic proofs and the sparse-resultant arguments in the text.

For a three-dimensional support, the program forms all independent pairs in $S - S$, normalizes their cross products to primitive normals up to sign, and removes duplicates. It then groups S by the integer values $\langle n, I\rangle$. In dimension two it uses the primitive normals to the nonzero difference vectors. These procedures enumerate exactly the hyperplanes in the definition of $\mathcal{H}_S$. They verify the 505 directions for $3\Delta_3 \cap \mathbb{Z}^3$, the 93 directions for the eight-point support, the four sparse facets and six edges, and each stage of the five-coefficient deletion sequence in Proposition 7.4.

Here are lattice coordinates for the two types of four-layer direction in Proposition 7.6. For $n = e_3$, the layer $i_3 = 3 - m$ has coordinates (i_1, i_2) and polygon $m\Delta_2$. For $n = e_1 + e_2$, the layer $i_1 + i_2 = m$ is based at $(0, m, 0)$. In the basis $(0, 0, 1)$, $(1, -1, 0)$ of $H \cap M$, its coordinates are (i_3, i_1) and its polygon is $[0, 3-m] \times [0, m]$. Permutations of the four homogeneous coordinates give the other five directions. Alternatively, for any of the seven primitive normals n (all of whose entries are 0 or 1), choose p with $n_p = 1$. The vectors $e_j - n_j e_p$, $j \neq p$, form a basis of $H \cap M$. The program uses this basis separately for every direction.

For each triple of retained layer supports with no point layer, the program checks the essential-subfamily dimension conditions and that the greatest common divisor of the two-by-two determinants of their within-layer differences is 1. The latter verifies the full-lattice condition in

the resultant multiplicity formula. Polygon areas are computed by the exact shoelace formula after taking convex hulls, and $\mathrm{MV}(P,Q) = \mathrm{Area}(P+Q) - \mathrm{Area}(P) - \mathrm{Area}(Q)$. The resulting reduced components have the following degrees:

component type	number	degree each	contribution
coordinate hyperplane	4	1	4
face-parallel noncoordinate eliminant	4	11	44
other noncoordinate eliminant	6	14	84
other noncoordinate eliminant	6	18	108
total	20		240

Repeated coordinate components are counted once. The omitted coefficient sets of the sixteen noncoordinate components are checked to be distinct; their use to distinguish the components is justified in the proof of Proposition 7.6. The program also verifies the layer bounds giving codimension at least 4 for other full-support directions, 2 on facets, and 3 on edges. The coefficient-chart boundary has codimension 10. Thus these additional sets do not change the divisorial degree.

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