Decreasing Filtrations, C_2-Algebra and Twisted Modules

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Abstract. We investigate a question posed by Gaberdiel and Gannon (2009) concerning the relationship between C_2-algebras and twisted modules. To each twisted module W of a vertex algebra V, we first associate a decreasing sequence of subspaces $\{E_n^T(W)\}_{n\in\mathbb{Z}}$ and demonstrate that the associated graded vector space $\mathrm{gr}_{\mathcal{E}}^T(W)$ is a twisted module of vertex Poisson algebra $\mathrm{gr}_{\mathcal{E}}^T(V)$. We introduce another decreasing sequence of subspaces $\{C_n^T(W)\}_{n\in\mathbb{Z}_{\geq 2}}$ and establish a connection between $\{E_n^T(W)\}_{n\in\mathbb{Z}}$ and $\{C_n^T(W)\}_{n\in\mathbb{Z}_{\geq 2}}$. By utilizing the twisted module $\mathrm{gr}_{\mathcal{E}}^T(W)$ of vertex Poisson algebra $\mathrm{gr}_{\mathcal{E}}^T(V)$, we prove that for any twisted module W of a vertex algebra V, C_2-cofiniteness implies C_n-cofiniteness for all $n \geq 2$. Furthermore, we employ $\mathrm{gr}_{\mathcal{E}}^T(W)$ to study generating subspaces of $\frac{1}{T}\mathbb{N}$-graded twisted modules of lower truncated $\mathbb{Z}$-graded vertex algebras.

Key words: vertex algebra; twisted module; decreasing filtrations; C_2-algebra

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1 Introduction

In [17], Li introduced and investigated decreasing filtrations for vertex algebras. To any vertex algebra V, a canonical decreasing sequence $\mathcal{E}$ of subspaces is associated and it was proven that the associated graded vector space $\mathrm{gr}_{\mathcal{E}}(V)$ naturally forms an $\mathbb{N}$-graded vertex Poisson algebra. The decreasing sequence $\mathcal{E}$ was related to Zhu's decreasing sequence $\mathcal{C} = \{C_n\}_{n\geq 2}$. The degree-zero subspace E_0/E_1 of $\mathrm{gr}_{\mathcal{E}}(V)$ was shown to coincide with Zhu's Poisson algebra $V/C_2(V)$. It was demonstrated that for any vertex algebra V, if V is C_2-cofinite, then V is E_n-cofinite and C_{n+2}-cofinite for all $n \geq 0$ (see also [5, 12, 18]). Additionally, [17] established that if V is a C_2-cofinite vertex algebra and W is a C_2-cofinite V-module, then W is C_n-cofinite for all $n \geq 2$. $\mathrm{gr}_{\mathcal{E}}(V)$ was also utilized to study generating subspaces of certain types for lower truncated $\mathbb{Z}$-graded vertex algebras in [17].

The Li filtration has played a pivotal role in the theory of vertex algebras. It was employed to clarify the equivalence of the two finiteness conditions on vertex algebras in [2] and to study W-algebras in [3]. Li filtration for any SUSY vertex algebra was introduced in [21] where it was proven that the associated graded superspace of the filtration has a structure of a SUSY vertex Poisson algebra.

In [11], Gaberdiel and Gannon raised a question about clarifying the relation between C_2-algebra and twisted modules. In this paper, we will investigate this question using decreasing filtrations. We first introduce and study decreasing filtrations for twisted modules of vertex algebras. Let V be a vertex algebra and let g be an automorphism of V of finite period T. In order to match our results well with the cases of twisted modules, we first construct a decreasing sequence $\mathcal{E}_V^T$ for each vertex algebra V, which is a slight generalization of the canonical decreasing

sequence $\mathcal{E}$ in [17], and demonstrate that the associated graded vector space $\mathrm{gr}_{\mathcal{E}}^T(V)$ is naturally a vertex Poisson algebra, where for $n \in \mathbb{Z}$, $E_n^T(V)$ is linearly spanned by the vectors

$$u^{(1)}_{-1-k_1} u^{(2)}_{-1-k_2} \cdots u^{(r)}_{-1-k_r} v$$

for $r \geq 1$, $u^{(1)}, u^{(2)}, \ldots, u^{(r)}, v \in V$, $k_1, k_2, \ldots, k_r \geq 0$ with $k_1 + k_2 + \cdots + k_r \geq \frac{n}{T}$.

To any g-twisted module W of a vertex algebra V we associate a decreasing sequence $\mathcal{E}_W^T$ of subspaces $E_n^T(W)$ for $n \in \mathbb{Z}$, which is linearly spanned by the vectors

$$u^{(1)}_{-1-k_1+\frac{r_1}{T}} u^{(2)}_{-1-k_2+\frac{r_2}{T}} \cdots u^{(s)}_{-1-k_s+\frac{r_s}{T}} w$$

for $s \geq 1$, $u^{(i)} \in V^{r_i}$, $0 \leq r_i \leq T-1$, $1 \leq i \leq s$, $w \in W$, $k_1, k_2, \ldots, k_s \geq 0$ with $k_1 + k_2 + \cdots + k_s - \frac{r_1+r_2+\cdots+r_s}{T} \geq \frac{n}{T}$. We define the notion of twisted modules for vertex Poisson algebras, which is a generalization of the concept of modules of vertex Poisson algebras. We then prove that the associated graded vector space $\mathrm{gr}_{\mathcal{E}}^T(W)$ is a twisted module for the vertex Poisson algebra $\mathrm{gr}_{\mathcal{E}}^T(V)$.

In this paper, we introduce the decreasing sequence $\mathcal{C}_W^T = \{C_n^T(W)\}_{n \in \mathbb{Z}_{\geq 2}}$ associated with the twisted module W, where for $n \in \mathbb{Z}_{\geq 2}$,

$$C_n^T(W) = \mathrm{span}\{u_{-n+\frac{p}{T}} w \mid u \in V^p,\ w \in W,\ p = 0, 1, \ldots, T-1\}.$$

We establish a relationship between the sequences $\mathcal{E}_W^T$ and $\mathcal{C}_W^T$. Using the twisted module $\mathrm{gr}_{\mathcal{E}}^T(W)$ of vertex Poisson algebra $\mathrm{gr}_{\mathcal{E}}^T(V)$, we prove that for any twisted module W of a vertex algebra V, C_2-cofiniteness implies C_n-cofiniteness for all $n \geq 2$. Recently, C_n-cofinite twisted modules for C_2-cofinite vertex operator algebras with general automorphisms were studied in [20].

As shown in this paper, for certain twisted V-modules W, both sequences $\mathcal{E}_W^T$ and $\mathcal{C}_W^T$ are trivial in the sense that $E_n^T(W) = C_{n+2}^T(W) = W$ for all $n \geq 0$. By using the connection between the two decreasing sequences, we prove that if $V = \coprod_{n \geq t} V_{(n)}$ is a lower truncated $\mathbb{Z}$-graded vertex algebra for some $t \in \mathbb{Z}$ and $W = \bigoplus_{n \in \frac{1}{T}\mathbb{N}} W_{(n)}$ is a $\frac{1}{T}\mathbb{N}$-graded g-twisted V-module, then

$$C_n^T(W) \subset \coprod_{k \geq n+t-\frac{2T-1}{T}} W_{(k)} \qquad \text{for } n \geq 2,$$

$$E_m^T(W) \subset \coprod_{k \geq n+t-\frac{2T-1}{T}} W_{(k)} \qquad \text{for } m \geq (n-2)T(T+1)2^{n-3}.$$

Consequently,

$$\bigcap_{n \geq 0} E_n^T(W) = \bigcap_{n \geq 2} C_n^T(W) = 0.$$

In this case, both sequences are filtrations. Furthermore, using this result and $\mathrm{gr}_{\mathcal{E}}^T(W)$ we show that if there exists a graded subspace U of V such that $V = U + C_2(V)$ and a graded subspace M of a $\frac{1}{T}\mathbb{N}$-graded g-twisted V-module W such that $W = M + C_2^T(W)$, then U and M generate W with a certain spanning property.

This paper is organized as follows: In Section 2, we review the concepts of vertex algebras and twisted modules. In Section 3, we recall the definition of a vertex Poisson algebra and introduce the notion of twisted modules for vertex Lie algebras and vertex Poisson algebras. We then construct a decreasing sequence $\mathcal{E}_V^T$, and show that the associated graded vector space $\mathrm{gr}_{\mathcal{E}}^T(V)$ is a vertex Poisson algebra. In Section 4, we first construct the decreasing filtration $\mathcal{E}_W^T$ of twisted modules and the associated graded vector space $\mathrm{gr}_{\mathcal{E}}^T(W)$, then we prove $\mathrm{gr}_{\mathcal{E}}^T(W)$ is

a twisted module of vertex Poisson algebra $\mathrm{gr}^T_{\mathcal{E}}(V)$. In Section 5, we introduce the decreasing sequence $\mathcal{C}^T_W$ and establish the relationship between the sequences $\mathcal{E}^T_W$ and $\mathcal{C}^T_W$. Finally, in Section 6, by utilizing the twisted module $\mathrm{gr}^T_{\mathcal{E}}(W)$ of vertex Poisson algebra $\mathrm{gr}^T_{\mathcal{E}}(V)$, we prove that for any twisted module W of a vertex algebra V, C_2-cofiniteness implies C_n-cofiniteness for all $n \geq 2$. We also study generating subspaces of $\frac{1}{T}\mathbb{N}$-graded g-twisted V-modules of lower truncated $\mathbb{Z}$-graded vertex algebras.

Throughout this paper, we denote by $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{Z}_{\geq 2}$ and $\mathbb{N}$ the sets of rational numbers, integers, integers greater than or equal to 2 and nonnegative integers, respectively.

2 Preliminaries

In this section, we recall the notions of vertex algebras and twisted modules. We begin by recalling the notion of vertex algebra (see [4, 9, 10, 15]).

Definition 2.1. A *vertex algebra* is a vector space V, equipped with a linear map

$$\begin{aligned} Y(\cdot,x)\colon\ V &\to \mathrm{Hom}(V, V((x))) \subset (\mathrm{End}\, V)\big[\big[x, x^{-1}\big]\big] \\ v &\mapsto Y(v,x) = \sum_{n\in\mathbb{Z}} v_n x^{-n-1} \qquad (\text{where } v_n \in \mathrm{End}\, V), \end{aligned}$$

and equipped with a vector $\mathbf{1} \in V$, satisfying the conditions that for $v \in V$,

$$Y(\mathbf{1},x)v = v, \qquad Y(v,x)\mathbf{1} \in V[[x]] \qquad \text{and} \qquad \lim_{x\to 0} Y(v,x)\mathbf{1} = v,$$

and for $u, v, w \in V$, the *Jacobi identity* holds

$$\begin{aligned} &x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y(u,x_1)Y(v,x_2)w - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y(v,x_2)Y(u,x_1)w \\ &\quad = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y(Y(u,x_0)v,x_2)w, \end{aligned} \tag{2.1}$$

where $\delta(x) = \sum_{n\in\mathbb{Z}} x^n$. Elementary properties of the delta function can be found in [15]. All binomial expressions (here and below) are to be expanded in nonnegative integral powers of the second variable, $(x+y)^n = \sum_{k\in\mathbb{N}} \binom{n}{k} x^{n-k}y^k$ with $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$ for $n \in \mathbb{Q}$.

Taking Res_{x_0} and Res_{x_1} of (2.1) and equating the related coefficients, we have Borcherds' commutator formula and iterate formula:

$$[u_m, v_n]w = \sum_{i\in\mathbb{N}} \binom{m}{i} (u_i v)_{m+n-i} w,$$

$$(u_m v)_n w = \sum_{i\in\mathbb{N}} (-1)^i \binom{m}{i} \left(u_{m-i}v_{n+i}w - (-1)^m v_{m+n-i}u_i w\right)$$

for $u, v, w \in V, m, n \in \mathbb{Z}$.

Define a (canonical) linear operator $\mathcal{D}$ on V by

$$\mathcal{D}(v) = v_{-2}\mathbf{1} \qquad \text{for } v \in V.$$

Then

$$Y(v,x)\mathbf{1} = \mathrm{e}^{x\mathcal{D}}v \qquad \text{for } v \in V.$$

Furthermore,

$$[\mathcal{D}, v_n] = (\mathcal{D}v)_n = -nv_{n-1} \qquad \text{for } v \in V,\ n \in \mathbb{Z}.$$

Next we recall the definitions of an automorphism of a vertex algebra V and a g-twisted V-module for a finite order automorphism g of V (see [6, 9]).

Definition 2.2. An *automorphism* of a vertex algebra V is a linear isomorphism g of V such that $gY(v,x)g^{-1} = Y(g(v),x)$ for any $v \in V$.

Let g be a finite order automorphism of V with a period T in the sense that T is a positive integer such that $g^T = 1$. Then

$$V = \bigoplus_{r=0}^{T-1} V^r,$$

where $V^r = \{v \in V | g(v) = \mathrm{e}^{\frac{2\pi \mathrm{i} r}{T}} v\}$ for $r \in \mathbb{Z}$. Note that for $r, s \in \mathbb{Z}$, $V^r = V^s$ if $r \equiv s \pmod{T}$.

Definition 2.3. Let V be a vertex algebra. A *g-twisted V-module* is a vector space M equipped with a linear map

$$\begin{aligned} Y_M(\cdot,x)\colon\ & V \to (\operatorname{End} M)\big[\big[x^{\frac{1}{T}}, x^{-\frac{1}{T}}\big]\big], \\ & v \mapsto Y_M(v,x) = \sum_{n \in \frac{1}{T}\mathbb{Z}} v_n x^{-n-1} \qquad (\text{where } v_n \in \operatorname{End} M), \end{aligned}$$

which satisfies the following conditions: For all $u \in V^r$, $v \in V$, $w \in M$ with $0 \le r \le T-1$,

$$\begin{aligned} & Y_M(u,x) = \sum_{n \in \frac{r}{T}+\mathbb{Z}} u_n x^{-n-1}, \\ & u_n w = 0 \qquad \text{for } n \in \tfrac{r}{T} + \mathbb{Z} \text{ sufficiently large}, \\ & Y_M(\mathbf{1},x) = \mathrm{Id}_M, \end{aligned}$$

and the *twisted Jacobi identity*

$$\begin{aligned} & x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_M(u,x_1)Y_M(v,x_2)w - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y_M(v,x_2)Y_M(u,x_1)w \\ & \quad = x_2^{-1}\left(\frac{x_1 - x_0}{x_2}\right)^{-\frac{r}{T}} \delta\left(\frac{x_1 - x_0}{x_2}\right) Y_M(Y(u,x_0)v,x_2)w. \end{aligned} \tag{2.2}$$

And for $u \in V^r, w \in M$, $0 \le r \le T-1$, we also have

$$Y_M(\mathcal{D}u,x)w = \frac{\mathrm{d}}{\mathrm{d}x} Y_M(u,x)w,$$

which implies

$$(\mathcal{D}u)_{-n+\frac{r}{T}} w = \left(n - \frac{r}{T}\right) u_{-n-1+\frac{r}{T}} w$$

for $n \in \mathbb{Z}$.

Taking Res_{x_0} of (2.2) we have the commutator relation in twisted case:

$$[Y_M(u,x_1), Y_M(v,x_2)]w = \operatorname{Res}_{x_0} x_2^{-1}\left(\frac{x_1 - x_0}{x_2}\right)^{-\frac{r}{T}} \delta\left(\frac{x_1 - x_0}{x_2}\right) Y_M(Y(u,x_0)v,x_2)w. \tag{2.3}$$

For $u \in V^r$, $v \in V^s$, $0 \le r, s \le T-1$, $m, n \in \mathbb{Z}$, equating the coefficients of $x_1^{-m-r/T-1} x_2^{-n-s/T-1}$ in (2.3), we get the twisted commutator formula (cf. [14]):

$$[u_{m+\frac{r}{T}}, v_{n+\frac{s}{T}}]w = \sum_{i \in \mathbb{N}} \binom{m + \frac{r}{T}}{i} (u_i v)_{m+n-i+\frac{r+s}{T}} w. \tag{2.4}$$

From (2.3), there is a nonnegative integer N such that

$$(x_1-x_2)^N[Y_M(u,x_1),Y_M(v,x_2)]w=0. \tag{2.5}$$

Then for $m\in\mathbb{Z}$, $n\in\frac{1}{T}\mathbb{Z}$, $u\in V^r$, $v\in V$, $w\in M$, $0\le r\le T-1$, we have the twisted iterate formula (cf. [14]):

$$\begin{aligned}(u_mv)_nw=&\sum_{i\in\mathbb{N}}\sum_{m\le j<N}((-1)^i\binom{-\frac{r}{T}}{j-m}\binom{j}{i}u_{j-i+\frac{r}{T}}v_{m+n+i-j-\frac{r}{T}}w\\&-(-1)^{i+j}\binom{-\frac{r}{T}}{j-m}\binom{j}{i}v_{m+n-i-\frac{r}{T}}u_{i+\frac{r}{T}}w).\end{aligned} \tag{2.6}$$

And we know that the twisted Jacobi identity is equivalent to the twisted commutator formula (2.3) and the following associativity relation:

$$(x_0+x_2)^{l+\frac{r}{T}}Y_M(u,x_0+x_2)Y_M(v,x_2)w=(x_2+x_0)^{l+\frac{r}{T}}Y_M(Y(u,x_0)v,x_2)w \tag{2.7}$$

for $u\in V^r$, $0\le r\le T-1$, $w\in M$, where l is a nonnegative integer such that $x^{l+\frac{r}{T}}Y_M(u,x)w$ involves only nonnegative integral powers of x.

A vertex algebra V equipped with a $\mathbb{Z}$-grading $V=\coprod_{n\in\mathbb{Z}}V_{(n)}$ is called a *$\mathbb{Z}$-graded vertex algebra* if $\mathbf{1}\in V_{(0)}$ and for $u\in V_{(k)}$, $k,m,n\in\mathbb{Z}$,

$$u_mV_{(n)}\subset V_{(n+k-m-1)}.$$

For $u\in V_{(k)}$ with $k\in\mathbb{Z}$, denote $\operatorname{wt}u=k$. We say that a $\mathbb{Z}$-graded vertex algebra $V=\coprod_{n\in\mathbb{Z}}V_{(n)}$ is *lower truncated* if $V_{(n)}=0$ for n sufficiently small. An *$\mathbb{N}$-graded vertex algebra* is defined in the obvious way.

Definition 2.4. A *$\frac{1}{T}\mathbb{N}$-graded g-twisted V-module* is a g-twisted module with a $\frac{1}{T}\mathbb{N}$-grading

$$M=\bigoplus_{n\in\frac{1}{T}\mathbb{N}}M_{(n)},$$

which satisfies the following

$$v_mM_{(n)}\subseteq M_{(n+k-m-1)}$$

for homogeneous vector $v\in V_{(k)}$, $k\in\mathbb{Z}$, $m\in\frac{1}{T}\mathbb{Z}$, $n\in\frac{1}{T}\mathbb{N}$.

For $w\in M_{(n)}$ with $n\in\frac{1}{T}\mathbb{N}$, we denote $\operatorname{wt}w=n$. We also need the following formula.

Lemma 2.5. *Let V be a vertex algebra and let M be a g-twisted V-module. For $u\in V^r$, $v\in V$, $w\in M$, $0\le r\le T-1$, $m\in\mathbb{Z}$, $n\in\frac{1}{T}\mathbb{Z}$, there exist nonnegative integers l and k such that*

$$(u_mv)_nw=\sum_{i=0}^{k}\sum_{j\in\mathbb{N}}\binom{-l-\frac{r}{T}}{i}\binom{m+i}{j}(-1)^ju_{m+l+i-j+\frac{r}{T}}v_{n-l-i+j-\frac{r}{T}}w. \tag{2.8}$$

Proof. For $u\in V^r$, $v\in V$, $w\in M$, $0\le r\le T-1$, from (2.7) there exists a nonnegative integer l such that

$$(x_0+x_2)^{l+\frac{r}{T}}Y_M(u,x_0+x_2)Y_M(v,x_2)w=(x_2+x_0)^{l+\frac{r}{T}}Y_M(Y(u,x_0)v,x_2)w.$$

For $m\in\mathbb{Z}$, $n\in\frac{1}{T}\mathbb{Z}$, we have

$$(u_mv)_nw=\operatorname{Res}_{x_0}\operatorname{Res}_{x_2}x_0^mx_2^nY_M(Y(u,x_0)v,x_2)w$$

$$= \mathrm{Res}_{x_0}\mathrm{Res}_{x_2} x_0^m x_2^n (x_2+x_0)^{-l-\frac{r}{T}}((x_2+x_0)^{l+\frac{r}{T}}Y_M(Y(u,x_0)v,x_2)w)$$
$$= \mathrm{Res}_{x_0}\mathrm{Res}_{x_2} \sum_{i\in\mathbb{N}} \binom{-l-\frac{r}{T}}{i} x_0^{m+i} x_2^{n-l-i-\frac{r}{T}}((x_2+x_0)^{l+\frac{r}{T}}Y_M(Y(u,x_0)v,x_2)w).$$

Let k be a nonnegative integer such that $x_0^{m+k}Y(u,x_0)v \in V[[x_0]]$. Then

$$\begin{aligned}(u_m v)_n w &= \mathrm{Res}_{x_0}\mathrm{Res}_{x_2} \sum_{i=0}^{k} \binom{-l-\frac{r}{T}}{i} x_0^{m+i} x_2^{n-l-i-\frac{r}{T}}((x_2+x_0)^{l+\frac{r}{T}}Y_M(Y(u,x_0)v,x_2)w)\\
&= \mathrm{Res}_{x_0}\mathrm{Res}_{x_2} \sum_{i=0}^{k} \binom{-l-\frac{r}{T}}{i} x_0^{m+i} x_2^{n-l-i-\frac{r}{T}}((x_0+x_2)^{l+\frac{r}{T}}\\
&\quad \times Y_M(u,x_0+x_2)Y_M(v,x_2)w)\\
&= \mathrm{Res}_{x_1}\mathrm{Res}_{x_2} \sum_{i=0}^{k} \binom{-l-\frac{r}{T}}{i} (x_1-x_2)^{m+i} x_2^{n-l-i-\frac{r}{T}} x_1^{l+\frac{r}{T}} Y_M(u,x_1)Y_M(v,x_2)w\\
&= \mathrm{Res}_{x_1}\mathrm{Res}_{x_2} \sum_{i=0}^{k} \binom{-l-\frac{r}{T}}{i} \sum_{j\in\mathbb{N}} \binom{m+i}{j} (-1)^j x_1^{m+l+i-j+\frac{r}{T}} x_2^{n-l-i+j-\frac{r}{T}}\\
&\quad \times Y_M(u,x_1)Y_M(v,x_2)w.\end{aligned}$$

Then we have

$$(u_m v)_n w = \sum_{i=0}^{k}\sum_{j\in\mathbb{N}} \binom{-l-\frac{r}{T}}{i}\binom{m+i}{j}(-1)^j u_{m+l+i-j+\frac{r}{T}} v_{n-l-i+j-\frac{r}{T}} w$$

for $u \in V^r$, $v \in V$, $w \in M$, $0 \le r \le T-1$, $m \in \mathbb{Z}$, $n \in \frac{1}{T}\mathbb{Z}$. This completes the proof. ■

3 Decreasing sequence $\mathcal{E}_V^T$ and the vertex Poisson algebra $\mathrm{gr}_{\mathcal{E}}^T(V)$

In this section, we first recall the definition of a vertex Lie algebra and a vertex Poisson algebra. In order to match our results well with the decreasing sequences of twisted modules, we then construct a decreasing sequence $\mathcal{E}_V^T$ for each vertex algebra V, which is a slight generalization of the canonical decreasing sequence $\mathcal{E}$ in [17], and show that the associated graded vector space $\mathrm{gr}_{\mathcal{E}}^T(V)$ is naturally a vertex Poisson algebra. Because the proofs are essentially the same as in the decreasing sequence $\mathcal{E}$ case (see the proofs in [17, Section 2]), we will omit the proofs.

A vertex algebra V is called a *commutative vertex algebra* if

$$[u_m, v_n] = 0 \qquad \text{for } u,v \in V,\ m,n \in \mathbb{Z}. \tag{3.1}$$

It is well known (see [4, 10]) that (3.1) is equivalent to

$$u_n = 0 \qquad \text{for } u \in V,\ n \ge 0.$$

A commutative vertex algebra exactly amounts to a unital commutative associative algebra equipped with a derivation, which is often called a *differential algebra*.

The following definition of the notion of vertex Lie algebra is due to [13, 19].

Definition 3.1. A *vertex Lie algebra* is a vector space V equipped with a linear operator D and a linear map

$$\begin{aligned} Y_-(\cdot,x)\colon\ & V \to \operatorname{Hom}\big(V, x^{-1}V\big[x^{-1}\big]\big), \\ & v \mapsto Y_-(v,x) = \sum_{n\geq 0} v_n x^{-n-1} \end{aligned}$$

such that for $u, v \in V$, $m, n \in \mathbb{N}$,

$$(Dv)_n = -nv_{n-1}, \qquad u_n v = \sum_{i=0}^{\infty} (-1)^{n+i+1} \frac{1}{i!} D^i(v_{n+i}u),$$

$$[u_m, v_n] = \sum_{i=0}^{m} \binom{m}{i} (u_i v)_{m+n-i}.$$

Definition 3.2. An *automorphism* of a vertex Lie algebra V is a linear isomorphism g of V such that $gY_-(v,x)g^{-1} = Y_-(g(v),x)$ for $v \in V$.

Let g be a finite order automorphism of a vertex Lie algebra V with a period T in the sense that T is a positive integer such that $g^T = 1$. Then

$$V = \bigoplus_{r=0}^{T-1} V^r,$$

where $V^r = \big\{v \in V \mid g(v) = \mathrm{e}^{\frac{2\pi i r}{T}} v\big\}$ for $r \in \mathbb{Z}$.

Next we give the definition of the notion of twisted module of vertex Lie algebra.

Definition 3.3. A *g-twisted module* for a vertex Lie algebra V is a vector space W equipped with a linear map

$$\begin{aligned} Y_-^W(\cdot,x)\colon\ & V \to (\operatorname{End} W)\big[\big[x^{\frac{1}{T}}, x^{-\frac{1}{T}}\big]\big], \\ & v \mapsto Y_-^W(v,x) = \sum_{n\in\frac{1}{T}\mathbb{Z}} v_n x^{-n-1}, \qquad v_n \in \operatorname{End} W, \end{aligned}$$

which satisfies the following conditions: For a linear operator D and $u \in V^p$, $v \in V^q$, $w \in W$ with $0 \leq p, q \leq T-1$, $m, n \in \mathbb{Z}$,

$$Y_-^W(u,x) = \begin{cases} \displaystyle\sum_{n\in\mathbb{N}} u_n x^{-n-1} & \text{for } p = 0, \\ \displaystyle\sum_{n\in\mathbb{Z}} u_{n+\frac{p}{T}} x^{-n-\frac{p}{T}-1} & \text{for } 1 \leq p \leq T-1, \end{cases}$$

$$u_{n+\frac{p}{T}} w = 0 \qquad \text{for } n \text{ sufficiently large},$$

$$(Du)_{n+\frac{p}{T}} = \Big(-n - \frac{p}{T}\Big) u_{n-1+\frac{p}{T}}, \tag{3.2}$$

$$\big[u_{m+\frac{p}{T}}, v_{n+\frac{q}{T}}\big] = \sum_{i\in\mathbb{N}} \binom{m+\frac{p}{T}}{i} (u_i v)_{m+n-i+\frac{p+q}{T}}. \tag{3.3}$$

Note that from Definition 3.3, we have

$$u_{-j+\frac{p}{T}} = \frac{1}{\big(j-1-\frac{p}{T}\big)\big(j-2-\frac{p}{T}\big)\cdots\big(-\frac{p}{T}\big)} \big(D^j u\big)_{\frac{p}{T}}$$

for $u \in V^p$, $j \geq 1$, $1 \leq p \leq T-1$.

Recall the following notion of vertex Poisson algebra from [7, 8, 16].

Definition 3.4. A *vertex Poisson algebra* is a commutative vertex algebra A, or equivalently, a (unital) commutative associative algebra with a derivation ∂, equipped with a vertex Lie algebra structure (Y_-, ∂) such that

$$Y_-(a,x) \in x^{-1}(\mathrm{Der}\, A)\big[x^{-1}\big] \qquad \text{for } a \in A.$$

Next we give the following definition of the notion of twisted module for vertex Poisson algebra.

Definition 3.5. Let A be a vertex Poisson algebra and let g be an automorphism of A of finite period T. A g-twisted A-module is a vector space W equipped with a g-twisted module structure for the underlying vertex Lie algebra (A, Y_-, ∂) in the sense of Definition 3.3, and a bilinear map

$$A \otimes W \longrightarrow W, \qquad a \otimes w \longmapsto a \cdot w$$

satisfying $\mathbf{1} \cdot w = w$ for every $w \in W$ and the compatibility condition

$$Y_-^W(u,x)(v \cdot w) = (Y_-(u,x)v) \cdot w + v \cdot Y_-^W(u,x)w \tag{3.4}$$

for $u, v \in A$ and $w \in W$.

Below for each vertex algebra, we construct a decreasing sequence $\mathcal{E}_V^T = \big\{E_n^T(V)\big\}_{n\in\mathbb{Z}}$, as a slight generalization of the canonical decreasing sequence $\mathcal{E}$ in [17].

Definition 3.6. Let V be a vertex algebra and let g be an automorphism of V with a period T. Define a sequence $\mathcal{E}_V^T = \big\{E_n^T(V)\big\}_{n\in\mathbb{Z}}$ of subspaces of V, where for $n \in \mathbb{Z}$, $E_n^T(V)$ is linearly spanned by the vectors

$$u^{(1)}_{-1-k_1} u^{(2)}_{-1-k_2} \cdots u^{(r)}_{-1-k_r} v$$

for $r \geq 1$, $u^{(1)}, u^{(2)}, \ldots, u^{(r)}, v \in V$, $k_1, k_2, \ldots, k_r \geq 0$ with $k_1 + k_2 + \cdots + k_r \geq \frac{n}{T}$.

The following are some immediate consequences.

Lemma 3.7. *Let V be a vertex algebra and let g be an automorphism of V with a period T. We have*

$$\begin{aligned}
&E_n^T(V) \supset E_{n+1}^T(V) \qquad \text{for } n \in \mathbb{Z},\\
&E_{1+kT}^T(V) = E_{2+kT}^T(V) = \cdots = E_{T+kT}^T(V) \qquad \text{for } k \in \mathbb{Z},\\
&E_n^T(V) = V \qquad \text{for } n \leq 0,\\
&u_{-1-k}E_n^T(V) \subset E_{n+kT}^T(V) \qquad \text{for } u \in V,\ k \geq 0,\ n \in \mathbb{Z}.
\end{aligned}$$

The following gives a stronger spanning property for $E_n^T(V)$.

Lemma 3.8. *Let V be a vertex algebra and let g be an automorphism of V with a period T. For $n \geq 1$, we have*

$$E_n^T(V) = \mathrm{span}\big\{u_{-1-i}v \mid u \in V,\ i \geq 1,\ v \in E_{n-iT}^T(V)\big\}. \tag{3.5}$$

Furthermore, for $n \geq 1$, $E_n^T(V)$ is linearly spanned by the vectors

$$u^{(1)}_{-1-k_1} u^{(2)}_{-1-k_2} \cdots u^{(r)}_{-1-k_r} v$$

for $r \geq 1$, $u^{(1)}, u^{(2)}, \ldots, u^{(r)}, v \in V$, $k_1, k_2, \ldots, k_r \geq 1$ with $k_1 + k_2 + \cdots + k_r \geq \frac{n}{T}$.

First, we have the following special case for $\mathcal{E}_V^T$.

Lemma 3.9. *Let V be a vertex algebra and let g be an automorphism of V with a period T. For $a \in V$, $m, n \in \mathbb{Z}$, we have*

$$a_m E_n^T(V) \subset E_{n-(m+1)T}^T(V).$$

Furthermore,

$$a_m E_n^T(V) \subset E_{n-(m+1)T+1}^T(V) \qquad \text{for } m \geq 0.$$

Now we have the following general case.

Proposition 3.10. *Let V be a vertex algebra and let g be an automorphism of V with a period T. Let $u \in E_r^T(V)$, $v \in E_s^T(V)$ with $r, s \in \mathbb{Z}$. Then*

$$u_n v \in E_{r+s-(n+1)T}^T(V) \qquad \text{for } n \in \mathbb{Z}.$$

Furthermore, we have

$$u_n v \in E_{r+s-(n+1)T+1}^T(V) \qquad \text{for } n \geq 0. \tag{3.6}$$

From Proposition 3.10, we immediately have the following.

Theorem 3.11. *Let V be a vertex algebra, let g be an automorphism of V with a period T and let $\mathcal{E}_V^T = \{E_n^T(V)\}_{n\in\mathbb{Z}}$ be the decreasing sequence defined in Definition 3.6 for V. Set*

$$\mathrm{gr}_{\mathcal{E}}^T(V) = \coprod_{n\in\mathbb{N}} E_{nT}^T(V)/E_{(n+1)T}^T(V).$$

Then $\mathrm{gr}_{\mathcal{E}}^T(V)$ is a vertex Poisson algebra if we define

$$\big(u + E_{(r+1)T}^T(V)\big) \cdot \big(v + E_{(s+1)T}^T(V)\big) = u_{-1}v + E_{(r+s+1)T}^T(V), \tag{3.7}$$

$$\partial\big(u + E_{(r+1)T}^T(V)\big) = \mathcal{D}(u) + E_{(r+2)T}^T(V), \tag{3.8}$$

$$Y_-\big(u + E_{(r+1)T}^T(V), x\big)\big(v + E_{(s+1)T}^T(V)\big) = \sum_{n\in\mathbb{N}} \big(u_n v + E_{(r+s-n+1)T}^T(V)\big)x^{-n-1}$$

for $u \in E_{rT}^T(V)$, $v \in E_{sT}^T(V)$ with $r, s \in \mathbb{N}$.

Remark 3.12. Note that vertex Poisson algebra $\mathrm{gr}_{\mathcal{E}}^T(V)$ in Theorem 3.11 is isomorphic to vertex Poisson algebra $\mathrm{gr}_{\mathcal{E}}(V)$ in [17, Theorem 2.12].

Next we recall the sequence $\mathcal{C}$ introduced by Zhu and we then give a relation between Zhu's Poisson algebra and vertex Poisson algebra $\mathrm{gr}_{\mathcal{E}}^T(V)$.

Definition 3.13. Let V be a vertex algebra. For $n \geq 2$, define

$$C_n(V) = \mathrm{span}\{u_{-n}v \mid u, v \in V\}, \qquad n \geq 2.$$

We say a vertex algebra V is *C_n-cofinite* if $V/C_n(V)$ is finite-dimensional. Similarly we define a vertex algebra V is *E_n-cofinite* if $V/E_n^T(V)$ is finite-dimensional.

Recall the following result of [22].

Proposition 3.14. *Let V be a vertex algebra. Then $V/C_2(V)$ is a Poisson algebra with*

$$(u+C_2(V))\cdot(v+C_2(V)) = u_{-1}v + C_2(V),$$
$$[u+C_2(V), v+C_2(V)] = u_0 v + C_2(V)$$

for $u, v \in V$, and with $\mathbf{1}+C_2(V)$ as the identity element.

From [17] and combining Definitions 3.6 and 3.13, we have

$$E_1^T(V) = E_2^T(V) = \cdots = E_T^T(V) = C_2(V),$$
$$E_{T+1}^T(V) = E_{T+2}^T(V) = \cdots = E_{2T}^T(V) = C_3(V).$$

The following is a relation between Zhu's Poisson algebra and the degree-zero subspace $E_0^T(V)/E_T^T(V)$ of vertex Poisson algebra $\mathrm{gr}_{\mathcal{E}}^T(V)$.

Proposition 3.15. *Let V be a vertex algebra and let g be an automorphism of V with a period T. Then $E_0^T(V)/E_T^T(V)$ is a Poisson algebra which coincides with Zhu's Poisson algebra $V/C_2(V)$, where*

$$\big(u+E_T^T(V)\big)\cdot\big(v+E_T^T(V)\big) = u_{-1}v + E_T^T(V),$$
$$\big[u+E_T^T(V), v+E_T^T(V)\big] = u_0 v + E_T^T(V)$$

for $u, v \in V$.

4 Li filtration of twisted modules

In this section, we give a construction of twisted modules for the vertex Poisson algebra $\mathrm{gr}_{\mathcal{E}}^T(V)$ from twisted modules for the vertex algebra V, using a decreasing filtration.

First, we construct the decreasing sequence $\mathcal{E}_W^T = \{E_n^T(W)\}_{n\in\mathbb{Z}}$ associated with a twisted module.

Definition 4.1. Let V be a vertex algebra and let W be a g-twisted V-module. Define a decreasing sequence $\{E_n^T(W)\}_{n\in\mathbb{Z}}$ of subspaces of W, where for $n\in\mathbb{Z}$, $E_n^T(W)$ is linearly spanned by the vectors

$$u^{(1)}_{-1-k_1+\frac{r_1}{T}} u^{(2)}_{-1-k_2+\frac{r_2}{T}} \cdots u^{(s)}_{-1-k_s+\frac{r_s}{T}} w \tag{4.1}$$

for $s\geq 1$, $u^{(i)}\in V^{r_i}$, $0\leq r_i\leq T-1$, $1\leq i\leq s$, $w\in W$, $k_1,k_2,\ldots,k_s\geq 0$ with $k_1+k_2+\cdots+k_s-\frac{r_1+r_2+\cdots+r_s}{T}\geq\frac{n}{T}$.

The following are some immediate consequences.

Lemma 4.2. *Let V be a vertex algebra, let W be a g-twisted V-module, and let $u\in V^r$ with $0\leq r\leq T-1$. Then we have*

$$E_n^T(W)\supset E_{n+1}^T(W) \qquad \text{for } n\in\mathbb{Z},$$
$$E_n^T(W) = W \qquad \text{for } n\leq 0,$$
$$u_{-1-k+\frac{r}{T}}E_n^T(W)\subset E_{n+kT-r}^T(W) \qquad \text{for } k\geq 0,\ n\in\mathbb{Z}. \tag{4.2}$$

We now give a stronger spanning property for $E_n^T(W)$.

Lemma 4.3. *Let V be a vertex algebra and let W be a g-twisted V-module. For any $n \geq 1$, we have*

$$E_n^T(W) = \mathrm{span}\{u_{-1-i+\frac{r}{T}}w \mid u \in V^r,\ i \geq 1,\ w \in E^T_{n-iT+r}(W)\}. \tag{4.3}$$

Furthermore, for $n \geq 1$, $E_n^T(W)$ is linearly spanned by the vectors

$$u^{(1)}_{-1-k_1+\frac{r_1}{T}} u^{(2)}_{-1-k_2+\frac{r_2}{T}} \cdots u^{(s)}_{-1-k_s+\frac{r_s}{T}} w \tag{4.4}$$

for $s \geq 1$, $u^{(i)} \in V^{r_i}$, $0 \leq r_i \leq T-1$, $1 \leq i \leq s$, $w \in W$, $k_1, k_2, \ldots, k_s \geq 1$ with $k_1 + k_2 + \cdots + k_s - \frac{r_1+r_2+\cdots+r_s}{T} \geq \frac{n}{T}$.

Proof. Notice that (4.4) follows from (4.3) and induction. Denote by $\big(E_n^T\big)'(W)$ the subspace on the right-hand side of (4.3). From (4.2), we have $\big(E_n^T\big)'(W) \subset E_n^T(W)$. In order to prove (4.3), we need to prove that each spanning vector of $E_n^T(W)$ in (4.1) lies in $\big(E_n^T\big)'(W)$. Now we use induction on s. If $s = 1$, we consider $u_{-1-k+\frac{r}{T}}w \in E_n^T(W)$, $u \in V^r$, $0 \leq r \leq T-1$, $k \geq 1$. From Definition 4.1, we have $k - \frac{r}{T} \geq \frac{n}{T}$ such that $n - kT + r \leq 0$ and $w \in W = E^T_{n-kT+r}(W)$, so that we have $u_{-1-k+\frac{r}{T}}w \in \big(E_n^T\big)'(W)$. Assume $s \geq 2$. If $k_1 \geq 1$, consider

$$u^{(1)}_{-1-k_1+\frac{r_1}{T}} u^{(2)}_{-1-k_2+\frac{r_2}{T}} \cdots u^{(s)}_{-1-k_s+\frac{r_s}{T}} w \in E_n^T(W).$$

From Definition 4.1, we have

$$k_2 + k_3 + \cdots + k_s - \frac{r_2 + r_3 + \cdots + r_s}{T} \geq \frac{n - k_1 T + r_1}{T}.$$

So we get

$$u^{(2)}_{-1-k_2+\frac{r_2}{T}} u^{(3)}_{-1-k_3+\frac{r_3}{T}} \cdots u^{(s)}_{-1-k_s+\frac{r_s}{T}} w \in E^T_{n-k_1T+r_1}(W).$$

Thus we have

$$u^{(1)}_{-1-k_1+\frac{r_1}{T}} u^{(2)}_{-1-k_2+\frac{r_2}{T}} \cdots u^{(s)}_{-1-k_s+\frac{r_s}{T}} w \in \big(E_n^T\big)'(W).$$

If $k_1 = 0$, from Definition 4.1 we have

$$u^{(2)}_{-1-k_2+\frac{r_2}{T}} u^{(3)}_{-1-k_3+\frac{r_3}{T}} \cdots u^{(s)}_{-1-k_s+\frac{r_s}{T}} w \in E^T_{n+r_1}(W).$$

By the inductive hypothesis, we have

$$u^{(2)}_{-1-k_2+\frac{r_2}{T}} u^{(3)}_{-1-k_3+\frac{r_3}{T}} \cdots u^{(s)}_{-1-k_s+\frac{r_s}{T}} w \in \big(E^T_{n+r_1}\big)'(W).$$

Furthermore, for any $b \in V^p$, $0 \leq p \leq T-1$, $k \geq 1$, $w' \in E^T_{n+r_1-kT+p}(W)$, we have

$$u^{(1)}_{-1+\frac{r_1}{T}} b_{-1-k+\frac{p}{T}} w' = b_{-1-k+\frac{p}{T}} u^{(1)}_{-1+\frac{r_1}{T}} w' + \sum_{i \in \mathbb{N}} \binom{-1+\frac{r_1}{T}}{i} (u_i^{(1)} b)_{-2-k-i+\frac{r_1+p}{T}} w'.$$

From Definition 4.1 we have $u^{(1)}_{-1+\frac{r_1}{T}} w' \in E^T_{n-kT+p}(W)$, so that $b_{-1-k+\frac{p}{T}} u^{(1)}_{-1+\frac{r_1}{T}} w' \in \big(E_n^T\big)'(W)$. On the other hand, if $0 \leq r_1 + p \leq T-1$, we have

$$\big(u_i^{(1)} b\big)_{-2-k-i+\frac{r_1+p}{T}} w' = (u_i^{(1)} b)_{-1-(k+i+1)+\frac{r_1+p}{T}} w',$$

with $w' \in E^T_{n-kT+r_1+p}(W) \subset E^T_{n-(1+k+i)T+r_1+p}(W)$. So that

$$\left(u_i^{(1)}b\right)_{-2-k-i+\frac{r_1+p}{T}} w' \in \left(E^T_n\right)'(W).$$

If $T \le r_1 + p \le 2T-2$, we have

$$\left(u_i^{(1)}b\right)_{-2-k-i+\frac{r_1+p}{T}} w' = \left(u_i^{(1)}b\right)_{-1-(k+i)+\frac{r_1+p-T}{T}} w'$$

with $w' \in E^T_{n-kT+r_1+p}(W) \subset E^T_{n-(k+i)T+r_1+p-T}(W)$, so that

$$\left(u_i^{(1)}b\right)_{-2-k-i+\frac{r_1+p}{T}} w' \in \left(E^T_n\right)'(W).$$

Therefore, $u^{(1)}_{-1+\frac{r_1}{T}} b_{-1-k+\frac{p}{T}} w' \in \left(E^T_n\right)'(W)$, this proves that

$$u^{(1)}_{-1+\frac{r_1}{T}} u^{(2)}_{-1-k_2+\frac{r_2}{T}} \cdots u^{(s)}_{-1-k_s+\frac{r_s}{T}} w \in \left(E^T_n\right)'(W),$$

completing the proof. ■

Lemma 4.4. *Let V be a vertex algebra and let W be a g-twisted V-module. For any $a \in V^p$, $0 \le p \le T-1$, $m, n \in \mathbb{Z}$, we have*

$$a_{m+\frac{p}{T}} E^T_n(W) \subset E^T_{n-(m+1)T-p}(W) \qquad \text{for } m \in \mathbb{Z}. \tag{4.5}$$

Furthermore,

$$a_{m+\frac{p}{T}} E^T_n(W) \subset E^T_{n-(m+1)T-p+T}(W) \qquad \text{for } m \ge 0. \tag{4.6}$$

Proof. If $m \le -1$, by (4.2) we have

$$a_{m+\frac{p}{T}} E^T_n(W) = a_{-1-(-1-m)+\frac{p}{T}} E^T_n(W) \subset E^T_{n-(m+1)T-p}(W).$$

Assume $m \ge 0$, since $E^T_{n-(m+1)T-p+T}(W) \subset E^T_{n-(m+1)T-p}(W)$, it suffices to prove (4.6). We now prove the assertion by induction on n. If $n \le 0$, we have $n-(m+1)T-p+T \le 0$, we have $a_{m+\frac{p}{T}} E^T_n(W) \subset W = E^T_{n-(m+1)T-p+T}(W)$. If $m \ge 0$, $n \ge 1$, from (4.3) we have $E^T_n(W)$ is spanned by the vectors $u_{-1-k+\frac{q}{T}} w$ with $u \in V^q$, $0 \le q \le T-1$, $k \ge 1$, $w \in E^T_{n-kT+q}(W)$. In view of (2.4), we have

$$a_{m+\frac{p}{T}} u_{-1-k+\frac{q}{T}} w = u_{-1-k+\frac{q}{T}} a_{m+\frac{p}{T}} w + \sum_{i \in \mathbb{N}} \binom{m+\frac{p}{T}}{i} (a_i u)_{m-k-i-1+\frac{p+q}{T}} w.$$

Since $w \in E^T_{n-kT+q}(W)$, $n-kT+q < n$, from the inductive hypothesis and (4.2) we have

$$a_{m+\frac{p}{T}} w \in E^T_{n-kT+q-(m+1)T-p+T}(W),$$
$$u_{-1-k+\frac{q}{T}} a_{m+\frac{p}{T}} w \in u_{-1-k+\frac{q}{T}} E^T_{n-kT+q-(m+1)T-p+T}(W) \subset E^T_{n-(m+1)T-p+T}(W).$$

If $0 \le p+q \le T-1$, since $i \ge 0$, from the inductive hypothesis we have

$$(a_i u)_{m-k-i-1+\frac{p+q}{T}} w \in E^T_{n-kT+q-(m-k-i-1+1)T-p-q}(W) \subset E^T_{n-(m+1)T-p+T}(W).$$

If $T \le p+q \le 2T-2$, since $i \ge 0$, similarly we have

$$(a_i u)_{m-k-i-1+\frac{p+q}{T}} w$$
$$= (a_i u)_{m-k-i+\frac{p+q-T}{T}} w \in E^T_{n-kT+q-(m-k-i+1)T-p-q+T}(W) \subset E^T_{n-(m+1)T-p+T}(W).$$

Therefore, $a_{m+\frac{p}{T}}u_{-1-k+\frac{q}{T}}w \in E^T_{n-(m+1)T-p+T}(W)$. This proves

$$a_{m+\frac{p}{T}}E^T_n(W) \subset E^T_{n-(m+1)T-p+T}(W),$$

completing the induction and the whole proof. ■

Now we have the following general case.

Proposition 4.5. *Suppose that V is a vertex algebra, let W be a g-twisted V-module and let $u \in E^T_{rT}(V) \cap V^p$, $w \in E^T_s(W)$ with $0 \le p \le T-1$, $r, s \in \mathbb{Z}$. Then,*

$$u_{n+\frac{p}{T}}w \in E^T_{rT+s-(n+1)T-p}(W) \qquad \text{for } n \in \mathbb{Z}. \tag{4.7}$$

Proof. We are going to use induction on r. By (4.5), we have $u_{n+\frac{p}{T}}w \in E^T_{s-(n+1)T-p}(W)$. If $r \le 0$, we have $rT+s-(n+1)T-p \le s-(n+1)T-p$, so that $u_{n+\frac{p}{T}}w \in E^T_{s-(n+1)T-p}(W) \subset E^T_{rT+s-(n+1)T-p}(W)$.

Assume $r \ge 1$ and $u \in E^T_{(r+1)T}(V)$, $u \in V^p$, $0 \le p \le T-1$. In view of (3.5), it suffices to consider $u = a_{-2-k}b$ with $a \in V^q$, $b \in V^t$, $b \in E^T_{(r-k)T}(V)$, $0 \le k \le r$, $0 \le q, t \le T-1$. And from (2.5), we know that for each $a \in V^q$ and $b \in V^t$, there exists a nonnegative integer N such that $(x_1-x_2)^N[Y_W(a,x_1), Y_W(b,x_2)]w = 0$.

If $0 \le q+t \le T-1$, then $q+t = p$. By (2.6), we have

$$\begin{aligned}(a_{-2-k}b)_{n+\frac{q+t}{T}}w = \sum_{i\in\mathbb{N}}\sum_{-2-k\le j<N}&\Big((-1)^i\binom{-\frac{q}{T}}{j+k+2}\binom{j}{i}a_{j-i+\frac{q}{T}}b_{n-2-k+i-j+\frac{t}{T}}w\\ &-(-1)^{i+j}\binom{-\frac{q}{T}}{j+k+2}\binom{j}{i}b_{n-2-k-i+\frac{t}{T}}a_{i+\frac{q}{T}}w\Big).\end{aligned}$$

For $n \in \mathbb{Z}$, $b \in E^T_{(r-k)T}(V)$ with $r-k \le r$, using the inductive hypothesis and (4.5) we have

$$\begin{aligned}a_{j-i+\frac{q}{T}}b_{n-2-k+i-j+\frac{t}{T}}w &\in a_{j-i+\frac{q}{T}}E^T_{(r-k)T+s-(n-2-k+i-j+1)T-t}(W)\\ &\subset E^T_{(r-k)T+s-(n-2-k+i-j+1)T-t-(j-i+1)T-q}(W)\\ &= E^T_{(r+1)T+s-(n+1)T-t-q}(W) = E^T_{(r+1)T+s-(n+1)T-p}(W),\\ b_{n-2-k-i+\frac{t}{T}}a_{i+\frac{q}{T}}w &\in b_{n-2-k-i+\frac{t}{T}}E^T_{s-(i+1)T-q}(W)\\ &\subset E^T_{(r-k)T+s-(i+1)T-q-(n-2-k-i+1)T-t}(W)\\ &= E^T_{(r+1)T+s-(n+1)T-q-t}(W) = E^T_{(r+1)T+s-(n+1)T-p}(W),\end{aligned}$$

from which we have $u_{n+\frac{p}{T}}w \in E^T_{(r+1)T+s-(n+1)T-p}(W)$.

If $T \le q+t \le 2T-2$, then $q+t-T = p$. By (2.6), we have

$$\begin{aligned}(a_{-2-k}b)_{n+\frac{q+t-T}{T}}w = \sum_{i\in\mathbb{N}}\sum_{-2-k\le j<N}&\Big((-1)^i\binom{-\frac{q}{T}}{j+k+2}\binom{j}{i}a_{j-i+\frac{q}{T}}b_{n-3-k+i-j+\frac{t}{T}}w\\ &-(-1)^{i+j}\binom{-\frac{q}{T}}{j+k+2}\binom{j}{i}b_{n-3-k-i+\frac{t}{T}}a_{i+\frac{q}{T}}w\Big).\end{aligned}$$

Similarly, using inductive hypothesis and (4.5), we have

$$\begin{aligned}a_{j-i+\frac{q}{T}}b_{n-3-k+i-j+\frac{t}{T}}w &\in a_{j-i+\frac{q}{T}}E^T_{(r-k)T+s-(n-3-k+i-j+1)T-t}(W)\\ &\subset E^T_{(r-k)T+s-(n-3-k+i-j+1)T-t-(j-i+1)T-q}(W)\end{aligned}$$

$$= E^T_{(r+1)T+s-(n+1)T+T-t-q}(W) = E^T_{(r+1)T+s-(n+1)T-p}(W),$$

$$\begin{aligned} b_{n-3-k-i+\frac{t}{T}}a_{i+\frac{q}{T}}w &\in b_{n-3-k-i+\frac{t}{T}}E^T_{s-(i+1)T-q}(W) \\ &\subset E^T_{(r-k)T+s-(i+1)T-q-(n-3-k-i+1)T-t}(W) \\ &= E^T_{(r+1)T+s-(n+1)T+T-q-t}(W) = E^T_{(r+1)T+s-(n+1)T-p}(W), \end{aligned}$$

from which we have $u_{n+\frac{p}{T}}w \in E^T_{(r+1)T+s-(n+1)T-p}(W)$. This concludes the proof. ∎

The case $T = 1$ is Li's result [17]. The following is the main result of this section.

Proposition 4.6. *Let V be a vertex algebra, let g have finite period T, and let W be a g-twisted V-module. For $T \geq 2$, let $E^T_n(W)$ be the decreasing sequence defined in Definition 4.1 for W, then the associated graded vector space $\mathrm{gr}^T_{\mathcal{E}}(W) = \coprod_{n\geq 0} E^T_n(W)/E^T_{n+1}(W)$ is a g-twisted module for the vertex Poisson algebra $\mathrm{gr}^T_{\mathcal{E}}(V)$ with*

$$\big(u + E^T_{(r+1)T}(V)\big) \cdot \big(w + E^T_{s+1}(W)\big) = u_{-1+\frac{p}{T}}w + E^T_{rT+s-p+1}(W), \tag{4.8}$$

$$\begin{aligned} &Y^W_-\big(u + E^T_{(r+1)T}(V), x\big)\big(w + E^T_{s+1}(W)\big) \\ &\quad = \begin{cases} \displaystyle\sum_{n\in\mathbb{N}} \big(u_n w + E^T_{rT+s-(n+1)T+1}(W)\big)x^{-n-1} & \text{for } p = 0, \\ \displaystyle\sum_{n\in\mathbb{Z}} \big(u_{n+\frac{p}{T}}w + E^T_{rT+s-(n+1)T-p+1}(W)\big)x^{-n-\frac{p}{T}-1} & \text{for } 1 \leq p \leq T-1, \end{cases} \end{aligned} \tag{4.9}$$

with $u \in E^T_{rT}(V) \cap V^p$, $w \in E^T_s(W)$, $r, s \in \mathbb{N}$, $0 \leq p \leq T-1$.

Proof. From Proposition 4.5, the actions given by (4.8) and (4.9) are well defined. The vacuum axiom gives $\mathbf{1}_{-1}w = w$, hence $\big(\mathbf{1} + E^T_T(V)\big) \cdot \big(w + E^T_{s+1}(W)\big) = w + E^T_{s+1}(W)$. Clearly, it is straightforward to check (3.2) and (3.3). Now we prove the compatibility (3.4). For $u \in E^T_{rT}(V) \cap V^p$, $v \in E^T_{sT}(V) \cap V^q$, $w \in E^T_k(W)$, $r, s, k \in \mathbb{N}$.

For $p = 0$, $0 \leq q \leq T-1$, by (4.9) we have $n \in \mathbb{N}$. From (2.4), we have

$$u_n v_{-1+\frac{q}{T}}w = v_{-1+\frac{q}{T}}u_n w + \sum_{i=0}^{n}\binom{n}{i}(u_i v)_{n-1-i+\frac{q}{T}}w.$$

Since $i \geq 0$, by (4.7) and (3.6) we have

$$\begin{aligned} &(u_i v)_{n-1-i+\frac{q}{T}}w \in E^T_{(r+s-n)T+k-q}(W) \subset E^T_{(r+s-n-1)T+k-q+1}(W), \\ &u_n v_{-1+\frac{q}{T}}w \in E^T_{(r+s-n-1)T+k-q}(W), \\ &v_{-1+\frac{q}{T}}u_n w \in E^T_{(r+s-n-1)T+k-q}(W). \end{aligned}$$

Then

$$u_n(v_{-1+\frac{q}{T}}w) = v_{-1+\frac{q}{T}}(u_n w) + (u_n v)_{-1+\frac{q}{T}}w + \sum_{i=0}^{n-1}\binom{n}{i}(u_i v)_{n-1-i+\frac{q}{T}}w.$$

Thus

$$\begin{aligned} &u_n(v_{-1+\frac{q}{T}}w) + E^T_{(r+s-n-1)T+k-q+1}(W) \\ &\quad = v_{-1+\frac{q}{T}}(u_n w) + (u_n v)_{-1+\frac{q}{T}}w + E^T_{(r+s-n-1)T+k-q+1}(W). \end{aligned}$$

For $1 \le p \le T-1$, $0 \le q \le T-1$, then by (4.9) we have $n \in \mathbb{Z}$. From (2.4), we have

$$u_{n+\frac{p}{T}} v_{-1+\frac{q}{T}} w = v_{-1+\frac{q}{T}} u_{n+\frac{p}{T}} w + \sum_{i \in \mathbb{N}} \binom{n+\frac{p}{T}}{i} (u_i v)_{n-1-i+\frac{p+q}{T}} w.$$

For $n \ge 0$, since $i \ge 0$, by (4.7) and (3.6) we have

$$\begin{aligned}
&(u_i v)_{n-1-i+\frac{p+q}{T}} w \in E^T_{(r+s-n)T+k-p-q}(W) \subset E^T_{(r+s-n-1)T+k-p-q+1}(W),\\
&u_{n+\frac{p}{T}} v_{-1+\frac{q}{T}} w \in E^T_{(r+s-n-1)T+k-p-q}(W),\\
&v_{-1+\frac{q}{T}} u_{n+\frac{p}{T}} w \in E^T_{(r+s-n-1)T+k-p-q}(W).
\end{aligned}$$

If $1 \le p+q \le T-1$, we have

$$(u_n v)_{-1+\frac{p+q}{T}} w \in E^T_{(r+s-n)T+k-p-q}(W) \subset E^T_{(r+s-n-1)T+k-p-q+1}(W).$$

Then

$$\begin{aligned}
&u_{n+\frac{p}{T}} (v_{-1+\frac{q}{T}} w) + E^T_{(r+s-n)T+k-p-q+1}(W)\\
&\quad = (u_n v)_{-1+\frac{p+q}{T}} w + v_{-1+\frac{q}{T}} (u_{n+\frac{p}{T}} w) + E^T_{(r+s-n)T+k-p-q+1}(W).
\end{aligned}$$

If $T \le p+q \le 2T-2$, we also have

$$(u_n v)_{-1+\frac{p+q-T}{T}} w \in E^T_{(r+s-n)T+k-p-q}(W) \subset E^T_{(r+s-n-1)T+k-p-q+1}(W).$$

Then

$$\begin{aligned}
&u_{n+\frac{p}{T}} (v_{-1+\frac{q}{T}} w) + E^T_{(r+s-n)T+k-p-q+1}(W)\\
&\quad = (u_n v)_{-1+\frac{p+q-T}{T}} w + v_{-1+\frac{q}{T}} (u_{n+\frac{p}{T}} w) + E^T_{(r+s-n)T+k-p-q+1}(W).
\end{aligned}$$

For $n \le -1$, by (2.4) we have

$$u_{n+\frac{p}{T}} v_{-1+\frac{q}{T}} w = v_{-1+\frac{q}{T}} u_{n+\frac{p}{T}} w + \sum_{i \in \mathbb{N}} \binom{n+\frac{p}{T}}{i} (u_i v)_{n-1-i+\frac{p+q}{T}} w.$$

Similarly, we have

$$\begin{aligned}
&(u_i v)_{n-1-i+\frac{p+q}{T}} w \in E^T_{(r+s-n)T+k-p-q}(W) \subset E^T_{(r+s-n-1)T+k-p-q+1}(W),\\
&u_{n+\frac{p}{T}} v_{-1+\frac{q}{T}} w \in E^T_{(r+s-n-1)T+k-p-q}(W),\\
&v_{-1+\frac{q}{T}} u_{n+\frac{p}{T}} w \in E^T_{(r+s-n-1)T+k-p-q}(W).
\end{aligned}$$

Then

$$\begin{aligned}
&u_{n+\frac{p}{T}} (v_{-1+\frac{q}{T}} w) + E^T_{(r+s-n)T+k-p-q+1}(W)\\
&\quad = (u_n v)_{-1+\frac{p+q}{T}} w + v_{-1+\frac{q}{T}} (u_{n+\frac{p}{T}} w) + E^T_{(r+s-n)T+k-p-q+1}(W).
\end{aligned}$$

To sum up, it proves $Y^W_-(u,x)(vw) = (Y_-(u,x)v)w + vY^W_-(u,x)w$. Therefore $\mathrm{gr}^T_{\mathcal{E}}(W)$ is a g-twisted module for the vertex Poisson algebra $\mathrm{gr}^T_{\mathcal{E}}(V)$. ■

In Section 6, we will exclude the possibility that the associated sequence $\mathcal{E}^T_W$ is trivial in the sense that $W = E^T_n(W)$ for all $n \ge 0$. Here we have the following lemma.

Lemma 4.7. *Let $V=\coprod_{n\geq 0}V_{(n)}$ be an $\mathbb{N}$-graded vertex algebra, let $W=\bigoplus_{n\in\frac{1}{T}\mathbb{N}}W_{(n)}$ be a $\frac{1}{T}\mathbb{N}$-graded g-twisted V-module and $\mathcal{E}_W^T=\{E_n^T(W)\}_{n\in\mathbb{Z}}$ be the decreasing sequence defined in Definition 4.1. Then*

$$E_n^T(W)\subset \coprod_{m\geq\frac{n}{T}} W_{(m)} \qquad \text{for } n\geq 0. \tag{4.10}$$

Furthermore, the associated decreasing sequence $\mathcal{E}_W^T=\{E_n^T(W)\}_{n\in\mathbb{Z}}$ for W is a filtration, i.e.,

$$\bigcap_{n\geq 0} E_n^T(W)=0. \tag{4.11}$$

Proof. For $n=0$, from Definition 4.1 we have $E_0^T(W)=\coprod_{m\geq 0}W_{(m)}$. For $n\geq 1$, $E_n^T(W)$ is linearly spanned by the vectors

$$u^{(1)}_{-1-k_1+\frac{r_1}{T}}u^{(2)}_{-1-k_2+\frac{r_2}{T}}\cdots u^{(s)}_{-1-k_s+\frac{r_s}{T}}w$$

with $s\geq 1$, $u^{(i)}\in V^{r_i}$, $w\in W_{(m_1)}$, $k_1,k_2,\dots,k_s\geq 1$, $k_1+k_2+\cdots+k_s-\frac{r_1+r_2+\cdots+r_s}{T}\geq\frac{n}{T}$, $m_1\geq 0$, $0\leq r_i\leq T-1$, $1\leq i\leq s$. If the vectors $u^{(1)},u^{(2)},\dots,u^{(s)},w$ are homogeneous, we have

$$\begin{aligned}&\operatorname{wt}(u^{(1)}_{-1-k_1+\frac{r_1}{T}}u^{(2)}_{-1-k_2+\frac{r_2}{T}}\cdots u^{(s)}_{-1-k_s+\frac{r_s}{T}}w)\\&\quad=\operatorname{wt}u^{(1)}+\operatorname{wt}u^{(2)}+\cdots+\operatorname{wt}u^{(s)}+k_1+k_2+\cdots+k_s-\frac{r_1+r_2+\cdots+r_s}{T}+m_1\geq\frac{n}{T}.\end{aligned}$$

This proves (4.10).

Clearly, each subspace $E_n^T(W)$ of W is graded, we immediately have (4.11). ■

5 The relation between the sequences $\mathcal{E}_W^T$ and $\mathcal{C}_W^T$

In this section, we introduce the decreasing sequence $\mathcal{C}_W^T=\{C_n^T(W)\}_{n\in\mathbb{Z}_{\geq 2}}$ of twisted modules. We define the subspace $C_n^T(W)$ of a twisted module W as follows.

Definition 5.1. Let V be a vertex algebra and let W be a g-twisted V-module. For $n\geq 2$, we define $C_n^T(W)$ to be the subspace of W, with

$$C_n^T(W)=\operatorname{span}\{u_{-n+\frac{p}{T}}w\mid u\in V^p,\ w\in W,\ p=0,1,\dots,T-1\}. \tag{5.1}$$

A twisted V-module W is said to be *C_n-cofinite* if $W/C_n^T(W)$ is finite-dimensional. A twisted V-module W is said to be *E_n-cofinite* if $W/E_n^T(W)$ is finite-dimensional.

Here are some consequences for $C_n^T(W)$.

Lemma 5.2. *Let V be a vertex algebra and let W be a g-twisted V-module. For $n\geq 2$, we have*

$$C_m^T(W)\subset C_n^T(W)\quad \text{for } m\geq n, \tag{5.2}$$

$$u_{-k+\frac{p}{T}}C_n^T(W)\subset C_n^T(W), \tag{5.3}$$

$$u_{-n+\frac{p}{T}}v_{-k+\frac{q}{T}}w\equiv v_{-k+\frac{q}{T}}u_{-n+\frac{p}{T}}w\ \big(\operatorname{mod}\ C_{n+k-1}^T(W)\big) \tag{5.4}$$

for $u\in V^p$, $v\in V^q$, $0\leq p,q\leq T-1$, $w\in W$, $k\geq 1$.

Proof. For $u \in V^p$, $0 \leq p \leq T-1$, $w \in W$, $r \geq 2$, we have

$$u_{-r-1+\frac{p}{T}}w = \frac{1}{r-\frac{p}{T}}(\mathcal{D}u)_{-r+\frac{p}{T}}w.$$

From this, we immediately have $C^T_{r+1}(W) \subset C^T_r(W)$ for $r \geq 2$, this implies (5.2).

For $u \in V^p$, $v \in V^q$, $0 \leq p, q \leq T-1$, $w \in W$, $k \geq 1$, $n \geq 2$, by (2.4) we have

$$u_{-k+\frac{p}{T}}v_{-n+\frac{q}{T}}w = v_{-n+\frac{q}{T}}u_{-k+\frac{p}{T}}w + \sum_{i\in\mathbb{N}}\binom{-k+\frac{p}{T}}{i}(u_iv)_{-n-k-i+\frac{p+q}{T}}w.$$

If $0 \leq p+q \leq T-1$, then

$$(u_iv)_{-n-k-i+\frac{p+q}{T}}w \in C^T_{n+k+i}(W) \subset C^T_n(W).$$

If $T \leq p+q \leq 2T-2$, then we have

$$(u_iv)_{-n-k-i+1+\frac{p+q-T}{T}}w \in C^T_{n+k+i-1}(W) \subset C^T_n(W),$$

this proves (5.3).

We also have

$$u_{-n+\frac{p}{T}}v_{-k+\frac{q}{T}}w - v_{-k+\frac{q}{T}}u_{-n+\frac{p}{T}}w = \sum_{i\in\mathbb{N}}\binom{-n+\frac{p}{T}}{i}(u_iv)_{-n-k-i+\frac{p+q}{T}}w. \tag{5.5}$$

If $0 \leq p+q \leq T-1$, we have

$$\sum_{i\in\mathbb{N}}\binom{-n+\frac{p}{T}}{i}(u_iv)_{-n-k-i+\frac{p+q}{T}}w \in C^T_{n+k}(W) \subset C^T_{n+k-1}(W), \tag{5.6}$$

and if $T \leq p+q \leq 2T-2$, we have

$$\begin{aligned}&\sum_{i\in\mathbb{N}}\binom{-n+\frac{p}{T}}{i}(u_iv)_{-n-k-i+\frac{p+q}{T}}w\\ &\quad=\sum_{i\in\mathbb{N}}\binom{-n+\frac{p}{T}}{i}(u_iv)_{-n-k-i+1+\frac{p+q-T}{T}}w \in C^T_{n+k-1}(W).\end{aligned} \tag{5.7}$$

Combining (5.5), (5.6) and (5.7), we have (5.4). ■

We have the following technical result.

Lemma 5.3. *Let V be a vertex algebra and let W be a g-twisted V-module. Then*

$$u^{(1)}_{-2+\frac{r_1}{T}}u^{(2)}_{-2+\frac{r_2}{T}}\cdots u^{(s)}_{-2+\frac{r_s}{T}}w \in C^T_3(W)$$

for $s \geq T+1$, $u^{(i)} \in V^{r_i}$, $w \in W$, $0 \leq r_i \leq T-1$, $1 \leq i \leq s$.

Proof. For $u \in V^p$, $0 \leq p \leq T-1$, since $u_{-2+\frac{p}{T}}C^T_3(W) \subset C^T_3(W)$, it is sufficient to prove the conclusion holds for $s = T+1$. Assume $s = T+1$, if there exist k_1, k_2 with $1 \leq k_1 < k_2 \leq s$ such that $0 \leq r_{k_1}+r_{k_2} \leq T-1$, then by (2.8), there exist nonnegative integers k and l such that

$$\left(u^{(k_1)}_{-1}u^{(k_2)}\right)_{-3+\frac{r_{k_1}+r_{k_2}}{T}}w$$

$$= \sum_{i=0}^{k} \sum_{j\in\mathbb{N}} \binom{-l-\frac{r_{k_1}}{T}}{i} \binom{i-1}{j} (-1)^j u^{(k_1)}_{-1+l+i-j+\frac{r_{k_1}}{T}} u^{(k_2)}_{-3-l-i+j+\frac{r_{k_2}}{T}} w.$$

Since $0 \le r_{k_1} + r_{k_2} \le T-1$, we have $\big(u^{(k_1)}_{-1} u^{(k_2)}\big)_{-3+\frac{r_{k_1}+r_{k_2}}{T}} w \in C_3^T(W)$. If $i-1 \ge 0$, then $j \le i-1$. For each i, j, from (2.4) we have

$$\begin{aligned} &u^{(k_1)}_{-1+l+i-j+\frac{r_{k_1}}{T}} u^{(k_2)}_{-3-l-i+j+\frac{r_{k_2}}{T}} w \\ &= u^{(k_2)}_{-3-l-i+j+\frac{r_{k_2}}{T}} u^{(k_1)}_{-1+l+i-j+\frac{r_{k_1}}{T}} w + \sum_{n\in\mathbb{N}} \binom{-1+l+i-j+\frac{r_{k_1}}{T}}{n} \big(u^{(k_1)}_n u^{(k_2)}\big)_{-4-n+\frac{r_{k_1}+r_{k_2}}{T}} w. \end{aligned}$$

Notice that $-4-n \le -3$ and $-3-l-i+j \le -3$, we have

$$\begin{aligned} &\big(u^{(k_1)}_n u^{(k_2)}\big)_{-4-n+\frac{r_{k_1}+r_{k_2}}{T}} w \in C_3^T(W), \\ &u^{(k_2)}_{-3-l-i+j+\frac{r_{k_2}}{T}} u^{(k_1)}_{-1+l+i-j+\frac{r_{k_1}}{T}} w \in C_3^T(W), \end{aligned}$$

which implies

$$u^{(k_1)}_{-1+l+i-j+\frac{r_{k_1}}{T}} u^{(k_2)}_{-3-l-i+j+\frac{r_{k_2}}{T}} w \in C_3^T(W).$$

For $i=0$, if $j \ge l+2$, we have $-1+l-j \le -3$, so that

$$u^{(k_1)}_{-1+l-j+\frac{r_{k_1}}{T}} u^{(k_2)}_{-3-l+j+\frac{r_{k_2}}{T}} w \in C_3^T(W).$$

If $j \le l$, we have $-3-l+j \le -3$. From (2.4), we have

$$\begin{aligned} &u^{(k_1)}_{-1+l-j+\frac{r_{k_1}}{T}} u^{(k_2)}_{-3-l+j+\frac{r_{k_2}}{T}} w \\ &\quad = u^{(k_2)}_{-3-l+j+\frac{r_{k_2}}{T}} u^{(k_1)}_{-1+l-j+\frac{r_{k_1}}{T}} w + \sum_{n\in\mathbb{N}} \binom{-1+l-j+\frac{r_{k_1}}{T}}{n} \big(u^{(k_1)}_n u^{(k_2)}\big)_{-4-n+\frac{r_{k_1}+r_{k_2}}{T}} w. \end{aligned}$$

Thus we have

$$u^{(k_1)}_{-1+l-j+\frac{r_{k_1}}{T}} u^{(k_2)}_{-3-l+j+\frac{r_{k_2}}{T}} w \in C_3^T(W).$$

Therefore, the only remaining case $i=0$ and $j=l+1$, the corresponding term $u^{(k_1)}_{-2+\frac{r_{k_1}}{T}} u^{(k_2)}_{-2+\frac{r_{k_2}}{T}} w$ must also lie in $C_3^T(W)$. For any $w \in W$, by (5.4) we have

$$\begin{aligned} u^{(1)}_{-2+\frac{r_1}{T}} u^{(2)}_{-2+\frac{r_2}{T}} \cdots u^{(s)}_{-2+\frac{r_s}{T}} w \equiv\ & u^{(k_1)}_{-2+\frac{r_{k_1}}{T}} u^{(k_2)}_{-2+\frac{r_{k_2}}{T}} u^{(1)}_{-2+\frac{r_1}{T}} \cdots u^{(k_1-1)}_{-2+\frac{r_{k_1-1}}{T}} u^{(k_1+1)}_{-2+\frac{r_{k_1+1}}{T}} \cdots \\ & \times u^{(k_2-1)}_{-2+\frac{r_{k_2-1}}{T}} u^{(k_2+1)}_{-2+\frac{r_{k_2+1}}{T}} \cdots u^{(s)}_{-2+\frac{r_s}{T}} w \pmod{C_3^T(W)}, \end{aligned}$$

which means

$$u^{(1)}_{-2+\frac{r_1}{T}} u^{(2)}_{-2+\frac{r_2}{T}} \cdots u^{(s)}_{-2+\frac{r_s}{T}} w \in C_3^T(W).$$

Suppose that no two of the original residues have sum at most $T-1$; otherwise the preceding argument applies directly. Since $s = T+1$, choose any ordering $u^{(1)}, \dots, u^{(T+1)}$ and put $a^{(1)} = u^{(1)} \in V^{q_1}$ with $q_1 = r_1$. At each stage $j \ge 2$, if $q_{j-1} + r_j \le T-1$, stop and set $t = j-1$. The two required vectors are then $a^{(t)} \in V^{q_t}$ and $u^{(t+1)} \in V^{r_{t+1}}$; their residues have sum at most

$T-1$. If $q_{j-1}+r_j \geq T$, instead set $a^{(j)} = a^{(j-1)}_{-1}u^{(j)} \in V^{q_j}$, where $q_j = q_{j-1}+r_j-T$. If this second case occurred at every stage $2 \leq j \leq T+1$, then

$$q_{T+1} = \sum_{i=1}^{T+1} r_i - T^2 \leq (T+1)(T-1) - T^2 = -1,$$

contradicting $q_{T+1} \geq 0$. Thus the process stops by stage $T+1$. It cannot stop at $j=2$ under the present assumption, so $2 \leq t \leq T$. This proves the required existence, including the boundary sums $2T-1$, $3T-2$, $3T-1$, and their analogues at later stages.

$$a^{(t)} \in V^{q_t}, \qquad u^{(t+1)} \in V^{r_{t+1}}, \qquad 0 \leq q_t + r_{t+1} \leq T-1. \tag{5.8}$$

Next, for the nonnegative integer t $(2 \leq t \leq T)$, assume the desired vectors are

$$\begin{cases} \big(\big(\cdots\big(\big(u^{(1)}_{-1}u^{(2)}\big)_{-1}u^{(3)}\big)_{-1}\cdots\big)_{-1}u^{(t-1)}\big)_{-1}u^{(t)} \in V^{r}, \\ u^{(t+1)} \in V^{r_{t+1}} \end{cases}$$

for $u^{(i)} \in V^{r_i}$, $1 \leq i \leq t+1$, $0 \leq r_i, r \leq T-1$ with $0 \leq r + r_{t+1} \leq T-1$. For $0 \leq q_j \leq T-1$, $1 \leq j \leq t$, set

$$a^{(j)} = \big(\big(\cdots\big(\big(u^{(1)}_{-1}u^{(2)}\big)_{-1}u^{(3)}\big)_{-1}\cdots\big)_{-1}u^{(j-1)}\big)_{-1}u^{(j)} \in V^{q_j},$$

then it follows $a^{(j)} = a^{(j-1)}_{-1}u^{(j)}$. From the above proof of the existence of nonnegative integer t and vectors in (5.8), for any j with $2 \leq j \leq t$ we have $T \leq q_{j-1}+r_j \leq 2T-2$, thus $q_j = q_{j-1}+r_j-T$. By (2.8), there exist nonnegative integers k_1 and l_1 such that

$$\begin{aligned} &\big(a^{(t)}_{-1}u^{(t+1)}\big)_{-3+\frac{q_t+r_{t+1}}{T}}w \\ &\quad = \sum_{i=0}^{k_1}\sum_{j\in\mathbb{N}} \binom{-l_1-\frac{q_t}{T}}{i}\binom{i-1}{j}(-1)^j a^{(t)}_{-1+l_1+i-j+\frac{q_t}{T}} u^{(t+1)}_{-3-l_1-i+j+\frac{r_{t+1}}{T}} w. \end{aligned} \tag{5.9}$$

Since $0 \leq q_t + r_{t+1} \leq T-1$, we have

$$\big(a^{(t)}_{-1}u^{(t+1)}\big)_{-3+\frac{q_t+r_{t+1}}{T}}w \in C_3^T(W).$$

If $i-1 \geq 0$, then $j \leq i-1$. For each i, j, from (2.4) we have

$$\begin{aligned} &a^{(t)}_{-1+l_1+i-j+\frac{q_t}{T}} u^{(t+1)}_{-3-l_1-i+j+\frac{r_{t+1}}{T}} w \\ &= u^{(t+1)}_{-3-l_1-i+j+\frac{r_{t+1}}{T}} a^{(t)}_{-1+l_1+i-j+\frac{q_t}{T}} w + \sum_{n\in\mathbb{N}} \binom{-1+l_1+i-j+\frac{q_t}{T}}{n} \big(u^{(t+1)}_n a^{(t)}\big)_{-4-n+\frac{q_t+r_{t+1}}{T}} w. \end{aligned}$$

Notice that $-4-n \leq -3$ and $-3-l_1-i+j \leq -3$, we have

$$\begin{aligned} &\big(u^{(t+1)}_n a^{(t)}\big)_{-4-n+\frac{q_t+r_{t+1}}{T}} w \in C_3^T(W), \\ &u^{(t+1)}_{-3-l_1-i+j+\frac{r_{t+1}}{T}} a^{(t)}_{-1+l_1+i-j+\frac{q_t}{T}} w \in C_3^T(W), \end{aligned}$$

which implies

$$a^{(t)}_{-1+l_1+i-j+\frac{q_t}{T}} u^{(t+1)}_{-3-l_1-i+j+\frac{r_{t+1}}{T}} w \in C_3^T(W).$$

For $i=0$, if $j \geq l_1+2$, we have $-1+l_1-j \leq -3$, so that

$$a^{(t)}_{-1+l_1-j+\frac{q_t}{T}} u^{(t+1)}_{-3-l_1+j+\frac{r_{t+1}}{T}} w \in C_3^T(W).$$

If $j \leq l_1$, we have $-3-l_1+j \leq -3$. From (2.4), we have

$$\begin{aligned} &a^{(t)}_{-1+l_1-j+\frac{q_t}{T}} u^{(t+1)}_{-3-l_1+j+\frac{r_{t+1}}{T}} w \\ &\quad = u^{(t+1)}_{-3-l_1+j+\frac{r_{t+1}}{T}} a^{(t)}_{-1+l_1-j+\frac{q_t}{T}} w + \sum_{n\in\mathbb{N}} \binom{-1+l_1-j+\frac{q_t}{T}}{n} (u^{(t+1)}_n a^{(t)})_{-4-n+\frac{q_t+r_{t+1}}{T}} w. \end{aligned}$$

Thus we have

$$a^{(t)}_{-1+l_1-j+\frac{q_t}{T}} u^{(t+1)}_{-3-l_1+j+\frac{r_{t+1}}{T}} w \in C_3^T(W).$$

So, for the only remaining case $i=0$ and $j=l_1+1$, the corresponding term $a^{(t)}_{-2+\frac{q_t}{T}} u^{(t+1)}_{-2+\frac{r_{t+1}}{T}} w$ must also lie in $C_3^T(W)$.

Next we set

$$w' = u^{(t+1)}_{-2+\frac{r_{t+1}}{T}} w.$$

Since $a^{(t)} = a^{(t-1)}_{-1} u^{(t)}$ and $q_t = q_{t-1}+r_t-T$, then by (2.8) there exist nonnegative integers k_2 and l_2 such that

$$\begin{aligned} &(a^{(t-1)}_{-1} u^{(t)})_{-2+\frac{q_t}{T}} w' \\ &\quad = \sum_{i=0}^{k_2} \sum_{j\in\mathbb{N}} \binom{-l_2-\frac{q_{t-1}}{T}}{i} \binom{i-1}{j} (-1)^j a^{(t-1)}_{-1+l_2+i-j+\frac{q_{t-1}}{T}} u^{(t)}_{-2+\frac{q_t}{T}-l_2-i+j-\frac{q_{t-1}}{T}} w' \\ &\quad = \sum_{i=0}^{k_2} \sum_{j\in\mathbb{N}} \binom{-l_2-\frac{q_{t-1}}{T}}{i} \binom{i-1}{j} (-1)^j a^{(t-1)}_{-1+l_2+i-j+\frac{q_{t-1}}{T}} u^{(t)}_{-3-l_2-i+j+\frac{r_t}{T}} w'. \end{aligned}$$

Refer to the proof of (5.9), similarly for the only remaining case $i=0$ and $j=l_2+1$, the corresponding term

$$a^{(t-1)}_{-2+\frac{q_{t-1}}{T}} u^{(t)}_{-2+\frac{r_t}{T}} w' = a^{(t-1)}_{-2+\frac{q_{t-1}}{T}} u^{(t)}_{-2+\frac{r_t}{T}} u^{(t+1)}_{-2+\frac{r_{t+1}}{T}} w$$

must also lie in $C_3^T(W)$. Therefore, repeat the above process t times, the only remaining term

$$u^{(1)}_{-2+\frac{r_1}{T}} u^{(2)}_{-2+\frac{r_2}{T}} \cdots u^{(t)}_{-2+\frac{r_t}{T}} u^{(t+1)}_{-2+\frac{r_{t+1}}{T}} w$$

must also lie in $C_3^T(W)$, which completes the whole proof. ■

From Lemma 5.3, we immediately have the following corollary.

Corollary 5.4. *Let V be a vertex algebra and let W be a g-twisted V-module. Then*

$$u^{(1)}_{-2+\frac{r_1}{T}} u^{(2)}_{-2+\frac{r_2}{T}} \cdots u^{(s)}_{-2+\frac{r_s}{T}} C_2^T(W) \subset C_3^T(W)$$

for $s \geq T$, $u^{(i)} \in V^{r_i}$, $w \in W$, $0 \leq r_i \leq T-1$, $1 \leq i \leq s$.

The following result provides the relation between $u_{-k+\frac{p}{T}} C_k^T(W)$ and $C_{k+1}^T(W)$ for $k \geq 3$.

Lemma 5.5. *Let V be a vertex algebra and let W be a g-twisted V-module. Let $k \in \mathbb{Z}$ and $k \geq 3$. Then*

$$u_{-k+\frac{p}{T}} C_k^T(W) \subset C_{k+1}^T(W)$$

for $u \in V^p$, $0 \leq p \leq T-1$.

Proof. Let $u \in V^p$, $0 \leq p \leq T-1$, from Definition 5.1 we have $C_n^T(W)$ is linearly spanned by the vectors $v_{-n+\frac{q}{T}} w$, with $v \in V^q$, $0 \leq q \leq T-1$. By (2.8), there exist nonnegative integers t and l such that

$$\begin{aligned}&(u_{-1}v)_{-2k+1+\frac{p+q}{T}} w \\ &\quad = \sum_{i=0}^{t} \sum_{j \in \mathbb{N}} \binom{-l-\frac{p}{T}}{i} \binom{i-1}{j} (-1)^j u_{-1+l+i-j+\frac{p}{T}} v_{-2k+1-l-i+j+\frac{q}{T}} w. \end{aligned} \tag{5.10}$$

For $0 \leq p+q \leq T-1$, $k \geq 3$, we have $-2k+1 \leq -k-1$ which implies

$$(u_{-1}v)_{-2k+1+\frac{p+q}{T}} w \in C_{k+1}^T(W).$$

If $i-1 \geq 0$, then $j \leq i-1$. For each i, j, from (2.4) we have

$$\begin{aligned}&u_{-1+l+i-j+\frac{p}{T}} v_{-2k+1-l-i+j+\frac{q}{T}} w \\ &\quad = v_{-2k+1-l-i+j+\frac{q}{T}} u_{-1+l+i-j+\frac{p}{T}} w + \sum_{r \in \mathbb{N}} \binom{i-j+l-1+\frac{p}{T}}{r} (u_r v)_{-2k-r+\frac{p+q}{T}} w. \end{aligned}$$

Notice that $-2k-r \leq -k-1$ and $-2k+1-l-i+j \leq -k-1$, we have

$$\begin{aligned}&(u_r v)_{-2k-r+\frac{p+q}{T}} w \in C_{k+1}^T(W), \\ &v_{-2k+1-l-i+j+\frac{q}{T}} u_{-1+l+i-j+\frac{p}{T}} w \in C_{k+1}^T(W), \end{aligned}$$

which implies

$$u_{-1+l+i-j+\frac{p}{T}} v_{-2k+1-l-i+j+\frac{q}{T}} w \in C_{k+1}^T(W).$$

For $i=0$, if $j \geq k+l$, we have $-1+l-j \leq -k-1$, so that

$$u_{-1+l-j+\frac{p}{T}} v_{-2k+1-l+j+\frac{q}{T}} w \in C_{k+1}^T(W).$$

If $j \leq k+l-2$, we have $-2k+1-l+j \leq -k-1$. From (2.4), we get

$$\begin{aligned}&u_{-1+l-j+\frac{p}{T}} v_{-2k+1-l+j+\frac{q}{T}} w \\ &\quad = v_{-2k+1-l+j+\frac{q}{T}} u_{-1+l-j+\frac{p}{T}} w + \sum_{r \in \mathbb{N}} \binom{-j+l-1+\frac{p}{T}}{r} (u_r v)_{-2k-r+\frac{p+q}{T}} w. \end{aligned}$$

Thus we have

$$u_{-1+l-j+\frac{p}{T}} v_{-2k+1-l+j+\frac{q}{T}} w \in C_{k+1}^T(W).$$

Therefore, the only remaining case $i=0$ and $j=k+l-1$, the corresponding term $u_{-k+\frac{p}{T}} v_{-k+\frac{q}{T}} w$ must also lie in $C_{k+1}^T(W)$. This proves $u_{-k+\frac{p}{T}} C_k^T(W) \subset C_{k+1}^T(W)$.

For $T \leq p+q \leq 2T-2$, we have

$$(u_{-1}v)_{-2k+1+\frac{p+q}{T}} w = (u_{-1}v)_{-2k+2+\frac{p+q-T}{T}} w \in C_{k+1}^T(W),$$

since $-2k+2 \le -k-1$ for $k \ge 3$. If $i-1 \ge 0$, we have $j \le i-1$. By (2.4), we have

$$u_{-1+l+i-j+\frac{p}{T}} v_{-2k+1-l-i+j+\frac{q}{T}} w = v_{-2k+1-l-i+j+\frac{q}{T}} u_{-1+l+i-j+\frac{p}{T}} w + \sum_{r\in\mathbb{N}} \binom{i-j+l-1+\frac{p}{T}}{r} (u_r v)_{-2k-r+1+\frac{p+q-T}{T}} w.$$

Since $-2k-r+1 \le -k-1$ and $-2k+1-l-i+j \le -k-1$, we have

$$(u_r v)_{-2k-r+1+\frac{p+q-T}{T}} w \in C_{k+1}^T(W),$$
$$v_{-2k+1-l-i+j+\frac{q}{T}} u_{-1+l+i-j+\frac{p}{T}} w \in C_{k+1}^T(W).$$

For $i=0$, if $j \ge k+l$, we have $-1+l-j \le -k-1$, so that we obtain

$$u_{-1+l-j+\frac{p}{T}} v_{-2k+1-l+j+\frac{q}{T}} w \in C_{k+1}^T(W).$$

If $j \le k+l-2$, we have $-2k+1-l+j \le -k-1$. From (2.4), we have

$$u_{-1+l-j+\frac{p}{T}} v_{-2k+1-l+j+\frac{q}{T}} w = v_{-2k+1-l+j+\frac{q}{T}} u_{-1+l-j+\frac{p}{T}} w + \sum_{r\in\mathbb{N}} \binom{-j+l-1+\frac{p}{T}}{r} (u_r v)_{-2k-r+1+\frac{p+q-T}{T}} w.$$

Thus we have

$$u_{-1+l-j+\frac{p}{T}} v_{-2k+1-l+j+\frac{q}{T}} w \in C_{k+1}^T(W).$$

Similarly, the only remaining term $u_{-k+\frac{p}{T}} v_{-k+\frac{q}{T}} w$ in (5.10) must also lie in $C_{k+1}^T(W)$.

To sum up, it proves $u_{-k+\frac{p}{T}} C_k^T(W) \subset C_{k+1}^T(W)$ for $k \in \mathbb{Z}$, $k \ge 3$, $u \in V^p$, $0 \le p \le T-1$. ■

Proposition 5.6. *Let V be a vertex algebra, let W be a g-twisted V-module and let $n \ge 0$. Then*

$$u^{(1)}_{-k_1+\frac{r_1}{T}} u^{(2)}_{-k_2+\frac{r_2}{T}} \cdots u^{(s)}_{-k_s+\frac{r_s}{T}} w \in C_{n+3}^T(W)$$

for $u^{(i)} \in V^{r_i}$, $0 \le r_i \le T-1$, $1 \le i \le s$, $w \in W$, $k_1, k_2, \ldots, k_s \ge 2$, $s \ge (T+1)2^n$.

Proof. From (5.3), we have

$$u_{-i+\frac{p}{T}} C_{n+3}^T(W) \subset C_{n+3}^T(W)$$

for $u \in V^p$, $0 \le p \le T-1$, $i \ge 1$, $n \ge 0$, it suffices to prove the assertion for $s = (T+1)2^n$. Since

$$u_{-k+\frac{p}{T}} = \frac{1}{\left(k-1-\frac{p}{T}\right)\left(k-2-\frac{p}{T}\right)\cdots\left(2-\frac{p}{T}\right)} \left(\mathcal{D}^{k-2}u\right)_{-2+\frac{p}{T}}$$

for $k \ge 3$, so we only need to prove the assertion for $k_1 = k_2 = \cdots = k_s = 2$. We are going to use induction on n. If $n=0$, by Lemma 5.3, for $s \ge T+1$, we have

$$u^{(1)}_{-2+\frac{r_1}{T}} u^{(2)}_{-2+\frac{r_2}{T}} \cdots u^{(s)}_{-2+\frac{r_s}{T}} w \in C_3^T(W).$$

Assume the assertion holds for $n=l$, some nonnegative integer. Set $s=(T+1)2^{l+1}$, $m=(T+1)2^l$. For $u^{(i)} \in V^{r_i}$, $0 \le r_i \le T-1$, $1 \le i \le s$, $w \in W$. By inductive hypothesis, we have

$$u^{(m+1)}_{-2+\frac{r_{m+1}}{T}} u^{(m+2)}_{-2+\frac{r_{m+2}}{T}} \cdots u^{(s)}_{-2+\frac{r_s}{T}} w \in C_{l+3}^T(W).$$

So that we have

$$u^{(1)}_{-2+\frac{r_1}{T}}u^{(2)}_{-2+\frac{r_2}{T}}\cdots u^{(s)}_{-2+\frac{r_s}{T}}w \in u^{(1)}_{-2+\frac{r_1}{T}}u^{(2)}_{-2+\frac{r_2}{T}}\cdots u^{(m)}_{-2+\frac{r_m}{T}}C^T_{l+3}(W). \tag{5.11}$$

From Definition 5.1, we know that $C^T_{l+3}(W)$ is linearly spanned by the vectors $a_{-l-3+\frac{r}{T}}w'$, with $a\in V^r$, $w'\in W$. Using (5.3) and (5.4), we have

$$\begin{aligned}&u^{(1)}_{-2+\frac{r_1}{T}}u^{(2)}_{-2+\frac{r_2}{T}}\cdots u^{(m)}_{-2+\frac{r_m}{T}}a_{-l-3+\frac{r}{T}}w'\\ &\quad\equiv a_{-l-3+\frac{r}{T}}u^{(1)}_{-2+\frac{r_1}{T}}u^{(2)}_{-2+\frac{r_2}{T}}\cdots u^{(m)}_{-2+\frac{r_m}{T}}w' \pmod{C^T_{l+4}(W)}.\end{aligned} \tag{5.12}$$

Furthermore, by inductive hypothesis, we have

$$u^{(1)}_{-2+\frac{r_1}{T}}u^{(2)}_{-2+\frac{r_2}{T}}\cdots u^{(m)}_{-2+\frac{r_m}{T}}w' \in C^T_{l+3}(W),$$

which together with Lemma 5.5 and $l\geq 0$ gives

$$a_{-l-3+\frac{r}{T}}u^{(1)}_{-2+\frac{r_1}{T}}u^{(2)}_{-2+\frac{r_2}{T}}\cdots u^{(m)}_{-2+\frac{r_m}{T}}w' \in a_{-l-3+\frac{r}{T}}C^T_{l+3}(W)\subset C^T_{l+4}(W).$$

Then by (5.12) we have

$$u^{(1)}_{-2+\frac{r_1}{T}}u^{(2)}_{-2+\frac{r_2}{T}}\cdots u^{(m)}_{-2+\frac{r_m}{T}}a_{-l-3+\frac{r}{T}}w' \in C^T_{l+4}(W),$$

proving that

$$u^{(1)}_{-2+\frac{r_1}{T}}u^{(2)}_{-2+\frac{r_2}{T}}\cdots u^{(m)}_{-2+\frac{r_m}{T}}C^T_{l+3}(W)\subset C^T_{l+4}(W).$$

Therefore, by (5.11) we have

$$u^{(1)}_{-2+\frac{r_1}{T}}u^{(2)}_{-2+\frac{r_2}{T}}\cdots u^{(s)}_{-2+\frac{r_s}{T}}w \in C^T_{l+4}(W).$$

This concludes the proof. ■

The relationship between $E^T_n(W)$ and $C^T_n(W)$ in twisted case is described as follows.

Theorem 5.7. *Let V be a vertex algebra, let W be a g-twisted V-module and let $\mathcal{E}^T_W = \{E^T_n(W)\}_{n\in\mathbb{Z}}$ be the associated decreasing sequence. Then for any $n\geq 2$*

$$C^T_n(W)\subset E^T_{(n-2)T+1}(W), \tag{5.13}$$

$$E^T_m(W)\subset C^T_n(W) \qquad \text{whenever } m\geq \max\{1,(n-2)T(T+1)2^{n-3}\}. \tag{5.14}$$

Furthermore,

$$\bigcap_{n\geq 0}E^T_n(W)=\bigcap_{n\geq 2}C^T_{n+2}(W). \tag{5.15}$$

Proof. For $u\in V^p$, $0\leq p\leq T-1$, $n\geq 2$, $w\in W$, from Definitions 5.1 and 4.1, we have

$$u_{-n+\frac{p}{T}}w = u_{-1-(n-1)+\frac{p}{T}}w \in E^T_{(n-1)T-p}(W)\subset E^T_{(n-2)T+1}(W).$$

This proves (5.13). For $n=2$, from Lemma 4.3 we have $E^T_1(W)\subset C^T_2(W)$. For $n\geq 3$, consider a generic spanning element of $E^T_m(W)$ with $m\geq 1$:

$$X = u^{(1)}_{-1-k_1+\frac{r_1}{T}}u^{(2)}_{-1-k_2+\frac{r_2}{T}}\cdots u^{(s)}_{-1-k_s+\frac{r_s}{T}}w,$$

where $s \geq 1$, $u^{(i)} \in V^{r_i}$, $0 \leq r_i \leq T-1$, $1 \leq i \leq s$, $w \in W$, $k_1, k_2, \ldots, k_s \geq 1$, with $k_1 + k_2 + \cdots + k_s - \frac{r_1+r_2+\cdots+r_s}{T} \geq \frac{m}{T}$. If there exists j with $1 \leq j \leq s$ such that $k_j \geq n-1$, by (5.1) we have

$$u^{(j)}_{-1-k_j+\frac{r_j}{T}} W \subset C_{1+k_j}^T(W) \subset C_n^T(W),$$

then by (5.3) we have $X \in C_n^T(W)$. If $s \geq (T+1)2^{n-3}$, then by Proposition 5.6 we have $X \in C_n^T(W)$. Next we demonstrate at least one of $k_i \geq n-1$ and $s \geq (T+1)2^{n-3}$ holds. Assume for any i with $1 \leq i \leq s$, we have $k_i \leq n-2$ and $s \leq (T+1)2^{n-3}-1$, thus $k_1+k_2+\cdots+k_s \leq s(n-2)$. Since $0 \leq r_i \leq T-1$, we have

$$\begin{aligned}(n-2)\frac{T(T+1)}{T}2^{n-3} &\leq \frac{m}{T} \leq k_1 + k_2 + \cdots + k_s - \frac{r_1+r_2+\cdots+r_s}{T} \\ &\leq s(n-2) \leq (n-2)\big((T+1)2^{n-3}-1\big).\end{aligned}$$

This leads to a contradiction. This proves (5.14).

Combining (5.13) and (5.14), we have (5.15). ■

From Theorem 5.7, we immediately have the following corollary.

Corollary 5.8. *Let V be a vertex algebra and W be a g-twisted V-module, we have*

$$E_1^T(W) = C_2^T(W).$$

Proof. From (5.13), we have $C_2^T(W) \subset E_1^T(W)$. On the other hand, from (5.14) we have $E_1^T(W) \subset C_2^T(W)$, which proves $E_1^T(W) = C_2^T(W)$. ■

6 Generating subspace of $\frac{1}{T}\mathbb{N}$-graded twisted modules of vertex algebras

In this section, we shall use the twisted module $\mathrm{gr}_{\mathcal{E}}^T(W)$ of vertex Poisson algebra $\mathrm{gr}_{\mathcal{E}}^T(V)$ and the relation between $\{E_n^T(W)\}_{n\in\mathbb{Z}}$ and $\{C_n^T(W)\}_{n\in\mathbb{Z}_{\geq 2}}$ to prove that for any twisted module W of a vertex algebra V, C_2-cofiniteness implies C_n-cofiniteness for all $n \geq 2$. Next we employ $\mathrm{gr}_{\mathcal{E}}^T(W)$ to study generating subspaces of $\frac{1}{T}\mathbb{N}$-graded twisted modules of lower truncated $\mathbb{Z}$-graded vertex algebras.

The following gives the bilinear action of the differential algebra $A = \mathrm{gr}_{\mathcal{E}}^T(V)$ on $\mathrm{gr}_{\mathcal{E}}^T(W)$ defined by (6.1).

Lemma 6.1. *Let V be a vertex algebra and let $A = \mathrm{gr}_{\mathcal{E}}^T(V)$ be the vertex Poisson algebra obtained in Theorem 3.11, which is in particular a $T\mathbb{N}$-graded (unital) differential algebra. Let W be a g-twisted V-module, the associated graded vector space $\mathrm{gr}_{\mathcal{E}}^T(W)$ is equipped with the bilinear map*

$$\begin{aligned}A \otimes \mathrm{gr}_{\mathcal{E}}^T(W) &\longrightarrow \mathrm{gr}_{\mathcal{E}}^T(W), \\ \big(u + E_{(m+1)T}^T(V)\big) \otimes \big(w + E_{n+1}^T(W)\big) &\longmapsto \big(u + E_{(m+1)T}^T(V)\big) \cdot \big(w + E_{n+1}^T(W)\big)\end{aligned}$$

where

$$\big(u + E_{(m+1)T}^T(V)\big) \cdot \big(w + E_{n+1}^T(W)\big) = u_{-1+\frac{p}{T}} w + E_{mT+n-p+1}^T(W) \tag{6.1}$$

for $u \in E_{mT}^T(V) \cap V^p$, $w \in E_n^T(W)$, with $m, n \in \mathbb{N}$, $0 \leq p \leq T-1$. Moreover, $\mathrm{gr}_{\mathcal{E}}^T(W)$ is generated under the iterated action of A via (6.1) by its degree-zero subspace $E_0^T(W)/E_1^T(W)$,

i.e., for $n \geq 0$, the degree-n subspace $E_n^T(W)/E_{n+1}^T(W)$ of $\mathrm{gr}_{\mathcal{E}}^T(W)$ is linearly spanned by the vectors

$$\partial^{k_1}\big(u^{(1)} + E_T^T(V)\big)\partial^{k_2}\big(u^{(2)} + E_T^T(V)\big)\cdots\partial^{k_s}\big(u^{(s)} + E_T^T(V)\big)\big(w + E_1^T(W)\big) \tag{6.2}$$

for $1 \leq s \leq n$, $u^{(i)} \in V^{r_i}$, $w \in W$, $k_1 \geq k_2 \geq \cdots \geq k_s \geq 0$ with $k_1+k_2+\cdots+k_s-\frac{r_1+r_2+\cdots+r_s}{T} = \frac{n}{T}$, $0 \leq r_i \leq T-1$, $1 \leq i \leq s$.

Proof. Notice that Proposition 4.5 guarantees that the operation given in (6.1) is well defined. The case $n = 0$ is trivial. For $n \geq 1$, from (4.3) we have $E_n^T(W)$ is linearly spanned by the vectors $u_{-2-i+\frac{p}{T}}w$, where $u \in E_0^T(V) \cap V^p$, $w \in E_{n-(i+1)T+p}^T(W)$ for $0 \leq i \leq \frac{n+p}{T} - 1$, then we have

$$\begin{aligned}
&u_{-2-i+\frac{p}{T}}w + E_{n+1}^T(W)\\
&\quad= \frac{1}{\big(i+1-\frac{p}{T}\big)\big(i-\frac{p}{T}\big)\cdots\big(1-\frac{p}{T}\big)}\big(\mathcal{D}^{i+1}u\big)_{-1+\frac{p}{T}}w + E_{n+1}^T(W)\\
&\quad= \frac{1}{\big(i+1-\frac{p}{T}\big)\big(i-\frac{p}{T}\big)\cdots\big(1-\frac{p}{T}\big)}\big(\mathcal{D}^{i+1}u + E_{(i+2)T}^T(V)\big)\big(w + E_{n-(i+1)T+p+1}^T(W)\big)\\
&\quad= \frac{1}{\big(i+1-\frac{p}{T}\big)\big(i-\frac{p}{T}\big)\cdots\big(1-\frac{p}{T}\big)}\big(\partial^{i+1}\big(u + E_T^T(V)\big)\big)\big(w + E_{n-(i+1)T+p+1}^T(W)\big).
\end{aligned} \tag{6.3}$$

We are going to prove by induction on n. For $n = 1$, from (4.3) we have $E_1^T(W)/E_2^T(W)$ is linearly spanned by $u_{-2-i+\frac{p}{T}}w + E_2^T(W)$ with $u \in E_0^T(V) \cap V^p$, $w \in E_{1-(i+1)T+p}^T(W)$ for $0 \leq i \leq \frac{1+p}{T} - 1$. Then it follows $i = 0$ and $p = T-1$, we have $E_1^T(W)/E_2^T(W)$ is linearly spanned by vectors

$$u_{-2+\frac{T-1}{T}}w + E_2^T(W) = \frac{1}{1-\frac{T-1}{T}}\big(\partial\big(u + E_T^T(V)\big)\big)\big(w + E_1^T(W)\big).$$

We assume the assumption holds for $n-1$. From (6.3), since $i \geq 0$, $0 \leq p \leq T-1$, we have $n-(i+1)T+p \leq n-1$, then it follows immediately from induction that $E_n^T(W)/E_{n+1}^T(W)$ is linearly spanned by those vectors in (6.2). For $n \geq 1$ and $1 \leq j \leq s$, by (4.4) we notice that k_j and r_j appear in pairs and $k_j \geq 1$, $0 \leq r_j \leq T-1$, we have $k_j - \frac{r_j}{T} \geq \frac{1}{T}$, which means $1 \leq s \leq n$. This concludes the proof. ∎

The following result is from [17].

Lemma 6.2. *Let V be a vertex algebra and let $\mathrm{gr}_{\mathcal{E}}^T(V)$ be the vertex Poisson algebra obtained in Theorem 3.11. Then $\mathrm{gr}_{\mathcal{E}}^T(V)$ is linearly spanned by the vectors*

$$\partial^{k_1}\big(v^{(1)} + E_T^T(V)\big)\partial^{k_2}\big(v^{(2)} + E_T^T(V)\big)\cdots\partial^{k_s}\big(v^{(s)} + E_T^T(V)\big)$$

for $s \geq 1$, $v^{(i)} \in V$, $k_1 > k_2 > \cdots > k_s \geq 0$. In particular, $\mathrm{gr}_{\mathcal{E}}^T(V)$ as a differential algebra is generated by the subspace $E_0^T(V)/E_T^T(V) (= V/C_2(V))$.

From Lemma 6.2 and the definition of $E_n^T(V)$, we have the following (see [1, 5, 12, 18]).

Lemma 6.3. *Let V be a vertex algebra. If V is C_2-cofinite, then V is E_n-cofinite and C_{n+2}-cofinite for any $n \geq 0$.*

Combining Theorem 5.7, Corollary 5.8, Lemmas 6.1 and 6.3, we have the following.

Proposition 6.4. *Let V be a vertex algebra and let W be a g-twisted V-module. If V and W are C_2-cofinite, then W is E_n-cofinite and C_{n+2}-cofinite for all $n \geq 0$.*

Proof. We are going to use induction on n. Since $C_2(V) = E_T^T(V)$ and $C_2^T(W) = E_1^T(W)$, we have

$$\dim V/E_T^T(V) = \dim V/C_2(V) < \infty.$$

From Lemma 6.1, we know that the degree-n subspace $E_n^T(W)/E_{n+1}^T(W)$ is finite-dimensional for $n \geq 0$. For $n = 0$, we have $\dim E_0^T(W)/E_1^T(W) = \dim W/C_2^T(W) < \infty$. For $E_0^T(W)/E_{n+1}^T(W)$, we have

$$E_0^T(W)/E_n^T(W) \cong \big(E_0^T(W)/E_{n+1}^T(W)\big)/\big(E_n^T(W)/E_{n+1}^T(W)\big).$$

By induction hypothesis, $\dim\big(E_0^T(W)/E_n^T(W)\big) < \infty$. Then $\dim\big(E_0^T(W)/E_{n+1}^T(W)\big) < \infty$, which proves W is E_n-cofinite. By (5.14), $E_m^T(W) \subset C_n^T(W)$ when $m \geq (n-2)T(T+1)2^{n-3}$. Then

$$\dim W/C_n^T(W) \leq \dim W/E_m^T(W) < \infty.$$

This completes the proof. ∎

The following result generalizes the result of Lemma 4.7.

Proposition 6.5. *Let $V = \coprod_{n\geq t} V_{(n)}$ be a lower truncated $\mathbb{Z}$-graded vertex algebra for some $t \in \mathbb{Z}$, and $W = \bigoplus_{n\in\frac{1}{T}\mathbb{N}} W_{(n)}$ be a $\frac{1}{T}\mathbb{N}$-graded g-twisted V-module. Then*

$$C_n^T(W) \subset \coprod_{k\geq n+t-\frac{2T-1}{T}} W_{(k)} \qquad \text{for } n \geq 2. \tag{6.4}$$

Furthermore, for $n \geq 2$, we have

$$\bigcap_{n\geq 0} E_n^T(W) = \bigcap_{n\geq 2} C_n^T(W) = 0, \tag{6.5}$$

$$E_m^T(W) \subset \coprod_{k\geq n+t-\frac{2T-1}{T}} W_{(k)} \qquad \text{for } m \geq (n-2)T(T+1)2^{n-3}. \tag{6.6}$$

Proof. For $n \geq 2$ and for homogeneous vectors $u \in V^p$, $w \in W_{(m)}$, $m \in \frac{1}{T}\mathbb{N}$, we have

$$\operatorname{wt}(u_{-n+\frac{p}{T}}w) = \operatorname{wt} u + \operatorname{wt} w + n - \frac{p}{T} - 1 \geq t + n - \frac{p}{T} - 1 \geq n + t - \frac{2T-1}{T}.$$

Thus we prove (6.4). Then we have $\bigcap_{n\geq 2} C_n^T(W) = 0$ which implies (6.5). Combining Theorem 5.7 and (6.4), we immediately have (6.6). ∎

Next we consider a special case of g-twisted V-module whose associated decreasing sequence $\mathcal{E}_W^T$ is trivial.

Lemma 6.6. *Let V be a vertex algebra and let W be a g-twisted V-module. If $W = C_2^T(W)$, then*

$$E_n^T(W) = C_{n+2}^T(W) = W \qquad \text{for all } n \geq 0.$$

Proof. We are going to use induction on n. Since $W = C_2^T(W)$, by Corollary 5.8 we have

$$W = C_2^T(W) = E_1^T(W) \subset W,$$

which implies $E_1^T(W) = W$. For some $k \geq 1$, assume $E_k^T(W) = W$. For $v \in V^p$, $0 \leq p \leq T-1$, then by Definitions 5.1 and 4.1 we have

$$v_{-2+\frac{p}{T}}W = v_{-2+\frac{p}{T}}E_k^T(W) \subset E_{k+T-p}^T(W) \subset E_{k+1}^T(W),$$

which implies $W = C_2^T(W) \subset E_{k+1}^T(W)$, proving that $E_{k+1}^T(W) = W$. By induction, we have $E_n^T(W) = W$ for all $n \geq 0$. From (5.14), we have $C_n^T(W) = W$ for all $n \geq 2$. ∎

Proposition 6.7. *Let $V = \coprod_{n\geq t} V_{(n)}$ be a lower truncated vertex algebra for some $t \in \mathbb{Z}$, let W be a nonzero g-twisted V-module such that $C_2^T(W) = W$. Then there does not exist a lower truncated $\frac{1}{T}\mathbb{N}$-grading $W = \bigoplus_{n\in\frac{1}{T}\mathbb{N}} W_{(n)}$ with which W becomes a $\frac{1}{T}\mathbb{N}$-graded g-twisted V-module.*

Proof. Suppose that W is a g-twisted V-module such that $C_2^T(W) = W$, then by Lemma 6.6 we have $W = C_{n+2}^T(W)$ for $n \geq 0$, so that $W = \bigcap_{n\geq 0} C_{n+2}^T(W)$. Furthermore, if W carries a $\frac{1}{T}\mathbb{N}$-grading such that W becomes a $\frac{1}{T}\mathbb{N}$-graded g-twisted V-module, then by (6.5) we have $\bigcap_{n\geq 0} C_{n+2}^T(W) = 0$, so that $W = \bigcap_{n\geq 0} C_{n+2}^T(W) = 0$. ■

The following proposition gives a spanning property of certain type for $\frac{1}{T}\mathbb{N}$-graded twisted modules of lower truncated $\mathbb{Z}$-graded vertex algebras.

Proposition 6.8. *Let $V = \coprod_{n\geq t} V_{(n)}$ be a $\mathbb{Z}$-graded vertex algebra for some $t \in \mathbb{Z}$ and let U be a graded subspace of V such that $V = U + C_2(V)$. Let $W = \bigoplus_{n\in\frac{1}{T}\mathbb{N}} W_{(n)}$ be a $\frac{1}{T}\mathbb{N}$-graded g-twisted V-module and let M be a graded subspace of W such that $W = M + C_2^T(W)$. Then W is linearly spanned by the vectors*

$$u^{(1)}_{-1-k_1+\frac{r_1}{T}} u^{(2)}_{-1-k_2+\frac{r_2}{T}} \cdots u^{(s)}_{-1-k_s+\frac{r_s}{T}} w \tag{6.7}$$

for $u^{(i)} \in U \cap V^{r_i}$, $0 \leq r_i \leq T-1$, $1 \leq i \leq s$, $w \in M$, $k_1 \geq k_2 \geq \cdots \geq k_s \geq 0$, $s \geq 1$.

Proof. Let $K^T(W)$ be a subspace of W, which is spanned by the vectors in (6.7). It is clear that $K^T(W)$ is a graded subspace. For $m \geq 0$, set $K_m^T(W) = K^T(W) \cap E_m^T(W)$. For $u \in E_{lT}^T(V) \cap V^p$, $l \geq 0$, $0 \leq p \leq T-1$, set

$$\partial^{(k)}(u + E_{(l+1)T}^T(V)) = \frac{1}{\left(k-\frac{p}{T}\right)\left(k-1-\frac{p}{T}\right)\cdots\left(1-\frac{p}{T}\right)} \partial^k\left(u + E_{(l+1)T}^T(V)\right),$$
$$\mathcal{D}^{(k)}u + E_{(l+k+1)T}^T(V) = \frac{1}{\left(k-\frac{p}{T}\right)\left(k-1-\frac{p}{T}\right)\cdots\left(1-\frac{p}{T}\right)} \left(\mathcal{D}^k u + E_{(l+k+1)T}^T(V)\right).$$

Since $C_2^T(W) = E_1^T(W)$, then $W = M + C_2^T(W) = M + E_1^T(W)$. From (6.2), for any $m \geq 0$, $E_m^T(W)/E_{m+1}^T(W)$ is linearly spanned by the vectors

$$\partial^{(k_1)}\left(u^{(1)} + E_T^T(V)\right)\partial^{(k_2)}\left(u^{(2)} + E_T^T(V)\right)\cdots\partial^{(k_s)}\left(u^{(s)} + E_T^T(V)\right)\left(w + E_1^T(W)\right)$$

for $s \geq 0$, $u^{(i)} \in U \cap V^{r_i}$, $w \in M$, $k_1 \geq k_2 \geq \cdots \geq k_s \geq 0$ with $k_1+k_2+\cdots+k_s - \frac{r_1+r_2+\cdots+r_s}{T} = \frac{m}{T}$. By (3.7) and (3.8), we have

$$\begin{aligned}
&\partial^{(k_1)}\left(u^{(1)} + E_T^T(V)\right)\partial^{(k_2)}\left(u^{(2)} + E_T^T(V)\right)\cdots\partial^{(k_s)}\left(u^{(s)} + E_T^T(V)\right)\left(w + E_1^T(W)\right)\\
&\quad = \left(\mathcal{D}^{(k_1)}u^{(1)} + E_{k_1T+T}^T(V)\right)\left(\mathcal{D}^{(k_2)}u^{(2)} + E_{k_2T+T}^T(V)\right)\cdots\\
&\qquad \times \left(\mathcal{D}^{(k_s)}u^{(s)} + E_{k_sT+T}^T(V)\right)\left(w + E_1^T(W)\right)\\
&\quad = u^{(1)}_{-1-k_1+\frac{r_1}{T}} u^{(2)}_{-1-k_2+\frac{r_2}{T}} \cdots u^{(s)}_{-1-k_s+\frac{r_s}{T}} w + E_{m+1}^T(W).
\end{aligned}$$

It follows that $E_m^T(W) = K_m^T(W) + E_{m+1}^T(W)$. Thus we have

$$W = E_0^T(W) = K_0^T(W) + K_1^T(W) + \cdots + K_n^T(W) + E_{n+1}^T(W) \subset K^T(W) + E_{n+1}^T(W)$$

for any $n \geq 0$. Since both $K^T(W)$ and $E_{n+1}^T(W)$ are graded subspaces of W and by (6.5) we have $\bigcap_{m\geq 0} E_m^T(W) = 0$, it forces

$$W = K^T(W) = \sum_{i\geq 0} K_i^T(W),$$

proving the desired spanning property. ■

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