Partition Functions of Two-Dimensional Coulomb Gases with Circular Root- and Jump-Type Singularities

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Received January 15, 2026, in final form October 10, 2026; Published online October 30, 2026
https://doi.org/10.3842/SIGMA.2026.106

Abstract. In this paper, we study the random polynomial $p_n(\rho) := \prod_{j=1}^n (|z_j| - \rho)$, where the points $\{z_j\}_{j=1}^n$ are the eigenvalues of random normal matrices with a radially symmetric potential. We establish precise large n asymptotic expansions for the moment generating function $\mathbb{E}\big[\mathrm{e}^{(u \operatorname{Im}\log p_n(\rho))/\pi} \mathrm{e}^{a \operatorname{Re}\log p_n(\rho)}\big]$, $u \in \mathbb{R}$, $a > -1$, where $\rho > 0$ lies in the bulk of the spectral droplet. The asymptotic expansion is expressed in terms of parabolic cylinder functions, which confirms a conjecture of Byun and Charlier. This also provides the first free energy expansion of two-dimensional Coulomb gases with general circular root- and jump-type singularities. While the $a = 0$ case has already been widely studied in the literature due to its relation to counting statistics, we also obtain new results for this special case.

Key words: random normal matrices; counting statistics; partition functions; asymptotics

2020 Mathematics Subject Classification: 41A60; 60B20; 60G55

1 Introduction and statement of results

For $\boldsymbol{z}_n = (z_1, \dots, z_n) \in \mathbb{C}^n$, consider the n-fold integral

$$Z_{n,u,a}[Q] := \int_{\mathbb{C}^n} \prod_{1 \le j < k \le n} |z_k - z_j|^2 \prod_{j=1}^n |z_j|^{2\alpha} \mathrm{e}^{-nQ(z_j)} \omega(z_j) \frac{\mathrm{d}^2 z_j}{\pi}, \qquad \alpha > -1, \tag{1.1}$$

where $\mathrm{d}^2 z$ is the Lebesgue measure on $\mathbb{C}$, and $Q\colon \mathbb{C} \to \mathbb{R}$ is called the external potential. The factor $|z|^{2\alpha} = \mathrm{e}^{\alpha \log |z|^2}$ is referred to as a *Fisher–Hartwig singularity* at the origin, see e.g., [7, Section 1.4]. The condition $\alpha > -1$ is necessary for the convergence of the integral in (1.1). Our assumptions on Q are stated in Assumptions 1.1 below. Here, $\omega(z) \equiv \omega(z; u, a)$ possesses a root singularity and a jump along the circle centered at 0 of radius $\rho > 0$; more precisely, it is defined by

$$\omega(z) := |x - \rho|^a \begin{cases} \mathrm{e}^u, & \text{if } x < \rho, \\ 1, & \text{if } x \ge \rho, \end{cases} \qquad x = |z|, \quad a > -1, \quad u \in \mathbb{R}. \tag{1.2}$$

Integrals of the form (1.1) are typically called partition functions in the literature. They find applications in random matrix theory and statistical physics, and have therefore been widely studied in [4, 7, 22, 26, 27, 28, 31, 60, 74]. $Z_{n,0,0}[Q]$ is also the normalization constant of the following probability measure:

$$d\mathbb{P}_n(\boldsymbol{z}_n) := \frac{1}{Z_{n,0,0}[Q]} \prod_{1 \le j < k \le n} |z_k - z_j|^2 \prod_{j=1}^n |z_j|^{2\alpha} \mathrm{e}^{-nQ(z_j)} \frac{\mathrm{d}^2 z_j}{\pi}, \qquad \alpha > -1, \tag{1.3}$$

which represents the joint probability distribution of the eigenvalues of a random normal matrix [24, 52, 67]. In particular, the choice $Q(z) = |z|^2$ and $\alpha = 0$ corresponds to the complex Ginibre ensemble [53]. The measure (1.3) is a *determinantal point process* [52]. For a review of recent developments on non-Hermitian random matrices, see [24].

The ratio $\frac{Z_{n,u,a}[Q]}{Z_{n,0,0}[Q]}$ gives the joint moment generating function of $(\operatorname{Re}\log p_n(\rho), \operatorname{Im}\log p_n(\rho))$, where p_n is the random polynomial given by

$$p_n(x) := \prod_{j=1}^{n} \big(|z_j| - x\big).$$

Note that the roots of p_n are the moduli $\{|z_j|\}_{j=1}^n$, where $\{z_j\}_{j=1}^n$ are distributed according to (1.3). For $\rho > 0$, we define

$$\log p_n(\rho) := \log|p_n(\rho)| + \pi \mathrm{i}\, \mathrm{N}_\rho, \qquad \mathrm{N}_\rho := \#\{z_j \mid |z_j| < \rho\}.$$

Here log denotes the principal branch of the logarithm. If we denote by $\mathbb{E}$ the expectation with respect to (1.3), then

$$\mathcal{E}_{n,u,a} := \mathbb{E}\big[\mathrm{e}^{\frac{u}{\pi}\operatorname{Im}\log p_n(\rho)} \mathrm{e}^{a \operatorname{Re}\log p_n(\rho)}\big] = \frac{Z_{n,u,a}[Q]}{Z_{n,0,0}[Q]}, \tag{1.4}$$

with $u \in \mathbb{R}$, $a > -1$. The partition function $Z_{n,u,a}[Q]$ for $a \neq 0$ was first introduced in the work [23] of Byun and Charlier, where the potential $Q(z)$ in (1.3) was chosen to be the Mittag-Leffler potential $Q(z) = |z|^{2b}$ with $b > 0$. In [23, Theorem 1.1], they established the large n asymptotic expansion of $\mathcal{E}_{n,u,a}$ up to and including the term of order 1, with a precise estimate for the error term. However, only the case $a \in \mathbb{N}$ was considered. An interesting phenomenon observed in their result is the appearance of associated Hermite polynomials in the coefficients of the terms of order $\sqrt{n}$ and 1. The large n asymptotics of $\mathcal{E}_{n,u,a}$ in the general case $a > -1$ was left open in [23], and was conjectured to involve parabolic cylinder functions in place of the associated Hermite polynomials, see [23, Remark 1.3]. We also mention that the weight function (1.2) was studied in the study for the correlation kernel in [70, equation (2.4)].

In this paper, we obtain precise large n asymptotics of $\mathcal{E}_{n,u,a}$ for general $u \in \mathbb{R}$, $a > -1$ and rotation-invariant Q, in the regime where ρ lies in the bulk. In particular, we confirm the conjecture from [23] that the asymptotics involve parabolic cylinder functions. This type of asymptotic behavior is completely new in random matrix theory, to the best of our knowledge. The case $a = 0$ of our main results is already of interest, as it generalizes previous results on counting statistics of random normal matrix eigenvalues for general rotation-invariant potentials, as explained below.

For $a = 0$, (1.4) reduces to the moment generating function of the disk counting statistics of random normal matrix eigenvalues, i.e.,

$$\mathcal{E}_{n,u,0} = \mathbb{E}\big[\mathrm{e}^{u\,\mathrm{N}_\rho}\big].$$

Counting statistics of random normal matrix eigenvalues have attracted considerable attention in recent years, see, e.g., [1, 2, 3, 8, 9, 35, 39, 50, 58, 59, 61, 71, 72]. In the work [35], Charlier established the precise large n asymptotic expansion of $\mathcal{E}_{n,u,0}$ for the Mittag-Leffler potential $Q(z) = |z|^{2b}$ with $b > 0$. The asymptotics of the *joint* moment generating functions, in the critical regime where all disk boundaries are merging at speed $n^{-1/2}$, were then obtained in the follow-up work [39]. In [8, 9], Ameur, Charlier, Cronvall, and Lenells then treated the more difficult hard-edge regime where all disk boundaries are merging at speed n^{-1} near a hard wall. In [9], the asymptotics contain an oscillatory term due to the fact that the particles accumulate on several components. The work [10] also treats a hard-edge case, but in a simpler

situation where there is no bulk. Leading order asymptotics of $\mathcal{E}_{n,u,0}$ were then obtained in [2] for general rotation invariant potentials, and leading order asymptotics of $\mathrm{Var}[\mathrm{N}_\rho]$ in [66] for general potentials and domains. Our main result, in the special case $a = 0$, improves on [2] by providing the next two terms in the large n asymptotics of $\mathcal{E}_{n,u,0}$. Since counting statistics have attracted considerable attention in recent years, for the convenience of the reader this particular case is stated separately in Theorem 1.3 below. In Corollary 1.5, we also provide precise large n asymptotics of all cumulants of N_ρ (not just the variance).

We will make the following assumptions on Q:

Assumptions 1.1. We suppose that the potential is rotation invariant, i.e., $Q(z) = q(|z|)$, and satisfies the following conditions:

(1) $\liminf_{|z|\to\infty} \frac{Q(z)}{2\log|z|} > 1$, which guarantees that $Z_{n,u,a}[Q] < +\infty$.

(2) Q is C^6-smooth in a neighborhood of the droplet, globally subharmonic in $\mathbb{C}$, and strictly subharmonic (i.e., $\Delta Q(z) > 0$) in a neighborhood of the droplet (the droplet is defined in (1.5) below).

Remark 1.2. Assumptions 1.1 cover a broad class of rotation-invariant potentials. However, they do not cover the power-law potentials $Q(z) = |z|^{2b}$ with $b \neq 1$, which form a distinguished subclass of the Mittag-Leffler potentials. Indeed,

$$\Delta Q(z) \asymp |z|^{2b-2}, \qquad z \to 0,$$

and hence $\Delta Q(z) \to 0$ for $b > 1$, whereas $\Delta Q(z) \to \infty$ for $b \in (0, 1)$. This power-law behavior of the Laplacian is associated with a conical singularity at the origin; see, e.g., [33, 55].

Such singularities occur naturally in the study of quantum Hall states on surfaces with conical points. In the presence of a constant magnetic field, the corresponding Coulomb-gas partition function naturally involves both conical and Fisher–Hartwig singularities, as in (1.1). Partition functions of this type admit descriptions in terms of Barnes double zeta functions [56] and are closely related to ζ-regularized determinants of Laplace operators on surfaces with conical singularities [55]. It would be of interest to extend the results of the present paper to classes of potentials allowing for such singular behavior at the origin; we do not pursue this direction here.

Moreover, Assumption 1.1 (2) excludes the disconnected-droplet considered in [6, 7, 11].

Under Assumptions 1.1, the empirical measure $\frac{1}{n}\sum_{j=1}^n \delta_{z_j}$ of (1.3) converges weakly to the measure σ_Q given by

$$\mathrm{d}\sigma_Q := \Delta Q \cdot \mathbf{1}_S \frac{\mathrm{d}^2 z}{\pi},$$

where $S \equiv S_Q$ is a compact subset of $\mathbb{C}$ called the *droplet*. Here we use $\Delta = \frac{1}{4}(\partial_x^2 + \partial_y^2)$ for $z = x + \mathrm{i}y$. Under parts (1) and (2) of Assumptions 1.1, S is of the form

$$S = \mathbb{A}_{r_0,r_1} := \{z \in \mathbb{C} \mid r_0 \le |z| \le r_1\}, \tag{1.5}$$

where r_0 is the largest solution to $rq'(r) = 0$, and r_1 is the smallest solution of $rq'(r) = 2$; see, e.g., [27] and [69, Section IV.6]. Throughout this paper, we assume that $r_0 = 0$, i.e., $S = \mathbb{D}_{r_1} := \{z \in \mathbb{C} \mid |z| \le r_1\}$. (The analysis done in this paper can be adapted to the case $r_0 > 0$, which is in fact simpler.)

We recall that the complementary error function is defined by

$$\mathrm{erfc}(t) := \frac{2}{\sqrt{\pi}} \int_t^{+\infty} \mathrm{e}^{-x^2}\, \mathrm{d}x, \qquad t \in \mathbb{R},$$

see, e.g., [68, equation (7.2.1)]. Following [35, equation (1.6)], we introduce

$$\mathcal{F}(t,s) := \log\Big(1+\frac{s-1}{2}\operatorname{erfc}(t)\Big), \qquad t\in\mathbb{R}, \quad s\in\mathbb{C}\backslash(-\infty,0], \tag{1.6}$$

where the principal branch is chosen for the log.

Our first main result corresponds to the case $a=0$.

Theorem 1.3 (counting statistics)**.** *Let $\rho\in(0,r_1)$, $\alpha>-1$, $u\in\mathbb{R}$, and $a=0$. Under Assumptions 1.1, for any fixed $x\in\mathbb{R}$, there exists $\delta>0$ such that, as $n\to+\infty$, we have*

$$\log\mathcal{E}_{n,u,0} = C_1(u)n + C_2(u)\sqrt{n} + C_3(u) + \mathcal{O}\left(\frac{(\log n)^3}{n^{1/12}}\right), \tag{1.7}$$

uniformly for $u\in\{z\in\mathbb{C}\mid |z-x|\le\delta\}$, where

$$\begin{aligned}
C_1(u) &:= u\int_{\mathbb{D}_\rho} \mathrm{d}\sigma_Q(z),\\
C_2(u) &:= \rho\sqrt{2\Delta Q(\rho)}\int_0^{+\infty}\big(\mathcal{F}(x,\mathrm{e}^{u})+\mathcal{F}(x,\mathrm{e}^{-u})\big)\,\mathrm{d}x,\\
C_3(u) &:= -\left(\alpha+\frac{1}{2}\right)u+\frac{1}{6}\left(2+\frac{\rho\partial_r\Delta Q(\rho)}{\Delta Q(\rho)}\right)u\\
&\quad+\frac{1}{3}\left(2+\frac{\rho\partial_r\Delta Q(\rho)}{\Delta Q(\rho)}\right)\int_0^{+\infty}x\big(\mathcal{F}(x,\mathrm{e}^{u})-\mathcal{F}(x,\mathrm{e}^{-u})\big)\,\mathrm{d}x.
\end{aligned}$$

Remark 1.4 (consistency with [35, Theorem 1.1])**.** First, note that $C_3(u)$ is given in a simpler form than C_3 in [35, Theorem 1.1].

Second, as mentioned in Remark 1.2, Assumptions 1.1 do not cover the case $Q(z)=|z|^{2b}$ for $b\neq 1$. However, surprisingly, substituting $Q(z)=|z|^{2b}$ into Theorem 1.3 recovers [23, Theorem 1.1] for any $b>0$, see Appendix A. It would be interesting to extend the results of the present paper to a broader class of potentials, including $Q(z)=|z|^{2b}$ for $b>0$.

Recall that the cumulants $\{\kappa_j\}_{j\in\mathbb{N}_{>0}}$ of the random variable N_ρ (see (1.4) above) are defined through the expansion

$$\log\mathbb{E}\big[\mathrm{e}^{u\,\mathrm{N}_\rho}\big] = \kappa_1 u + \frac{\kappa_2 u^2}{2!} + \frac{\kappa_3 u^3}{3!} + \cdots, \qquad u\to 0,$$

or equivalently by

$$\kappa_j = \partial_u^j \log\mathbb{E}\big[\mathrm{e}^{u\,\mathrm{N}_\rho}\big]\Big|_{u=0}, \qquad j\in\mathbb{N}.$$

In particular, since $\mathbb{E}\big[\mathrm{e}^{u\,\mathrm{N}_\rho}\big]$ is analytic for $u\in\mathbb{C}$ and positive for any $u\in\mathbb{R}$, and Theorem 1.3 is valid uniformly for $u\in\{z\in\mathbb{C}\mid|z-x|\le\delta\}$ for some $\delta>0$, Cauchy's formula implies that for any $u\in\mathbb{R}$, $j\in\mathbb{N}$, we have

$$\begin{aligned}
&\partial_u^j\big\{\log\mathbb{E}\big[\mathrm{e}^{u\mathrm{N}_\rho}\big]-\big(C_1(u)n+C_2(u)\sqrt{n}+C_3(u)\big)\big\}\\
&\quad=\frac{j!}{2\pi\mathrm{i}}\oint_{|\zeta-u|=\frac{\delta}{2}}\frac{\log\mathbb{E}\big[\mathrm{e}^{\zeta\mathrm{N}_\rho}\big]-\big(C_1(\zeta)n+C_2(\zeta)\sqrt{n}+C_3(\zeta)\big)}{(\zeta-u)^{j+1}}\,\mathrm{d}\zeta\\
&\quad=\mathcal{O}\left(\frac{(\log n)^3}{n^{1/12}}\right), \qquad n\to+\infty.
\end{aligned}$$

The above equation shows that (1.7) can be differentiated with respect to u any fixed number of times without increasing the error term. Therefore, Theorem 1.3 combined with [35, Corollary 1.6] provides the following corollary, generalizing the result in [2, Proposition 2.14].

Corollary 1.5. *Let $\rho \in (0, r_1)$, $\alpha > -1$, $u \in \mathbb{R}$, and $a = 0$. Under Assumptions 1.1, as $n \to +\infty$, we have*

$$\kappa_j = \begin{cases} C_1'(0)n + C_3'(0) + \mathcal{O}\left(\dfrac{(\log n)^3}{n^{1/12}}\right) & \text{if } j = 1, \\ \partial_u^j C_3(0) + \mathcal{O}\left(\dfrac{(\log n)^3}{n^{1/12}}\right) & \text{if } j \neq 1 \text{ is odd}, \\ \partial_u^j C_2(0)\sqrt{n} + \mathcal{O}\left(\dfrac{(\log n)^3}{n^{1/12}}\right) & \text{if } j \text{ is even}. \end{cases}$$

The central limit theorem for the disk counting statistic N_ρ follows from Corollary 1.5 by the same argument as in [35, proof of Corollary 1.6 (e)].

We now turn our attention to (1.4) in the general case $a > -1$. For any $x \in \mathbb{R}$, $u \in \mathbb{C}$, and $a > -1$, define

$$\mathcal{H}_{a,u}(x) := \frac{\Gamma(a+1)}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{4}x^2} g_{a,u}(x), \qquad g_{a,u}(x) := \mathrm{e}^u D_{-a-1}(x) + D_{-a-1}(-x), \tag{1.8}$$

where $D_{-\nu}(x)$ is the parabolic cylinder function [68, equation (12.5.1)], which is defined by

$$D_{-\nu}(z) = U\left(\nu - \tfrac{1}{2}, z\right), \qquad \operatorname{Re}\nu > 0. \tag{1.9}$$

Here,

$$U(\nu, z) := \frac{\mathrm{e}^{-\frac{1}{4}z^2}}{\Gamma\left(\nu + \frac{1}{2}\right)} \int_0^{+\infty} \mathrm{e}^{-zt} t^{\nu - \frac{1}{2}} \mathrm{e}^{-\frac{1}{2}t^2} \, \mathrm{d}t, \qquad \operatorname{Re}\nu > -\frac{1}{2}.$$

For the relationship between (1.9) and the associated Hermite polynomials, see Appendix B. Note that for any $a > -1$, the integrand of $D_{-a-1}(x)$ is positive for all $x \in \mathbb{R}$. Consequently, $\mathcal{H}_{a,u}(x)$ is positive for all $x \in \mathbb{R}$ whenever $a > -1$ and $u \in \mathbb{R}$. Therefore, it follows that the logarithm $\log \mathcal{H}_{a,u}(x)$ is well defined for all $x \in \mathbb{R}$ and $u \in \mathbb{R}$.

The following is the main result of this paper.

Theorem 1.6 (counting statistics and root-type statistics)**.** *Let $\rho \in (0, r_1)$, $\alpha > -1$, $u \in \mathbb{R}$, and $a > -1$. Under Assumptions 1.1, for any fixed $x \in \mathbb{R}$, there exists $\delta > 0$ such that as $n \to +\infty$, we have*

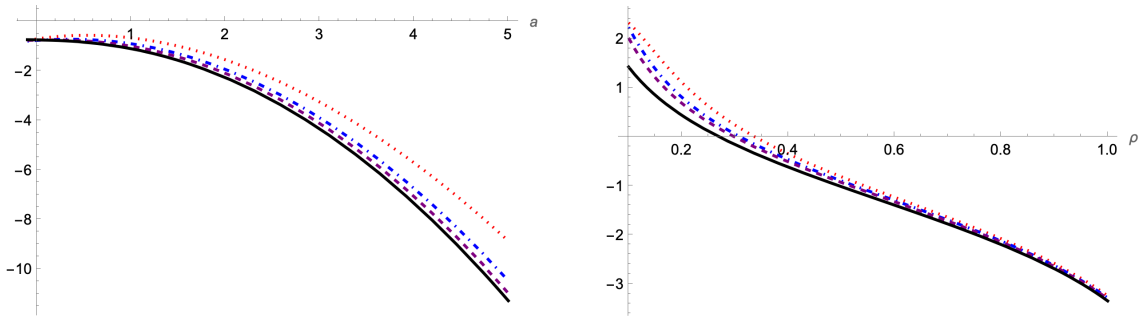
$$\log \mathcal{E}_{n,u,a} = C_1(u,a)n + C_2(u,a)\sqrt{n} + C_3(u,a) + \mathcal{O}\left(\frac{(\log n)^3}{n^{1/12}}\right), \tag{1.10}$$

uniformly for $u \in \{z \in \mathbb{C} \mid |z - x| \le \delta\}$ and a in compact subsets of $(-1, +\infty)$, where

$$\begin{aligned} C_1(u,a) &:= \int_{\mathbb{C}} \log \omega(z) \, \mathrm{d}\sigma_Q(z) \\ &= \int_{\mathbb{D}_\rho} (u + a \log(\rho - |z|)) \, \mathrm{d}\sigma_Q(z) + \int_{\mathbb{A}_{\rho, r_1}} a \log(|z| - \rho) \, \mathrm{d}\sigma_Q(z), \end{aligned} \tag{1.11}$$

$$C_2(u,a) := \rho\sqrt{\Delta Q(\rho)} \int_{-\infty}^{+\infty} \left(\log[\mathcal{H}_{a,u}(x)] - a \log|x| - u\mathbf{1}_{(-\infty,0)}(x)\right) \mathrm{d}x, \tag{1.12}$$

$$\begin{aligned} C_3(u,a) := &-\frac{a}{2} \log\left(\frac{r_1}{\rho} - 1\right) - \frac{a(a-1)}{4} \frac{r_1}{r_1 - \rho} \\ &- \frac{a}{4} \int_0^{r_1} \frac{1}{x - \rho} \left(\frac{x \partial_x \Delta Q(x)}{\Delta Q(x)} - \frac{\rho \partial_r \Delta Q(\rho)}{\Delta Q(\rho)}\right) \mathrm{d}x \end{aligned}$$



(a) $a \mapsto C_3$ **(b)** $\rho \mapsto C_3$

Figure 1. Plot (a) shows $a \mapsto C_3$ (black line) and its comparison $a \mapsto \log \mathcal{E}_{n,u,a} - \big(C_1 n + C_2\sqrt{n}\big)$, where $Q(z) = 0.2|z|^2 + 0.2345|z|^3$, $\alpha = 0.667$, $u = 1.56$, $\rho = 0.71 r_1$, $n = 10$ (red, dotted line), $n = 40$ (blue, dot-dashed line), and $n = 160$ (purple, dashed line). Plot (b) shows $\rho \mapsto C_3$ (black line) and its comparison $\rho \mapsto \log \mathcal{E}_{n,u,a} - (C_1 n + C_2\sqrt{n})$, where Q, α, u are same as before, $a = 1.25$, and $n = 100$ (red, dotted line), $n = 300$ (blue, dot-dashed line), and $n = 600$ (purple, dashed line). This numerical computation is based on the fact that, as shown in Section 2, (1.10) can be rewritten in terms of (1.4).

$$
\begin{aligned}
&+ \frac{a}{4}\left(4\alpha + a + 2 - \frac{\rho\partial_r \Delta Q(\rho)}{\Delta Q(\rho)}\right)\log\left(\frac{r_1}{\rho} - 1\right) - \left(\alpha + \frac{1}{2}\right)u \\
&- \frac{a}{12}\left(1 - \frac{\rho\partial_r \Delta Q(\rho)}{\Delta Q(\rho)}\right)u + \frac{1}{6}\left(2 + \frac{\rho\partial_r \Delta Q(\rho)}{\Delta Q(\rho)}\right)u + \frac{1}{6}\left(2 + \frac{\rho\partial_r \Delta Q(\rho)}{\Delta Q(\rho)}\right) \\
&\times \int_{-\infty}^{+\infty}\left[x\big(\log[\mathcal{H}_{a,u}(x)] - u\mathbf{1}_{(-\infty,0)}(x)\big) - ax\log|x| - \frac{a(a-1)x}{2(x^2+1)}\right]\mathrm{d}x. \qquad (1.13)
\end{aligned}
$$

Remark 1.7. Again, as explained in Remark 1.2, Assumptions 1.1 do not cover the case $Q(z) = |z|^{2b}$ if $b \neq 1$. However, in a similar way as in Remark 1.4, substituting $Q(z) = |z|^{2b}$ into Theorem 1.6 recovers [23, Theorem 1.1] for any $b > 0$, see Appendix B.

As mentioned earlier, Theorem 1.6 confirms the conjecture stated in [23, Remark 1.3]. Moreover, we obtained $C_3(u, a)$ in a simpler form than C_3 in [23, Theorem 1.1]. Figure 1 provides a numerical comparison to verify the asymptotic formula (1.13). We currently do not have a conformal theoretic or geometric interpretation for (1.11), (1.12), and (1.13) as in [27, 79]. We also believe that the error estimate in (1.10) is not optimal, but we do not pursue this issue here. It is expected that the error can be improved to $n^{-1/2}$, see [23, Theorem 1.1 and Remark 1.2].

We finally deduce the large n asymptotics of (1.1) by combining Theorem 1.6 with [7, Theorem 1.4].

We define

$$
I_Q[\mu] := \int_{\mathbb{C}^2} \log\frac{1}{|z-w|}\,\mathrm{d}\mu(z)\mathrm{d}\mu(w) + \int_{\mathbb{C}} Q(z)\,\mathrm{d}\mu(z),
$$

which is referred to as the weighted logarithmic energy, defined over all compactly supported Borel probability measures μ. In particular, if Q is lower semi-continuous and finite on a set of positive capacity, Frostman's theorem guarantees the existence of a unique equilibrium measure σ_Q minimizing the weighted logarithmic energy, see [69]. In particular, for the potential Q satisfying Assumptions 1.1, we have

$$
I_Q[\sigma_Q] = q(r_1) - \log r_1 - \frac{1}{4}\int_0^{r_1} r q'(r)^2\,\mathrm{d}r.
$$

We define

$$
E_Q[\sigma_Q] := \int_{\mathbb{C}} \log \Delta Q\,\mathrm{d}\sigma_Q,
$$

$$F_Q[\sigma_Q] := \frac{1}{12}\log\frac{1}{r_1^2\Delta Q(r_1)} - \frac{1}{16}\frac{r_1\partial_r\Delta Q(r_1)}{\Delta Q(r_1)} + \frac{1}{24}\int_0^{r_1}\left(\frac{\partial_r\Delta Q(r)}{\Delta Q(r)}\right)^2 r\,\mathrm{d}r,$$

and following [7, equation (1.14)], we also define, for $\ell_\alpha(z) := 2\alpha\log|z|$,

$$\mathsf{e}_{\ell_\alpha} := \frac{1}{2}\int_S \ell_\alpha(z)\Delta\log\Delta Q(z)\,\mathrm{d}A(z) + \frac{1}{8\pi}\int_{\partial S}\partial_{\mathrm{n}}\ell_\alpha(z)\,|\mathrm{d}z| - \frac{1}{8\pi}\int_{\partial S}\ell_\alpha(z)\frac{\partial_{\mathrm{n}}\Delta Q(z)}{\Delta Q(z)}\,|\mathrm{d}z|,$$

where "∂_{n}" designates differentiation in the normal direction to ∂S pointing out from the droplet S. Here, $E_Q[\sigma_Q]$ represents the negative entropy of the equilibrium measure, while $F_Q[\sigma_Q]$ can be interpreted in terms of ζ-regularized determinants associated with certain pseudo-differential operators, see [27], [7, p. 284, Remark], and the references therein.

For smooth linear statistics, the asymptotic cumulant generating function, including its mean correction, is implicit in the physical predictions of Zabrodin and Wiegmann [79] and was rigorously established by Ameur, Hedenmalm, and Makarov [12] in a general connected-droplet setting. Ameur and Cronvall [11] extended this fluctuation theory to a class of disconnected droplets separated by ring-shaped spectral gaps, see also [7]. The free energy expansion with a Fisher–Hartwig insertion at the origin is provided by [7, Theorem 1.4], where the quantity e_{ℓ_α} above is obtained.

We conclude with the following result.

Theorem 1.8 (partition function with circular- and root-type singularities)**.** *Let $\rho \in (0, r_1)$, $\alpha > -1$, $u \in \mathbb{R}$, and $a > -1$. Under Assumptions 1.1, for any fixed $x \in \mathbb{R}$, there exists $\delta > 0$ such that as $n \to +\infty$, we have*

$$\log Z_{n,u,a}[Q] = \widetilde{C}_1 n^2 + \widetilde{C}_2 n\log n + \widetilde{C}_3 n + \widetilde{C}_4\sqrt{n} + \widetilde{C}_5\log n + \widetilde{C}_6 + \mathcal{O}\left(\frac{(\log n)^3}{n^{1/12}}\right),$$

uniformly for $u \in \{z \in \mathbb{C} \mid |z - x| \le \delta\}$ and a in compact subsets of $(-1, +\infty)$, where

$$\begin{aligned}
&\widetilde{C}_1 := -I_Q[\sigma_Q], \qquad \widetilde{C}_2 := \frac{1}{2},\\
&\widetilde{C}_3 := \frac{\log 2\pi}{2} - 1 - \frac{E_Q[\sigma_Q]}{2} + \int_{\mathbb{C}}\ell_\alpha(z)\,\mathrm{d}\sigma_Q(z) + C_1(u,a),\\
&\widetilde{C}_4 := C_2(u,a), \qquad \widetilde{C}_5 := \frac{5}{12} + \frac{\alpha^2}{2},\\
&\widetilde{C}_6 := \zeta'(-1) - \log G(1+\alpha) + F_Q[\sigma_Q] + \frac{1+\alpha}{2}\log(2\pi) + \mathsf{e}_{\ell_\alpha} - \frac{\alpha}{2}\\
&\qquad + \frac{\alpha^2}{2}\log\big(r_1^2\Delta Q(0)\big) + C_3(u,a),
\end{aligned}$$

where $C_1(u,a)$, $C_2(u,a)$, $C_3(u,a)$ are given by (1.11), (1.12), (1.13), respectively, and $\zeta(z)$ is the Riemann zeta function.

Remark 1.9. The coefficient $\widetilde{C}_5$ can be rewritten as $\frac{6-\chi}{12} + \frac{\alpha^2}{2}$, where χ is the Euler characteristic of the droplet. In our case, since the droplet is a disk, $\chi = 1$. The fact that $\widetilde{C}_5$ can be rewritten in this way is consistent with the general conjecture [79]. Similar results have been established in [27] for centered disks and annuli, in [7] for multiple disjoint connected annuli possibly including a central disk, and in [28] for the (generalized) spherical ensemble.

Remark 1.10. Throughout this paper, we do not address the case where $r_0 > 0$ (see (1.5)). Nevertheless, the asymptotic expansion of $Z_{n,u,a}[Q]$ in this case can be established by arguments similar to those in [4, Section 5] and [5, Section 5.2]. To avoid introducing additional notation and technicalities, we do not pursue this direction here.

Related works

We conclude by briefly mentioning recent developments in the literature related to Theorems 1.3, 1.6, and 1.8.

The study of structured determinants with singularities in one-dimensional point processes has a long history. In the seminal work [51], Fisher and Hartwig conjectured the asymptotic behavior of Toeplitz determinants in the large-size limit when the weight function is supported on the unit circle and possesses root- and jump-type singularities, which is now commonly referred to as Fisher–Hartwig singularities. For early works and more historical background, we refer to [47] and references therein.

Recent results on structured determinants with singularities in connection with random matrix theory include [19, 41, 42, 46, 48, 49] for Toeplitz determinants, [40, 43, 44, 65, 76, 77, 78] for Fredholm determinants, [15, 34, 38, 54, 57] for Hankel determinants, [14, 46] for Toeplitz+Hankel determinants.

There has been significant progress on the precise large-n asymptotics of partition functions, moment generating functions, and hole probabilities for the random normal matrices mentioned earlier. A seminal development is the works of Charlier [35, 37] on the moment generating function and hole probabilities for random normal matrices with Mittag-Leffler potentials $Q(z) = |z|^{2b}$ for $b > 0$.

Further progress was then made by Byun, Kang, and Seo [27], who obtained precise large-n asymptotics for the partition functions of two-dimensional Coulomb gases with rotation-invariant potentials. Subsequent developments include the study of multi-component droplets [7] by Ameur, Charlier, and Cronvall, who removed the global subharmonicity assumption on the potential and established new asymptotic results for droplets with multiple connected components, thereby extending the results of [79].

Other developments concern hole probabilities [30, 36], partition functions with a fixed hard wall [4], and potentials for which ΔQ vanishes along a circle inside the droplet [5]. Related free energy expansions have been obtained for spherical Coulomb gases [28, 79] and for lemniscate-type potentials with a conical singularity at the origin [22].

Precise large-n asymptotics for the partition function of the Ginibre ensemble with a big point-charge insertion that breaks rotational symmetry were obtained in [31]. This work is motivated by strong asymptotics of planar orthogonal polynomials for the Ginibre ensemble with a point insertion [13]. For related developments, see [17, 18, 21, 29, 32, 62, 63, 64], and for connections with Gaussian multiplicative chaos, see [74].

Functional central limit theorems for hard-edge disk counting statistics in Mittag-Leffler ensembles ($Q(z) = |z|^{2b}$) with a hard wall were obtained in [16]. For point insertions in other ensembles, including the truncated unitary and spherical ensembles, see [25, 26, 45]. In contrast to point insertions, the factor (1.2) introduces circular root- and jump-type singularities. As in [23], our analysis does not require Riemann–Hilbert methods. Free energy asymptotics for determinantal Coulomb gases on compact Riemann surfaces were studied in [20].

Outline

It suffices to prove Theorem 1.6, as Theorems 1.3 and 1.8 follow as special cases. The strategy for proving Theorem 1.6 combines the Laplace method, the precise Riemann sum approximation of [37], and the decomposition of a large sum into global and local analysis parts as in [23].

We first split the logarithmic sum in (1.4) into a global part and a local part. In Section 2, we establish the large n asymptotics of the global part (Lemmas 2.2, 2.4, and 2.5), where the proofs of Lemmas 2.4 and 2.5 require careful analysis of the error. In Section 3, we establish the large n asymptotics of the summand corresponding to the local part. This is first stated in

Lemma 3.1, and subsequently, each sum is approximated using Lemma C.1 with more refined error estimates than in the global part.

2 Global analysis part for the proof of Theorem 1.6

Since $|z|^{2\alpha}\mathrm{e}^{-nq(|z|)}\omega(z)$ is rotation invariant, $\mathcal{E}_n \equiv \mathcal{E}_{n,u,a}$ can be identically expressed in terms of one-fold integrals. This fact is well known as *Andréief identity* and has already been used in different contexts, see, e.g., [7, 8, 9, 10, 23, 27, 30, 35]. For fixed $u \in \mathbb{R}, a \in (-1, +\infty)$, and $\rho \in (0, r_1)$, we have

$$\mathcal{E}_n \equiv \mathbb{E}\big[\mathrm{e}^{\frac{u}{\pi}\operatorname{Im}\log p_n(\rho)}\mathrm{e}^{a\operatorname{Re}\log p_n(\rho)}\big] = \frac{\mathcal{D}_n}{\mathcal{Z}_n}, \tag{2.1}$$

where

$$\begin{aligned}\mathcal{D}_n &:= \prod_{j=0}^{n-1}\int_0^{+\infty} 2v^{2j+2\alpha+1}\mathrm{e}^{-nq(v)}\omega(v)\,\mathrm{d}v, \\ \mathcal{Z}_n &:= \prod_{j=0}^{n-1}\int_0^{+\infty} 2v^{2j+2\alpha+1}\mathrm{e}^{-nq(v)}\,\mathrm{d}v.\end{aligned} \tag{2.2}$$

Here, $\omega(|z|)$ is defined by (1.2). Note that (2.2) is simply written as

$$\mathcal{D}_n = \prod_{j=0}^{n-1}\left(\mathrm{e}^{u}\int_0^{\rho} 2v^{2j+2\alpha+1}\mathrm{e}^{-nq(v)}(\rho - v)^a\,\mathrm{d}v + \int_{\rho}^{+\infty} 2v^{2j+2\alpha+1}\mathrm{e}^{-nq(v)}(v-\rho)^a\,\mathrm{d}v\right).$$

In order to analyze the precise large n-asymptotics of (2.1), we follow the robust strategy done in [7, 23, 27]. We define

$$V_\tau(r) := q(r) - 2\tau\log r, \qquad \tau \equiv \tau(j) := \frac{j}{n}. \tag{2.3}$$

By differentiating (2.3) with respect to r, from [27, equation (2.4)], we have

$$\begin{aligned}&V_\tau'(r) = q'(r) - \frac{2\tau}{r}, \qquad V_\tau''(r) = 4\Delta Q(r) - \frac{1}{r}V_\tau'(r), \\ &V_\tau^{(3)}(r) = 4\partial_r\Delta Q(r) - \frac{4}{r}\Delta Q(r) + \frac{2}{r^2}V_\tau'(r), \\ &V_\tau^{(4)}(r) = 4\partial_r^2\Delta Q(r) + \frac{12}{r^2}\Delta Q(r) - \frac{4}{r}\partial_r\Delta Q(r) - \frac{6}{r^3}V_\tau'(r).\end{aligned} \tag{2.4}$$

For $0 \le \tau \le 1$ we denote by r_τ the solution to

$$r_\tau q'(r_\tau) = 2\tau,$$

which gives rise to $V_\tau'(r_\tau) = 0$. Note that r_τ satisfies the following differential equation [27, equation (2.6)]:

$$\frac{\mathrm{d}r_\tau}{\mathrm{d}\tau} = \frac{1}{2r_\tau\Delta Q(r_\tau)} > 0, \tag{2.5}$$

where we have used part (2) of Assumptions 1.1. By [27, equation (3.7)], r_τ satisfies the following asymptotics:

$$r_\tau = \left(\frac{\tau}{\Delta Q(0)}\right)^{1/2} + \mathcal{O}(\tau), \qquad \tau \to 0. \tag{2.6}$$

Let τ_ρ be a solution so that

$$\rho q'(\rho) = 2\tau_\rho. \tag{2.7}$$

To split the logarithmic sum of (1.4), we define critical indices

$$g_{1,-} := \lceil n(\tau_\rho - \delta_n)\rceil, \qquad g_{1,+} := \lfloor n(\tau_\rho + \delta_n)\rfloor, \qquad \delta_n' := \frac{M}{\sqrt{n}}, \qquad M := n^{1/8}(\log n)^{-1/8}, \tag{2.8}$$

where $\lceil x\rceil$ denotes the smallest integer $\geq x$, and $\lfloor x\rfloor$ denotes the largest integer $\leq x$. We also define

$$m_n := \lfloor n^{1/6}\rfloor, \qquad \delta_n := \frac{\log n}{\sqrt{n}}.$$

See Remark 2.6 for an explanation of the choice of the parameter M in (2.8). See also [27, Lemma 3.1] for a similar technical choice of the parameter m_n in a different context. We write

$$\begin{aligned} h_{n,j}^{(\mathrm{in})}(\rho) &:= \int_0^\rho 2v\mathrm{e}^{\mathsf{k}(v)}\mathrm{e}^{-nV_\tau(v)}|\rho - v|^a\,\mathrm{d}v, \\ h_{n,j}^{(\mathrm{out})}(\rho) &:= \int_\rho^{+\infty} 2v\mathrm{e}^{\mathsf{k}(v)}\mathrm{e}^{-nV_\tau(v)}|v - \rho|^a\,\mathrm{d}v, \\ h_{n,j} &:= \int_0^{+\infty} 2v\mathrm{e}^{\mathsf{k}(v)}\mathrm{e}^{-nV_\tau(v)}\,\mathrm{d}v, \end{aligned} \tag{2.9}$$

where $\mathsf{k}(v) := 2\alpha\log v$ for $\alpha > -1$. Then, we can rewrite $\mathcal{E}_n$ as

$$\log\mathcal{E}_n = \sum_{j=0}^{n-1}\log\left[\mathrm{e}^u\frac{h_{n,j}^{(\mathrm{in})}(\rho)}{h_{n,j}} + \frac{h_{n,j}^{(\mathrm{out})}(\rho)}{h_{n,j}}\right] = S_0 + S_1 + S_2 + S_3, \tag{2.10}$$

where

$$\begin{aligned} S_0 &:= \sum_{j=0}^{m_n-1}\log\left[\mathrm{e}^u\frac{h_{n,j}^{(\mathrm{in})}(\rho)}{h_{n,j}} + \frac{h_{n,j}^{(\mathrm{out})}(\rho)}{h_{n,j}}\right], & S_1 &:= \sum_{j=m_n}^{g_{1,-}-1}\log\left[\mathrm{e}^u\frac{h_{n,j}^{(\mathrm{in})}(\rho)}{h_{n,j}} + \frac{h_{n,j}^{(\mathrm{out})}(\rho)}{h_{n,j}}\right], \\ S_2 &:= \sum_{j=g_{1,-}}^{g_{1,+}}\log\left[\mathrm{e}^u\frac{h_{n,j}^{(\mathrm{in})}(\rho)}{h_{n,j}} + \frac{h_{n,j}^{(\mathrm{out})}(\rho)}{h_{n,j}}\right], & S_3 &:= \sum_{j=g_{1,+}+1}^{n-1}\log\left[\mathrm{e}^u\frac{h_{n,j}^{(\mathrm{in})}(\rho)}{h_{n,j}} + \frac{h_{n,j}^{(\mathrm{out})}(\rho)}{h_{n,j}}\right]. \end{aligned}$$

For the later purpose, we split S_2 into

$$S_2 = S_2^{(\mathrm{in})} + S_2^{(\mathrm{out})},$$

where

$$\begin{aligned} S_2^{(\mathrm{in})} &:= \sum_{j=g_{1,-}}^{\lfloor n\tau_\rho\rfloor-1}\log\left[\mathrm{e}^u\frac{h_{n,j}^{(\mathrm{in})}(\rho)}{h_{n,j}} + \frac{h_{n,j}^{(\mathrm{out})}(\rho)}{h_{n,j}}\right], \\ S_2^{(\mathrm{out})} &:= \sum_{j=\lfloor n\tau_\rho\rfloor}^{g_{1,+}}\log\left[\mathrm{e}^u\frac{h_{n,j}^{(\mathrm{in})}(\rho)}{h_{n,j}} + \frac{h_{n,j}^{(\mathrm{out})}(\rho)}{h_{n,j}}\right]. \end{aligned}$$

Lemma 2.1. *For $0 \leq j \leq m_n - 1$, there exists $c > 0$ independent of n such that as $n \to +\infty$, we have*

$$h_{n,j}^{(\mathrm{in})}(\rho) = \mathrm{e}^{-nq(0)}\left(\frac{2}{nq''(0)}\right)^{j+\alpha+1}\rho^a\Gamma(j+\alpha+1)\left[1 + \mathcal{O}\left(\frac{(j+1)^{3/2}(\log n)^3}{\sqrt{n}}\right)\right], \tag{2.11}$$

$$h_{n,j}^{(\mathrm{out})}(\rho) = \mathrm{e}^{-nq(0)}\cdot\mathcal{O}(\mathrm{e}^{-cn}), \tag{2.12}$$

uniformly for a in compact subsets of $(-1, +\infty)$.

Proof. We recall [7, Lemma 4.1]

$$h_{n,j} = \mathrm{e}^{-nq(0)} \left(\frac{2}{nq''(0)}\right)^{j+\alpha+1} \Gamma(j+\alpha+1) \left[1 + \mathcal{O}\left(\frac{(j+1)^{3/2}(\log n)^3}{\sqrt{n}}\right)\right].$$

By a similar manner to the proof of [7, Lemma 4.1], as $n \to +\infty$, we obtain (2.11) and (2.12). In particular, the error terms do not depend on a. This completes the proof. ■

Lemma 2.2. *For any fixed $x \in \mathbb{R}$, there exists $\delta > 0$ such that as $n \to +\infty$, we have*

$$S_0 = m_n \log(\rho^a \mathrm{e}^u) + \mathcal{O}\big((\log n)^3 n^{-1/12}\big), \tag{2.13}$$

uniformly for $u \in \{z \in \mathbb{C} \mid |z - x| \le \delta\}$ and a in compact subsets of $(-1, +\infty)$.

Proof. By Lemma 2.1 and [7, Lemma 4.1], there exists $\delta > 0$ such that we obtain (2.13), uniformly for $u \in \{z \in \mathbb{C} \mid |z - x| \le \delta\}$ and a in compact subsets of $(-1, +\infty)$. ■

We next turn to S_1 and S_3. We use the following lemma from [7, Lemma 4.3].

Lemma 2.3. *For $m_n \le j \le n-1$ and $\mathsf{k}(r) = 2\alpha \log r$, as $n \to +\infty$, we have*

$$h_{n,j} = \sqrt{\frac{2\pi}{n}} \frac{r_\tau}{\sqrt{\Delta Q(r_\tau)}} \mathrm{e}^{\mathsf{k}(r_\tau)} \mathrm{e}^{-nV_\tau(r_\tau)} \cdot \left(1 + \frac{\mathcal{A}(r_\tau)}{n} + \mathcal{O}\left(\frac{(\log n)^\nu}{j^{3/2}}\right)\right)$$

for some $\nu > 0$, where

$$\mathcal{A}(r_\tau) := \mathcal{B}(r_\tau) + \frac{\mathsf{k}'(r_\tau)^2}{2}\frac{1}{d_2} + \frac{\mathsf{k}''(r_\tau)}{2}\frac{1}{d_2} + \frac{\mathsf{k}'(r_\tau)}{r_\tau}\frac{1}{d_2} - \frac{\mathsf{k}'(r_\tau)}{2}\frac{d_3}{d_2^2},$$
$$\mathcal{B}(r) := -\frac{1}{32}\frac{\partial_r^2 \Delta Q(r)}{(\Delta Q(r))^2} - \frac{19}{96}\frac{\partial_r \Delta Q(r)}{(\Delta Q(r))^2} + \frac{5}{96}\frac{(\partial_r \Delta Q(r))^2}{(\Delta Q(r))^3} + \frac{1}{12}\frac{1}{r^2 \Delta Q(r)},$$
$$d_m := V_\tau^{(m)}(r_\tau).$$

Here, $\mathcal{O}\big(j^{-3/2}(\log n)^\nu\big)$ can be replaced with $\mathcal{O}\big(n^{-2}\big)$ for large j, i.e., for $j \ge c_0 n$ with some $c_0 > 0$.

We first establish the large n-asymptotics of S_1. To this end, for a sufficiently small $\epsilon > 0$, let $j_{1,-} := \lceil n(\tau_\rho - \epsilon) \rceil$. We write

$$\theta_{1,-}^{(\epsilon)} := j_{1,-} - n(\tau_\rho - \epsilon), \qquad \theta_{m_n} := n^{1/6} - m_n,$$
$$\theta_{1,-} := \lceil n(\tau_\rho - \delta_n) \rceil - n(\tau_\rho - \delta_n), \qquad \theta_{1,+} := n(\tau_\rho + \delta_n) - \lfloor n(\tau_\rho + \delta_n) \rfloor.$$

Let us denote

$$\Delta_{n,+} := \sqrt{\frac{n}{\Delta Q(\rho)}} \frac{\delta_n' - \frac{\theta_{1,+}}{n}}{\rho} = \frac{1}{\rho\sqrt{\Delta Q(\rho)}} \left(M - \frac{\theta_{1,+}}{\sqrt{n}}\right), \tag{2.14}$$
$$\Delta_{n,-} := \sqrt{\frac{n}{\Delta Q(\rho)}} \frac{\delta_n' - \frac{\theta_{1,-}}{n}}{\rho} = \frac{1}{\rho\sqrt{\Delta Q(\rho)}} \left(M - \frac{\theta_{1,-}}{\sqrt{n}}\right).$$

By the implicit function theorem, Assumptions 1.1, and (2.5), as $n \to +\infty$, $r_{\tau(g_\pm)}$ are expanded as

$$r_{\tau(g_{1,+})} = \rho + \frac{1}{2\sqrt{\Delta Q(\rho)}} \frac{\Delta_{n,+}}{\sqrt{n}} - \frac{\Delta Q(\rho) + \rho \partial_r \Delta Q(\rho)}{8\rho \Delta Q(\rho)^2} \frac{\Delta_{n,+}^2}{n} + \mathcal{O}\left(\frac{\Delta_{n,+}^3}{n^{3/2}}\right), \tag{2.15}$$
$$r_{\tau(g_{1,-})} = \rho - \frac{1}{2\sqrt{\Delta Q(\rho)}} \frac{\Delta_{n,-}}{\sqrt{n}} - \frac{\Delta Q(\rho) + \rho \partial_r \Delta Q(\rho)}{8\rho \Delta Q(\rho)^2} \frac{\Delta_{n,-}^2}{n} + \mathcal{O}\left(\frac{\Delta_{n,-}^3}{n^{3/2}}\right). \tag{2.16}$$

We first establish the large n-expansion of S_1.

Lemma 2.4. *For any fixed $x \in \mathbb{R}$, there exists $\delta > 0$ such that as $n \to +\infty$, we have*

$$S_1 = C_1^{(1)} n + C_2^{(1)} \sqrt{n} + C_3^{(1)} + C_n^{(1)} + \mathcal{O}\left(\frac{(\log n)^3}{n^{1/12}} \right), \tag{2.17}$$

uniformly for $u \in \{z \in \mathbb{C} \mid |z - x| \le \delta\}$ and a in compact subsets of $(-1, +\infty)$, where

$$\begin{aligned}
C_1^{(1)} &:= \tau_\rho u + \int_0^\rho 2au \Delta Q(u) \log(\rho - u)\, \mathrm{d}u \qquad C_2^{(1)} := 0, \\
C_3^{(1)} &:= \frac{a}{2} \log 2 + \frac{a}{4} \log \Delta Q(\rho) + \frac{a}{2} \log \rho - \frac{a(a-1)}{8} \left(3 + \frac{\rho \partial_r \Delta Q(\rho)}{\Delta Q(\rho)} \right) \\
&\quad + \frac{a}{4} \int_0^\rho \left(\frac{1}{\rho - u} \frac{u \partial_r \Delta Q(u)}{\Delta Q(u)} - \frac{1}{\rho - u} \frac{\rho \partial_r \Delta Q(\rho)}{\Delta Q(\rho)} \right) \mathrm{d}u \\
&\quad - \frac{a}{4} \log\bigl(2\rho \sqrt{\Delta Q(\rho)}\bigr) \left(4\alpha + a + 2 - \frac{\rho \partial_r \Delta Q(\rho)}{\Delta Q(\rho)} \right), \\
C_n^{(1)} &:= -\rho \sqrt{\Delta Q(\rho)} \sqrt{n} \Delta_{n,-} u - m_n u - a m_n \log \rho \\
&\quad - \frac{a}{2} \rho \sqrt{\Delta Q(\rho)} \bigl(2 \log \Delta_{n,-} - 2 \log 2 - \log \Delta Q(\rho) - \log n - 2 \bigr) \Delta_{n,-} \sqrt{n} \\
&\quad - \frac{a}{8} \left(1 + \frac{\rho \partial_r \Delta Q(\rho)}{\Delta Q(\rho)} \right) \Delta_{n,-}^2 - \frac{a}{2} \log \Delta_{n,-} + \frac{a}{4} \log n + \frac{a(a-1)}{2} \frac{\sqrt{n}}{\Delta_{n,-}} \rho \sqrt{\Delta Q(\rho)} \\
&\quad - \frac{a(a-1)(2a-3)}{12} \rho \sqrt{\Delta Q(\rho)} \frac{\sqrt{n}}{\Delta_{n,-}^3} \\
&\quad + \frac{a}{4} \left(\log \Delta_{n,-} - \frac{1}{2} \log n \right) \left(4\alpha + a + 2 - \frac{\rho \partial_r \Delta Q(\rho)}{\Delta Q(\rho)} \right).
\end{aligned}$$

Proof. By Assumptions 1.1 on the potential Q and for $j \in \{m_n, \dots, g_{1,-} - 1\}$, and a sufficiently large $n \in \mathbb{N}$, there exists a unique critical point $r_\tau \in (0, \rho)$, which satisfies $r_\tau q'(r_\tau) = 2\tau$. This implies that there is no critical point inside $(\rho, +\infty)$. By the integrability of the exponential term $\mathrm{e}^{-nq(r)}$, we find that there exists $c > 0$ such that

$$h_{n,j}^{(\mathrm{out})}(\rho) = \mathrm{e}^{-nV_\tau(r_\tau)} \cdot \mathcal{O}(\mathrm{e}^{-cn}).$$

Therefore, we focus on (2.9). We split the integral

$$\begin{aligned}
h_{n,j}^{(\mathrm{in})}(\rho) &= \int_{[0,\rho] \cap \{r \mid |r - r_\tau| < \delta_n\}} 2r^{2\alpha+1} \mathrm{e}^{-nV_\tau(r)} (\rho - r)^a \, \mathrm{d}r \\
&\quad + \int_{[0,\rho] \cap \{r \mid |r - r_\tau| \ge \delta_n\}} 2r^{2\alpha+1} \mathrm{e}^{-nV_\tau(r)} (\rho - r)^a \, \mathrm{d}r \\
&= \int_{[0,\rho] \cap \{r \mid |r - r_\tau| < \delta_n\}} 2r \mathrm{e}^{\mathsf{k}(r)} \mathrm{e}^{-nV_\tau(r)} (\rho - r)^a \, \mathrm{d}r + \mathrm{e}^{-nV_\tau(r_\tau)} \cdot \mathcal{O}\bigl(\mathrm{e}^{-c(\log n)^2}\bigr),
\end{aligned}$$

where $\mathsf{k}(r) := 2\alpha \log r$. Here, the exponential small error follows from a similar manner to [27, proof of Lemma 2.1]. For the first term in the last line, we apply the Laplace method, and then we have

$$\begin{aligned}
&\int_{[0,\rho] \cap \{r \mid |r - r_\tau| < \delta_n\}} 2r \mathrm{e}^{\mathsf{k}(r)} \mathrm{e}^{-nV_\tau(r)} |r - \rho|^a \, \mathrm{d}r \\
&\quad = \frac{2r_\tau \mathrm{e}^{\mathsf{k}(r_\tau)} \mathrm{e}^{-nV_\tau(r_\tau)}}{\sqrt{nd_2}} \int_{-\sqrt{d_2 n}\delta_n}^{\sqrt{d_2 n}\delta_n} \mathrm{e}^{-\frac{1}{2}u^2} \\
&\qquad \times \left| r_\tau - \rho + \frac{u}{\sqrt{nd_2}} \right|^a \left(1 + \frac{c_1(u)}{\sqrt{n}} + \frac{c_2(u)}{n} + \frac{c_3(u)}{n^{3/2}} + \mathcal{O}\left(\frac{c_4(u)}{n^2} \right) \right) \mathrm{d}u
\end{aligned}$$

$$= \frac{\sqrt{2\pi} r_\tau \mathrm{e}^{\mathsf{k}(r_\tau)} \mathrm{e}^{-nV_\tau(r_\tau)}}{\sqrt{n\Delta Q(r_\tau)}} (\rho - r_\tau)^a \Bigg[1 + \frac{1}{n} \big(\mathcal{M}_1^{(\mathrm{in})}(r_\tau) + \mathcal{A}(r_\tau) \big) + \frac{1}{n^2} \left\{ \frac{a(a-1)(a-2)(a-3)u^4}{24 d_2^2 (\rho - r_\tau)^4} + \widetilde{\mathcal{A}}(r_\tau) + \mathcal{O}\left(\frac{1}{(\rho - r_\tau)^3} \right) \right\} \Bigg] + \mathcal{O}\big(\mathrm{e}^{-c(\log n)^2} \big),$$

where the error term depends on a, but it does not affect the order of the error. Here, $\widetilde{\mathcal{A}}$ is a function with respect to r_τ but does not have the singularity at ρ, and

$$\mathcal{M}_1^{(\mathrm{in})}(r_\tau) = \frac{a(a-1)(\rho - r_\tau)^{-2}}{8\Delta Q(r_\tau)} - \frac{a(\rho - r_\tau)^{-1}}{4\Delta Q(r_\tau)} \left(-\frac{\partial_r \Delta Q(r_\tau)}{2\Delta Q(r_\tau)} + \frac{4\alpha + 3}{2r_\tau} \right).$$

and, we safely extended the integral region to $(-\infty, \infty)$ with the exponential error $\mathcal{O}\big(\mathrm{e}^{-c(\log n)^2}\big)$ for some $c > 0$, and for $d_j := V_\tau^{(j)}(r_\tau)$ given by (2.4), $c_k(u)$ for $k = 1, 2, 3$ are given by

$$c_1(u) := -\frac{d_3}{6 d_2^{3/2}} u^3 + \left(\mathsf{k}'(r_\tau) + \frac{1}{r_\tau} \right) \frac{1}{d_2^{1/2}} u, \tag{2.18}$$

$$c_2(u) := \frac{d_3^2}{72 d_2^3} u^6 - \left(d_4 + 4 d_3 \mathsf{k}'(r_\tau) + \frac{4 d_3}{r_\tau} \right) \frac{u^4}{24 d_2^2} + \left(\frac{\mathsf{k}''(r_\tau)}{2} + \frac{\mathsf{k}'(r_\tau)^2}{2} + \frac{\mathsf{k}'(r_\tau)}{r_\tau} \right) \frac{u^2}{d_2}, \tag{2.19}$$

$$\begin{aligned} c_3(u) := {} & -\frac{d_3^3}{1296 d_2^{9/2}} u^9 + \left(d_3 d_4 + 2 d_3^2 \mathsf{k}'(r_\tau) + \frac{2 d_3^2}{r_\tau} \right) \frac{u^7}{144 d_2^{7/2}} \\ & - \left(d_5 + \frac{5}{r_\tau} d_4 + 5 d_4 \mathsf{k}'(r_\tau) + \frac{20 d_3}{r_\tau} \mathsf{k}'(r_\tau) + 10 d_3 \mathsf{k}'(r_\tau)^2 + 10 d_3 \mathsf{k}''(r_\tau) \right) \frac{u^5}{120 d_2^{5/2}} \\ & + \left(\frac{3\mathsf{k}'(r_\tau)^2}{r_\tau} + \frac{3\mathsf{k}''(r_\tau)}{r_\tau} + \mathsf{k}'(r_\tau)^3 + 3\mathsf{k}'(r_\tau)\mathsf{k}''(r_\tau) + \mathsf{k}^{(3)}(r_\tau) \right) \frac{u^3}{6 d_2^{3/2}}, \end{aligned} \tag{2.20}$$

and $c_4(u)$ is a polynomial consisting of u^{12}, u^{10}, u^8, u^6, u^4. Since $\rho - r_\tau > 0$ and $\rho - r_\tau \searrow cn^{-1/2}M$ with some $c > 0$ for $m_n \le j \le g_{1,-} - 1$, by Lemma 2.3, we have

$$\begin{aligned} S_1 = {} & \sum_{j=D_n}^{g_{1,-}-1} \log\big(\mathrm{e}^u (\rho - r_\tau)^a \big) \\ & + \frac{1}{n} \sum_{j=j_{1,-}}^{g_{1,-}-1} \left[\frac{a(a-1)(\rho - r_\tau)^{-2}}{8\Delta Q(r_\tau)} - \frac{a(\rho - r_\tau)^{-1}}{4\Delta Q(r_\tau)} \left(-\frac{\partial_r \Delta Q(r_\tau)}{2\Delta Q(r_\tau)} + \frac{4\alpha + 3}{2r_\tau} \right) \right] \\ & - \frac{1}{n^2} \sum_{j=D_n}^{g_{1,-}-1} \frac{a(a-1)(2a-3)}{64 \Delta Q(r_\tau)^2 (\rho - r_\tau)^4} + \sum_{j=D_n}^{g_{1,-}-1} \mathcal{O}\left(\frac{1}{n^2} \frac{1}{(\rho - r_\tau)^3} \right) + \mathcal{O}\left(\frac{(\log n)^3}{n^{1/12}} \right), \end{aligned}$$

where the second error term is independent of a, u. By Lemma C.1 and by change of variable $r_{\tau(j)} = t$, we have

$$\sum_{j=m_n}^{g_{1,-}-1} \mathcal{O}\left(\frac{1}{n^2} \frac{1}{(\rho - r_\tau)^3} \right) = \mathcal{O}\left(\frac{(\log n)^{3/8}}{n^{7/8}} \right).$$

Consequently, the expansion of S_1 takes the following form:

$$\begin{aligned} S_1 = {} & (g_{1,-} - m_n) u + a \int_{m_n}^{g_{1,-}} \log(\rho - r_{\tau(t)}) \, \mathrm{d}t - \frac{a}{2} \log(\rho - r_{\tau(g_{1,-})}) + \frac{a}{2} \log(\rho - r_{\tau(D_n)}) \\ & + \frac{1}{n} \sum_{j=D_n}^{g_{1,-}-1} \left[\frac{a(a-1)(\rho - r_\tau)^{-2}}{8\Delta Q(r_\tau)} - \frac{a(\rho - r_\tau)^{-1}}{4\Delta Q(r_\tau)} \left(-\frac{\partial_r \Delta Q(r_\tau)}{2\Delta Q(r_\tau)} + \frac{4\alpha + 3}{2r_\tau} \right) \right] \end{aligned}$$

$$-\frac{1}{n^2}\sum_{j=m_n}^{g_{1,-}-1}\frac{a(a-1)(2a-3)}{64\Delta Q(r_\tau)^2(\rho-r_\tau)^4}+\mathcal{O}\left(\frac{(\log n)^3}{n^{1/12}}\right),$$

where we have used Lemma C.1, and the error term is independent of a, u. By (2.6) and (2.16), we have

$$\begin{aligned}
&a\int_{m_n}^{g_{1,-}}\log(\rho-r_{\tau(t)})\,\mathrm{d}t\\
&\quad=n\int_0^\rho 2au\Delta Q(u)\log(\rho-u)\,\mathrm{d}u-am_n\log\rho\\
&\qquad-\frac{a}{2}\rho\sqrt{\Delta Q(\rho)}\big(2\log\Delta_{n,-}-2\log 2-\log\Delta Q(\rho)-\log n-2\big)\Delta_{n,-}\sqrt{n}\\
&\qquad-\frac{a}{8}\left(1+\frac{\rho\partial_r\Delta Q(\rho)}{\Delta Q(\rho)}\right)\Delta_{n,-}^2+\mathcal{O}\left(\frac{(\log n)^{1/4}}{n^{1/4}}\right),
\end{aligned}$$

and

$$\begin{aligned}
&-\frac{a}{2}\log(\rho-r_{\tau(g_{1,-})})+\frac{a}{2}\log(\rho-r_{\tau(D_n)})\\
&\quad=-\frac{a}{2}\log\Delta_{n,-}+\frac{a}{2}\log 2+\frac{a}{4}\log\Delta Q(\rho)+\frac{a}{4}\log n+\frac{a}{2}\log\rho+\mathcal{O}\left(\frac{(\log n)^{1/4}}{n^{1/4}}\right).
\end{aligned}$$

The Taylor theorem gives rise to

$$\begin{aligned}
&a\int_{m_n}^{g_{1,-}}\log(\rho-r_{\tau(t)})\,\mathrm{d}t-\frac{a}{2}\log(\rho-r_{\tau(g_{1,-})})+\frac{a}{2}\log(\rho-r_{\tau(m_n)})\\
&\quad=n\int_0^\rho 2au\Delta Q(u)\log(\rho-u)\,\mathrm{d}u+\frac{a}{2}\log 2+\frac{a}{4}\log\Delta Q(\rho)+\frac{a}{2}\log\rho\\
&\qquad-\frac{a}{2}\rho\sqrt{\Delta Q(\rho)}\big(2\log\Delta_{n,-}-2\log 2-\log\Delta Q(\rho)-\log n-2\big)\Delta_{n,-}\sqrt{n}\\
&\qquad-am_n\log\rho-\frac{a}{8}\left(1+\frac{\rho\partial_r\Delta Q(\rho)}{\Delta Q(\rho)}\right)\Delta_{n,-}^2-\frac{a}{2}\log\Delta_{n,-}+\frac{a}{4}\log n+\mathcal{O}\left(\frac{(\log n)^{1/4}}{n^{1/4}}\right).
\end{aligned}$$

By Lemma C.1, we have

$$-\frac{1}{n^2}\sum_{j=m_n}^{g_{1,-}-1}\frac{a(a-1)(2a-3)}{64\Delta Q(r_\tau)^2(\rho-r_\tau)^4}=-\frac{1}{n}\int_{r_{\tau(m_n)}}^{r_{\tau(g_{1,-})}}\frac{a(a-1)(2a-3)}{32\Delta Q(u)(\rho-u)^4}u\,\mathrm{d}u+\mathcal{O}(\Delta_{n,-}^{-4}).$$

Integration by parts leads to

$$-\frac{1}{n}\int_{r_{\tau(m_n)}}^{r_{\tau(g_{1,-})}}\frac{a(a-1)(2a-3)}{32\Delta Q(u)(\rho-u)^4}u\,\mathrm{d}u=-\frac{1}{n}\frac{a(a-1)(2a-3)r_{\tau(g_{1,-})}}{96\Delta Q(r_{\tau(g_{1,-})})(\rho-r_{\tau(g_{1,-})})^3}+\widetilde{\epsilon}_{n,-}+\mathcal{O}(n^{-1}),$$

where

$$\widetilde{\epsilon}_{n,-}:=\frac{1}{n}\int_{r_{\tau(m_n)}}^{r_{\tau(g_{1,-})}}\frac{a(a-1)(2a-3)}{96(\rho-u)^3}\frac{\Delta Q(u)-u\partial_u\Delta Q(u)}{\Delta Q(u)^2}\,\mathrm{d}u.$$

Note that

$$-\frac{1}{n}\frac{a(a-1)(2a-3)r_{\tau(g_{1,-})}}{96\Delta Q(r_{\tau(g_{1,-})})(\rho-r_{\tau(g_{1,-})})^3}=-\frac{a(a-1)(2a-3)}{12}\rho\sqrt{\Delta Q(\rho)}\frac{\sqrt{n}}{\Delta_{n,-}^3}+\mathcal{O}(\Delta_{n,-}^{-2}),$$

$$|\widetilde{\epsilon}_{n,-}| \lesssim \frac{1}{n}\left|\int_{r_{\tau(m_n)}}^{r_{\tau(g_{1,-})}} \frac{\mathrm{d}u}{(\rho-u)^3}\right| = \mathcal{O}(\Delta_{n,-}^{-2}).$$

By Lemma C.1, we have

$$\begin{aligned}
&\frac{1}{n}\sum_{j=m_n}^{g_{1,-}-1}\left[\frac{a(a-1)(\rho-r_\tau)^{-2}}{8\Delta Q(r_\tau)} - \frac{a(\rho-r_\tau)^{-1}}{4\Delta Q(r_\tau)}\left(-\frac{\partial_r\Delta Q(r_\tau)}{2\Delta Q(r_\tau)}+\frac{4\alpha+3}{2r_\tau}\right)\right]\\
&\quad= \frac{a(a-1)}{4}\left[\frac{\rho}{\rho-r_{\tau(g_{1,-})}}+\log(\rho-r_{\tau(g_{1,-})})-\frac{\rho}{\rho-r_{\tau(m_n)}}-\log(\rho-r_{\tau(m_n)})\right]\\
&\qquad+\frac{a}{4}\big(\log(\rho-r_{\tau(g_{1,-})})-\log(\rho-r_{\tau(m_n)})\big)\left(4\alpha+3-\frac{\rho\partial_r\Delta Q(\rho)}{\Delta Q(\rho)}\right)\\
&\qquad+\frac{a}{4}\int_{r_{\tau(m_n)}}^{r_{\tau(g_{1,-})}}\left(\frac{1}{\rho-u}\frac{u\partial_r\Delta Q(u)}{\Delta Q(u)}-\frac{1}{\rho-u}\frac{\rho\partial_r\Delta Q(\rho)}{\Delta Q(\rho)}\right)\mathrm{d}u+\mathcal{O}(\Delta_{n,-}^{-2}).
\end{aligned}$$

Using the asymptotics by the Taylor theorem,

$$\begin{aligned}
&\frac{a}{4}\int_{r_{\tau(m_n)}}^{r_{\tau(g_{1,-})}}\left(\frac{1}{\rho-u}\frac{u\partial_r\Delta Q(u)}{\Delta Q(u)}-\frac{1}{\rho-u}\frac{\rho\partial_r\Delta Q(\rho)}{\Delta Q(\rho)}\right)\mathrm{d}u\\
&\quad=\frac{a}{4}\int_0^{\rho}\left(\frac{1}{\rho-u}\frac{u\partial_r\Delta Q(u)}{\Delta Q(u)}-\frac{1}{\rho-u}\frac{\rho\partial_r\Delta Q(\rho)}{\Delta Q(\rho)}\right)\mathrm{d}u+\mathcal{O}\left(\frac{\Delta_{n,-}}{\sqrt{n}}+\frac{m_n}{n}\right),\\
&\frac{\rho}{\rho-r_{\tau(g_{1,-})}}+\log(\rho-r_{\tau(g_{1,-})})-\frac{\rho}{\rho-r_{\tau(D_n)}}-\log(\rho-r_{\tau(m_n)})\\
&\quad=2\rho\sqrt{\Delta Q(\rho)}\frac{\sqrt{n}}{\Delta_{n,-}}-\frac{1}{2}\left(3+\frac{\rho\partial_r\Delta Q(\rho)}{\Delta Q(\rho)}\right)-\log\big(2\rho\sqrt{\Delta Q(\rho)}\big)\\
&\qquad-\frac{1}{2}\log n+\log\Delta_{n,-}+\mathcal{O}\left(\sqrt{\frac{m_n}{n}}\right),
\end{aligned}$$

and

$$\log(\rho-r_{\tau(g_{1,-})})-\log(\rho-r_{\tau(D_n)}) = -\frac{1}{2}\log n+\log\Delta_{n,-}-\log\big(2\rho\sqrt{\Delta Q(\rho)}\big)+\mathcal{O}\left(\sqrt{\frac{m_n}{n}}\right),$$

we have

$$\begin{aligned}
&\frac{1}{n}\sum_{j=m_n}^{g_{1,-}-1}\left[\frac{a(a-1)(\rho-r_\tau)^{-2}}{8\Delta Q(r_\tau)} - \frac{a(\rho-r_\tau)^{-1}}{4\Delta Q(r_\tau)}\left(-\frac{\partial_r\Delta Q(r_\tau)}{2\Delta Q(r_\tau)}+\frac{4\alpha+3}{2r_\tau}\right)\right]\\
&\quad=-\frac{a(a-1)}{4}\left[\frac{1}{2}\left(3+\frac{\rho\partial_r\Delta Q(\rho)}{\Delta Q(\rho)}\right)+\log\big(2\rho\sqrt{\Delta Q(\rho)}\big)\right]\\
&\qquad-\frac{a}{4}\log\big(2\rho\sqrt{\Delta Q(\rho)}\big)\left(4\alpha+3-\frac{\rho\partial_r\Delta Q(\rho)}{\Delta Q(\rho)}\right)\\
&\qquad+\frac{a}{4}\int_0^{\rho}\left(\frac{1}{\rho-u}\frac{u\partial_r\Delta Q(u)}{\Delta Q(u)}-\frac{1}{\rho-u}\frac{\rho\partial_r\Delta Q(\rho)}{\Delta Q(\rho)}\right)\mathrm{d}u\\
&\qquad+\frac{a}{4}\left(-\frac{1}{2}\log n+\log\Delta_{n,-}\right)\left(4\alpha+a+2-\frac{\rho\partial_r\Delta Q(\rho)}{\Delta Q(\rho)}\right)\\
&\qquad+\frac{a(a-1)}{2}\rho\sqrt{\Delta Q(\rho)}\frac{\sqrt{n}}{\Delta_{n,-}}+\mathcal{O}(\Delta_{n,-}^{-2}).
\end{aligned}$$

Combining all the above, we obtain (2.17). ■

Next, we compute the large n-asymptotics of S_3.

Lemma 2.5. *For any fixed $x \in \mathbb{R}$, there exists $\delta > 0$ such that as $n \to +\infty$, we have*

$$S_3 = C_1^{(3)} n + C_2^{(3)} \sqrt{n} + C_3^{(3)} + C_n^{(3)} + \mathcal{O}\left(\frac{(\log n)^{1/4}}{n^{1/4}} \right), \tag{2.21}$$

uniformly for $u \in \{z \in \mathbb{C} \mid |z - x| \le \delta\}$ and a in compact subsets of $(-1, +\infty)$, where

$$\begin{aligned}
C_1^{(3)} &:= a \int_\rho^{r_1} \log(t-\rho) 2t \Delta Q(t)\, \mathrm{d}t, \qquad C_2^{(3)} := 0, \\
C_3^{(3)} &:= -\frac{a}{2} \log(r_1 - \rho) + \frac{a}{2} \log 2 + \frac{a}{4} \log(\Delta Q(\rho)) \\
&\quad + \frac{a(a-1)}{4} \left[\frac{1}{2} \left(1 + \frac{\rho \partial_r \Delta Q(\rho)}{\Delta Q(\rho)} \right) - \frac{\rho}{r_1 - \rho} \right] \\
&\quad + \frac{a}{4} \left(4\alpha + a + 2 - \frac{\rho \partial_r \Delta Q(\rho)}{\Delta Q(\rho)} \right) \log\left(2(r_1 - \rho) \sqrt{\Delta Q(\rho)} \right) \\
&\quad - \frac{a}{4} \int_\rho^{r_1} \left(\frac{1}{t-\rho} \frac{t \partial_t \Delta Q(t)}{\Delta Q(t)} - \frac{1}{t-\rho} \frac{\rho \partial_r \Delta Q(\rho)}{\Delta Q(\rho)} \right) \mathrm{d}t, \\
C_n^{(3)} &:= -\frac{a}{2} \rho \sqrt{\Delta Q(\rho)} \left(2 \log \Delta_{n,+} - 2 \log 2 - \log \Delta Q(\rho) - \log n - 2 \right) \sqrt{n} \Delta_{n,+} \\
&\quad + \frac{a}{8} \left(1 + \frac{\rho \partial_r \Delta Q(\rho)}{\Delta Q(\rho)} \right) \Delta_{n,+}^2 - \frac{a}{2} \log(\Delta_{n,+}) + \frac{a}{4} \log n + \frac{a(a-1)}{2} \rho \sqrt{\Delta Q(\rho)} \frac{\sqrt{n}}{\Delta_{n,+}} \\
&\quad - \frac{a}{4} \left(\log \Delta_{n,+} - \frac{1}{2} \log n \right) \left(4\alpha + a + 2 - \frac{\rho \partial_r \Delta Q(\rho)}{\Delta Q(\rho)} \right) \\
&\quad - \frac{a(a-1)(2a-3)}{12} \rho \sqrt{\Delta Q(\rho)} \frac{\sqrt{n}}{\Delta_{n,+}^3}.
\end{aligned}$$

Proof. Similar manner to Lemma 2.4, there exists $c > 0$ such that as $n \to +\infty$, we have

$$\begin{aligned}
h_{n,j}^{(\mathrm{out})}(\rho) &= \frac{\sqrt{2\pi} r_\tau \mathrm{e}^{\mathsf{k}(r_\tau)} \mathrm{e}^{-nV_\tau(r_\tau)}}{\sqrt{n \Delta Q(r_\tau)}} (r_\tau - \rho)^a \left[1 + \frac{1}{n} \left(\mathcal{A}_1(r_\tau) + \mathcal{M}_1^{(\mathrm{out})}(r_\tau) \right) \right. \\
&\quad \left. + \frac{1}{n^2} \left\{ \frac{a(a-1)(a-2)(a-3) u^4}{24 d_2^2 (r_\tau - \rho)^4} + \mathcal{A}_2(r_\tau) + \mathcal{O}\left(\frac{1}{(r_\tau - \rho)^3} \right) \right\} \right] \\
&\quad + \mathcal{O}\left(\mathrm{e}^{-c(\log n)^2} \right),
\end{aligned} \tag{2.22}$$

where the constant in the error term might depend on a, but the order of the error term is independent of a. Here,

$$\mathcal{M}_1^{(\mathrm{out})}(r_\tau) = \frac{a(a-1)(r_\tau - \rho)^{-2}}{8 \Delta Q(r_\tau)} + \frac{a(r_\tau - \rho)^{-1}}{8 \Delta Q(r_\tau)} \left(-\frac{\partial_r \Delta Q(r_\tau)}{\Delta Q(r_\tau)} + \frac{4\alpha + 3}{r_\tau} \right).$$

Also, there exists $c > 0$ independent of n such that we have

$$\mathrm{e}^u \frac{h_{n,j}^{(\mathrm{in})}(\rho)}{h_{n,j}} = \mathrm{e}^{-nV_\tau(r_\tau)} \cdot \mathcal{O}(\mathrm{e}^{-cn}), \tag{2.23}$$

where we can take the error term to be independent of u. Thus, by Lemma 2.3, (2.22), and (2.23), let us choose $\delta > 0$ sufficiently small so that

$$\mathrm{e}^u \frac{h_{n,j}^{(\mathrm{in})}(\rho)}{h_{n,j}} + \frac{h_{n,j}^{(\mathrm{out})}(\rho)}{h_{n,j}}$$

remains bounded away from the interval $(-\infty, 0]$ as $n \to +\infty$ uniformly for $u \in \{z \in \mathbb{C} \mid |z - x| \le \delta\}$ and a in compact subsets of $(-1, +\infty)$. Therefore, as $n \to +\infty$, we have

$$\begin{aligned}\sum_{j=g_{1,+}+1}^{n-1} \log\left[\mathrm{e}^{u}\frac{h_{n,j}^{(\mathrm{in})}(\rho)}{h_{n,j}} + \frac{h_{n,j}^{(\mathrm{out})}(\rho)}{h_{n,j}}\right] &= a\sum_{j=g_{1,+}+1}^{n-1}\log(r_\tau - \rho)\\ &+ \frac{1}{n}\sum_{j=g_{1,+}+1}^{n-1}\left\{\frac{a(a-1)(r_\tau-\rho)^{-2}}{8\Delta Q(r_\tau)} + \frac{a(r_\tau-\rho)^{-1}}{8\Delta Q(r_\tau)}\left(-\frac{\partial_r \Delta Q(r_\tau)}{\Delta Q(r_\tau)} + \frac{4\alpha+3}{r_\tau}\right)\right\}\\ &- \frac{1}{n^2}\sum_{j=g_{1,+}+1}^{n-1}\frac{a(a-1)(2a-3)}{64\Delta Q(r_\tau)^2(r_\tau-\rho)^4} + \sum_{j=g_{1,+}+1}^{n-1}\mathcal{O}\left(\frac{(r_\tau-\rho)^{-3}}{n^2}\right),\end{aligned}$$

uniformly for $u \in \{z \in \mathbb{C} \mid |z - x| \le \delta\}$ and a in compact subsets of $(-1, +\infty)$. By change of variable $r_{\tau(j)} = t$ and (2.5), we have

$$\sum_{j=g_{1,+}+1}^{n-1}\mathcal{O}\left(\frac{(r_\tau-\rho)^{-3}}{n^2}\right) = \mathcal{O}\left(\frac{(\log n)^{3/8}}{n^{7/8}}\right). \tag{2.24}$$

By Lemma C.1, we have

$$\begin{aligned}&\sum_{j=g_{1,+}+1}^{n-1} a\log(r_\tau - \rho)\\ &\quad= \int_{g_{1,+}}^{n} a\log(r_{\tau(t)} - \rho)\,\mathrm{d}t - \frac{a}{2}\log(r_1 - \rho) - \frac{a}{2}\log(r_{\tau(g_{1,+})} - \rho) + \mathcal{O}\big(n^{-3/8}\big).\end{aligned}$$

Note that by (2.15), as $n \to +\infty$, we have

$$\begin{aligned}\int_{g_{1,+}}^{n} a\log(r_{\tau(t)} - \rho)\,\mathrm{d}t &= na\int_{\rho}^{r_1}\log(u-\rho)2u\Delta Q(u)\,\mathrm{d}u\\ &- a\rho\sqrt{\Delta Q(\rho)}\left(\log\Delta_{n,+} - \log 2 - \frac{1}{2}\log\Delta Q(\rho) - \frac{1}{2}\log n - 1\right)\sqrt{n}\Delta_{n,+}\\ &+ \frac{a}{8}\left(1 + \frac{\rho\partial_r\Delta Q(\rho)}{\Delta Q(\rho)}\right)\Delta_{n,+}^2 + \mathcal{O}\left(\frac{\Delta_{n,+}^3}{\sqrt{n}}\right).\end{aligned} \tag{2.25}$$

Thus, the Taylor theorem together with (2.15) gives rise to

$$\begin{aligned}\sum_{j=g_{1,+}+1}^{n-1} a\log(r_\tau - \rho) &= na\int_{r_\tau(g_{1,+})}^{r_1}\log(u-\rho)2u\Delta Q(u)\,\mathrm{d}u\\ &+ \frac{a}{2}\rho\sqrt{\Delta Q(\rho)}\left(2\log 2 + 2 + \log\Delta Q(\rho) + \log n - 2\log\Delta_{n,+}\right)\sqrt{n}\Delta_{n,+}\\ &+ \frac{a}{8}\left(1 + \frac{\rho\partial_r\Delta Q(\rho)}{\Delta Q(\rho)}\right)\Delta_{n,+}^2 - \frac{a}{2}\log(r_1-\rho) - \frac{a}{2}\log(\Delta_{n,+}) + \frac{a}{2}\log 2\\ &+ \frac{a}{4}\log(\Delta Q(\rho)) + \frac{a}{4}\log n + \mathcal{O}\left(\frac{1}{\Delta_{n,+}}\right).\end{aligned} \tag{2.26}$$

The term of order n^{-1} can be calculated as

$$\frac{1}{n}\sum_{j=g_{1,+}+1}^{n-1}\left\{\frac{a(a-1)(r_\tau-\rho)^{-2}}{8\Delta Q(r_\tau)} + \frac{a(r_\tau-\rho)^{-1}}{8\Delta Q(r_\tau)}\left(-\frac{\partial_r\Delta Q(r_\tau)}{\Delta Q(r_\tau)} + \frac{4\alpha+3}{r_\tau}\right)\right\}$$

$$= \frac{a(a-1)}{4}\left[\frac{1}{2}\left(1+\frac{\rho\partial_r\Delta Q(\rho)}{\Delta Q(\rho)}\right)+\log(2(r_1-\rho)\sqrt{\Delta Q(\rho)})-\frac{\rho}{r_1-\rho}\right]$$
$$+\frac{a}{4}\left(4\alpha+3-\frac{\rho\partial_r\Delta Q(\rho)}{\Delta Q(\rho)}\right)\log\bigl(2(r_1-\rho)\sqrt{\Delta Q(\rho)}\bigr)$$
$$-\frac{a}{4}\int_\rho^{r_1}\left(\frac{1}{u-\rho}\frac{u\partial_u\Delta Q(u)}{\Delta Q(u)}-\frac{1}{u-\rho}\frac{\rho\partial_r\Delta Q(\rho)}{\Delta Q(\rho)}\right)\mathrm{d}u+\frac{a(a-1)}{2}\rho\sqrt{\Delta Q(\rho)}\frac{\sqrt{n}}{\Delta_{n,+}}$$
$$+\frac{a}{4}\left(\frac{1}{2}\log n-\log\Delta_{n,+}\right)\left(4\alpha+a+2-\frac{\rho\partial_r\Delta Q(\rho)}{\Delta Q(\rho)}\right)+\mathcal{O}\bigl(\Delta_{n,+}^{-2}\bigr). \tag{2.27}$$

Finally, it is straightforward to see that by (2.15) and Lemma C.1,

$$-\frac{1}{n^2}\sum_{j=g_{1,+}+1}^{n-1}\frac{a(a-1)(2a-3)}{64\Delta Q(r_\tau)^2(r_\tau-\rho)^4}$$
$$=-\frac{a(a-1)(2a-3)}{12}\rho\sqrt{\Delta Q(\rho)}\frac{\sqrt{n}}{\Delta_{n,+}^3}+\mathcal{O}\left(\frac{(\log n)^{1/4}}{n^{1/4}}\right). \tag{2.28}$$

Combining (2.24) with (2.26), (2.27), and (2.28), we obtain (2.21). ∎

Remark 2.6. The reader might wonder whether the choice of the parameter M in (2.8) is merely technical. Indeed, as shown in Lemma 2.5, the constant $C_n^{(3)}$ contains the divergent term $\sqrt{n}/\Delta_{n,+}^3 = \mathcal{O}\bigl(n^{1/8}(\log n)^{3/8}\bigr)$ as $n\to+\infty$ (see (2.14) for the definition of $\Delta_{n,+}$). On the other hand, the proof of Lemma 2.5 also produces a term of order $\Delta_{n,+}^3/\sqrt{n}$ in (2.25). Although the divergent contribution $\sqrt{n}/\Delta_{n,+}^3$ is eventually cancelled in the final step of the proof (see Section 3.2), the remaining term $\Delta_{n,+}^3/\sqrt{n}$ must still be absorbed into the error term. The parameter M in (2.8) is therefore chosen precisely so that both of these requirements are satisfied simultaneously. A similar technical choice of parameters also appeared in a different context in [37, Remark 3.17].

3 Local asymptotic analysis for the proof of Theorem 1.6

3.1 Asymptotic expansion of S_2

We begin with the asymptotic expansion of the summand (2.10). We recall (1.9) here.

Lemma 3.1. *Let*

$$\xi\equiv\xi_j := \frac{\sqrt{n}}{\sqrt{\Delta Q(\rho)}}\frac{\tau-\tau_\rho}{\rho}, \tag{3.1}$$

where τ_ρ is given by (2.7). There exists $\delta>0$ such that as $n\to+\infty$, we have

$$S_2=\sum_{j=g_{1,-}}^{g_{1,+}}\log\left(\frac{1}{(4n)^{a/2}}\right)+\frac{a}{2}\sum_{j=g_{1,-}}^{g_{1,+}}\log\left(\frac{1}{\Delta Q(\rho)}\right)+\sum_{j=g_{1,-}}^{g_{1,+}}\log[\mathcal{H}_{a,u}(\xi)]$$
$$+\frac{1}{\sqrt{n}}\sum_{j=g_{1,-}}^{g_{1,+}}\frac{\widetilde{g}_{a,u}(\xi)}{12\rho\sqrt{\Delta Q(\rho)}g_{a,u}(\xi)}+\mathcal{O}\left(\frac{(\log n)^{7/8}}{n^{1/8}}\right), \tag{3.2}$$

uniformly for $u\in\{z\in\mathbb{C}\mid|z-x|\le\delta\}$ and a in compact subsets of $(-1,+\infty)$. Here, $\mathcal{H}_{a,u}(\xi), g_{a,u}(\xi)$ are given by (1.8), and

$$\widetilde{g}_{a,u}(\xi) := -a\xi\left(1-\frac{\rho\partial_r\Delta Q(\rho)}{\Delta Q(\rho)}\right)(\mathrm{e}^u D_{-a-1}(\xi)+D_{-a-1}(-\xi))$$

$$+ (\mathrm{e}^{u} D_{-a}(\xi) - D_{-a}(-\xi))\xi^2 \left(2 + \frac{\rho \partial_r \Delta Q(\rho)}{\Delta Q(\rho)}\right)$$
$$- (\mathrm{e}^{u} D_{-a}(\xi) - D_{-a}(-\xi)) \left[6(\rho \mathsf{k}'(\rho) + 1) + (2+a)\left(1 - \frac{\rho \partial_r \Delta Q(\rho)}{\Delta Q(\rho)}\right)\right]. \tag{3.3}$$

Proof. We begin with analyzing $S_2^{(\mathrm{in})}$. There exists $c > 0$ such that

$$\begin{aligned} h_{n,j}^{(\mathrm{in})}(\rho) &= \int_{r_\tau - \delta_n'}^{\rho} 2r \mathrm{e}^{\mathsf{k}(r)} \mathrm{e}^{-nV_\tau(r)} |\rho - r|^a \, \mathrm{d}r + \int_0^{r_\tau - \delta_n'} 2r \mathrm{e}^{\mathsf{k}(r)} \mathrm{e}^{-nV_\tau(r)} |\rho - r|^a \, \mathrm{d}r \\ &= \frac{2r_\tau \mathrm{e}^{\mathsf{k}(r_\tau)} \mathrm{e}^{-nV_\tau(r_\tau)}}{(nd_2)^{(a+1)/2}} \int_0^{\sqrt{d_2 n}(\rho - r_\tau + \delta_n')} v^a \mathrm{e}^{-\frac{1}{2}(v - \sqrt{nd_2}(\rho - r_\tau))^2} \\ &\quad \times \left(1 + \frac{c_1(u_v)}{\sqrt{n}} + \frac{c_2(u_v)}{n} + \frac{c_3(u_v)}{n^{3/2}} + \mathcal{O}\left(\frac{c_4(u_v)}{n^2}\right)\right) \mathrm{d}v + \mathrm{e}^{-nV_\tau(r_\tau)} \cdot \mathcal{O}\big(\mathrm{e}^{-cM^2}\big), \end{aligned} \tag{3.4}$$

where $u_v = \sqrt{nd_2}(\rho - r_\tau) - v = \chi(\tau) - v$ with $\chi(\tau) := \sqrt{nd_2}(\rho - r_\tau)$. Since $\sqrt{d_2 n}(\rho - r_\tau + \delta_n') > \sqrt{nd_2}(\rho - r_\tau)$ for $g_{1,-} \le j \le \lfloor n\tau_\rho \rfloor - 1$, one can safely extend the integral region to $[0, +\infty)$ with an error $\mathcal{O}(\mathrm{e}^{-cM^2})$ for some $c > 0$. Note that for $a > -1$ and $k \in \mathbb{Z}_{\ge 0}$, by (1.9), we have

$$\int_0^{+\infty} v^{a+k} \mathrm{e}^{-\frac{1}{2}(v - \chi(\tau))^2} \, \mathrm{d}v = \vartheta_{k,a} \mathrm{e}^{-\frac{\chi(\tau)^2}{4}} D_{-a-k-1}(-\chi(\tau)).$$

Here, $\vartheta_{k,a} := \Gamma(a + k + 1)$. Therefore, we have

$$\begin{aligned} &\int_0^{\sqrt{d_2 n}(\rho - r_\tau + \delta_n')} v^a \mathrm{e}^{-\frac{1}{2}(v - \sqrt{nd_2}(\rho - r_\tau))^2} \left(1 + \frac{c_1(u_v)}{\sqrt{n}} + \frac{c_2(u_v)}{n} + \frac{c_3(u_v)}{n^{3/2}} + \mathcal{O}\left(\frac{c_4(u_v)}{n^2}\right)\right) \mathrm{d}v \\ &\quad = \mathsf{c}_0 + \frac{\mathsf{c}_1}{\sqrt{n}} + \frac{\mathsf{c}_2}{n} + \frac{\mathsf{c}_3}{n^{3/2}} + \mathcal{O}\left(\frac{\mathsf{c}_4}{n^2}\right), \end{aligned} \tag{3.5}$$

where c_1, c_2, c_3 are given by (2.18), (2.19), and (2.20), respectively, and

$$\mathsf{c}_0 := \vartheta_{0,a} \mathrm{e}^{-\frac{\chi(\tau)^2}{4}} D_{-a-1}(-\chi(\tau)) = \vartheta_{0,a} \mathrm{e}^{-\frac{\phi(\tau)^2}{4}} D_{-a-1}(\phi(\tau)), \tag{3.6}$$

$$\begin{aligned} \mathsf{c}_1 &:= \int_0^{+\infty} v^a \mathrm{e}^{-\frac{1}{2}(v - \sqrt{nd_2}(\rho - r_\tau))^2} c_1(u_v) \, \mathrm{d}v \\ &= \frac{d_3}{6d_2^{3/2}} \sum_{\ell=0}^{3} \binom{3}{\ell} \phi(\tau)^{3-\ell} \vartheta_{\ell,a} \mathrm{e}^{-\frac{\phi(\tau)^2}{4}} D_{-a-\ell-1}(\phi(\tau)) \\ &\quad - \frac{1}{d_2^{1/2}} \left(\mathsf{k}'(r_\tau) + \frac{1}{r_\tau}\right) \sum_{\ell=0}^{1} \binom{1}{\ell} \phi(\tau)^{1-\ell} \vartheta_{\ell,a} \mathrm{e}^{-\frac{\phi(\tau)^2}{4}} D_{-a-\ell-1}(\phi(\tau)), \end{aligned} \tag{3.7}$$

$$\begin{aligned} \mathsf{c}_2 &:= \int_0^{+\infty} v^a \mathrm{e}^{-\frac{1}{2}(v - \sqrt{nd_2}(\rho - r_\tau))^2} c_2(u_v) \, \mathrm{d}v \\ &= \frac{d_3^2}{72d_2^3} \sum_{\ell=0}^{6} \binom{6}{\ell} \phi(\tau)^{6-\ell} \vartheta_{\ell,a} \mathrm{e}^{-\frac{\chi(\tau)^2}{4}} D_{-a-\ell-1}(\phi(\tau)) \\ &\quad - \frac{1}{24d_2^2} \left(d_4 + 4d_3 \mathsf{k}'(r_\tau) + \frac{4d_3}{r_\tau}\right) \sum_{\ell=0}^{4} \binom{4}{\ell} \phi(\tau)^{4-\ell} \vartheta_{\ell,a} \mathrm{e}^{-\frac{\phi(\tau)^2}{4}} D_{-a-\ell-1}(\phi(\tau)) \\ &\quad + \frac{1}{d_2} \left(\frac{\mathsf{k}''(r_\tau)}{2} + \frac{\mathsf{k}'(r_\tau)^2}{2} + \frac{\mathsf{k}'(r_\tau)}{r_\tau}\right) \sum_{\ell=0}^{2} \binom{2}{\ell} \phi(\tau)^{2-\ell} \vartheta_{\ell,a} \mathrm{e}^{-\frac{\phi(\tau)^2}{4}} D_{-a-\ell-1}(\phi(\tau)), \end{aligned} \tag{3.8}$$

$$
\begin{aligned}
\mathsf{c}_3 &:= \int_0^{+\infty} v^a \mathrm{e}^{-\frac{1}{2}(v-\sqrt{nd_2}(\rho-r_\tau))^2} c_3(u_v)\,\mathrm{d}v \\
&= \frac{d_3^3}{1296 d_2^{9/2}} \sum_{\ell=0}^{9} \binom{9}{\ell} \phi(\tau)^{9-\ell} \vartheta_{\ell,a} \mathrm{e}^{-\frac{\phi(\tau)^2}{4}} D_{-a-\ell-1}(\phi(\tau)) \\
&\quad - \left(d_3 d_4 + 2d_3^2 \mathsf{k}'(r_\tau) + \frac{2d_3^2}{r_\tau} \right) \frac{1}{144 d_2^{7/2}} \sum_{\ell=0}^{7} \binom{7}{\ell} \phi(\tau)^{7-\ell} \vartheta_{\ell,a} \mathrm{e}^{-\frac{\phi(\tau)^2}{4}} D_{-a-\ell-1}(\phi(\tau)) \\
&\quad + \left(d_5 + \frac{5}{r_\tau} d_4 + 5 d_4 \mathsf{k}'(r_\tau) + \frac{20 d_3}{r_\tau} \mathsf{k}'(r_\tau) + 10 d_3 \mathsf{k}'(r_\tau)^2 + 10 d_3 \mathsf{k}''(r_\tau) \right) \\
&\quad \times \sum_{\ell=0}^{5} \binom{5}{\ell} \frac{\phi(\tau)^{5-\ell} \vartheta_{\ell,a} \mathrm{e}^{-\frac{\phi(\tau)^2}{4}}}{120 d_2^{5/2}} D_{-a-\ell-1}(\phi(\tau)) \\
&\quad - \left(\frac{3\mathsf{k}'(r_\tau)^2}{r_\tau} + \frac{3\mathsf{k}''(r_\tau)}{r_\tau} + \mathsf{k}'(r_\tau)^3 + 3\mathsf{k}'(r_\tau)\mathsf{k}''(r_\tau) + \mathsf{k}^{(3)}(r_\tau) \right) \\
&\quad \times \sum_{\ell=0}^{3} \binom{3}{\ell} \frac{\phi(\tau)^{3-\ell} \vartheta_{\ell,a} \mathrm{e}^{-\frac{\phi(\tau)^2}{4}}}{6 d_2^{3/2}} D_{-a-\ell-1}(\phi(\tau)).
\end{aligned} \tag{3.9}
$$

We next consider $h_{n,j}^{(\mathrm{out})}(\rho)$. In this integral region and for $g_{1,-} \le j \le \lfloor n\tau_\rho \rfloor - 1$, there is no critical point, but we still proceed with the expansion in terms of r_τ. Indeed,

$$
\begin{aligned}
h_{n,j}^{(\mathrm{out})}(\rho) &= \int_\rho^{+\infty} 2v \mathrm{e}^{\mathsf{k}(v)} \mathrm{e}^{-nV_\tau(v)} |v-\rho|^a \,\mathrm{d}v \\
&= \int_0^{+\infty} 2v \mathrm{e}^{\mathsf{k}(v)} \mathrm{e}^{-nV_\tau(v)} |v-\rho|^a \,\mathrm{d}v - \int_0^{\rho} 2v \mathrm{e}^{\mathsf{k}(v)} \mathrm{e}^{-nV_\tau(v)} |v-\rho|^a \,\mathrm{d}v.
\end{aligned}
$$

We already know the asymptotics of the second terms in the last line. For the first term, we split the integral $[0,+\infty)$ into $[0,+\infty) = \{r \in [0,+\infty) \mid |r-r_\tau| < \delta_n'\} \cup \{r \in [0,+\infty) \mid |r-r_\tau| \ge \delta_n'\}$. In the latter region, the corresponding integral can be neglected with an error $\mathrm{e}^{-nV_\tau(r_\tau)} \cdot \mathcal{O}\big(\mathrm{e}^{-cM^2}\big)$ for some $c > 0$. The Gaussian integral gives

$$
\begin{aligned}
&\frac{2r_\tau \mathrm{e}^{\mathsf{k}(r_\tau)} \mathrm{e}^{-nV_\tau(r_\tau)}}{(nd_2)^{1/2}} \int_{-\sqrt{d_2 n}\delta_n'}^{\sqrt{d_2 n}\delta_n'} \mathrm{e}^{-\frac{1}{2}u^2} \left| r_\tau - \rho + \frac{u}{\sqrt{nd_2}} \right|^a \\
&\qquad \times \left(1 + \frac{c_1(u)}{\sqrt{n}} + \frac{c_2(u)}{n} + \frac{c_3(u)}{n^{3/2}} + \mathcal{O}\left(\frac{c_4(u)}{n^2} \right) \right) \mathrm{d}u \\
&= \frac{2r_\tau \mathrm{e}^{\mathsf{k}(r_\tau)} \mathrm{e}^{-nV_\tau(r_\tau)}}{(nd_2)^{(a+1)/2}} \int_{-\sqrt{d_2 n}\delta_n' + \sqrt{nd_2}(r_\tau-\rho)}^{\sqrt{nd_2}\delta_n' + \sqrt{nd_2}(r_\tau-\rho)} \mathrm{e}^{-\frac{1}{2}(v-\sqrt{nd_2}(r_\tau-\rho))^2} |v|^a \\
&\qquad \times \left(1 + \frac{c_1(u_v)}{\sqrt{n}} + \frac{c_2(u_v)}{n} + \frac{c_3(u_v)}{n^{3/2}} + \mathcal{O}\left(\frac{c_4(u_v)}{n^2} \right) \right) \mathrm{d}v,
\end{aligned}
$$

where $u_v = v - \sqrt{nd_2}(r_\tau - \rho)$. One can safely extend the integral region to $(-\infty, +\infty)$ with an exponential error $\mathcal{O}\big(\mathrm{e}^{-cM^2}\big)$ for some $c > 0$. Each integral is given by the following: for $\phi(\tau) := \sqrt{nd_2}(r_\tau - \rho)$,

$$
\begin{aligned}
\int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{1}{2}(v-\sqrt{nd_2}(r_\tau-\rho))^2} |v|^a \,\mathrm{d}v &= \int_0^{+\infty} \mathrm{e}^{-\frac{1}{2}(v-\phi(\tau))^2} v^a \,\mathrm{d}v + \int_0^{+\infty} \mathrm{e}^{-\frac{1}{2}(v+\phi(\tau))^2} v^a \,\mathrm{d}v \\
&= \vartheta_{0,a} \mathrm{e}^{-\frac{1}{4}\phi(\tau)^2} \big(D_{-a-1}(-\phi(\tau)) + D_{-a-1}(\phi(\tau)) \big).
\end{aligned}
$$

Since by (1.9),

$$
\int_0^{+\infty} t^a \mathrm{e}^{-\frac{1}{2}(t+z)^2} \,\mathrm{d}t = \Gamma(a+1) \mathrm{e}^{-\frac{1}{4}z^2} D_{-a-1}(z),
$$

$$\int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{1}{2}(u-\phi(\tau))^2}|v|^a v^\ell \,\mathrm{d}v$$
$$= \Gamma(a+\ell+1)\mathrm{e}^{-\frac{1}{4}\phi(\tau)^2}\big((-1)^\ell D_{-a-\ell-1}(\phi(\tau)) + D_{-a-\ell-1}(-\phi(\tau))\big),$$

we have

$$\mathsf{c}_0' = \int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{1}{2}(u-\phi(\tau))^2}|v|^a \,\mathrm{d}v = \vartheta_{0,a}\mathrm{e}^{-\frac{1}{4}\phi(\tau)^2}\big(D_{-a-1}(\phi(\tau)) + D_{-a-1}(-\phi(\tau))\big), \tag{3.10}$$

$$\begin{aligned}\mathsf{c}_1' &= \int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{1}{2}(v-\sqrt{nd_2}(r_\tau-\rho))^2}|v|^a c_1(u_v)\,\mathrm{d}v \\ &= -\frac{d_3}{6d_2^{3/2}}\sum_{\ell=0}^{3}\binom{3}{\ell}(-\phi(\tau))^{3-\ell}\vartheta_{\ell,a}\mathrm{e}^{-\frac{1}{4}\phi(\tau)^2}\big((-1)^\ell D_{-a-\ell-1}(\phi(\tau)) + D_{-a-\ell-1}(-\phi(\tau))\big) \\ &\quad + \frac{1}{d_2^{1/2}}\Big(\mathsf{k}'(r_\tau)+\frac{1}{r_\tau}\Big) \\ &\quad \times \sum_{\ell=0}^{1}\binom{1}{\ell}(-\phi(\tau))^{1-\ell}\vartheta_{\ell,a}\mathrm{e}^{-\frac{1}{4}\phi(\tau)^2}\big((-1)^\ell D_{-a-\ell-1}(\phi(\tau)) + D_{-a-\ell-1}(-\phi(\tau))\big),\end{aligned} \tag{3.11}$$

$$\begin{aligned}\mathsf{c}_2' &= \int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{1}{2}(v-\sqrt{nd_2}(r_\tau-\rho))^2}|v|^a c_2(u_v)\,\mathrm{d}v \\ &= \frac{d_3^2}{72d_2^3}\sum_{\ell=0}^{6}\binom{6}{\ell}(-\phi(\tau))^{6-\ell}\vartheta_{\ell,a}\mathrm{e}^{-\frac{1}{4}\phi(\tau)^2}\big((-1)^\ell D_{-a-\ell-1}(\phi(\tau)) + D_{-a-\ell-1}(-\phi(\tau))\big) \\ &\quad - \frac{1}{24d_2^2}\Big(d_4 + 4d_3\mathsf{k}'(r_\tau) + \frac{4d_3}{r_\tau}\Big) \\ &\quad \times \sum_{\ell=0}^{4}\binom{4}{\ell}(-\phi(\tau))^{4-\ell}\vartheta_{\ell,a}\mathrm{e}^{-\frac{1}{4}\phi(\tau)^2}\big((-1)^\ell D_{-a-\ell-1}(\phi(\tau)) + D_{-a-\ell-1}(-\phi(\tau))\big) \\ &\quad + \frac{1}{d_2}\Big(\frac{\mathsf{k}''(r_\tau)}{2} + \frac{\mathsf{k}'(r_\tau)^2}{2} + \frac{\mathsf{k}'(r_\tau)}{r_\tau}\Big) \\ &\quad \times \sum_{\ell=0}^{2}\binom{2}{\ell}(-\phi(\tau))^{2-\ell}\vartheta_{\ell,a}\mathrm{e}^{-\frac{1}{4}\phi(\tau)^2}\big((-1)^\ell D_{-a-\ell-1}(\phi(\tau)) + D_{-a-\ell-1}(-\phi(\tau))\big).\end{aligned} \tag{3.12}$$

Note that c_3' is given as well as (3.9), but we omit the explicit expression. Therefore, we obtain

$$\begin{aligned}h_{n,j}^{(\mathrm{out})}(\rho) &= \int_0^{+\infty} 2v\mathrm{e}^{\mathsf{k}(v)}\mathrm{e}^{-nV_\tau(v)}|v-\rho|^a\,\mathrm{d}v - \int_0^{\rho} 2v\mathrm{e}^{\mathsf{k}(v)}\mathrm{e}^{-nV_\tau(v)}|v-\rho|^a\,\mathrm{d}v \\ &= \frac{2r_\tau \mathrm{e}^{\mathsf{k}(r_\tau)}\mathrm{e}^{-nV_\tau(r_\tau)}}{(nd_2)^{(a+1)/2}}\left(\mathsf{c}_0' - \mathsf{c}_0 + \frac{\mathsf{c}_1'-\mathsf{c}_1}{\sqrt{n}} + \frac{\mathsf{c}_2'-\mathsf{c}_2}{n} + \mathcal{O}\left(\frac{\mathsf{c}_3'-\mathsf{c}_3}{n^{3/2}}\right)\right),\end{aligned} \tag{3.13}$$

where by (3.6), (3.7), (3.8), (3.9), (3.10), (3.11), and (3.12),

$$\mathsf{c}_0' - \mathsf{c}_0 = \vartheta_{0,a}\mathrm{e}^{-\frac{1}{4}\phi(\tau)^2}D_{-a-1}(-\phi(\tau)),$$

$$\begin{aligned}\mathsf{c}_1' - \mathsf{c}_1 &= -\frac{d_3}{6d_2^{3/2}}\sum_{\ell=0}^{3}\binom{3}{\ell}(-\phi(\tau))^{3-\ell}\vartheta_{\ell,a}\mathrm{e}^{-\frac{1}{4}\phi(\tau)^2}D_{-a-\ell-1}(-\phi(\tau)) \\ &\quad + \frac{1}{d_2^{1/2}}\Big(\mathsf{k}'(r_\tau)+\frac{1}{r_\tau}\Big)\sum_{\ell=0}^{1}\binom{1}{\ell}(-\phi(\tau))^{1-\ell}\vartheta_{\ell,a}\mathrm{e}^{-\frac{1}{4}\phi(\tau)^2}D_{-a-\ell-1}(-\phi(\tau)),\end{aligned}$$

$$
\begin{aligned}
\mathsf{c}_2' - \mathsf{c}_2 = {} & \frac{d_3^2}{72d_2^3}\sum_{\ell=0}^{6}\binom{6}{\ell}(-\phi(\tau))^{6-\ell}\vartheta_{\ell,a}\mathrm{e}^{-\frac{1}{4}\phi(\tau)^2}D_{-a-\ell-1}(-\phi(\tau)) \\
& - \frac{1}{24d_2^2}\left(d_4 + 4d_3\mathsf{k}'(r_\tau) + \frac{4d_3}{r_\tau}\right)\sum_{\ell=0}^{4}\binom{4}{\ell}(-\phi(\tau))^{4-\ell}\vartheta_{\ell,a}\mathrm{e}^{-\frac{1}{4}\phi(\tau)^2}D_{-a-\ell-1}(-\phi(\tau)) \\
& + \frac{1}{d_2}\left(\frac{\mathsf{k}''(r_\tau)}{2} + \frac{\mathsf{k}'(r_\tau)^2}{2} + \frac{\mathsf{k}'(r_\tau)}{r_\tau}\right) \\
& \times \sum_{\ell=0}^{2}\binom{2}{\ell}(-\phi(\tau))^{2-\ell}\vartheta_{\ell,a}\mathrm{e}^{-\frac{1}{4}\phi(\tau)^2}D_{-a-\ell-1}(-\phi(\tau)).
\end{aligned}
$$

By (3.4), (3.5), and (3.13), let us choose $\delta > 0$ sufficiently small so that $\mathrm{e}^u\frac{h_{n,j}^{(\mathrm{in})}(\rho)}{h_{n,j}} + \frac{h_{n,j}^{(\mathrm{out})}(\rho)}{h_{n,j}}$ remains bounded away from the interval $(-\infty, 0]$ as $n \to +\infty$ uniformly for $u \in \{z \in \mathbb{C} \mid |z - x| \le \delta\}$ and a in compact subsets of $(-1, +\infty)$. Then, we have

$$
\begin{aligned}
& \sum_{j=g_{1,-}}^{\lfloor n\tau_\rho\rfloor - 1}\log\left(\mathrm{e}^u\frac{h_{n,j}^{(\mathrm{in})}(\rho)}{h_{n,j}} + \frac{h_{n,j}^{(\mathrm{out})}(\rho)}{h_{n,j}}\right) \\
& \quad = \sum_{j=g_{1,-}}^{\lfloor n\tau_\rho\rfloor - 1}\log\left(\frac{1}{(4n)^{a/2}}\right) + \frac{a}{2}\sum_{j=g_{1,-}}^{\lfloor n\tau_\rho\rfloor - 1}\log\left(\frac{1}{\Delta Q(\rho)}\right) + \sum_{j=g_{1,-}}^{\lfloor n\tau_\rho\rfloor - 1}\log[\mathcal{H}_{a,u}(\xi)] \\
& \qquad + \frac{1}{\sqrt{n}}\sum_{j=g_{1,-}}^{\lfloor n\tau_\rho\rfloor - 1}\frac{\widetilde{g}_{a,u}(\xi)}{12\rho\sqrt{\Delta Q(\rho)}g_{a,u}(\xi)} + \mathcal{O}\left(\frac{(\log n)^{7/8}}{n^{1/8}}\right),
\end{aligned}
\tag{3.14}
$$

uniformly for $u \in \{z \in \mathbb{C} \mid |z - x| \le \delta\}$ and a in compact subsets of $(-1, +\infty)$. Here, $g_{a,u}(x), \widetilde{g}_{a,u}(x)$ are given by (1.8) and (3.3), respectively.

We consider the case $\lfloor n\tau_\rho\rfloor \le j \le g_{1,+}$. Similar manner to (3.13), there exists $c > 0$ such that we have

$$
\begin{aligned}
h_{n,j}^{(\mathrm{out})}(\rho) = {} & \frac{2r_\tau\mathrm{e}^{\mathsf{k}(r_\tau)}\mathrm{e}^{-nV_\tau(r_\tau)}}{(nd_2)^{(a+1)/2}}\int_0^{\sqrt{nd_2}\delta_n' + \sqrt{nd_2}(r_\tau - \rho)} y^a\mathrm{e}^{-\frac{1}{2}(y - \sqrt{nd_2}(r_\tau-\rho))^2} \\
& \times\left(1 + \frac{c_1(u_y)}{\sqrt{n}} + \frac{c_2(u_y)}{n} + \frac{c_3(u)}{n^{3/2}} + \mathcal{O}\left(\frac{c_4(u_y)}{n^2}\right)\right)\mathrm{d}y + \mathcal{O}\big(\mathrm{e}^{-cM^2}\big),
\end{aligned}
$$

where $u_y := y - \sqrt{nd_2}(r_\tau - \rho) = y - \phi(\tau)$ with $\phi(\tau) := \sqrt{nd_2}(r_\tau - \rho)$. Since $\sqrt{nd_2}\delta_n' + \sqrt{nd_2}(r_\tau - \rho) > \sqrt{nd_2}(r_\tau - \rho)$ for $\lfloor n\tau_\rho\rfloor \le j \le g_{1,+}$, we can extend the integral region to $[0, +\infty)$ with an exponential error $\mathcal{O}\big(\mathrm{e}^{-cM^2}\big)$. The Gaussian integral gives

$$
h_{n,j}^{(\mathrm{out})}(\rho) = \frac{2r_\tau\mathrm{e}^{\mathsf{k}(r_\tau)}\mathrm{e}^{-nV_\tau(r_\tau)}}{(nd_2)^{(a+1)/2}}\left(\widetilde{\mathsf{c}}_0 + \frac{\widetilde{\mathsf{c}}_1}{\sqrt{n}} + \frac{\widetilde{\mathsf{c}}_2}{n} + \frac{\widetilde{\mathsf{c}}_3}{n^{3/2}} + \mathcal{O}\left(\frac{\widetilde{\mathsf{c}}_4}{n^2}\right)\right) + \mathcal{O}\big(\mathrm{e}^{-cM^2}\big),
\tag{3.15}
$$

where

$$
\begin{aligned}
\widetilde{\mathsf{c}}_0 &= \int_0^{+\infty} y^a\mathrm{e}^{-\frac{1}{2}(y-\phi(\tau))^2}\,\mathrm{d}y = \vartheta_{0,a}\mathrm{e}^{-\frac{\phi(\tau)^2}{4}}D_{-a-1}(-\phi(\tau)), \\
\widetilde{\mathsf{c}}_1 &= \int_0^{+\infty} y^a\mathrm{e}^{-\frac{1}{2}(y-\phi(\tau))^2}c_1(u_y)\,\mathrm{d}y \\
&= -\frac{d_3}{6d_2^{3/2}}\sum_{\ell=0}^{3}\binom{3}{\ell}(-\phi(\tau))^{3-\ell}\vartheta_{\ell,a}\mathrm{e}^{-\frac{\phi(\tau)^2}{4}}D_{-a-\ell-1}(-\phi(\tau))
\end{aligned}
$$

$$+\frac{1}{d_2^{1/2}}\Big(\mathsf{k}'(r_\tau)+\frac{1}{r_\tau}\Big)\sum_{\ell=0}^{1}\binom{1}{\ell}(-\phi(\tau))^{1-\ell}\vartheta_{\ell,a}\mathrm{e}^{-\frac{\phi(\tau)^2}{4}}D_{-a-\ell-1}(-\phi(\tau)),$$

$$\begin{aligned}\widetilde{\mathsf{c}}_2&=\int_0^{+\infty}y^a\mathrm{e}^{-\frac{1}{2}(y-\phi(\tau))^2}c_2(u_y)\,\mathrm{d}y\\&=\frac{d_3^2}{72d_2^3}\sum_{\ell=0}^{6}\binom{6}{\ell}(-\phi(\tau))^{6-\ell}\vartheta_{\ell,a}\mathrm{e}^{-\frac{\phi(\tau)^2}{4}}D_{-a-\ell-1}(-\phi(\tau))\\&\quad-\frac{1}{24d_2^2}\Big(d_4+4d_3\mathsf{k}'(r_\tau)+\frac{4d_3}{r_\tau}\Big)\sum_{\ell=0}^{4}\binom{4}{\ell}(-\phi(\tau))^{4-\ell}\vartheta_{\ell,a}\mathrm{e}^{-\frac{\phi(\tau)^2}{4}}D_{-a-\ell-1}(-\phi(\tau))\\&\quad+\frac{1}{d_2}\Big(\frac{\mathsf{k}''(r_\tau)}{2}+\frac{\mathsf{k}'(r_\tau)^2}{2}+\frac{\mathsf{k}'(r_\tau)}{r_\tau}\Big)\sum_{\ell=0}^{2}\binom{2}{\ell}(-\phi(\tau))^{2-\ell}\vartheta_{\ell,a}\mathrm{e}^{-\frac{\phi(\tau)^2}{4}}D_{-a-\ell-1}(-\phi(\tau)).\end{aligned}$$

On the other hand,

$$\begin{aligned}h_{n,j}^{(\mathrm{in})}(\rho)&:=\int_0^{+\infty}2v\mathrm{e}^{\mathsf{k}(v)}\mathrm{e}^{-nV_\tau(v)}|v-\rho|^a\,\mathrm{d}v-\int_\rho^{+\infty}2v\mathrm{e}^{\mathsf{k}(v)}\mathrm{e}^{-nV_\tau(v)}(v-\rho)^a\,\mathrm{d}v\\&=\frac{2r_\tau\mathrm{e}^{\mathsf{k}(r_\tau)}\mathrm{e}^{-nV_\tau(r_\tau)}}{(nd_2)^{(a+1)/2}}\Big(\mathsf{c}_0'-\widetilde{\mathsf{c}}_0+\frac{\mathsf{c}_1'-\widetilde{\mathsf{c}}_1}{\sqrt{n}}+\frac{\mathsf{c}_2'-\widetilde{\mathsf{c}}_2}{n}+\mathcal{O}\Big(\frac{\mathsf{c}_3'-\widetilde{\mathsf{c}}_3}{n^{3/2}}\Big)\Big),\end{aligned}\tag{3.16}$$

where

$$\mathsf{c}_0'-\widetilde{\mathsf{c}}_0=\vartheta_{0,a}\mathrm{e}^{-\frac{1}{4}\phi(\tau)^2}D_{-a-1}(\phi(\tau)),$$

$$\begin{aligned}\mathsf{c}_1'-\widetilde{\mathsf{c}}_1&=-\frac{d_3}{6d_2^{3/2}}\sum_{\ell=0}^{3}\binom{3}{\ell}(-\phi(\tau))^{3-\ell}\vartheta_{\ell,a}\mathrm{e}^{-\frac{1}{4}\phi(\tau)^2}(-1)^\ell D_{-a-\ell-1}(\phi(\tau))\\&\quad+\frac{1}{d_2^{1/2}}\Big(\mathsf{k}'(r_\tau)+\frac{1}{r_\tau}\Big)\sum_{\ell=0}^{1}\binom{1}{\ell}(-\phi(\tau))^{1-\ell}\vartheta_{\ell,a}\mathrm{e}^{-\frac{1}{4}\phi(\tau)^2}(-1)^\ell D_{-a-\ell-1}(\phi(\tau)),\end{aligned}$$

$$\begin{aligned}\mathsf{c}_2'-\widetilde{\mathsf{c}}_2&=\frac{d_3^2}{72d_2^3}\sum_{\ell=0}^{6}\binom{6}{\ell}(-\phi(\tau))^{6-\ell}\vartheta_{\ell,a}\mathrm{e}^{-\frac{1}{4}\phi(\tau)^2}(-1)^\ell D_{-a-\ell-1}(\phi(\tau))\\&\quad-\frac{1}{24d_2^2}\Big(d_4+4d_3\mathsf{k}'(r_\tau)+\frac{4d_3}{r_\tau}\Big)\\&\quad\times\sum_{\ell=0}^{4}\binom{4}{\ell}(-\phi(\tau))^{4-\ell}\vartheta_{\ell,a}\mathrm{e}^{-\frac{1}{4}\phi(\tau)^2}(-1)^\ell D_{-a-\ell-1}(\phi(\tau))\\&\quad+\frac{1}{d_2}\Big(\frac{\mathsf{k}''(r_\tau)}{2}+\frac{\mathsf{k}'(r_\tau)^2}{2}+\frac{\mathsf{k}'(r_\tau)}{r_\tau}\Big)\\&\quad\times\sum_{\ell=0}^{2}\binom{2}{\ell}(-\phi(\tau))^{2-\ell}\vartheta_{\ell,a}\mathrm{e}^{-\frac{1}{4}\phi(\tau)^2}(-1)^\ell D_{-a-\ell-1}(\phi(\tau)).\end{aligned}$$

Rewriting the above expansion in terms of (3.1), by (3.15) and (3.16) we obtain the same expansion with (3.14). ■

To apply Lemma C.1 to each sum in Lemma 3.1, we will use

$$\partial_x\log[\mathcal{H}_{a,u}(x)]=-\frac{\mathrm{e}^uD_{-a}(x)-D_{-a}(-x)}{\mathrm{e}^uD_{-a-1}(x)+D_{-a-1}(-x)},\tag{3.17}$$

where we have used [68, Sections 12.8 and 12.9]. Now, we establish the asymptotic expansion of S_2.

Lemma 3.2. *For any fixed $x \in \mathbb{R}$, there exists $\delta > 0$ such that as $n \to +\infty$, we have*

$$S_2 = C_2^{(2)}\sqrt{n} + C_3^{(2)} + C_n^{(2)} + \mathcal{O}\left(\frac{(\log n)^{1/4}}{n^{1/4}}\right),$$

uniformly for $u \in \{z \in \mathbb{C} \mid |z - x| \le \delta\}$ and a in compact subsets of $(-1, +\infty)$, where

$$\begin{aligned}
C_2^{(2)} &:= \rho\sqrt{\Delta Q(\rho)} \int_{-\infty}^{\infty} \big(\log[\mathcal{H}_{a,u}(x)] - a\log|x| - u\mathbf{1}_{(-\infty,0)}(x)\big)\,\mathrm{d}x,\\
C_3^{(2)} &:= -\left(\alpha + \frac{1}{2}\right)u + \frac{1}{6}\left(2 + \frac{\rho\partial_r \Delta Q(\rho)}{\Delta Q(\rho)}\right)u - \frac{a}{12}\left(1 - \frac{\rho\partial_r \Delta Q(\rho)}{\Delta Q(\rho)}\right)u\\
&\quad + \frac{1}{6}\left(2 + \frac{\rho\partial_r \Delta Q(\rho)}{\Delta Q(\rho)}\right)\\
&\quad \times \int_{-\infty}^{+\infty} \left[x\big(\log[\mathcal{H}_{a,u}(x)] - u\mathbf{1}_{(-\infty,0)}\big) - ax\log|x| - \frac{a(a-1)x}{2(x^2+1)}\right]\mathrm{d}x,\\
C_n^{(2)} &:= -\frac{a}{2}\big(\rho\sqrt{\Delta Q(\rho)}\sqrt{n}\Delta_{n,+} + \rho\sqrt{\Delta Q(\rho)}\sqrt{n}\Delta_{n,-} + 1\big)\log(4n)\\
&\quad + \sqrt{n}\rho\sqrt{\Delta Q(\rho)}\big(a\Delta_{n,+}(\log\Delta_{n,+} - 1) + a\Delta_{n,-}(\log\Delta_{n,-} - 1) + u\Delta_{n,-}\big)\\
&\quad + \frac{a}{2}\log\Delta_{n,+} + \frac{a}{2}\log\Delta_{n,-}.
\end{aligned}$$

Proof. By (2.8), as $n \to +\infty$, we have

$$g_{1,+} - g_{1,-} + 1 = \rho\sqrt{\Delta Q(\rho)}\sqrt{n}\Delta_{n,+} + \rho\sqrt{\Delta Q(\rho)}\sqrt{n}\Delta_{n,-} + 1.$$

Therefore, the first and second terms of (3.2) satisfy

$$\begin{aligned}
&\sum_{j=g_{1,-}}^{g_{1,+}} \log\left(\frac{1}{(4n)^{a/2}}\right) + \frac{a}{2}\sum_{j=g_{1,-}}^{g_{1,+}} \log\left(\frac{1}{\Delta Q(\rho)}\right)\\
&\quad = -\big(\rho\sqrt{\Delta Q(\rho)}\sqrt{n}\Delta_{n,+} + \rho\sqrt{\Delta Q(\rho)}\sqrt{n}\Delta_{n,-} + 1\big)a\log 2\\
&\qquad - \big(\rho\sqrt{\Delta Q(\rho)}\sqrt{n}\Delta_{n,+} + \rho\sqrt{\Delta Q(\rho)}\sqrt{n}\Delta_{n,-} + 1\big)\frac{a}{2}\log n\\
&\qquad - \big(\rho\sqrt{\Delta Q(\rho)}\sqrt{n}\Delta_{n,+} + \rho\sqrt{\Delta Q(\rho)}\sqrt{n}\Delta_{n,-} + 1\big)\frac{a}{2}\log\Delta Q(\rho).
\end{aligned}$$

By Lemma C.1, as $n \to +\infty$, we have

$$\begin{aligned}
\sum_{j=g_{1,-}}^{g_{1,+}} \log[\mathcal{H}_{a,u}(\xi)] &= \int_{g_{1,-}}^{g_{1,+}} \log\big[\mathcal{H}_{a,u}(\xi_t)\big]\,\mathrm{d}t + \frac{1}{2}\log[\mathcal{H}_{a,u}(-\Delta_{n,-})]\\
&\quad + \frac{1}{2}\log[\mathcal{H}_{a,u}(\Delta_{n,+})] + \mathcal{O}\left(\frac{(\log n)^{1/4}}{n^{1/4}}\right).
\end{aligned}$$

By change of variables (2.5), we get

$$\int_{g_{1,-}}^{g_{1,+}} \log[\mathcal{H}_{a,u}(\xi_t)]\,\mathrm{d}t = \sqrt{n}\,\rho\sqrt{\Delta Q(\rho)} \int_{-\Delta_{n,-}}^{\Delta_{n,+}} \log[\mathcal{H}_{a,u}(x)]\,\mathrm{d}x.$$

From [68, Section 12.9], we have

$$\log[\mathcal{H}_{a,u}(x)] = a\log|x| + u\mathbf{1}_{(-\infty,0)}(x) + \frac{a(a-1)}{2x^2} - \frac{a(a-1)(2a-3)}{4x^4} + \mathcal{O}\big(x^{-6}\big), \tag{3.18}$$

for $x \to \pm\infty$. By the above and regularizing the integral, we have

$$\begin{aligned}
&\sqrt{n}\,\rho\sqrt{\Delta Q(\rho)}\int_{-\Delta_{n,-}}^{\Delta_{n,+}} \log[\mathcal{H}_{a,u}(x)]\,\mathrm{d}x \\
&\quad= \sqrt{n}\rho\sqrt{\Delta Q(\rho)}\int_{-\infty}^{+\infty}\big(\log[\mathcal{H}_{a,u}(x)] - a\log|x| - u\mathbf{1}_{(-\infty,0)}(x)\big)\,\mathrm{d}x \\
&\qquad+ \sqrt{n}\rho\sqrt{\Delta Q(\rho)}\big(a\Delta_{n,+}(\log\Delta_{n,+}-1) + a\Delta_{n,-}(\log\Delta_{n,-}-1) + u\Delta_{n,-}\big) \\
&\qquad- \sqrt{n}\rho\sqrt{\Delta Q(\rho)}\bigg(\frac{a(a-1)}{2\Delta_{n,+}} + \frac{a(a-1)}{2\Delta_{n,-}} - \frac{a(a-1)(2a-3)}{12\Delta_{n,+}^3} - \frac{a(a-1)(2a-3)}{12\Delta_{n,-}^3} \\
&\qquad+ \mathcal{O}\big(\Delta_{n,+}^{-5} + \Delta_{n,-}^{-5}\big)\bigg).
\end{aligned}$$

Next we observe that

$$\frac{1}{\sqrt{n}}\sum_{j=g_{1,-}}^{g_{1,+}} \frac{\widetilde{g}_{a,u}(\xi)}{12\rho\sqrt{\Delta Q(\rho)}g_{a,u}(\xi)} = \frac{1}{12}\int_{-\Delta_{n,-}}^{\Delta_{n,+}} \frac{\widetilde{g}_{a,u}(x)}{g_{a,u}(x)}\,\mathrm{d}x + \mathcal{O}\left(\frac{(\log n)^{1/4}}{n^{1/4}}\right),$$

Note that by (3.3), (1.8), and (3.17), we have

$$\begin{aligned}
&\frac{1}{12}\int_{-\Delta_{n,-}}^{\Delta_{n,+}} \frac{\widetilde{g}_{a,u}(x)}{g_{a,u}(x)}\,\mathrm{d}x \\
&\quad= \frac{1}{6}\left(2 + \frac{\rho\partial_r\Delta Q(\rho)}{\Delta Q(\rho)}\right)\int_{-\Delta_{n,-}}^{\Delta_{n,+}} x\log[\mathcal{H}_{a,u}(x)\,]\,\mathrm{d}x - \frac{a}{24}\left(1 - \frac{\rho\partial_r\Delta Q(\rho)}{\Delta Q(\rho)}\right)\big(\Delta_{n,+}^2 - \Delta_{n,-}^2\big) \\
&\qquad- \frac{1}{12}\left(2 + \frac{\rho\partial_r\Delta Q(\rho)}{\Delta Q(\rho)}\right)\big[\Delta_{n,+}^2\log[\mathcal{H}_{a,u}(\Delta_{n,+})] - \Delta_{n,-}^2\log[\mathcal{H}_{a,u}(-\Delta_{n,-})]\big] \\
&\qquad+ \left[\alpha + \frac{1}{2} + \frac{2+a}{12}\left(1 - \frac{\rho\partial_r\Delta Q(\rho)}{\Delta Q(\rho)}\right)\right]\big[\log[\mathcal{H}_{a,u}(\Delta_{n,+})] - \log\big[\mathcal{H}_{a,u}(-\Delta_{n,-})\big]\big].
\end{aligned}$$

By regularizing the integral, we obtain

$$\begin{aligned}
&\frac{1}{12}\int_{-\Delta_{n,-}}^{\Delta_{n,+}} \frac{\widetilde{g}_{a,u}(x)}{g_{a,u}(x)}\,\mathrm{d}x = -\frac{a}{24}\left(1 + \frac{2\rho\partial_r\Delta Q(\rho)}{\Delta Q(\rho)}\right)\big(\Delta_{n,+}^2 - \Delta_{n,-}^2\big) \\
&\quad+ \left(\alpha + \frac{1}{2}\right)\log[\mathcal{H}_{a,u}(\Delta_{n,+})] - \left(\alpha + \frac{1}{2}\right)\log[\mathcal{H}_{a,u}(-\Delta_{n,+})] \\
&\quad+ \frac{2+a}{12}\left(1 - \frac{\rho\partial_r\Delta Q(\rho)}{\Delta Q(\rho)}\right)\log[\mathcal{H}_{a,u}(\Delta_{n,+})] \\
&\quad- \frac{2+a}{12}\left(1 - \frac{\rho\partial_r\Delta Q(\rho)}{\Delta Q(\rho)}\right)\log[\mathcal{H}_{a,u}(-\Delta_{n,-})] \\
&\quad- \frac{1}{12}\left(2 + \frac{\rho\partial_r\Delta Q(\rho)}{\Delta Q(\rho)}\right)\Delta_{n,+}^2\log[\mathcal{H}_{a,u}(\Delta_{n,+})] \\
&\quad+ \frac{1}{12}\left(2 + \frac{\rho\partial_r\Delta Q(\rho)}{\Delta Q(\rho)}\right)\Delta_{n,-}^2\log[\mathcal{H}_{a,u}(-\Delta_{n,-})] \\
&\quad+ \frac{1}{6}\left(2 + \frac{\rho\partial_r\Delta Q(\rho)}{\Delta Q(\rho)}\right)\int_{-\infty}^{+\infty}\left[x\left(\log[\mathcal{H}_{a,u}(x)] - u\mathbf{1}_{(-\infty,0)}\right) - ax\log|x| - \frac{a(a-1)x}{2(x^2+1)}\right]\mathrm{d}x \\
&\quad- \frac{u}{12}\left(2 + \frac{\rho\partial_r\Delta Q(\rho)}{\Delta Q(\rho)}\right)\Delta_{n,-}^2 + \frac{1}{6}\left(2 + \frac{\rho\partial_r\Delta Q(\rho)}{\Delta Q(\rho)}\right)\int_{-\Delta_{n,-}}^{\Delta_{n,+}}\left(ax\log|x| + \frac{a(a-1)x}{2(x^2+1)}\right)\mathrm{d}x \\
&\quad+ \mathcal{O}\left(\frac{(\log n)^{1/4}}{n^{1/4}}\right).
\end{aligned}$$

Combining all of the above together with (3.18), we obtain the result. ■

3.2 Proof of Theorem 1.6

We now complete the proof of Theorem 1.6. Combining the error terms from Lemmas 2.4, 3.2, and 2.5, we have $C_n^{(1)} + C_n^{(2)} + C_n^{(3)} = -a\log 2 - \frac{a}{2}\log \Delta Q(\rho)$, which is added to the term of order $\mathcal{O}(1)$. Thus, by Lemmas 2.2, 2.4, 3.2, and 2.5, we get $C_3^{(1)} + C_3^{(2)} + C_3^{(3)} - a\log 2 - \frac{a}{2}\log \Delta Q(\rho) = C_3(u,a)$. By Lemmas 2.2, 2.4, 3.2, and 2.5, $C_1(u,a), C_2(u,a)$ are similarly computed. This completes the proof of Theorem 1.6.

A Consistency between Theorem 1.3 and [35, Theorem 1.1]

In this appendix, we confirm the consistency between Theorem 1.3 and [35, Theorem 1.1] mentioned in Remark 1.4. We recall that C_1, C_2, C_3 for $Q(z) = |z|^{2b}$ $(b > 0)$ $r_1 = \rho$, and $r_k \equiv 0$ for $k = 2, 3, \dots, m+1$ in [35, Theorem 1.1] are given by

$$C_1 = b\rho^{2b}u, \tag{A.1}$$

$$C_2 = \sqrt{2}b\rho^b \int_0^{+\infty} \big(\mathcal{F}(t, \mathrm{e}^u) + \mathcal{F}(t, \mathrm{e}^{-u})\big)\,\mathrm{d}t, \tag{A.2}$$

$$C_3 = -\left(\frac{1}{2} + \alpha\right)u + 4b\int_0^{+\infty} t\big(\mathcal{F}(t, \mathrm{e}^u) - \mathcal{F}(t, \mathrm{e}^{-u})\big)\,\mathrm{d}t + b\int_{-\infty}^{+\infty} \mathcal{G}(t, \mathrm{e}^u)\frac{5t^2-1}{3}\,\mathrm{d}t, \tag{A.3}$$

where $\mathcal{F}(t,s)$ for $t \in \mathbb{R}$ and $s \in \mathbb{C}\backslash(-\infty, 0]$ is given by (1.6) and

$$\mathcal{G}(t,s) := \frac{1-s}{1 + \frac{s-1}{2}\mathrm{erfc}(t)}\frac{\mathrm{e}^{-t^2}}{\sqrt{\pi}} = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(t,s).$$

It is straightforward to see that $C_1(u)$ and $C_2(u)$ in Theorem 1.3 are consistent with (A.1) and (A.2), respectively. To see that C_3 in [35, Theorem 1.1] is consistent with (A.3), by integration by parts, the last term in (A.3) can be rewritten as

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \mathcal{G}(t, \mathrm{e}^u)\frac{5t^2-1}{3}\,\mathrm{d}t \\
&\quad = b\lim_{M\to+\infty}\int_0^M \mathcal{G}(t, \mathrm{e}^u)\frac{5t^2-1}{3}\,\mathrm{d}t - b\lim_{M\to+\infty}\int_0^M \mathcal{G}(t, \mathrm{e}^{-u})\frac{5t^2-1}{3}\,\mathrm{d}t \\
&\quad = \frac{b}{3}u - \frac{10b}{3}\int_0^{+\infty} t\big(\mathcal{F}(t, \mathrm{e}^u) - \mathcal{F}(t, \mathrm{e}^{-u})\big)\,\mathrm{d}t.
\end{aligned}$$

Therefore, (A.3) can be deduced to

$$C_3 = -\left(\frac{1}{2} + \alpha\right)u + \frac{b}{3}u + \frac{2b}{3}\int_0^{+\infty} t\big(\mathcal{F}(t, \mathrm{e}^u) - \mathcal{F}(t, \mathrm{e}^{-u})\big)\,\mathrm{d}t.$$

If we informally apply Theorem 1.3 to the case $Q(z) = |z|^{2b}$ $(b > 0)$, by $2 + \frac{\rho\partial_r \Delta Q(\rho)}{\Delta Q(\rho)} = 2b$, we obtain $C_3(u) = C_3$. This shows the consistency between Theorem 1.3 and [35, Theorem 1.1].

B Consistency between Theorem 1.6 and [23, Theorem 1.1]

We confirm the consistency between Theorem 1.6 and [23, Theorem 1.1]. We begin with collecting some facts on the parabolic cylinder function and the associated Hermite polynomials from [23, 68, 73, 75]. The ν-th associated Hermite polynomials $\{\mathrm{He}_k^{(\nu)}; k = 0, 1, \dots\}$ are defined recursively by

$$\mathrm{He}_{k+1}^{(\nu)}(x) = x\mathrm{He}_k^{(\nu)}(x) - (k+\nu)\mathrm{He}_{k-1}^{(\nu)}(x), \qquad k \geq 1,$$

$$\mathrm{He}_0^{(\nu)}(x)=1,\qquad \mathrm{He}_1^{(\nu)}(x)=x,$$

and satisfy the orthogonality relations

$$\int_{-\infty}^{+\infty}\mathrm{He}_k^{(\nu)}(x)\mathrm{He}_\ell^{(\nu)}(x)\,\frac{\mathrm{d}x}{|D_{-\nu}(\mathrm{i}x)|^2}=\sqrt{2\pi}(k+\nu)!\delta_{k,\ell},$$

see [68, equation (12.7.2)], where the parabolic cylinder function $D_{-\nu}(x)$ is given by (1.9). It is known that the parabolic cylinder function is related to a family of Hermite polynomials $\{H_n\}_{n\in\mathbb{N}}$, i.e.,

$$D_n(z)=\mathrm{e}^{-\frac{1}{4}z^2}2^{-n/2}H_n\left(\frac{z}{\sqrt{2}}\right)=\mathrm{e}^{-\frac{1}{4}z^2}\mathrm{He}_n(z),\qquad n=0,1,2,\dots,$$

where $H_n(x)=2^{n/2}\mathrm{He}_n\big(\sqrt{2}x\big)$ is defined by

$$H_n(x)=(-1)^n\mathrm{e}^{x^2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathrm{e}^{-x^2},$$

see [68, equation (18.5.5)]. For $\nu\in\mathbb{R}$ with $\nu\notin\mathbb{Z}_{<0}$ and $n\in\mathbb{Z}_{\geq 0}$,

$$D_{-\nu-n}(z):=\frac{\Gamma(-\nu-n+1)}{\Gamma(-\nu+1)}\big[(-\mathrm{i})^n\mathrm{He}_n^{(\nu-1)}(\mathrm{i}z)D_{-\nu}(z)-(-\mathrm{i})^{n-1}\mathrm{He}_{n-1}^{(\nu)}(\mathrm{i}z)D_{-\nu+1}(z)\big].$$

Particularly,

$$D_0(z):=U\big(-\tfrac{1}{2},z\big)=\mathrm{e}^{-\frac{z^2}{4}},\qquad D_{-1}(z):=U\big(\tfrac{1}{2},z\big)=\mathrm{e}^{\frac{z^2}{4}}\sqrt{\frac{\pi}{2}}\mathrm{erfc}\left(\frac{z}{\sqrt{2}}\right).$$

If $a\in\mathbb{Z}_{>0}$ and $\nu=1$, then by $\lim_{\nu\to 1}\frac{\Gamma(-\nu-a+1)}{\Gamma(-\nu+1)}=\frac{(-1)^a}{a!}$, one can write

$$\mathrm{e}^{-\frac{z^2}{4}}D_{-1-a}(z)=\frac{(-1)^a}{a!}\left[\frac{1}{\mathrm{i}^a}\mathrm{He}_a^{(0)}(\mathrm{i}z)\sqrt{\frac{\pi}{2}}\mathrm{erfc}\left(\frac{z}{\sqrt{2}}\right)-\frac{1}{\mathrm{i}^{a-1}}\mathrm{He}_{a-1}^{(1)}(\mathrm{i}z)\mathrm{e}^{-\frac{z^2}{2}}\right].$$

Next, we recall some functionals from [23, equations (1.8), (1.9), (1.10), (1.11), (1.13), and (1.14)] (here we mainly focus on $a\geq 3$ for the simplicity),

$$\begin{aligned}
&p_{0,a}(x):=\frac{1}{\mathrm{i}^a}\mathrm{He}_a(\mathrm{i}x),\qquad q_{0,a}(x):=\frac{1}{\mathrm{i}^{a-1}}\mathrm{He}_{a-1}^{(1)}(\mathrm{i}x),\\
&p_{1,a}(x):=-\frac{a}{2}p_{0,a+1}(x)-ab\left(p_{0,a+1}(x)-(3a-1)p_{0,a-1}(x)+\frac{5}{3}(a-1)(a-2)p_{0,a-3}(x)\right),\\
&q_{1,a}(x):=-\frac{a}{2}q_{0,a+1}(x)-ab\left(q_{0,a+1}(x)-(3a-1)q_{0,a-1}(x)+\frac{5}{3}(a-1)(a-2)q_{0,a-3}(x)\right),\\
&\mathcal{G}_0(y;u,a):=p_{0,a}\big(-\sqrt{2}y\big)\left((-1)^a+\frac{\mathrm{e}^u-(-1)^a}{2}\mathrm{erfc}(y)\right)+q_{0,a}\big(-\sqrt{2}y\big)(\mathrm{e}^u-(-1)^a)\frac{\mathrm{e}^{-y^2}}{\sqrt{2\pi}},\\
&\mathcal{G}_1(y;u,a):=p_{1,a}\big(-\sqrt{2}y\big)\left((-1)^a+\frac{\mathrm{e}^u-(-1)^a}{2}\mathrm{erfc}(y)\right)+q_{1,a}\big(-\sqrt{2}y\big)(\mathrm{e}^u-(-1)^a)\frac{\mathrm{e}^{-y^2}}{\sqrt{2\pi}}.
\end{aligned}$$

The fact that $\mathrm{He}_a^{(\nu)}(-\mathrm{i}y)=(-1)^a\mathrm{He}_a^{(\nu)}(\mathrm{i}y)$ for $y\in\mathbb{R}$ and $a\in\mathbb{N}$ gives rise to

$$\begin{aligned}
&\mathrm{e}^u\frac{\Gamma(a+1)}{\sqrt{2\pi}}\mathrm{e}^{-\frac{y^2}{4}}D_{-1-a}(y)+\frac{\Gamma(a+1)}{\sqrt{2\pi}}\mathrm{e}^{-\frac{y^2}{4}}D_{-1-a}(-y)\\
&\quad=\frac{1}{\mathrm{i}^a}\mathrm{He}_a^{(0)}(-\mathrm{i}y)\left[(-1)^a+(\mathrm{e}^u-(-1)^a)\frac{1}{2}\mathrm{erfc}\left(\frac{y}{\sqrt{2}}\right)\right]
\end{aligned}$$

$$+ (\mathrm{e}^{u} - (-1)^{a}) \frac{1}{\mathrm{i}^{a-1}} \mathrm{He}^{(1)}_{a-1}(-\mathrm{i}y) \frac{\mathrm{e}^{-\frac{y^2}{2}}}{\sqrt{2\pi}}.$$

This shows the consistency between $\mathcal{H}_{a,u}(y)$ given by (1.8) and [23, equation (1.13)], i.e.,

$$\mathcal{G}_0\left(\frac{y}{\sqrt{2}}; u, a\right) = \mathrm{e}^{u} \frac{\Gamma(a+1)}{\sqrt{2\pi}} \mathrm{e}^{-\frac{y^2}{4}} D_{-1-a}(y) + \frac{\Gamma(a+1)}{\sqrt{2\pi}} \mathrm{e}^{-\frac{y^2}{4}} D_{-1-a}(-y) = \mathcal{H}_{a,u}(y).$$

By a similar manner to the above, since by [68, equation (12.8.1)],

$$\begin{aligned} &D_{2-a}(y) = \big(y^2 + a - 1\big) D_{-a}(y) + ayD_{-a-1}(y), \\ &(a+1) D_{-a-2}(y) = -yD_{-a-1}(y) + D_{-a}(y), \end{aligned}$$

we have

$$\begin{aligned} \mathcal{G}_1\left(\frac{y}{\sqrt{2}}; u, a\right) &= -\frac{\Gamma(a+1)}{\sqrt{2\pi}} \mathrm{e}^{-\frac{y^2}{4}} \bigg[\left(\frac{a}{2} + ab\right)(a+1)\big(\mathrm{e}^{u} D_{-2-a}(y) - D_{-2-a}(-y)\big) \\ &\quad - b(3a-1)\big(\mathrm{e}^{u} D_{-a}(y) - D_{-a}(-y)\big) + \frac{5}{3} b\big(\mathrm{e}^{u} D_{2-a}(y) - D_{2-a}(-y)\big) \bigg] \\ &= -\frac{\Gamma(a+1)}{\sqrt{2\pi}} \mathrm{e}^{-\frac{y^2}{4}} \bigg[\left(\frac{a}{2} + ab - b(3a-1) + \frac{5}{3} b(y^2 + a - 1)\right) \\ &\quad \times \big(\mathrm{e}^{u} D_{-a}(y) - D_{-a}(-y)\big) \\ &\quad + \left(\frac{2}{3} ab - \frac{a}{2}\right) \big(\mathrm{e}^{u} D_{-a-1}(y) + D_{-a-1}(-y)\big) \bigg]. \end{aligned}$$

This gives rise to

$$\begin{aligned} \frac{\mathcal{G}_1\big(\frac{y}{\sqrt{2}}; u, a\big)}{\mathcal{G}_0\big(\frac{y}{\sqrt{2}}; u, a\big)} &= -\left[\frac{a}{2} + ab - b(3a-1) + \frac{5}{3} b(y^2 + a - 1)\right] \partial_y \big(\log[\mathcal{H}_{a,u}(y)] - u \mathbf{1}_{(-\infty,0)}(y)\big) \\ &\quad - \left(-\frac{a}{2} + \frac{2}{3} ab\right) y, \end{aligned} \tag{B.1}$$

where we have used (3.17). By substituting (B.1) into [23, $\frac{\mathcal{G}_1(y;u,a)}{\mathcal{G}_0(y;u,a)}$ of C_3 in Theorem 1.1] and straightforward computations, we find that [23, C_3 in Theorem 1.1] recovers C_3 in Theorem 1.6.

C Euler–Maclaurin formula

In this work, we have used Euler–Maclaurin formula [4, 7, 27] to establish the precise large n-asymptotics of (1.4), which we state below for reference.

Lemma C.1 (Euler–Maclaurin formula)**.** *Let $f(x)$ be $2m$ times differentiable function on the interval $[p, q]$. Then we have*

$$\begin{aligned} \sum_{j=p+1}^{q-1} f(j) &= \int_p^q f(x)\,\mathrm{d}x - \frac{f(p) + f(q)}{2} + \sum_{j=1}^{m-1} \frac{B_{2j}}{(2j)!} \big(f^{(2j-1)}(q) - f^{(2j-1)}(p)\big) + R_{2m}, \\ \sum_{j=p}^{q} f(j) &= \int_p^q f(x)\,\mathrm{d}x + \frac{f(p) + f(q)}{2} + \sum_{j=1}^{m-1} \frac{B_{2j}}{(2j)!} \big(f^{(2j-1)}(q) - f^{(2j-1)}(p)\big) + R_{2m}, \\ \sum_{j=p}^{q-1} f(j) &= \int_p^q f(x)\,\mathrm{d}x - \frac{f(q) - f(p)}{2} + \sum_{j=1}^{m-1} \frac{B_{2j}}{(2j)!} \big(f^{(2j-1)}(q) - f^{(2j-1)}(p)\big) + R_{2m}, \end{aligned}$$

where the sequence $\{B_{2k}\}_k$ *are even indexed Bernoulli numbers* $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}, \dots$, *and the remainder term* R_{2m} *satisfies the bound* $|R_{2m}| \leq c_{2m} \int_p^q |f^{(2m)}(x)|\,\mathrm{d}x$ *with* $c_{2m} = \frac{2\zeta(2m)}{(2\pi)^{2m}}$. *Here,* $\zeta(s)$ *is the Riemann zeta function.*

Acknowledgements

The author is grateful to Sung-Soo Byun and Seong-Mi Seo for valuable feedbacks. The author acknowledges support from the European Research Council (ERC), Grant Agreement No. 101115687. I thank the referees for the careful reading and additional comment, which has improved our presentation.

References

[1] Akemann G., Byun S.-S., Ebke M., Universality of the number variance in rotational invariant two-dimensional Coulomb gases, *J. Stat. Phys.* **190** (2023), 9, 34 pages, arXiv:2206.08815.

[2] Akemann G., Byun S.-S., Ebke M., Schehr G., Universality in the number variance and counting statistics of the real and symplectic Ginibre ensemble, *J. Phys. A* **56** (2023), 495202, 53 pages, arXiv:2308.05519.

[3] Akemann G., Duits M., Molag L.D., Fluctuations in various regimes of non-Hermiticity and a holographic principle, arXiv:2412.15854.

[4] Allard M., Forrester P.J., Lahiry S., Shen B.J., Partition function of 2D Coulomb gases with radially symmetric potentials and a hard wall, arXiv:2506.14738.

[5] Allard M., Lahiry S., Birth of a gap: Critical phenomena in 2D Coulomb gas, arXiv:2509.24529.

[6] Ameur Y., Charlier C., Cronvall J., The two-dimensional Coulomb gas: fluctuations through a spectral gap, *Arch. Ration. Mech. Anal.* **249** (2025), 63, 62 pages, arXiv:2210.13959.

[7] Ameur Y., Charlier C., Cronvall J., Free energy and fluctuations in the random normal matrix model with spectral gaps, *Constr. Approx.* **63** (2026), 279–335, arXiv:2312.13904.

[8] Ameur Y., Charlier C., Cronvall J., Lenells J., Exponential moments for disk counting statistics at the hard edge of random normal matrices, *J. Spectr. Theory* **13** (2023), 841–902, arXiv:2207.11092.

[9] Ameur Y., Charlier C., Cronvall J., Lenells J., Disk counting statistics near hard edges of random normal matrices: the multi-component regime, *Adv. Math.* **441** (2024), 109549, 55 pages, arXiv:2210.13962.

[10] Ameur Y., Charlier C., Moreillon P., Eigenvalues of truncated unitary matrices: disk counting statistics, *Monatsh. Math.* **204** (2024), 197–216, arXiv:2305.08976.

[11] Ameur Y., Cronvall J., On fluctuations of Coulomb systems and universality of the Heine distribution, *J. Funct. Anal.* **290** (2026), 111301, 46 pages, arXiv:2411.10288.

[12] Ameur Y., Hedenmalm H., Makarov N., Random normal matrices and Ward identities, *Ann. Probab.* **43** (2015), 1157–1201, arXiv:1109.5941.

[13] Balogh F., Bertola M., Lee S.-Y., McLaughlin K.D.T.-R., Strong asymptotics of the orthogonal polynomials with respect to a measure supported on the plane, *Comm. Pure Appl. Math.* **68** (2015), 112–172, arXiv:1209.6366.

[14] Basor E., Ehrhardt T., Asymptotic formulas for determinants of a special class of Toeplitz + Hankel matrices, in Large Truncated Toeplitz Matrices, Toeplitz Operators, and Related Topics, *Oper. Theory Adv. Appl.*, Vol. 259, Birkhäuser, Cham, 2017, 125–154, arXiv:1603.00506.

[15] Berestycki N., Webb C., Wong M.D., Random Hermitian matrices and Gaussian multiplicative chaos, *Probab. Theory Related Fields* **172** (2018), 103–189, arXiv:1701.03289.

[16] Berezin S., Functional central limit theorems for constrained Mittag-Leffler ensemble in hard edge scaling, *Electron. J. Probab.* **30** (2025), 138, 27 pages, arXiv:2308.12658.

[17] Berezin S., Kuijlaars A.B.J., Parra I., Planar orthogonal polynomials as type I multiple orthogonal polynomials, *SIGMA* **19** (2023), 020, 18 pages, arXiv:2212.06526.

[18] Bertola M., Elias Rebelo J.G., Grava T., Painlevé IV critical asymptotics for orthogonal polynomials in the complex plane, *SIGMA* **14** (2018), 091, 34 pages, arXiv:1802.01153.

[19] Blackstone E., Charlier C., Lenells J., Toeplitz determinants with a one-cut regular potential and Fisher–Hartwig singularities I. Equilibrium measure supported on the unit circle, *Proc. Roy. Soc. Edinburgh Sect. A* **154** (2024), 1431–1472, arXiv:2212.06763.

[20] Burgoin L., Free energy of the Coulomb gas in the determinantal case on Riemann surfaces, arXiv:2508.20598.

[21] Byun S.-S., Planar equilibrium measure problem in the quadratic fields with a point charge, *Comput. Methods Funct. Theory* **24** (2024), 303–332, arXiv:2301.00324.

[22] Byun S.-S., Anomalous free energy expansions of planar Coulomb gases: multi-component and conformal singularity, arXiv:2508.00316.

[23] Byun S.-S., Charlier C., On the characteristic polynomial of the eigenvalue moduli of random normal matrices, *Constr. Approx.* **62** (2025), 471–521, arXiv:2205.04298.

[24] Byun S.-S., Forrester P.J., Progress on the study of the Ginibre ensembles, *KIAS Springer Ser. Math.*, Vol. 3, Springer, Singapore, 2025.

[25] Byun S.-S., Forrester P.J., Kuijlaars A.B.J., Lahiry S., Orthogonal polynomials in the spherical ensemble with two insertions, *SIAM J. Math. Anal.* **58** (2026), 3472–3509, arXiv:2503.15732.

[26] Byun S.-S., Forrester P.J., Lahiry S., Properties of the one-component Coulomb gas on a sphere with two macroscopic external charges, arXiv:2501.05061.

[27] Byun S.-S., Kang N.-G., Seo S.-M., Partition functions of determinantal and Pfaffian Coulomb gases with radially symmetric potentials, *Comm. Math. Phys.* **401** (2023), 1627–1663, arXiv:2210.02799.

[28] Byun S.-S., Kang N.-G., Seo S.-M., Yang M., Free energy of spherical Coulomb gases with point charges, *J. Lond. Math. Soc.* **112** (2025), e70294, 36 pages, arXiv:2501.07284.

[29] Byun S.-S., Lee S.-Y., Yang M., Lemniscate ensembles with spectral singularity, arXiv:2107.07221.

[30] Byun S.-S., Park S., Large gap probabilities of complex and symplectic spherical ensembles with point charges, *J. Funct. Anal.* **290** (2026), 111260, 63 pages, arXiv:2405.00386.

[31] Byun S.-S., Seo S.-M., Yang M., Free energy expansions of a conditional GinUE and large deviations of the smallest eigenvalue of the LUE, *Comm. Pure Appl. Math.* **78** (2025), 2247–2304, arXiv:2402.18983.

[32] Byun S.-S., Yang M., Determinantal Coulomb gas ensembles with a class of discrete rotational symmetric potentials, *SIAM J. Math. Anal.* **55** (2023), 6867–6897, arXiv:2210.04019.

[33] Can T., Laskin M., Wiegmann P.B., Geometry of quantum Hall states: gravitational anomaly and transport coefficients, *Ann. Physics* **362** (2015), 752–794, arXiv:1411.3105.

[34] Charlier C., Asymptotics of Hankel determinants with a one-cut regular potential and Fisher–Hartwig singularities, *Int. Math. Res. Not.* **2019** (2019), 7515–7576, arXiv:1706.03579.

[35] Charlier C., Asymptotics of determinants with a rotation-invariant weight and discontinuities along circles, *Adv. Math.* **408** (2022), 108600, 36 pages, arXiv:2109.03660.

[36] Charlier C., Hole probabilities and balayage of measures for planar Coulomb gases, arXiv:2311.15285.

[37] Charlier C., Large gap asymptotics on annuli in the random normal matrix model, *Math. Ann.* **388** (2024), 3529–3587, arXiv:2110.06908.

[38] Charlier C., Fahs B., Webb C., Wong M.D., Asymptotics of Hankel determinants with a multi-cut regular potential and Fisher–Hartwig singularities, *Mem. Amer. Math. Soc.* **310** (2025), v+138 pages, arXiv:2111.08395.

[39] Charlier C., Lenells J., Exponential moments for disk counting statistics of random normal matrices in the critical regime, *Nonlinearity* **36** (2023), 1593–1616, arXiv:2205.00721.

[40] Charlier C., Moreillon P., On the generating function of the Pearcey process, *Ann. Appl. Probab.* **33** (2023), 3240–3277, arXiv:2107.01859.

[41] Claeys T., Glesner G., Minakov A., Yang M., Asymptotics for averages over classical orthogonal ensembles, *Int. Math. Res. Not.* **2022** (2022), 7922–7966, arXiv:2008.07785.

[42] Claeys T., Krasovsky I., Toeplitz determinants with merging singularities, *Duke Math. J.* **164** (2015), 2897–2987, arXiv:1403.3639.

[43] Dai D., Xu S.-X., Zhang L., On the deformed Pearcey determinant, *Adv. Math.* **400** (2022), 108291, 64 pages, arXiv:2007.12691.

[44] Dai D., Xu S.-X., Zhang L., Gap probability for the hard edge Pearcey process, *Ann. Henri Poincaré* **24** (2023), 2067–2136, arXiv:2204.04625.

[45] Deaño A., McLaughlin K., Molag L., Simm N., Asymptotics for a class of planar orthogonal polynomials and truncated unitary matrices, arXiv:2505.12633.

[46] Deift P., Its A., Krasovsky I., Asymptotics of Toeplitz, Hankel, and Toeplitz+Hankel determinants with Fisher–Hartwig singularities, *Ann. of Math.* **174** (2011), 1243–1299, arXiv:0905.0443.

[47] Deift P., Its A., Krasovsky I., Toeplitz matrices and Toeplitz determinants under the impetus of the Ising model: some history and some recent results, *Comm. Pure Appl. Math.* **66** (2013), 1360–1438, arXiv:1207.4990.

[48] Deift P., Its A., Krasovsky I., On the asymptotics of a Toeplitz determinant with singularities, in Random Matrix Theory, Interacting Particle Systems, and Integrable Systems, *Math. Sci. Res. Inst. Publ.*, Vol. 65, Cambridge University Press, New York, 2014, 93–146, arXiv:1206.1292.

[49] Fahs B., Uniform asymptotics of Toeplitz determinants with Fisher–Hartwig singularities, *Comm. Math. Phys.* **383** (2021), 685–730, arXiv:1909.07362.

[50] Fenzl M., Lambert G., Precise deviations for disk counting statistics of invariant determinantal processes, *Int. Math. Res. Not.* **2022** (2022), 7420–7494, arXiv:2003.07776.

[51] Fisher M.E., Hartwig R.E., Toeplitz determinants: Some applications, theorems, and conjectures, *Adv. Chem. Phys.* **15** (1969), 333–353.

[52] Forrester P.J., Log-gases and random matrices, *London Math. Soc. Monogr. Ser.*, Vol. 34, Princeton University Press, Princeton, NJ, 2010.

[53] Ginibre J., Statistical ensembles of complex, quaternion, and real matrices, *J. Math. Phys.* **6** (1965), 440–449.

[54] Its A., Krasovsky I., Hankel determinant and orthogonal polynomials for the Gaussian weight with a jump, in Integrable Systems and Random Matrices, *Contemp. Math.*, Vol. 458, American Mathematical Society, Providence, RI, 2008, 215–247, arXiv:0706.3192.

[55] Kalvin V., Polyakov–Alvarez type comparison formulas for determinants of Laplacians on Riemann surfaces with conical singularities, *J. Funct. Anal.* **280** (2021), 108866, 44 pages, arXiv:1910.00104.

[56] Klevtsov S., Lowest Landau level on a cone and zeta determinants, *J. Phys. A* **50** (2017), 234003, 16 pages, arXiv:1609.08587.

[57] Krasovsky I.V., Correlations of the characteristic polynomials in the Gaussian unitary ensemble or a singular Hankel determinant, *Duke Math. J.* **139** (2007), 581–619, arXiv:math-ph/0411016.

[58] Lacroix-A-Chez-Toine B., Garzón J.A.M., Calva C.S.H., Castillo I.P., Kundu A., Majumdar S.N., Schehr G., Intermediate deviation regime for the full eigenvalue statistics in the complex Ginibre ensemble, *Phys. Rev. E* **100** (2019), 012137, 10 pages.

[59] Lacroix-A-Chez-Toine B., Majumdar S.N., Schehr G., Rotating trapped fermions in two dimensions and the complex Ginibre ensemble: Exact results for the entanglement entropy and number variance, *Phys. Rev. A* **99** (2019), 021602, 6 pages.

[60] Leblé T., Serfaty S., Large deviation principle for empirical fields of log and Riesz gases, *Invent. Math.* **210** (2017), 645–757, arXiv:1502.02970.

[61] Lee S.-Y., Riser R., Fine asymptotic behavior for eigenvalues of random normal matrices: ellipse case, *J. Math. Phys.* **57** (2016), 023302, 29 pages, arXiv:1501.02781.

[62] Lee S.-Y., Yang M., Discontinuity in the asymptotic behavior of planar orthogonal polynomials under a perturbation of the Gaussian weight, *Comm. Math. Phys.* **355** (2017), 303–338, arXiv:1607.02821.

[63] Lee S.-Y., Yang M., Planar orthogonal polynomials as Type II multiple orthogonal polynomials, *J. Phys. A* **52** (2019), 275202, 14 pages, arXiv:1801.01084.

[64] Lee S.-Y., Yang M., Strong asymptotics of planar orthogonal polynomials: Gaussian weight perturbed by finite number of point charges, *Comm. Pure Appl. Math.* **76** (2023), 2888–2956, arXiv:2003.04401.

[65] Liu J., Yao L., Zhang L., Large gap asymptotics of the hard edge tacnode process, *J. Lond. Math. Soc.* **112** (2025), e70314, 82 pages, arXiv:2412.12920.

[66] Marzo J., Molag L., Ortega-Cerdà J., Universality for fluctuations of counting statistics of random normal matrices, *J. Lond. Math. Soc.* **113** (2026), e70462, 35 pages, arXiv:2508.04386.

[67] Mehta M.L., Random matrices, 3rd ed., *Pure Appl. Math.*, Vol. 142, Elsevier, Amsterdam, 2004.

[68] Olver F.W.J., Lozier D.W., Boisvert R.F., Clark C.W. (Editors), NIST handbook of mathematical functions, Cambridge University Press, Cambridge, 2010.

[69] Saff E.B., Totik V., Logarithmic potentials with external fields, *Grundlehren Math. Wiss.*, Vol. 316, Springer, Berlin, 1997.

[70] Seo S.-M., Edge behavior of two-dimensional Coulomb gases near a hard wall, *Ann. Henri Poincaré* **23** (2022), 2247–2275, arXiv:2010.08818.

[71] Smith N.R., Le Doussal P., Majumdar S.N., Schehr G., Counting statistics for noninteracting fermions in a d-dimensional potential, *Phys. Rev. E* **103** (2021), L030105, 8 pages, arXiv:2008.01045.

[72] Smith N.R., Le Doussal P., Majumdar S.N., Schehr G., Counting statistics for noninteracting fermions in a rotating trap, *Phys. Rev. A* **105** (2022), 043315, 19 pages, arXiv:2112.13355.

[73] Temme N.M., Special functions: An introduction to the classical functions of mathematical physics, Wiley-Intersci. Publ., John Wiley & Sons, Inc., New York, 1996.

[74] Webb C., Wong M.D., On the moments of the characteristic polynomial of a Ginibre random matrix, *Proc. Lond. Math. Soc.* **118** (2019), 1017–1056, arXiv:1704.04102.

[75] Wünsche A., Generalized Hermite polynomials associated with functions of parabolic cylinder, *Appl. Math. Comput.* **141** (2003), 197–213, arXiv:1603.07657.

[76] Xia J., Hao Y.-F., Xu S.-X., Zhang L., Zhao Y.-Q., Asymptotics of the deformed higher order Airy-kernel determinants and applications, *Nonlinearity* **36** (2023), 4384–4424, arXiv:2306.14835.

[77] Xu S.-X., Zhao S.-Q., Zhao Y.-Q., On the Fredholm determinant of the confluent hypergeometric kernel with discontinuities, *Phys. D* **461** (2024), 134101, 23 pages, arXiv:2402.11214.

[78] Yao L., Zhang L., On the gap probability of the tacnode process, *Adv. Math.* **438** (2024), 109474, 85 pages, arXiv:2307.05622.

[79] Zabrodin A., Wiegmann P., Large-N expansion for the 2D Dyson gas, *J. Phys. A* **39** (2006), 8933–8963, arXiv:hep-th/0601009.