

Integral Points and Volume of Integral-Integral Affine Manifolds

Oded ELISHA ^a, Yael KARSHON ^a and Yiannis LOIZIDES ^b

^{a)} School of Mathematical Sciences, Tel Aviv University, Israel

E-mail: odedelisha@mail.tau.ac.il, yaelkarshon@tauex.tau.ac.il

^{b)} Department of Mathematical Sciences, George Mason University, USA

E-mail: yloizide@gmu.edu

Received February 17, 2026, in final form October 11, 2026; Published online October 29, 2026

<https://doi.org/10.3842/SIGMA.2026.102>

Abstract. Using rational Ehrhart theory and Fourier analysis, we give an elementary proof that the volume of a closed manifold equipped with an integral-integral affine structure equals the number of its integral points.

Key words: integral affine structure; Ehrhart theory

2020 Mathematics Subject Classification: 53C15; 52B20; 53D50

1 Introduction

On an n -dimensional topological manifold M , an atlas whose transition maps are locally of the form

$$x \mapsto Ax + b \quad \text{with } A \in \mathrm{GL}(n, \mathbb{Z}) \text{ and } b \in \mathbb{Z}^n \quad (1.1)$$

defines an *integral-integral affine structure* [10] (also known as a *strongly integral affine structure* [16]). This structure determines a (smooth) manifold structure on M , a measure on M , and a subset $M_{\mathbb{Z}} \subset M$ of *integral points*. For details see Definition 3.2, Remarks 3.3 and 3.4, and Definition 3.5 below.

For example, if Γ is a group of transformations of $\mathbb{R}^n$ of the form (1.1) whose action on $\mathbb{R}^n$ is free and proper, then the quotient $M := \mathbb{R}^n/\Gamma$ inherits an integral-integral affine structure. We do not know if every closed integral-integral affine manifold is of this form; this is a special case of the Markus conjecture [13] (see the discussion in [7, Chapter 11]).

When M is closed, its volume (total measure) is a positive real number and its number of integral points is a non-negative integer. In this paper, we give an elementary proof of the following result, first proved in [10, Theorem 3.1].

Theorem 1.1. *Let M be a closed integral-integral affine manifold. Then its number of integral points is equal to its volume.*

The motivation behind Theorem 1.1 comes from geometric quantization: this theorem is a key ingredient in an instance of the “independence of polarization” phenomenon for Lagrangian torus fibrations (see [10, Section 2]). In [10], Theorem 1.1 appeared as the “downstairs theorem”; its proof was surprisingly non-trivial and was inspired by ideas that were sketched in papers on geometric quantization and on mirror symmetry (see [1, proof of Theorem 4.1] and [8, Conjecture 4.9]). The proof involved intersection theory in a bundle of tori over the integral-integral affine manifold and required a subtle cohomological argument [10, Theorem 7.6].

In this paper, we provide a more elementary proof of Theorem 1.1 that relies on rational Ehrhart theory in $\mathbb{R}^n$ [2, 6] and on Fourier analysis. Here is a sketch of our argument. We introduce the *lattice-counting* function, which associates to each $m \in \mathbb{N}$ the number $L_M(m)$ of points in M whose coordinates are in $m^{-1}\mathbb{Z}$. By rational Ehrhart theory, for a rational convex polytope P in $\mathbb{R}^n$, the number of points in $P \cap m^{-1}\mathbb{Z}^n$ is quasi-polynomial in m (see Definition 2.8). For a closed integral-integral affine manifold M , we do not know how to decompose M into rational polytopes; however, we can cover M by overlapping rational polytopes and apply an inclusion-exclusion argument, to conclude that $L_M(m)$ is quasi-polynomial in m . On the other hand, by “decomposing” M using a smooth partition of unity supported in charts, we conclude that $L_M(m) - \text{vol}(M)m^n = o(1)$ as $m \rightarrow \infty$. From these two results, we conclude that $L_M(m) = \text{vol}(M)m^n$ on the nose (Theorem 4.4), yielding Theorem 1.1 as the case $m = 1$.

2 Rational polytopes

In this section, we introduce rational polytopes and the rational Ehrhart theorem.

Notation 2.1. The set of natural numbers is $\mathbb{N} := \{1, 2, \dots\}$.

Notation 2.2. We denote by $\text{Aff}_{\mathbb{Z}}(\mathbb{R}^n)$ the set of maps $\mathbb{R}^n \rightarrow \mathbb{R}^n$ of the form $x \mapsto Ax + b$ with $A \in \text{GL}(n, \mathbb{Z})$ and $b \in \mathbb{Z}^n$.

Definition 2.3. A set $P \subseteq \mathbb{R}^n$ is called a *rational polytope* if there exist $v_1, \dots, v_k \in \mathbb{Q}^n$ such that $P = \text{conv}\{v_1, \dots, v_k\} = \{\sum_{i=1}^k \alpha_i v_i \mid \alpha_i \geq 0, \sum_{i=1}^k \alpha_i = 1\}$.

Definition 2.4. Let $P \subseteq \mathbb{R}^n$ be a rational polytope. A point $x \in P$ is called a *vertex* of P if x is an extreme point of P .

Remark 2.5. If $P = \text{conv}\{v_1, \dots, v_k\}$, then the set of vertices of P is a subset of $\{v_1, \dots, v_k\}$.

Proposition 2.6. Let $P \subseteq \mathbb{R}^n$ be a rational polytope, and let $f \in \text{Aff}_{\mathbb{Z}}(\mathbb{R}^n)$. Then $f(P)$ is a rational polytope.

Proof. Let $P = \text{conv}\{q_1, \dots, q_k\}$ for $q_i \in \mathbb{Q}^n$. Let $A \in \text{GL}_n(\mathbb{Z})$ and $b \in \mathbb{Z}^n$ be such that $f(x) = b + Ax$. Since f is an affine map, $f(P) = \text{conv}\{f(q_1), \dots, f(q_k)\} = \text{conv}\{b + Aq_1, \dots, b + Aq_k\}$. Because $A \in \text{GL}_n(\mathbb{Z})$ and $b \in \mathbb{Z}^n$, we have $b + Aq_i \in \mathbb{Q}^n$ for all $1 \leq i \leq k$. Thus, $f(P)$ is a rational polytope. ■

Proposition 2.7. Let $P_1, P_2 \subseteq \mathbb{R}^n$ be rational polytopes. Then $P_1 \cap P_2$ is either empty or a rational polytope.

Proof. See [5, Proposition 1.69]. ■

Definition 2.8. A function $f: \mathbb{Z} \rightarrow \mathbb{R}$ is *quasi-polynomial* if there exist $n \in \mathbb{N} \cup \{0\}$ and functions $\{a_k: \mathbb{Z} \rightarrow \mathbb{R}\}_{k=0}^n$ such that

- each a_k is periodic with integer period, and
- $f(m) = \sum_{k=0}^n a_k(m)m^k$, for all $m \in \mathbb{Z}$.

We say that a function $p: \mathbb{N} \rightarrow \mathbb{R}$ is quasi-polynomial if it is the restriction to $\mathbb{N}$ of a quasi-polynomial function $f: \mathbb{Z} \rightarrow \mathbb{R}$.

Proposition 2.9. Let $p: \mathbb{N} \rightarrow \mathbb{R}$ be a quasi-polynomial function such that $p(m) = o(1)$ as $m \rightarrow \infty$. Then $p \equiv 0$.

Proof. By assumption, we can write

$$p(m) = \sum_{k=0}^n a_k(m) m^k$$

for all $m \in \mathbb{N}$, where each a_k is a periodic function with some integer period d_k . Let $d = \prod_{k=0}^n d_k$. Then each a_k is a periodic function with integer period d . For each $1 \leq i \leq d$, the function $g_i(m) := p(i+md)$ is then (the restriction to $\mathbb{N} \cup \{0\}$ of) a polynomial in m such that $g_i(m) = o(1)$ as $m \rightarrow \infty$. So $g_i \equiv 0$ for every $1 \leq i \leq d$, and hence $p \equiv 0$. ■

Definition 2.10. For a rational polytope $P \subseteq \mathbb{R}^n$, we define its *lattice-counting function* by

$$L_P: \mathbb{N} \rightarrow \mathbb{N} \cup \{0\}, \quad L_P(m) := \#((m^{-1}\mathbb{Z}^n) \cap P) = \#\{x \in P \mid mx \in \mathbb{Z}^n\}.$$

Proposition 2.11 (rational Ehrhart). *Let $P \subseteq \mathbb{R}^n$ be a rational polytope. Then its lattice-counting function is quasi-polynomial.*

Proof. See [2, Proposition 3.23]. ■

3 Integral-integral affine geometry

In this section, we define integral-integral affine manifolds, their integral points, and rational polytopes in them. We then prove that every closed integral-integral affine manifold can be covered by a finite collection of rational polytopes whose intersections are either empty or rational polytopes; see Corollary 3.15.

Definition 3.1. A map $f: \Omega \rightarrow \mathbb{R}^n$ on an open subset $\Omega \subseteq \mathbb{R}^n$ is called *locally integral-integral affine* if each point has a neighbourhood on which the map agrees with an element of $\text{Aff}_{\mathbb{Z}}(\mathbb{R}^n)$.

Definition 3.2. Let M be a topological n -dimensional manifold. An atlas

$$\{\varphi_i: U_i \rightarrow \Omega_i \subseteq \mathbb{R}^n\}_{i \in I}$$

for M is called an *integral-integral affine atlas* if $\varphi_j \circ \varphi_i^{-1}$ is locally integral-integral affine on $\varphi_i(U_i \cap U_j)$ for every $i, j \in I$. An *integral-integral affine structure* on M is an integral-integral affine atlas that is maximal (with respect to inclusion of integral-integral affine atlases). An *integral-integral affine manifold* is a manifold equipped with an integral-integral affine structure. An *integral chart* is an element of the maximal integral-integral affine atlas of M .

Remark 3.3. Every integral-integral affine atlas is contained in a unique maximal integral-integral affine atlas. This follows from the fact that locally integral-integral affine homeomorphisms between open subsets of $\mathbb{R}^n$ form a pseudogroup.

Remark 3.4. Let M be an n -dimensional integral-integral affine manifold. Since locally integral-integral affine maps preserve the standard Lebesgue measure on $\mathbb{R}^n$, there exists a unique (Borel) measure on M such that each integral chart is measure preserving. The (affine) volume $\text{vol}(M)$ is the measure of M and integration on M is done with respect to this measure.

Definition 3.5. Let M be an n -dimensional integral-integral affine manifold. Its set of *integral points* is

$$M_{\mathbb{Z}} := \{x \in M \mid \text{there exists an integral chart } \psi: U \rightarrow \Omega \subseteq \mathbb{R}^n \text{ such that } x \in U \text{ and } \psi(x) \in \mathbb{Z}^n\}.$$

Moreover, for $m \in \mathbb{N}$, denote

$$M_{m^{-1}\mathbb{Z}} := \{x \in M \mid \text{there exists an integral chart } \psi: U \rightarrow \Omega \subseteq \mathbb{R}^n \text{ such that } x \in U \text{ and } \psi(x) \in m^{-1}\mathbb{Z}^n\}.$$

Lemma 3.6. *For an n -dimensional integral-integral affine manifold M , and for $m \in \mathbb{N}$,*

$$M_{m^{-1}\mathbb{Z}} = \{x \in M \mid \text{for every integral chart } \phi: U \rightarrow \Omega \subseteq \mathbb{R}^n \\ \text{such that } x \in U, \text{ we have } \phi(x) \in m^{-1}\mathbb{Z}^n\}.$$

Proof. Let $x \in M_{m^{-1}\mathbb{Z}}$. So there exists an integral chart $\psi: U' \rightarrow \Omega' \subseteq \mathbb{R}^n$ whose domain contains x and such that $\psi(x) \in m^{-1}\mathbb{Z}^n$. Let $\phi: U \rightarrow \Omega \subseteq \mathbb{R}^n$ be another integral chart whose domain contains x . By compatibility, there exists a neighbourhood of $\psi(x)$ on which the composition $\phi \circ \psi^{-1}$ has the form $y \mapsto b + Ay$ for $b \in \mathbb{Z}^n$ and $A \in \text{GL}_n(\mathbb{Z})$. Thus $m\phi(x) = m\phi(\psi^{-1}(\psi(x))) = m(b + A(\psi(x))) = mb + A(m\psi(x)) \in \mathbb{Z}^n$. ■

Definition 3.7. Let M be an integral-integral affine manifold. Its *lattice-counting function* is

$$L_M: \mathbb{N} \rightarrow \mathbb{N} \cup \{0, \infty\}, \quad L_M(m) := \#M_{m^{-1}\mathbb{Z}}.$$

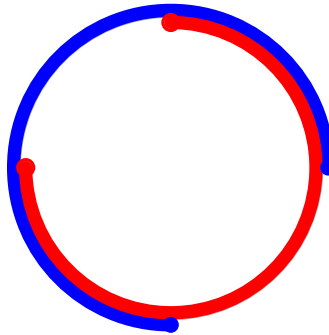
Proposition 3.8. *Let M be a closed n -dimensional integral-integral affine manifold. Then its lattice-counting function is finite: $L_M(m) < \infty$ for every $m \in \mathbb{N}$.*

Proof. Assume not; let $(x_i)_{i=0}^\infty$ be a sequence of distinct points in $M_{m^{-1}\mathbb{Z}}$. Since M is compact, this sequence has a converging subsequence $(x_{i_j})_{j=0}^\infty \rightarrow x^*$. Let $\phi: U \rightarrow \Omega \subseteq \mathbb{R}^n$ be an integral chart such that $x^* \in U$. Then for some $N > 0$, we have $x_{i_j} \in U$ for all $j \geq N$. Then $\phi(x_{i_j})$ is a convergent sequence of distinct elements of $m^{-1}\mathbb{Z}^n$, which is a contradiction. ■

In order to use Ehrhart theory to study the lattice-counting function of an integral-integral affine manifold M , we cover M by rational polytopes. However, if we are not careful, nonempty intersections of polytopes in the cover may be disconnected.

Definition 3.9. A subset $P \subseteq M$ of an n -dimensional integral-integral affine manifold M is called a *rational polytope* if there exists an integral chart ϕ defined in a neighbourhood of P such that $\phi(P) \subseteq \mathbb{R}^n$ is a rational polytope.

Example 3.10. In the integral-integral affine manifold $\mathbb{R}/4\mathbb{Z}$, the images of $[0, 3]$ and of $[2, 5]$ are rational polytopes, and their intersection is disconnected.



In the definition of an integral-integral affine structure, the transition maps are *locally* integral-integral affine (see Remark 3.3). In Lemma 3.12 below, we use a transition map that is *globally* integral-integral affine.

Proposition 3.11. *Let $f: \Omega \rightarrow \mathbb{R}^n$ be a locally integral-integral affine map on an open subset Ω of $\mathbb{R}^n$. If Ω is connected, then f is the restriction to Ω of some element of $\text{Aff}_{\mathbb{Z}}(\mathbb{R}^n)$.*

Proof. Because $A := \text{df}|_x \in \text{GL}_n(\mathbb{Z})$ is locally constant, it is constant. So $b := f(x) - Ax \in \mathbb{Z}^n$ is locally constant, hence constant. ■

Lemma 3.12. *Let M be an n -dimensional integral-integral affine manifold, $P \subseteq M$ a rational polytope and $\psi: W \rightarrow \Omega \subseteq \mathbb{R}^n$ any integral chart such that $P \subseteq W$. Then $\psi(P) \subseteq \mathbb{R}^n$ is a rational polytope.*

Proof. By definition, there is an integral chart $\phi: U \rightarrow \Omega' \subseteq \mathbb{R}^n$ such that $P \subseteq U$ and $\phi(P)$ is a rational polytope. In particular, $\phi(P)$ is connected. Let $\mathcal{O}$ be the connected component of $\phi(U \cap W)$ that contains $\phi(P)$. By Proposition 3.11, there exists $f \in \text{Aff}_{\mathbb{Z}}(\mathbb{R}^n)$ that coincides on $\mathcal{O}$ with $\psi \circ \phi^{-1}$. By Proposition 2.6, $\psi(P) = f(\phi(P))$ is a rational polytope. ■

Lemma 3.13. *Let M be an integral-integral affine manifold and $P_1, P_2 \subseteq M$ rational polytopes, and let $\phi_1: U_1 \rightarrow \Omega_1 \subseteq \mathbb{R}^n$ and $\phi_2: U_2 \rightarrow \Omega_2 \subseteq \mathbb{R}^n$ be integral charts such that $P_1 \subseteq U_1$ and $P_2 \subseteq U_2$. If $U_1 \cap U_2$ is connected, then $P_1 \cap P_2$ is either a rational polytope or is empty.*

Proof. By Lemma 3.12, $\phi_1(P_1)$ and $\phi_2(P_2)$ are rational polytopes. The intersection $U_1 \cap U_2$ is connected, so by Proposition 3.11, the composition $\phi_1 \circ \phi_2^{-1}$ has the form $x \mapsto Ax + b$ where $b \in \mathbb{Z}^n$ and $A \in \text{GL}_n(\mathbb{Z})$. By Propositions 2.7 and 2.6, the set

$$Q := \phi_1(P_1) \cap (b + A\phi_2(P_2)),$$

is a rational polytope. Moreover,

$$\phi_1(P_1 \cap P_2) = \phi_1(P_1 \cap P_2 \cap U_1 \cap U_2) = \phi_1(P_1) \cap \phi_1(P_2 \cap U_1 \cap U_2),$$

and

$$\begin{aligned} \phi_1(P_2 \cap U_1 \cap U_2) &= (\phi_1 \circ \phi_2^{-1})(\phi_2(P_2 \cap U_1 \cap U_2)) = b + A(\phi_2(P_2) \cap \phi_2(U_1 \cap U_2)) \\ &= b + (A(\phi_2(P_2)) \cap A\phi_2(U_1 \cap U_2)) \\ &= (b + A\phi_2(P_2)) \cap (b + A\phi_2(U_1 \cap U_2)). \end{aligned}$$

Therefore,

$$\phi_1(P_1 \cap P_2) = (\phi_1(P_1) \cap (b + A\phi_2(P_2))) \cap (b + A\phi_2(U_1 \cap U_2)) = Q \cap (b + A\phi_2(U_1 \cap U_2)).$$

The set $\phi_1(P_1 \cap P_2)$ is closed in Q , because P_1 and P_2 are compact. It is also open in Q , because $b + A\phi_2(U_1 \cap U_2)$ is open in $\mathbb{R}^n$. Because Q is (convex, hence) connected, $\phi_1(P_1 \cap P_2)$ is either all of Q or is empty. ■

By repeated application of Lemma 3.13, we obtain the following corollary.

Corollary 3.14. *Let M be an integral-integral affine manifold, let $\{P_i\}_{i \in I}$ be a finite collection of rational polytopes in M , and let $\{\phi_i: U_i \rightarrow \Omega_i \subseteq \mathbb{R}^n\}_{i \in I}$ be integral charts such that $P_i \subseteq U_i$ for every $i \in I$. Suppose that, for every subset $I' \subseteq I$, the intersection $\bigcap_{i \in I'} U_i$ is connected. Then $\bigcap_{i \in I} P_i$ is either a rational polytope or is empty.*

Corollary 3.15. *Let M be a closed integral-integral affine manifold. Then there exists a finite collection $\{P_i\}_{i=1}^k$ of rational polytopes such that*

- (1) $M = \bigcup_{i=1}^k P_i$,
- (2) for every $I \subseteq \{1, \dots, k\}$, the intersection $\bigcap_{i \in I} P_i$ is either a rational polytope or is empty.

Proof. Recall that a *good cover* of M is an open cover where every nonempty finite intersection is diffeomorphic to $\mathbb{R}^n$, and that every open cover of M has a refinement that is a good open cover of M [4, Propositions 5.1 and 5.2].

Thus, there exists a good open cover $\mathcal{W}$ of M such that every element of $\mathcal{W}$ is contained in the domain of an integral chart.

For each point $x \in M$, let W_x be an element of $\mathcal{W}$ that contains x , let $\varphi: U \rightarrow \Omega$ be an integral chart whose domain contains W_x , and let $\tilde{Q}_x \subseteq \mathbb{R}^n$ be a closed cube with rational vertices that is contained in $\varphi(W_x)$ and whose interior contains $\varphi(x)$. Then $Q_x := \varphi^{-1}(\tilde{Q}_x)$ is a rational polytope in M .

Because M is compact, there exist $x_1, \dots, x_k \in M$ such that $M = \bigcup_{i=1}^k \text{Int}(Q_{x_i})$.

For each $i \in \{1, \dots, k\}$, let $P_i := Q_{x_i}$. Then $M = \bigcup_{i=1}^k P_i$. Because $\mathcal{W}$ is a good cover, for each $I' \subseteq \{1, \dots, k\}$, the intersection $\bigcap_{i \in I'} W_{x_i}$ is connected. By Corollary 3.14, each intersection $\bigcap_{i \in I} P_i$ is either a rational polytope or is empty. ■

4 Main result

In this section, we calculate the lattice-counting function of a closed integral-integral affine manifold; see Theorem 4.4.

Proposition 4.1 (Ehrhart theorem for closed integral-integral affine manifolds). *The lattice-counting function of a closed integral-integral affine manifold is quasi-polynomial.*

Proof. Let M be a closed n -dimensional integral-integral affine manifold and let $\{P_i\}_{i=1}^k$ be a rational polytope cover from Corollary 3.15. For each set of indices $I \subseteq \{1, \dots, k\}$, denote $P_I := \bigcap_{i \in I} P_i$. By inclusion-exclusion,

$$L_M(m) = \sum_{x \in M_{m^{-1}\mathbb{Z}}} 1 = \sum_{j=1}^k (-1)^{j+1} \sum_{\substack{I \subseteq \{1, \dots, k\} \\ |I|=j}} \#(M_{m^{-1}\mathbb{Z}} \cap P_I).$$

For every $I \subseteq \{1, \dots, k\}$, take an integral chart $\psi: U \rightarrow \Omega \subseteq \mathbb{R}^n$ such that $P_I \subseteq U$. By Lemma 3.12, $\psi(P_I)$ is a rational polytope. By Lemma 3.6,

$$\#(M_{m^{-1}\mathbb{Z}} \cap P_I) = \#(m^{-1}\mathbb{Z}^n \cap \psi(P_I)).$$

By this and Proposition 2.11, L_M is an alternating sum of quasi-polynomials, hence a quasi-polynomial. ■

The fact that the lattice-counting function is quasi-polynomial allows us to use asymptotic methods.

Proposition 4.2. *Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be a compactly supported smooth function. Then*

$$\sum_{x \in m^{-1}\mathbb{Z}^n} f(x) = m^n \int_{\mathbb{R}^n} f(x) dx + o(1)$$

as $m \rightarrow \infty$.

Proof. Because f is a compactly supported smooth function, it is a Schwartz function. Because the Fourier transform maps Schwartz functions to Schwartz functions [17, p. 181, Corollary 2.2], the Fourier transform

$$\hat{f}: \mathbb{R}^n \rightarrow \mathbb{C},$$

defined by

$$\hat{f}(\zeta) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \zeta \rangle} dx,$$

is also a Schwartz function. In particular, for every polynomial $P \in \mathbb{R}[y_1, \dots, y_n]$,

$$\sup_{y \in \mathbb{R}^n} |\hat{f}(y)P(y)| < \infty.$$

Hence, the series $\sum_{y \in m\mathbb{Z}^n} \hat{f}(y)$ is absolutely convergent.

By the Poisson summation formula [15, p. 88, Theorem 3.44],

$$\sum_{x \in m^{-1}\mathbb{Z}^n} f(x) = m^n \sum_{y \in m\mathbb{Z}^n} \hat{f}(y) = m^n \left(\int_{\mathbb{R}^n} f(x) dx + \sum_{0 \neq y \in m\mathbb{Z}^n} \hat{f}(y) \right). \quad (4.1)$$

Thus we would like to prove that

$$\sum_{0 \neq y \in m\mathbb{Z}^n} \hat{f}(y) = o(m^{-n}) \quad (4.2)$$

as $m \rightarrow \infty$. (Although $\hat{f}$ is complex-valued, by (4.1) the sum in (4.2) is real.)

For any positive integers $r \geq m$, set

$$A_r^m := \{y \in m\mathbb{Z}^n \mid r \leq |y| < r+1\}.$$

Then

$$\#A_r^m \leq \#\{y \in \mathbb{Z}^n \mid |y_j| \leq r \text{ for all } j\} \leq (2r+1)^n \leq (3r)^n,$$

and so, for any integer N such that $2N-2 \geq n$,

$$\sum_{y \in A_r^m} \frac{1}{|y|^{2N}} \leq (3r)^n \frac{1}{r^{2N}} = 3^n \cdot \frac{r^n}{r^{2N-2}} \cdot \frac{1}{r^2} \leq 3^n \cdot m^{n+2-2N} \cdot \frac{1}{r^2}. \quad (4.3)$$

Because

$$\bigcup_{r=m}^{\infty} A_r^m = m\mathbb{Z}^n \setminus \{0\},$$

we have

$$\begin{aligned} \left| \sum_{0 \neq y \in m\mathbb{Z}^n} \hat{f}(y) \right| &\leq \sum_{0 \neq y \in m\mathbb{Z}^n} |\hat{f}(y)| = \sum_{0 \neq y \in m\mathbb{Z}^n} |\hat{f}(y)| |y|^{2N} \left| \frac{1}{|y|^{2N}} \right| \\ &\stackrel{\hat{f} \text{ is Schwartz}}{\leq} C \cdot \sum_{0 \neq y \in m\mathbb{Z}^n} \frac{1}{|y|^{2N}} = C \cdot \sum_{r=m}^{\infty} \sum_{y \in A_r^m} \frac{1}{|y|^{2N}} \stackrel{(4.3)}{\leq} m^{n+2-2N} \cdot 3^n C \cdot \sum_{r=m}^{\infty} \frac{1}{r^2} \\ &= o(m^{n+2-2N}) \text{ as } m \rightarrow \infty. \end{aligned}$$

To obtain (4.2), take $N = n+1$. ■

Remark 4.3. Although we do not require it, we have actually shown that

$$\sum_{x \in m^{-1}\mathbb{Z}^n} f(x) = m^n \int_{\mathbb{R}^n} f(x) dx + o(m^{-\infty})$$

as $m \rightarrow \infty$.

Combining the above asymptotics with the fact that L_M is quasi-polynomial (Proposition 4.1), we obtain our main result.

Theorem 4.4. *Let M be an n -dimensional closed integral-integral affine manifold. Then its lattice-counting function is*

$$L_M(m) = \text{vol}(M)m^n.$$

Proof. Because M is locally compact, it has an integral-integral affine atlas whose chart domains have compact closures in M . Because M is compact, it has such an atlas that is finite. Let $\{\rho_i\}_{i=1}^k$ be a partition of unity that is subordinate to the cover of M by the domains of the charts in such an atlas. For each $i \in \{1, \dots, k\}$, let $\psi_i: U_i \rightarrow \Omega_i \subseteq \mathbb{R}^n$ be an integral chart whose domain contains the support $\text{supp } \rho_i$ of ρ_i in M . Then the subset $\psi_i(\text{supp } \rho_i)$ of Ω_i is compact, so $\rho_i \circ \psi_i^{-1}$ extends by zero to a compactly supported smooth function on $\mathbb{R}^n$. By Proposition 4.2, we have (see Remark 3.4)

$$\begin{aligned} \sum_{x \in M_{m^{-1}\mathbb{Z}}} \rho_i(x) &= \sum_{y \in m^{-1}\mathbb{Z}^n \cap \psi_i(U_i)} \rho_i(\psi_i^{-1}(y)) = m^n \int_{\psi_i(U_i)} (\rho_i \circ \psi_i^{-1})(y) dy + o(1) \\ &= m^n \int_M \rho_i(x) dx + o(1). \end{aligned}$$

Thus

$$\begin{aligned} L_M(m) &= \#M_{m^{-1}\mathbb{Z}} = \sum_{x \in M_{m^{-1}\mathbb{Z}}} 1 = \sum_{x \in M_{m^{-1}\mathbb{Z}}} \sum_{i=1}^k \rho_i(x) = \sum_{i=1}^k \sum_{x \in M_{m^{-1}\mathbb{Z}}} \rho_i(x) \\ &= \sum_{i=1}^k \left(m^n \int_M \rho_i(x) dx + o(1) \right) = m^n \left(\sum_{i=1}^k \int_M \rho_i(x) dx \right) + o(1) \\ &= \text{vol}(M)m^n + o(1). \end{aligned}$$

By Proposition 4.1, L_M is quasi-polynomial; by Proposition 2.9, we obtain the desired equality. ■

Remark 4.5. Asymptotic expansions of a similar nature appeared in [3, 9, 12]. The idea to take advantage of quasi-polynomial behavior to show that an asymptotic result is exact appeared in [14]. In the future, it might be interesting to seek generalizations to integral-integral affine manifolds of Euler–Maclaurin formulas relating sums to integrals [3, 11].

Finally, the case $m = 1$ of Theorem 4.4 yields Theorem 1.1.

Acknowledgements

We are grateful to Roei Raveh for providing the idea of the proof of Proposition 4.2. We thank the anonymous referees for their helpful comments on an earlier version of this paper.

Y. Karshon’s research is partly funded by the Natural Sciences and Engineering Research Council of Canada Discovery Grant RGPIN-2024-05798 and by the United States – Israel Binational Science Foundation Grant 2021730. O. Elisha is partly supported by Yaron Ostrover’s ISF Grant 938/22.

References

- [1] Andersen J.E., Geometric quantization of symplectic manifolds with respect to reducible non-negative polarizations, *Comm. Math. Phys.* **183** (1997), 401–421.
- [2] Beck M., Robins S., Computing the continuous discretely: Integer-point enumeration in polyhedra, 2nd ed., *Undergrad. Texts Math.*, Springer, New York, 2015.

- [3] Berline N., Vergne M., Local asymptotic Euler–Maclaurin expansion for Riemann sums over a semi-rational polyhedron, in Configuration Spaces, *Springer INdAM Ser.*, Vol. 14, [Springer](#), Cham, 2016, 67–105, [arXiv:1502.01671](#).
- [4] Bott R., Tu L.W., Differential forms in algebraic topology, *Grad. Texts in Math.*, Vol. 82, [Springer](#), New York, 1982.
- [5] Bruns W., Gubeladze J., Polytopes, rings, and K -theory, *Springer Monogr. Math.*, [Springer](#), Dordrecht, 2009.
- [6] Ehrhart E., Sur les polyèdres rationnels homothétiques à n dimensions, *C. R. Acad. Sci. Paris* **254** (1962), 616–618.
- [7] Goldman W.M., Geometric structures on manifolds, *Grad. Stud. Math.*, Vol. 227, [American Mathematical Society](#), Providence, RI, 2022.
- [8] Gross M., Special Lagrangian fibrations. I. Topology, in Integrable Systems and Algebraic Geometry (Kobe/Kyoto, 1997), World Scientific Publishing, River Edge, NJ, 1998, 156–193, [arXiv:alg-geom/9710006](#).
- [9] Guillemin V., Sternberg S., Riemann sums over polytopes, *Ann. Inst. Fourier (Grenoble)* **57** (2007), 2183–2195, [arXiv:math/0608171](#).
- [10] Hamilton M.D., Karshon Y., Yoshida T., Integral-integral affine geometry, geometric quantization, and Riemann–Roch, *J. Geom. Phys.* **222** (2026), 105745, 17 pages, [arXiv:2411.10348](#).
- [11] Karshon Y., Sternberg S., Weitsman J., Euler–Maclaurin with remainder for a simple integral polytope, *Duke Math. J.* **130** (2005), 401–434, [arXiv:math/0307125](#).
- [12] Loizides Y., Paradan P.E., Vergne M., Semi-classical analysis of piecewise quasi-polynomial functions and applications to geometric quantization, *Indag. Math. (N.S.)* **32** (2021), 151–192, [arXiv:1907.01428](#).
- [13] Markus L., Cosmological models in differential geometry, University of Minnesota, 1963.
- [14] Meinrenken E., On Riemann–Roch formulas for multiplicities, *J. Amer. Math. Soc.* **9** (1996), 373–389, [arXiv:alg-geom/9405014](#).
- [15] Robins S., Fourier analysis on polytopes and the geometry of numbers. Part I. A friendly introduction, *Stud. Math. Libr.*, Vol. 107, [American Mathematical Society](#), Providence, RI, 2024.
- [16] Sepe D., Universal Lagrangian bundles, *Geom. Dedicata* **165** (2013), 53–93, [arXiv:1304.2855](#).
- [17] Stein E.M., Shakarchi R., Fourier analysis. An introduction, *Princeton Lect. Anal.*, Vol. 1, Princeton University Press, Princeton, NJ, 2003.