Symmetry, Integrability and Geometry: Methods and Applications SIGMA **22** (2026), 062, 50 pages

Affine Jacobi–Trudi Identities and q,t-Rogers–Ramanujan Identities

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Received November 24, 2025, in final form June 05, 2026; Published online June 25, 2026
https://doi.org/10.3842/SIGMA.2026.062

Abstract. We conjecture affine or Hall–Littlewood analogues of the dual Jacobi–Trudi identities for orthogonal and symplectic Schur functions indexed by rectangular partitions of maximal height. These conjectures are then used to derive t-analogues of many known Rogers–Ramanujan identities for the characters of standard modules of affine Lie algebras. This includes t-analogues of the classical Rogers–Ramanujan identities, (some of) the Andrews–Gordon identities and the $\mathrm{C}_n^{(1)}$, $\mathrm{A}_{2n}^{(2)}$ and $\mathrm{D}_{n+2}^{(2)}$ GOW identities. We also prove an affine analogue of the dual Jacobi–Trudi identity for Schur functions indexed by rectangular partitions of arbitrary height.

Key words: affine root systems; character formulas for standard modules; cylindric Schur functions; Hall–Littlewood polynomials; Jacobi–Trudi identities; Rogers–Ramanujan identities; theta function identities

2020 Mathematics Subject Classification: 05E05; 05E10; 11P84; 17B67; 33D15; 33D52

This paper is dedicated to Jim Lepowsky, a pioneer in the study of Rogers–Ramanujan-type identities from the point of view of representation theory.

1 Introduction

1.1 The Jacobi–Trudi and dual Jacobi–Trudi identities

A partition λ of length $l(\lambda)$ equal to ℓ is a weakly decreasing sequence $(\lambda_1,\lambda_2,\dots)$ of nonnegative integers such that $\lambda_i>0$ for $i\leqslant\ell$ and $\lambda_i=0$ for $i>\ell$. For $x=(x_1,\dots,x_n)$ and λ a partition, the Schur function $s_\lambda(x)$ is defined as

$$s_\lambda(x)=\frac{\det_{1\leqslant i,j\leqslant n}\bigl(x_i^{\lambda_j+n-j}\bigr)}{\prod_{1\leqslant i<j\leqslant n}(x_i-x_j)}$$

if $l(\lambda)\leqslant n$, while $s_\lambda(x)=0$ if $l(\lambda)>n$. It is a standard result in the theory of symmetric functions that $s_\lambda(x)$ for $l(\lambda)\leqslant n$ corresponds to the character of the polynomial representation of $\mathrm{GL}(n,\mathbb{C})$ indexed by λ and admits the combinatorial description

$$s_\lambda(x)=\sum_{T\in\mathrm{SSYT}_n(\lambda)}x^T. \tag{1.1}$$

Here $\mathrm{SSYT}_n(\lambda)$ is the set of semistandard Young tableaux of shape λ on $\{1,\dots,n\}$, and $x^T:=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, where α_i is the number of boxes or squares of T with filling i.

This paper is a contribution to the Special Issue on Recent Advances in Vertex Operator Algebras in honor of James Lepowsky. The full collection is available at https://sigma-journal.com/Lepowsky.html

The complete and elementary symmetric functions h_r and e_r are the Schur functions for partitions whose Young diagram consists of a single row or column of r boxes, respectively. That is,

$$h_r = s_{(r)} \qquad \text{and} \qquad e_r = s_{\underbrace{(1,\dots,1)}_{r\text{ times}}} = s_{(1^r)},$$

or, more explicitly, $h_0 = e_0 = 1$ and

$$h_r(x) = \sum_{1\leqslant i_1\leqslant i_2\leqslant\cdots\leqslant i_r\leqslant n} x_{i_1}\cdots x_{i_r}, \qquad e_r(x) = \sum_{1< i_1< i_2<\cdots< i_r\leqslant n} x_{i_1}\cdots x_{i_r}$$

for r a positive integer. Hence $e_r(x) = 0$ for $r > n$. Also defining $e_r = h_r = 0$ for $r < 0$, two classical results in the theory of symmetric functions, known as the Jacobi–Trudi identity and dual Jacobi–Trudi (or Nägelsbach–Kostka) identity, express the Schur functions in terms of determinants with entries given by the complete and elementary symmetric functions, respectively,

$$s_\lambda(x) = \det_{1\leqslant i,j\leqslant n}\big(h_{\lambda_i-i+j}(x)\big) = \det_{1\leqslant i,j\leqslant k}\big(e_{\lambda'_i-i+j}(x)\big). \tag{1.2}$$

In (1.2), the partition λ' is the conjugate of λ and k is an arbitrary integer such that $\lambda'_1 \leqslant k$.

Surprisingly little appears to be known about Jacobi–Trudi identities for important generalisations of the Schur functions such as the Hall–Littlewood, Jack and Macdonald polynomials. Matsumoto [53, Theorem 5.1] discovered Jacobi–Trudi-like formulas for the Jack polynomials for partitions of rectangular shape, provided the Jack parameter α is a positive integer or the reciprocal of a positive integer. In Matsumoto's formulas, determinants are replaced by hyperdeterminants of (even) order depending on the value of α. This was subsequently generalised to partitions of near rectangular shape by Belbachir, Boussicault and Luque [8]. In this paper, we add to these results by proving a dual Jacobi–Trudi formula for ordinary (or $\mathrm{GL}(n,\mathbb{C})$) Hall–Littlewood polynomials indexed by rectangular shapes and by conjecturing dual Jacobi–Trudi formulas for B_n, C_n and BC_n Hall–Littlewood polynomials indexed by rectangular partitions of length n.

1.2 Main results and conjectures

For $x = (x_1,\dots,x_n)$ and λ a partition, let $P_\lambda(x;t)$ be the Hall–Littlewood polynomial indexed by λ, see Section 2.2 for details. In particular, $P_\lambda(x;0) = s_\lambda(x)$ and $P_\lambda(x;1) = m_\lambda(x)$, where $m_\lambda(x)$ is the monomial symmetric function.

Our first result is an affine (dual) Jacobi–Trudi formula for Hall–Littlewood polynomials indexed by partitions of rectangular shape

$$(k^r) = (\underbrace{k,k,\dots,k}_{r\text{ times}}).$$

Theorem 1.1. *Let k be a positive integer, r a nonnegative integer and $x = (x_1,\dots,x_n)$. Then*

$$P_{(k^r)}(x;t) = \sum_{\substack{y_1,\dots,y_k\in\mathbb{Z}\\ y_1+\cdots+y_k=0}} \det_{1\leqslant i,j\leqslant k}\Big(t^{k\binom{y_i}{2}+iy_i} e_{r-i+j-ky_i}(x)\Big). \tag{1.3}$$

For $y = (y_1,\dots,y_k)\in\mathbb{Z}^k$, let $|y| := y_1+\cdots+y_k$. Then the right-hand side of (1.3) may also be stated as

$$P_{(k^r)}(x;t) = \sum_{y\in Q}\sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)\prod_{i=1}^k t^{k\binom{y_i}{2}+iy_i} e_{r-i+\sigma_i-ky_i}(x),$$

where $Q = \{y \in \mathbb{Z}^k \mid |y| = 0\}$ and $S_k \ltimes Q \cong W\big(\mathrm{A}_{k-1}^{(1)}\big)$, the Weyl group of the affine root system $\mathrm{A}_{k-1}^{(1)}$ [29, 50]. For integers i, k, u such that $1 \leqslant i \leqslant k$, let $s_{i,k}(u) := k\binom{u}{2} + iu$. Then (1) $s_{i,k}(u) \geqslant 0$, (2) for $i < k$, $s_{i,k}(u) = 0$ if and only if $u = 0$, and (3) $s_{k,k}(u) = 0$ if and only if $u = 0$ or $u = -1$. These facts imply that for $y \in Q$, $\sum_{i=1}^k \big(k\binom{y_i}{2} + iy_i\big) = \sum_{i=1}^k s_{i,k}(y_i)$ is strictly positive unless $y = (0, \dots, 0)$. Consequently, for $t = 0$ the only nonzero contribution to the sum over y in (1.3) comes from the zero vector. Theorem 1.1 thus generalises the $\lambda = (k^r)$ case of the dual Jacobi–Trudi identity (1.2).

For $x = (x_1, \dots, x_n)$ and $\lambda = (\lambda_1, \dots, \lambda_n)$ a partition, the odd orthogonal, even orthogonal and symplectic Schur functions $\mathrm{so}_{2n+1,\lambda}(x)$, $\mathrm{o}_{2n,\lambda}(x)$ and $\mathrm{sp}_{2n,\lambda}(x)$ are defined as [49][1]

$$\begin{aligned}
\mathrm{so}_{2n+1,\lambda}(x) &= \frac{\det_{1\leqslant i,j\leqslant n}\big(x_i^{-\lambda_j+j-1} - x_i^{\lambda_j+2n-j}\big)}{\prod_{i=1}^n (1-x_i) \prod_{1\leqslant i<j\leqslant n}(x_i - x_j)(x_i x_j - 1)}, \\
\mathrm{o}_{2n,\lambda}(x) &= f_\lambda \frac{\det_{1\leqslant i,j\leqslant n}\big(x_i^{-\lambda_j+j-1} + x_i^{\lambda_j+2n-j-1}\big)}{\prod_{i=1}^n \prod_{1\leqslant i<j\leqslant n}(x_i - x_j)(x_i x_j - 1)}, \\
\mathrm{sp}_{2n,\lambda}(x) &= \frac{\det_{1\leqslant i,j\leqslant n}\big(x_i^{-\lambda_j+j-1} - x_i^{\lambda_j+2n-j+1}\big)}{\prod_{i=1}^n (1-x_i^2) \prod_{1\leqslant i<j\leqslant n}(x_i - x_j)(x_i x_j - 1)},
\end{aligned} \tag{1.4}$$

where $f_\lambda = 1$ if $l(\lambda) = n$ and $f_\lambda = 1/2$ if $l(\lambda) < n$. In (1.4), we have followed the convention of writing $\mathrm{so}_{2n+1,\lambda}(x)$ instead of $\mathrm{o}_{2n+1,\lambda}(x)$, which stems from the fact that the characters of the irreducible polynomial representations of $\mathrm{O}(2n+1,\mathbb{C})$ and $\mathrm{SO}(2n+1,\mathbb{C})$ coincide. In the even case, a similar coincidence between the characters of $\mathrm{O}(2n,\mathbb{C})$ and $\mathrm{SO}(2n,\mathbb{C})$ only occurs for $l(\lambda) < n$. From the definition, it follows that $\mathrm{so}_{2n+1,\lambda}(x)$, $\mathrm{o}_{2n,\lambda}(x)$ and $\mathrm{sp}_{2n,\lambda}(x)$ are BC_n-symmetric Laurent polynomials in the sense of [63] in the variables $x_1, \dots, x_n$ with integer coefficients, whose top-degree homogeneous components are given by the Schur function $s_\lambda(x)$. Assuming $\lambda \subseteq (k^n)$, the symplectic and odd-orthogonal Schur functions admit analogues of the dual Jacobi–Trudi identity (1.2) as follows (see, e.g., [75, p. 123] or [24, equations (3.10) and (3.27)]):

$$\begin{aligned}
\mathrm{so}_{2n+1,\lambda}(x) &= \det_{1\leqslant i,j\leqslant k}\big(\dot{e}_{\lambda'_i-i+j}(x) + \dot{e}_{\lambda'_i-i-j+1}(x)\big), \\
\mathrm{o}_{2n,\lambda}(x) &= \frac{1}{2}\det_{1\leqslant i,j\leqslant k}\big(\dot{e}_{\lambda'_i-i+j}(x) + \dot{e}_{\lambda'_i-i-j+2}(x)\big), \\
\mathrm{sp}_{2n,\lambda}(x) &= \det_{1\leqslant i,j\leqslant k}\big(\dot{e}_{\lambda'_i-i+j}(x) - \dot{e}_{\lambda'_i-i-j}(x)\big),
\end{aligned}$$

where $\dot{e}_r(x)$ is shorthand for $e_r(x^{\pm}) = e_r\big(x_1, x_1^{-1}, \dots, x_n, x_n^{-1}\big)$. For the rectangular partition of maximal height, i.e., for $\lambda = (k^n)$, this yields

$$\mathrm{so}_{2n+1,(k^n)}(x) = \det_{1\leqslant i,j\leqslant k}(\dot{e}_{n-i+j}(x) + \dot{e}_{n+i+j-1}(x)), \tag{1.5a}$$

$$\mathrm{o}_{2n,(k^n)}(x) = \frac{1}{2}\det_{1\leqslant i,j\leqslant k}(\dot{e}_{n-i+j}(x) + \dot{e}_{n+i+j-2}(x)), \tag{1.5b}$$

$$\mathrm{sp}_{2n,(k^n)}(x) = \det_{1\leqslant i,j\leqslant k}(\dot{e}_{n-i+j}(x) - \dot{e}_{n+i+j}(x)), \tag{1.5c}$$

where we have also used the symmetry $\dot{e}_r(x) = \dot{e}_{2n-r}(x)$.

For $\lambda \subseteq (k^n)$ and x as above, let $P_\lambda^{\mathrm{B}_n}(x;t,s)$ denote the B_n Hall–Littlewood polynomial indexed by λ, see Section 2.3. Then $P_\lambda^{\mathrm{B}_n}(x;0,0) = \mathrm{so}_{2n+1,\lambda}(x)$ and $P_\lambda^{\mathrm{B}_n}(x;0,1) = \mathrm{o}_{2n,\lambda}(x)$. For the B_n Hall–Littlewood polynomial, we conjecture two affine analogues of (1.5a), the right-hand

[1] The full set of odd-orthogonal Schur functions includes (1.4) for half-partitions λ, i.e., weakly decreasing n-tuples $(\lambda_1, \dots, \lambda_n)$ such that $\lambda_i \in \mathbb{Z} + 1/2$ and $\lambda_n \geqslant \frac{1}{2}$.

sides of which may be identified with the affine root systems $\mathrm{A}_{2k}^{(2)}$ and $\mathrm{D}_{k+1}^{(2)}$, respectively. We also have one analogue of (1.5b), for which we do not have an identification in terms of affine root systems.

Conjecture 1.2. *Let k be a nonnegative integer, $x=(x_1,\dots,x_n)$ and $\dot{e}_r(x)=e_r(x^{\pm})$. Then*

$$P^{\mathrm{B}_n}_{(k^n)}(x;t,0)=\sum_{y\in\mathbb{Z}^k}\det_{1\leqslant i,j\leqslant k}\Big((-1)^{y_i}t^{\frac{1}{2}Ky_i^2-(j-\frac{1}{2})y_i}\big(\dot{e}_{n-i+j-Ky_i}(x)+\dot{e}_{n+i+j-Ky_i-1}(x)\big)\Big),\tag{1.6a}$$

where $K:=2k+1$, and

$$P^{\mathrm{B}_n}_{(k^n)}\big(x;t,-t^{1/2}\big)=\sum_{y\in\mathbb{Z}^k}\det_{1\leqslant i,j\leqslant k}\Big(t^{\frac{1}{2}Ky_i^2-(j-\frac{1}{2})y_i}\big(\dot{e}_{n-i+j-Ky_i}(x)+\dot{e}_{n+i+j-Ky_i-1}(x)\big)\Big),\tag{1.6b}$$

$$P^{\mathrm{B}_n}_{(k^n)}(x;t,1)=\frac{1}{2}\sum_{y\in\mathbb{Z}^k}\det_{1\leqslant i,j\leqslant k}\Big((-1)^{y_i}t^{\frac{1}{2}Ky_i^2-(j-1)y_i}\big(\dot{e}_{n-i+j-Ky_i}(x)+\dot{e}_{n+i+j-Ky_i-2}(x)\big)\Big),\tag{1.6c}$$

where $K:=2k$.

For $\lambda\subseteq(k^n)$ and x as above, let $P^{\mathrm{C}_n}_{\lambda}(x;t,s)$ denote the C_n Hall–Littlewood polynomial indexed by λ. Then $P^{\mathrm{C}_n}_{\lambda}(x;0,0)=\mathrm{sp}_{2n,\lambda}(x)$. This time we have two conjectural analogues of (1.5c), corresponding to the affine root systems $\mathrm{C}_k^{(1)}$ and $\mathrm{A}_{2k-1}^{(2)}$, respectively.

Conjecture 1.3. *Let k be a nonnegative integer, $x=(x_1,\dots,x_n)$ and $\dot{e}_r(x)=e_r(x^{\pm})$. Then*

$$P^{\mathrm{C}_n}_{(k^n)}(x;t,0)=\sum_{y\in\mathbb{Z}^k}\det_{1\leqslant i,j\leqslant k}\Big(t^{\frac{1}{2}Ky_i^2-jy_i}\big(\dot{e}_{n-i+j-Ky_i}(x)-\dot{e}_{n+i+j-Ky_i}(x)\big)\Big),\tag{1.7a}$$

where $K=2k+2$, and

$$P^{\mathrm{C}_n}_{(k^n)}(x;t,t)=\sum_{\substack{y\in\mathbb{Z}^k\\ |y|\text{ even}}}\det_{1\leqslant i,j\leqslant k}\Big(t^{\frac{1}{2}Ky_i^2-jy_i}\big(\dot{e}_{n-i+j-Ky_i}(x)-\dot{e}_{n+i+j-Ky_i}(x)\big)\Big),\tag{1.7b}$$

where $K=2k$.

Finally, let $P^{\mathrm{BC}_n}_{\lambda}(x;t,s_1,s_2)$ denote the BC_n Hall–Littlewood polynomial indexed by λ. Then $P^{\mathrm{BC}_n}_{\lambda}(x;0,0,0)=\mathrm{sp}_{2n,\lambda}(x)$ and $P^{\mathrm{BC}_n}_{\lambda}(x;0,1,-1)=\mathrm{o}_{2n,\lambda}(x)$. Conjecturally, we have one more generalisation of (1.5c) corresponding to the affine root system $\mathrm{A}_{2k}^{(2)}$.

Conjecture 1.4. *Let k be a nonnegative integer, $x=(x_1,\dots,x_n)$ and $\dot{e}_r(x)=e_r(x^{\pm})$. Then*

$$P^{\mathrm{BC}_n}_{(k^n)}\big(x;t,-t^{1/2},0\big)=\sum_{y\in\mathbb{Z}^k}\det_{1\leqslant i,j\leqslant k}\Big((-1)^{y_i}t^{\frac{1}{2}Ky_i^2-jy_i}\big(\dot{e}_{n-i+j-Ky_i}(x)-\dot{e}_{n+i+j-Ky_i}(x)\big)\Big),\tag{1.8}$$

where $K=2k+1$.

Once again this gives (1.5c) for $t=0$.

In Section 4, Conjectures 1.2–1.4 will be proved for $k=1$ using bounded Littlewood identities. In the same section we also present proofs of (1.6a) and (1.8) for $t=1$ based on cylindric Schur functions.

Conjectures 1.2–1.4 have some remarkable consequences in that (with the exception of (1.6c)) they imply new Rogers–Ramanujan type identities which contain not just the standard parameter q but also a Hall–Littlewood parameter t, while still admitting a product form. These q,t-Rogers–Ramanujan identities generalise many of the classical Rogers–Ramanujan identities, such as the two original identities of Rogers and Ramanujan [64, 65, 66], (a subset of) the Andrews–Gordon identities [1], their even modulus analogue of Bressoud [12, 13], and the $\mathrm{C}_n^{(1)}$, $\mathrm{A}_{2n}^{(2)}$ and $\mathrm{D}_{n+1}^{(2)}$ GOW identities of Griffin, Ono and the author [26].

Let $(a;q)_\infty=\prod_{i\geqslant 0}(1-aq^i)$ be a q-shifted factorial, $\theta(a;p)=(a;p)_\infty(p/a;p)_\infty$ a modified Jacobi theta function and $\theta(a_1,\dots,a_k;p)=\prod_{i=1}^k\theta(a_i;p)$. For λ a partition, λ is said to be even if all of its parts are even.

Theorem 1.5 (q,t-Rogers–Ramanujan-type identity for $\mathrm{C}_k^{(1)}$)**.** *For k a positive integer, let $p=tq^{2k+2}$. Then*

$$\sum_{\substack{\lambda \text{ even}\\ \lambda_1\leqslant 2k}} q^{(\sigma+1)|\lambda|/2}P_\lambda\big(1,q,q^2,\dots;t\big)=\frac{(p;p)_\infty^k}{(q;q)_\infty^k}\prod_{i=1}^k\theta\big(q^{(2-\sigma)i};p\big)\prod_{1\leqslant i<j\leqslant k}\theta\big(q^{j-i},q^{i+j};p\big),\tag{1.9}$$

where $\sigma\in\{0,1\}$.

The labelling of the theorem by the affine root system $\mathrm{C}_k^{(1)}$ reflects the fact that $k+1$ is the dual Coxeter number of $\mathrm{C}_k^{(1)}$ and the product on the right of (1.9) is a specialisation of the Weyl–Kac denominator for $\mathrm{C}_k^{(1)}$. For $t=0$, the theorem simplifies to

$$\begin{aligned}\sum_{\substack{\lambda \text{ even}\\ \lambda_1\leqslant 2k}} q^{(\sigma+1)|\lambda|/2}s_\lambda\big(1,q,q^2,\dots\big)&=\prod_{i\geqslant 1}\frac{1-q^{(\sigma+1)(k+i)}}{1-q^{(\sigma+1)i}}\prod_{1\leqslant i<j}\frac{1-q^{2k+i+j}}{1-q^{i+j}}\\&=\lim_{n\to\infty}\phi_\sigma\big(\mathrm{e}^{-k\omega_n}\operatorname{ch}L(k\omega_n)\big),\end{aligned}\tag{1.10}$$

where $L(k\omega_n)$ is the irreducible $\mathrm{Sp}(2n,\mathbb{C})$-module of highest weight $k\omega_n$ (with $\omega_1,\dots,\omega_n$ the fundamental weights of $\mathrm{Sp}(2n,\mathbb{C})$), and ϕ_σ is the specialisation

$$\phi_\sigma:\ \mathbb{Z}[\mathrm{e}^{-\alpha_1},\dots,\mathrm{e}^{-\alpha_n}]\to\mathbb{Z}[q],\qquad \mathrm{e}^{-\alpha_i}\mapsto\begin{cases}q & \text{for } 1\leqslant i\leqslant n-1,\\ q^{\sigma+1} & \text{for } i=n.\end{cases}$$

For the affine root system $\mathrm{A}_{2k}^{(2)}$, with (dual) Coxeter number $2k+1$, we have two identities. Given a partition λ, let λ^{o} be the partition consisting of the odd parts of λ, so that $l(\lambda^{\mathrm{o}})$ is the number of parts of λ that are odd.

Theorem 1.6 (q,t-Rogers–Ramanujan-type identities for $\mathrm{A}_{2k}^{(2)}$)**.** *For k a positive integer, let $p=tq^{2k+1}$. Then*

$$\begin{aligned}&\sum_{\substack{\lambda\\ \lambda_1\leqslant 2k}} q^{(\sigma+1)|\lambda|/2}P_\lambda\big(1,q,q^2,\dots;t\big)\\&\quad=\frac{(p;p)_\infty^k}{(\sigma+1)(q;q)_\infty^k}\prod_{i=1}^k\theta\big(-q^{i-(\sigma+1)/2};p\big)\theta\big(pq^{2i-\sigma-1};p^2\big)\prod_{1\leqslant i<j\leqslant k}\theta\big(q^{j-i},q^{i+j-\sigma-1};p\big)\end{aligned}\tag{1.11a}$$

and

$$\begin{aligned}&\sum_{\substack{\lambda\\ \lambda_1\leqslant 2k}} q^{(\sigma+1)|\lambda|/2}t^{l(\lambda^{\mathrm{o}})/2}P_\lambda\big(1,q,q^2,\dots;t\big)\\&\quad=\frac{(p;p)_\infty^k}{(q;q)_\infty^k}\prod_{i=1}^k\theta\big(-p^{1/2}q^{i-\sigma/2};p\big)\theta\big(q^{2i-\sigma};p^2\big)\prod_{1\leqslant i<j\leqslant k}\theta\big(q^{j-i},q^{i+j-\sigma};p\big),\end{aligned}\tag{1.11b}$$

where $\sigma\in\{0,1\}$.

We remark that by

$$\prod_{i=1}^{k}\theta\big(-q^{i-1};p\big)\theta\big(pq^{2i-2};p^2\big)\prod_{1\leqslant i<j\leqslant k}\theta\big(q^{i+j-2};p\big)=2\prod_{1\leqslant i<j\leqslant k}\theta\big(q^{i+j-1};p\big)$$

the identity (1.11a) for $\sigma=1$ can be written as

$$\sum_{\substack{\lambda\\ \lambda_1\leqslant 2k}} q^{|\lambda|}P_\lambda\big(1,q,q^2,\dots;t\big)=\frac{(p;p)_\infty^k}{(q;q)_\infty^k}\prod_{1\leqslant i<j\leqslant k}\theta\big(q^{j-i},q^{i+j-1};p\big).$$

For $t=0$, (1.11a) yields

$$\begin{aligned}\sum_{\substack{\lambda\\ \lambda_1\leqslant 2k}} q^{(\sigma+1)|\lambda|/2}s_\lambda\big(1,q,q^2,\dots\big)&=\prod_{i\geqslant 1}\frac{1-q^{(\sigma+1)(k+i-1/2)}}{1-q^{(\sigma+1)(i-1/2)}}\prod_{1\leqslant i<j}\frac{1-q^{2k+i+j}}{1-q^{i+j}}\\ &=\lim_{n\to\infty}\hat{\phi}_\sigma\big(\mathrm{e}^{-k\omega_n}\operatorname{ch}L(k\omega_n)\big),\end{aligned}\tag{1.12}$$

where $L(k\omega_n)$ is the irreducible $\mathrm{SO}(2n+1,\mathbb{C})$-module of highest weight $k\omega_n$ and $\hat{\phi}_\sigma$ is the specialisation

$$\hat{\phi}_\sigma\colon\ \mathbb{Z}\big[\mathrm{e}^{-\alpha_1},\dots,\mathrm{e}^{-\alpha_n}\big]\to\mathbb{Z}\big[q^{(\sigma+1)/2}\big],\qquad \mathrm{e}^{-\alpha_i}\mapsto\begin{cases}q & \text{for } 1\leqslant i\leqslant n-1,\\ q^{(\sigma+1)/2} & \text{for } i=n.\end{cases}$$

The $t=0$ case of (1.11b) once again yields (1.10).

We conclude with two theorems for $\mathrm{D}_{k+1}^{(2)}$ and $\mathrm{A}_{2k-1}^{(2)}$, respectively. For both these root systems, the dual Coxeter number is $2k$. For λ a partition, we denoted by $m_i(\lambda)$ the multiplicity of parts of size i.

Theorem 1.7 (q,t-Rogers–Ramanujan identity for $\mathrm{D}_{k+1}^{(2)}$)**.** *For k a positive integer, let $p=tq^{2k}$. Then*

$$\begin{aligned}&\sum_{\substack{\lambda\\ \lambda_1\leqslant 2k}} q^{(\sigma+1)|\lambda|/2}\Bigg(\prod_{i=1}^{2k-1}\big(-t^{1/2};t^{1/2}\big)_{m_i(\lambda)}\Bigg)P_\lambda\big(1,q,q^2,\dots;t\big)\\ &\quad=\frac{\big(p^{1/2};p^{1/2}\big)_\infty(p;p)_\infty^{k-1}}{(\sigma+1)(q;q)_\infty^k}\prod_{i=1}^{k}\theta\big(-q^{i-(\sigma+1)/2};p^{1/2}\big)\prod_{1\leqslant i<j\leqslant k}\theta\big(q^{j-i},q^{i+j-\sigma-1};p\big),\end{aligned}$$

where $\sigma\in\{0,1\}$.

Theorem 1.8 (q,t-Rogers–Ramanujan identity for $\mathrm{A}_{2k-1}^{(2)}$)**.** *For k a positive integer, let $p=tq^{2k}$. Then*

$$\begin{aligned}&{\sum_{\substack{\lambda\\ \lambda_1\leqslant 2k}}}' q^{(\sigma+1)|\lambda|/2}t^{l(\lambda^{\mathrm{o}})/2}\Bigg(\prod_{i=1}^{2k-1}\big(t;t^2\big)_{\lceil m_i(\lambda)/2\rceil}\Bigg)P_\lambda\big(1,q,q^2,\dots;t\big)\\ &\quad=\frac{(p;p)_\infty^2(p;p)_\infty^{k-1}}{(q;q)_\infty^k}\prod_{i=1}^{k}\theta\big(q^{2i-\sigma};p^2\big)\prod_{1\leqslant i<j\leqslant k}\theta\big(q^{j-i},q^{i+j-\sigma};p\big),\end{aligned}$$

where $\sigma\in\{0,1\}$ and the prime in the sum over λ denotes the restriction that the odd parts of λ have even multiplicity.

In the $t=0$ limit, these theorems simplify to (1.12) and (1.10), respectively.

It will be shown in Section 5 how Conjectures 1.3 and 1.4 imply the above results. Of course, since the latter are conjectures, they do not imply proofs of Theorems 1.5–1.8, but once discovered it is not hard to provide a proof that does not rely on the conjectures, see Section 5.3. In Section 5, it will also be shown how the Andrews–Gordon and GOW identities follow from (1.9) by a novel manifestation of level-rank duality. For now, we remark that by

$$P_{(2^r)}(1,q,q^2,\dots;q)=\frac{q^{r^2-r}}{(q;q)_r}$$

the identity (1.9) for $t=q$ and $k=1$ $\big($and thus $p=q^5\big)$ simplifies to the Rogers–Ramanujan identities [64, 65, 66]

$$\sum_{r=0}^{\infty}\frac{q^{r^2}}{(q;q)_r}=\frac{\big(q^2,q^3,q^5;q^5\big)_\infty}{(q;q)_\infty}\quad\text{and}\quad\sum_{r=0}^{\infty}\frac{q^{r^2+r}}{(q;q)_r}=\frac{\big(q,q^4,q^5;q^5\big)_\infty}{(q;q)_\infty},$$

where $(a;q)_r=\prod_{i=0}^{r-1}\big(1-aq^i\big)$ and $(a_1,\dots,a_k;q)_\infty=\prod_{i=1}^{k}(a_i;q)_\infty$.

1.3 Outline

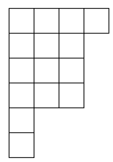
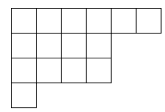
The remainder of this paper is organised as follows. In Section 2, we prepare the necessary symmetric function material needed for the rest of the paper. Most of what is covered in this section is well known, and apart from some easy to prove, marginally new results for Hall–Littlewood polynomials, the only new results of some depth are the two determinant identities for elementary symmetric functions stated in Lemma 2.9. These identities will be important in our applications of Conjectures 1.2–1.4. In Section 3, we prove Theorem 1.1 using results from crystal base theory due to Schilling and Shimozono. We also discuss a number of known special cases of the theorem as well as higher-level generalisations of Theorem 1.1 related to level-restricted Kostka polynomials. In Section 4, we use bounded Littlewood identities and Bailey pairs to prove the Conjectures 1.2–1.4 for $k=1$. We further apply known results for cylindric Schur functions to prove two of the six identities from the conjectures for $t=1$. In Section 5, we show that the q,t-Rogers–Ramanujan identities stated in Theorems 1.5–1.8 arise by specialising Conjectures 1.2–1.4. It is then shown that by specialising t to arbitrary positive powers of q the q,t-Rogers–Ramanujan identities lead to many new and old Rogers–Ramanujan identities for specialised characters of standard modules of affine Lie algebras. We use these Rogers–Ramanujan identities for standard modules together with Ismail's analytic argument to provide a proof of the q,t-Rogers–Ramanujan identities that is not reliant on the validity of Conjectures 1.2–1.4. In Section 6, we conclude the main part of the paper with a list of open problems related to our work. Finally, in the appendix a proof is given to a new family of Rogers–Ramanujan-type identities related to one of our applications of Theorem 1.8 discussed in Section 5.2. This family of identities was discovered by Matthew Russell after reading an earlier version of this paper.

2 Symmetric functions

2.1 (Cylindric) Schur functions

Let $\lambda=(\lambda_1,\lambda_2,\dots)$ be a partition, i.e., a weakly decreasing sequence of nonnegative integers such that $|\lambda|:=\lambda_1+\lambda_2+\cdots$ is finite. If $|\lambda|=m$, we say that λ is a partition of m, denoted as $\lambda\vdash m$. The parts of λ are the positive λ_i in the sequence, and the number of parts of λ is its length, denoted by $l(\lambda)$. For all $i\geqslant 1$, $m_i=m_i(\lambda)$ is the multiplicity of parts of size i in λ.

We alternatively write λ in multiplicity notation as $\lambda = (r^{m_r}, \dots, 1^{m_1})$, where r is the largest part of λ. Here we typically omit i^{m_i} if $m_i = 0$. For example, the partition $(4,3,3,3,1,1)$ is also written as $\big(4^1, 3^3, 1^2\big)$. A partition of the form (k^r) is referred to as a rectangular partition or a partition of rectangular shape. Given partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\mu = (\mu_1, \mu_2, \dots)$, we write $\mu \subseteq \lambda$ if $\lambda_i - \mu_i \geqslant 0$ for all $i \geqslant 1$, and say that μ is contained in λ. Given $\mu \subseteq \lambda$, we write $\mu \prec \lambda$ if the interlacing conditions $\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \mu_2 \geqslant \cdots$ hold. The set of partitions of length at most n will be denoted by Par_n and the set of partitions contained in the rectangle (k^n) by $\mathrm{Par}_{n,k}$. Given a partition λ, its Young diagram corresponds to the diagram obtained by drawing $l(\lambda)$ rows of left-aligned boxes or squares, such that the i-th row contains λ_i squares. The Young diagram of the partition $\nu = (4,3,3,3,1,1)$ is given by the left-most of the following two diagrams:

$\nu =$  $\nu' =$ 

We typically do not distinguish between a partition and its Young diagram. The conjugate of the partition λ, denoted λ', is obtained by reflecting the diagram of λ in the main diagonal, so that the conjugate of the partition ν in our example corresponds to the above Young diagram on the right. In our subsequent discussion of cylindric tableaux, it will be convenient to assume that Young diagrams are made up of unit squares, so that the length of the i-th row of λ is λ_i and the length of the j-th column is λ'_j. If $\mu \subseteq \lambda$, we denote by λ/μ the skew diagram obtained by removing those boxes of λ that are also contained in μ. If $\mu \prec \lambda$, then the skew diagram λ/μ has at most one box in each column and is known as a horizontal strip.

For partitions $\mu \subseteq \lambda$ and (weak) composition $\alpha = (\alpha_1, \dots, \alpha_n)$ such that $|\alpha| = |\lambda/\mu|$, let $\mathrm{SSYT}_n(\lambda/\mu, \alpha)$ denote the set of semistandard Young tableaux of shape λ/μ and filling α. That is, a tableaux T in $\mathrm{SSYT}_n(\lambda/\mu, \alpha)$ correspond to a filling of the Young diagram of λ/μ such that α_i squares are filled with the number i and such that rows are weakly increasing from left to right and columns are strictly increasing from top to bottom. Also set $\mathrm{SSYT}_n(\lambda/\mu) = \bigcup_\alpha \mathrm{SSYT}_n(\lambda/\mu, \alpha)$. An example of tableau in $\mathrm{SSYT}_7((6,6,6,4,1)/(3,2), (1,2,3,3,4,3,2))$ is given by

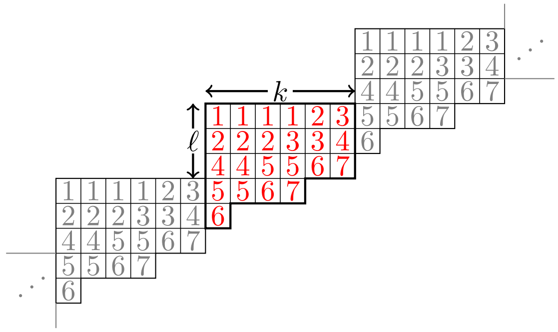
			1	2	3
		2	3	3	4
4	4	5	5	6	7
5	5	6	7		
6					

Recall from (1.1) that, for $\lambda \in \mathrm{Par}_n$, the Schur function $s_\lambda(x_1, \dots, x_n)$ may be expressed as a sum over tableaux in $\mathrm{SSYT}_n(\lambda)$. We now wish to extend this to the case of cylindric Schur functions [27, 39, 55, 60]. For positive integers k, n and nonnegative integer ℓ, let $\mathrm{Par}^{\ell}_{n,k}$ be the set of partitions λ contained in (k^n) such that $\lambda'_1 - \lambda'_k \leqslant \ell$. For $\lambda \in \mathrm{Par}^{\ell}_{n,k}$ and $T \in \mathrm{SSYT}_n(\lambda, \alpha)$, let $\overline{T}$ be the copy of T obtained by translating T by k units to the right and ℓ units up. Since $\lambda'_1 - \lambda'_k \leqslant \ell$, the union of T and $\overline{T}$ is a column strict skew tableaux of shape $\nu/\big(k^\ell\big)$, where

$$\nu_i = \begin{cases} \lambda_i + k & \text{for } 1 \leqslant i \leqslant \ell, \\ \lambda_i + \lambda_{i-\ell} & \text{for } \ell < i \leqslant n + \ell \end{cases}$$

and $\lambda_i = 0$ for $i > n$. If, beyond column-strictness, $T \cup \overline{T}$ is semistandard, T is said to be cylindric. The set of all cylindric semistandard Young tableaux of shape λ and filling α will be denoted by $\mathrm{CSSYT}_{n;k,\ell}(\lambda, \alpha)$, and once again we set $\mathrm{CSSYT}_{n;k,\ell}(\lambda) = \bigcup_\alpha \mathrm{CSSYT}_{n;k,\ell}(\lambda, \alpha)$.

Alternatively, $T \in \mathrm{CSSYT}_{n;k,\ell}(\lambda,\alpha)$ may be viewed as a tableau of shape $\lambda \in \mathrm{Par}^{\ell}_{n,k}$ wrapped around a cylinder of width k with vertical offset ℓ as in the following diagram for $\lambda = (6,6,6,4,1)$, $k=6$, $\ell=3$, $n=7$ and $\alpha=(4,4,3,3,4,3,2)$:



Combining (1.1) and (1.2), we have

$$\sum_{T \in \mathrm{SSYT}_n(\lambda)} x^T = \det_{1 \leqslant i,j \leqslant k} \bigl(e_{\lambda'_i - i + j}(x)\bigr) \tag{2.1}$$

for $\lambda \in \mathrm{Par}_{n,k}$ and $x=(x_1,\dots,x_n)$. Huh et al. [27, Proposition 2.6] generalised this as follows.

Proposition 2.1. *Let $\lambda \in \mathrm{Par}^{\ell}_{n,k}$ and $x=(x_1,\dots,x_n)$. Then*

$$\sum_{T \in \mathrm{CSSYT}_{n;k,\ell}(\lambda)} x^T = \sum_{\substack{y \in \mathbb{Z}^k \\ |y|=0}} \det_{1 \leqslant i,j \leqslant k} \bigl(e_{\lambda'_i - i + j - (k+\ell)y_i}(x)\bigr). \tag{2.2}$$

The symmetric function in this proposition is known as a cylindric Schur function and is denoted as $s_{\mu[k,\ell]'}(x)$ for $\mu = \lambda'$ in [27]. For $\ell \geqslant l(\lambda)$, $\mathrm{CSSYT}_{n;k,\ell}(\lambda) = \mathrm{SSYT}_n(\lambda)$ and $\det\bigl(e_{\lambda'_i - i + j - (k+\ell)y_i}(x)\bigr) = 0$ unless $y_1 = \cdots = y_n = 0$. Hence (2.2) includes (2.1) as a special case.

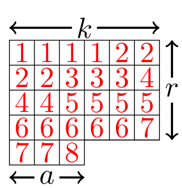
For later reference, we consider (2.2) for $\ell=0$ and $\ell=1$. Since $\mathrm{Par}^0_{n,k} = \{(k^r) \mid 0 \leqslant r \leqslant n\}$ and since the cyclic condition for $\ell=0$ implies that all boxes in row j of $T \in \mathrm{CSSYT}_{n;k,0}((k^r))$ have the same filling, say i_j, column-strictness implies

$$\sum_{T \in \mathrm{CSSYT}_{n;k,0}((k^r))} x^T = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n} x^k_{i_1} \cdots x^k_{i_r} = e_r\bigl(x^k_1,\dots,x^k_n\bigr) = e_r\bigl(x^k\bigr). \tag{2.3}$$

For $\ell=1$, the set of admissible partitions is the set of near-rectangles

$$\mathrm{Par}^1_{n,k} = \{(k^r,a) \mid r \geqslant 0,\ 0 \leqslant a < k\}.$$

Since $\ell=1$, the set $\mathrm{CSSYT}_{n;k,1}((k^r,a),\alpha)$ for $|\alpha| = kr+a$ contains at most one tableau. Column-strictness imposes the obvious condition $\alpha_i \leqslant k$ for all $1 \leqslant i \leqslant n$. Moreover, if a given row of $T \in \mathrm{SSYT}_n((k^r,a),\alpha)$ ends with a box with filling i and the next row begins with a box with filling $j<i$, then $T \notin \mathrm{CSSYT}_{n;k,1}((k^r,a),\alpha)$. Hence, assuming $|\alpha| = kr+a$ and $\alpha_i \leqslant k$, the only $T \in \mathrm{SSYT}_n((k^r,a),\alpha)$ contained in $\mathrm{CSSYT}_{n;k,1}((k^r,a),\alpha)$ is given by the tableau whose word obtained by reading rows from left to right starting with the top row and ending with the bottom row is given by $1^{\alpha_1}, 2^{\alpha_2}, \dots, n^{\alpha_n}$ as in



Hence for $\lambda \in \mathrm{Par}^1_{n,k}$,

$$\sum_{T\in \mathrm{CSSYT}_{n;k,1}(\lambda)} x^T = \sum_{\substack{\mu\in\mathrm{Par}_{n,k}\\ |\mu|=|\lambda|}} \sum_{\alpha\in S_n\cdot\mu} x^\alpha = \sum_{\substack{\mu\in\mathrm{Par}_{n,k}\\ |\mu|=|\lambda|}} m_\mu(x), \tag{2.4}$$

where $S_n\cdot\mu$ denotes the S_n-orbit of $\mu=(\mu_1,\dots,\mu_n)$ and $m_\mu(x)$ is the monomial symmetric function indexed by μ.

2.2 Hall–Littlewood polynomials for $\mathrm{GL}(n,\mathbb{C})$

Let $x=(x_1,\dots,x_n)$ and S_n the symmetric group of degree n with natural action on polynomials in x. Then the ordinary or $\mathrm{GL}(n,\mathbb{C})$ Hall–Littlewood polynomial $P_\lambda(x;t)$ is defined as [51]

$$P_\lambda(x;t)=\frac{1}{v_\lambda(t)}\sum_{w\in S_n} w\Bigg(x_1^{\lambda_1}\cdots x_n^{\lambda_n}\prod_{1\leqslant i<j\leqslant n}\frac{x_i-tx_j}{x_i-x_j}\Bigg),$$

where

$$v_\lambda(t)=\frac{(t;t)_{n-l(\lambda)}}{(1-t)^{n-l(\lambda)}}\prod_{i\geqslant 1}\frac{(t;t)_{m_i(\lambda)}}{(1-t)^{m_i(\lambda)}}.$$

The Hall–Littlewood polynomials $\{P_\lambda(x;t)\}_{l(\lambda)\leqslant n}$ form a basis of the ring of symmetric functions with coefficients in $\mathbb{Z}[t]$. In analogy with the Schur functions $s_\lambda(x)=P_\lambda(x;0)$, we set $P_\lambda(x;t)=0$ if λ is a partition such that $l(\lambda)>n$. By the stability property,

$$P_\lambda(x_1,\dots,x_n,0;t)=\begin{cases} P_\lambda(x_1,\dots,x_n;t) & \text{if } l(\lambda)\leqslant n,\\ 0 & \text{otherwise,}\end{cases}$$

the Hall–Littlewood polynomials may be extended to symmetric functions in countably many variables, $P_\lambda(x_1,x_2,\dots;t)$. Then the number of variables in a result such as Theorem 1.1 becomes irrelevant, and alternatively it may be stated as a result in the algebra of symmetric functions:

$$P_{(k^r)}(t)=\sum_{\substack{y\in\mathbb{Z}^k\\ |y|=0}} \det_{1\leqslant i,j\leqslant k}\Big(t^{k\binom{y_i}{2}+iy_i} e_{r-i+j-ky_i}\Big), \tag{2.5}$$

with no reference to a particular choice of alphabet.

Special cases of $P_\lambda(x;t)$ other than the Schur function are

$$P_\lambda(x;1)=m_\lambda(x) \tag{2.6}$$

and $P_{(1^r)}(x)=e_r(x)$.

For α a composition, let $e_\alpha(x)=\prod_{i\geqslant 1} e_{\alpha_i}(x)$. Then the transition coefficients $\mathcal{R}_{\lambda,\alpha}(t)$ between elementary symmetric functions and Hall–Littlewood polynomials are defined by

$$e_\alpha(x)=\sum_\lambda \mathcal{R}_{\lambda,\alpha}(t)P_\lambda(x;t). \tag{2.7}$$

For integers k, n, let

$$\begin{bmatrix} n\\ k\end{bmatrix}_q=\begin{cases}\dfrac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} & \text{if } 0\leqslant k\leqslant n,\\ 0 & \text{otherwise,}\end{cases}$$

be a Gaussian polynomial or q-binomial coefficient.

Theorem 2.2 (Kirillov [35, Theorem 3.4])**.** *For $\alpha = (\alpha_1, \dots, \alpha_k)$,*

$$\mathcal{R}_{\lambda,\alpha}(t) = \sum_{\substack{0=\nu^{(0)}\subseteq\nu^{(1)}\subseteq\cdots\subseteq\nu^{(k)}=\lambda' \\ |\nu^{(i)}/\nu^{(i-1)}|=\alpha_i}} \prod_{i=2}^{k}\prod_{j=1}^{i-1} \begin{bmatrix} \nu_j^{(i)} - \nu_{j+1}^{(i)} \\ \nu_j^{(i)} - \nu_j^{(i-1)} \end{bmatrix}_t. \tag{2.8}$$

The summand on the right vanishes unless the interlacing condition $\nu^{(a-1)} \prec \nu^{(a)}$ holds for all $1 \leqslant a \leqslant k$, so that one may also think of the sum over the $\nu^{(a)}$ as a sum over semistandard Young tableaux of shape λ' and filling α. This implies that $\mathcal{R}_{\lambda,\alpha}(t) = 0$ if $\lambda_1 > k$, where $\alpha = (\alpha_1, \dots, \alpha_k)$, but also that if $\lambda = (k^r, \mu)$ where μ is a partition such that $\mu_1 < k$, then

$$\mathcal{R}_{(k^r,\mu),(\alpha_1,\dots,\alpha_k)}(t) = \mathcal{R}_{\mu,(\alpha_1-r,\dots,\alpha_k-r)}(t). \tag{2.9}$$

The expression (2.8) for $k = 2$ simplifies to

$$\mathcal{R}_{(2^r,1^s),(\alpha_1,\alpha_2)}(t) = \delta_{2r+s,\alpha_1+\alpha_2} \begin{bmatrix} s \\ \alpha_1 - r \end{bmatrix}_t, \tag{2.10}$$

where $\delta_{i,j}$ is a Kronecker delta. This yields the following simple result, which will be used in the proof of Conjectures 1.2–1.4 for $k = 1$. Recall that $\dot{e}_r(x) = e_r(x^{\pm})$.

Lemma 2.3. *Let $x = (x_1, \dots, x_n)$ and k an integer such that $-n \leqslant k \leqslant n$. Then*

$$(x_1 \cdots x_n)\dot{e}_{n-k}(x) = \sum_{\substack{r,s\geqslant 0 \\ s-k \text{ even}}} \begin{bmatrix} s \\ \frac{1}{2}(s-k) \end{bmatrix}_t P_{(2^r,1^s)}(x;t).$$

Since the summand on the right vanishes unless $|k| \leqslant s \leqslant n - r$, only finitely many terms contribute to the sum, as it should.

Proof. By

$$e_r(x_1, \dots, x_n, y_1, \dots, y_m) = \sum_{i=0}^{r} e_i(x_1, \dots, x_n) e_{r-i}(y_1, \dots, y_m) \tag{2.11}$$

and

$$e_r\big(x^{-1}\big) = (x_1 \cdots x_n)^{-1}\, e_{n-r}(x),$$

we have

$$(x_1 \cdots x_n)\dot{e}_{n-k}(x) = \sum_{i\geqslant 0} e_i(x) e_{i+k}(x),$$

where the summand is nonzero for $\max\{0, -k\} \leqslant i \leqslant \min\{n, n-k\}$ only. Hence

$$\begin{aligned}
(x_1 \cdots x_n)\dot{e}_{n-k}(x) &= \sum_{i\geqslant 0} e_{(i,i+k)}(x) \\
&= \sum_{i,r,s\geqslant 0} \delta_{2r+s,k+2i} \begin{bmatrix} s \\ i - r \end{bmatrix}_t P_{(2^r,1^s)}(x;t) \\
&= \sum_{\substack{r,s\geqslant 0 \\ s-k \text{ even}}} \begin{bmatrix} s \\ \frac{1}{2}(s-k) \end{bmatrix}_t P_{(2^r,1^s)}(x;t),
\end{aligned}$$

where the second equality follows from (2.10). ∎

For m a nonnegative integer, let $H_m(x;t)$ be the Rogers–Szegő polynomial [2, p. 49]

$$H_m(x;t)=\sum_{i=0}^{m} x^i \begin{bmatrix} m \\ i \end{bmatrix}_t \tag{2.12}$$

and, for k a positive integer, let $h_\lambda^{(k)}(a,b;t)$ be the Rogers–Szegő polynomial indexed by the partition λ [7, 63]:

$$h_\lambda^{(k)}(a,b;t)=\prod_{\substack{i=1\\ i \text{ odd}}}^{k-1}(-a)^{m_i(\lambda)}H_{m_i(\lambda)}(b/a;t)\prod_{\substack{i=1\\ i \text{ even}}}^{k-1}H_{m_i(\lambda)}(ab;t).$$

This polynomial is symmetric in a and b, and

$$h_{(1^r)}^{(k)}(a,b;t)=(-a)^r H_r(b/a;t)=(-1)^r\sum_{i=0}^{r}a^{r-i}b^i\begin{bmatrix} r \\ i \end{bmatrix}_t$$

provided that $k\geqslant 2$. For notational convenience, we also define

$$\tilde{h}_\lambda^{(k)}(a,b;t):=a^{\lambda'_{2k+1}}h_\lambda^{(2k+1)}(-a,-b;t)=a^{l(\lambda^{\text{o}})}\prod_{i=1}^{2k}H_{m_{2i-1}(\lambda)}(b/a;t)H_{m_{2i}(\lambda)}(ab;t),$$

which is *not* symmetric in a and b.

Proposition 2.4. *For $x=(x_1,\dots,x_n)$ and k a nonnegative integer,*

$$\bigg(\prod_{i=1}^{n}(1+ax_i)\bigg)\sum_{\substack{\mu\\ \mu_1\leqslant 2k}} b^{l(\mu^{\text{o}})}P_\mu(x;t)=\sum_{\substack{\lambda\\ \lambda_1\leqslant 2k+1}}\tilde{h}_\lambda^{(k)}(a,b;t)P_\lambda(x;t). \tag{2.13}$$

Proof. The proof is essentially a repeat of the proof of [82, equation (3.3)], which is an identity equivalent to the $k\to\infty$ limit of (2.13).

We begin by recalling the e-Pieri rule for Hall–Littlewood polynomials [51, p. 215]

$$P_\mu(x;t)e_r(x)=\sum_{\lambda\vdash|\mu|+r}P_\lambda(x;t)\prod_{i\geqslant 1}\begin{bmatrix} \lambda'_i-\lambda'_{i+1} \\ \lambda'_i-\mu'_i \end{bmatrix}_t.$$

The summand on the right vanishes unless $\mu'\prec\lambda'$. Multiplying both sides by a^r and then summing over r, we obtain

$$\bigg(\prod_{i=1}^{n}(1+ax_i)\bigg)P_\mu(x;t)=\sum_{\lambda}a^{|\lambda/\mu|}P_\lambda(x;t)\prod_{i\geqslant 1}\begin{bmatrix} \lambda'_i-\lambda'_{i+1} \\ \lambda'_i-\mu'_i \end{bmatrix}_t.$$

Further multiplying this by $b^{l(\mu^{\text{o}})}$ and summing over μ such that $\mu_1\leqslant 2k$, we thus find

$$\bigg(\prod_{i=1}^{n}(1+ax_i)\bigg)\sum_{\substack{\mu\\ \mu_1\leqslant 2k}} b^{l(\mu^{\text{o}})}P_\mu(x;t)=\sum_{\substack{\lambda\\ \lambda_1\leqslant 2k+1}}P_\lambda(x;t)\sum_{\substack{\mu\\ \mu_1\leqslant 2k}}a^{|\lambda/\mu|}b^{l(\mu^{\text{o}})}\prod_{i=1}^{2k}\begin{bmatrix} \lambda'_i-\lambda'_{i+1} \\ \lambda'_i-\mu'_i \end{bmatrix}_t.$$

Here we have used the fact that the summand vanishes unless $\mu'\prec\lambda'$, which, given that $\mu_1\leqslant 2k$, implies that $\lambda_1\leqslant 2k+1$. In the sum over μ we now make the substitutions $\mu'_{2i-1}\mapsto j_i+\lambda'_{2i}$

and $\mu'_{2i} \mapsto \lambda'_{2i} - l_i$ for $1 \leqslant i \leqslant k$. Then $|\lambda/\mu| \mapsto l(\lambda^{\mathrm{o}}) + \sum_{i=1}^{k}(l_i - j_i)$ and $l(\mu^{\mathrm{o}}) \mapsto \sum_{i=1}^{k}(j_i + l_i)$, so that

$$\begin{aligned}
&\bigg(\prod_{i=1}^{n}(1+ax_i)\bigg) \sum_{\substack{\mu \\ \mu_1 \leqslant 2k}} b^{l(\mu^{\mathrm{o}})} P_\mu(x;t) \\
&\quad = \sum_{\substack{\lambda \\ \lambda_1 \leqslant 2k+1}} a^{l(\lambda^{\mathrm{o}})} P_\lambda(x;t) \sum_{\substack{j_1,\dots,j_k \geqslant 0 \\ l_1,\dots,l_k \geqslant 0}} \prod_{i=1}^{k} (b/a)^{j_i} \begin{bmatrix} \lambda'_{2i-1} - \lambda'_{2i} \\ j_i \end{bmatrix}_t (ab)^{l_i} \begin{bmatrix} \lambda'_{2i} - \lambda'_{2i+1} \\ l_i \end{bmatrix}_t \\
&\quad = \sum_{\substack{\lambda \\ \lambda_1 \leqslant 2k+1}} a^{l(\lambda^{\mathrm{o}})} P_\lambda(x;t) \prod_{i=1}^{k} H_{m_{2i-1}(\lambda)}(b/a;t) H_{m_{2i}(\lambda)}(ab;t) \\
&\quad = \sum_{\substack{\lambda \\ \lambda_1 \leqslant 2k+1}} \tilde{h}_\lambda^{(k)}(a,b;t) P_\lambda(x;t),
\end{aligned}$$

as claimed. ∎

According to [63, equation (2.3.7)],

$$\tilde{h}_\lambda^{(k)}(a,0;t) = a^{l(\lambda^{\mathrm{o}})} \qquad \text{and} \qquad \tilde{h}_\lambda^{(k)}\big(1,t^{1/2};t\big) = \prod_{i=1}^{2k} \big(-t^{1/2};t^{1/2}\big)_{m_i(\lambda)}.$$

This implies the following corollary of Proposition 2.4.

Corollary 2.5. *For $x = (x_1,\dots,x_n)$ and k a nonnegative integer,*

$$\bigg(\prod_{i=1}^{n}(1+ax_i)\bigg) \sum_{\substack{\mu \text{ even} \\ \mu_1 \leqslant 2k}} P_\mu(x;t) = \sum_{\substack{\lambda \\ \lambda_1 \leqslant 2k+1}} a^{l(\lambda^{\mathrm{o}})} P_\lambda(x;t), \tag{2.14a}$$

$$\bigg(\prod_{i=1}^{n}(1+x_i)\bigg) \sum_{\substack{\mu \\ \mu_1 \leqslant 2k}} t^{l(\mu^{\mathrm{o}})/2} P_\mu(x;t) = \sum_{\substack{\lambda \\ \lambda_1 \leqslant 2k+1}} \bigg(\prod_{i=1}^{2k} \big(-t^{1/2};t^{1/2}\big)_{m_i(\lambda)}\bigg) P_\lambda(x;t). \tag{2.14b}$$

To conclude our discussion of the Hall–Littlewood polynomials for $\mathrm{GL}(n,\mathbb{C})$, we define its modified analogue. Consider the expansion of the Hall–Littlewood symmetric functions in terms of the Newton power sums

$$P_\lambda(x_1,x_2,\dots;t) = \sum_\mu c_{\lambda\mu}(t) p_\mu(x_1,x_2,\dots),$$

where $p_\mu = \prod_{i \geqslant 1} p_{\mu_i}$ and $p_0 = 1$, $p_r(x_1,x_2,\dots) = x_1^r + x_2^r + \cdots$ for $r \geqslant 1$. Then the modified Hall–Littlewood polynomial $P'_\lambda(x_1,x_2,\dots,x_n;t)$ is defined as

$$P'_\lambda(x_1,\dots,x_n;t) = \sum_\mu c_{\lambda\mu}(t) p_\mu(X_1(t),\dots,X_n(t)),$$

where $X_i(t) = \big(x_i, x_i t, x_i t^2, \dots\big)$. Alternatively, using plethystic notation, $P'_\lambda(x;t) = P'_\lambda(x/(1-t);t)$. From the definition, it follows immediately that

$$P'_\lambda\big(1,q,\dots,q^{n-1};q^n\big) = P_\lambda\big(1,q,q^2,\dots;q^n\big). \tag{2.15}$$

2.3 The B_n, C_n and BC_n Hall–Littlewood polynomials

Let $W = S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ be the hyperoctahedral group (or group of signed permutations) with standard action on Laurent polynomials in $x = (x_1, \dots, x_n)$. Then, for λ a partition of length at most n, the BC_n Hall–Littlewood polynomial $P_\lambda^{\mathrm{BC}_n}(x; t, s_1, s_2)$ is defined as [52, 76]

$$P_\lambda^{(\mathrm{BC}_n)}(x; t, s_1, s_2) = \frac{1}{(s_1 s_2; t)_{n-l(\lambda)} v_\lambda(t)} \times \sum_{w \in W} w\left(\prod_{i=1}^n x_i^{-\lambda_i} \frac{(1 - s_1 x_i)(1 - s_2 x_i)}{1 - x_i^2} \prod_{1 \leqslant i < j \leqslant n} \frac{(t x_i - x_j)(1 - t x_i x_j)}{(x_i - x_j)(1 - x_i x_j)}\right). \tag{2.16}$$

The BC_n Hall–Littlewood polynomials are polynomials in $\mathbb{Z}[t, s_1, s_2]\big[x_1^{\pm}, \dots, x_n^{\pm}\big]^W$, normalised such that $P_0^{\mathrm{BC}_n}(x; t, s_1, s_2) = 1$. The top-degree homogeneous component of the polynomials $P_\lambda^{(\mathrm{BC}_n)}(x; t, s_1, s_2)$ is given by the ordinary Hall–Littlewood polynomial $P_\lambda(x; t)$.

The B_n and C_n Hall–Littlewood polynomials arise as special cases of the BC_n Hall–Littlewood polynomials:

$$P_\lambda^{(\mathrm{B}_n)}(x; t, s) = P_\lambda^{(\mathrm{BC}_n)}(x; t, s, -1), \tag{2.17a}$$

$$P_\lambda^{(\mathrm{C}_n)}(x; t, s) = P_\lambda^{(\mathrm{BC}_n)}\big(x; t, s^{1/2}, -s^{1/2}\big). \tag{2.17b}$$

(For B_n, there is once again an analogue for half-partitions λ which is not needed in this paper.) The odd-orthogonal, even-orthogonal and symplectic Schur functions correspond to the special cases

$$\mathrm{so}_{2n+1,\lambda}(x) = P_\lambda^{(\mathrm{B}_n)}(x; 0, 0), \qquad \mathrm{o}_{2n,\lambda}(x) = P_\lambda^{(\mathrm{B}_n)}(x; 0, 1), \qquad \mathrm{sp}_{2n,\lambda}(x) = P_\lambda^{(\mathrm{C}_n)}(x; 0, 0).$$

For $\lambda = (k^n)$, the summand in (2.16) has S_n symmetry so that the sum over W simplifies to a sum over $(\mathbb{Z}/2\mathbb{Z})^n$, leading to [63, Lemma 2.5]

$$P_{(k^n)}^{(\mathrm{BC}_n)}(x; t, s_1, s_2) = \sum_{\varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}} \Phi\big(x_1^{\varepsilon_1}, \dots, x_n^{\varepsilon_n}; t, s_1, s_2\big) \prod_{i=1}^n x_i^{-\varepsilon_i k},$$

where

$$\Phi(x_1, \dots, x_n; t, s_1, s_2) := \prod_{i=1}^n \frac{(1 - s_1 x_i)(1 - s_2 x_i)}{1 - x_i^2} \prod_{1 \leqslant i < j \leqslant n} \frac{1 - t x_i x_j}{1 - x_i x_j}.$$

In particular,

$$P_{(k^n)}^{(\mathrm{BC}_n)}(x; 1, s_1, s_2) = \prod_{i=1}^n \frac{(1 - s_1 x_i)(1 - s_2 x_i) x_i^{-k} - (s_1 - x_i)(s_2 - x_i) x_i^k}{1 - x_i^2}. \tag{2.18}$$

In the proofs of special cases of our conjectures as well as in the derivation of q,t-Rogers–Ramanujan identities, we require a number of bounded Littlewood identities, expressing the B_n, C_n and BC_n Hall–Littlewood polynomials indexed by rectangular partitions of length n in terms of ordinary Hall–Littlewood polynomials.

Theorem 2.6 (bounded Littlewood identities for B_n)**.** *For k a nonnegative integer,*

$$\sum_{\substack{\lambda \\ \lambda_1 \leqslant 2k}} P_\lambda(x; t) = (x_1 \cdots x_n)^k \, P_{(k^n)}^{\mathrm{B}_n}(x; t, 0), \tag{2.19a}$$

$$\sum_{\substack{\lambda\\ \lambda_1\leqslant 2k}}\Bigg(\prod_{i=1}^{2k-1}\big({-t^{1/2}};t^{1/2}\big)_{m_i(\lambda)}\Bigg)P_\lambda(x;t)=(x_1\cdots x_n)^k\,P^{\mathrm{B}_n}_{(k^n)}\big(x;t,-t^{1/2}\big), \tag{2.19b}$$

$$\sideset{}{''}\sum_{\substack{\lambda\\ \lambda_1\leqslant 2k}}\Bigg(\prod_{i=1}^{2k-1}\big(t;t^2\big)_{m_i(\lambda)/2}\Bigg)P_\lambda(x;t)=(x_1\cdots x_n)^k\,P^{\mathrm{B}_n}_{(k^n)}(x;t,1), \tag{2.19c}$$

where the double prime denotes the restriction that all parts of λ that are strictly less than $2k$ have even multiplicity.

These three results (which also hold for $k+\frac{1}{2}$ a nonnegative integer) are the special cases $t_2=0$, $t_2=-t^{1/2}$ and $t_2=1$ of the integral-k case of [63, Theorem 4.8]. The identity (2.19a) is equivalent to a result of Macdonald [51, pp. 232–233], although his right-hand side is not identified as a B_n Hall–Littlewood polynomial, a fact that is crucial in our use of the result. Since $P^{\mathrm{B}_n}_\lambda(x;t,1)=P^{\mathrm{C}_n}_\lambda(x;t,1)$, the identity (2.19c) may also be regarded as a result for C_n. This, however, does not apply to its extension to half-integer values of k.

Theorem 2.7 (bounded Littlewood identities for C_n)**.** *For k a nonnegative integer,*

$$\sum_{\substack{\lambda \text{ even}\\ \lambda_1\leqslant 2k}}P_\lambda(x;t)=(x_1\cdots x_n)^k\,P^{\mathrm{C}_n}_{(k^n)}(x;t,0), \tag{2.20a}$$

$$\sideset{}{'}\sum_{\substack{\lambda\\ \lambda_1\leqslant 2k}}t^{l(\lambda^{\mathrm{o}})/2}\Bigg(\prod_{i=1}^{2k-1}\big(t;t^2\big)_{\lceil m_i(\lambda)/2\rceil}\Bigg)P_\lambda(x;t)=(x_1\cdots x_n)^k\,P^{\mathrm{C}_n}_{(k^n)}(x;t,t), \tag{2.20b}$$

where the prime denotes the restriction that the odd parts of λ have even multiplicity.

In the above form, the identity (2.20a) was first stated in [63, equation (4.1.7)] and is the special $t_2=t_3=0$ instance of [63, Theorem 4.7]. An equivalent identity, which makes no reference to the symplectic Hall–Littlewood polynomials, was previously found by Stembridge [74, Theorem 1.2].

Finally, we need one BC_n bounded Littlewood identity, given by the $(t_2,t_3)=\big({-t^{1/2}},0\big)$ special case of [63, equation (4.1.15)].

Theorem 2.8. *For k a nonnegative integer,*

$$\sum_{\substack{\lambda\\ \lambda_1\leqslant 2k}}t^{l(\lambda^{\mathrm{o}})/2}P_\lambda(x;t)=(x_1\cdots x_n)^k\,P^{\mathrm{BC}_n}_{(k^n)}\big(x;t,-t^{1/2},0\big). \tag{2.21}$$

2.4 Determinant identities for elementary symmetric functions

The aim of this section is to show that, with the exception of (1.6c), each determinant in Conjectures 1.2–1.4 admits an alternative expression in terms of the elementary symmetric functions on the alphabet $\big(x^{\pm},1\big)=\big(x_1,x_1^{-1},\dots,x_n,x_n^{-1},1\big)$.

Let $\ddot{e}_r(x):=e_r\big(x^{\pm},1\big)$. By (2.11),

$$\ddot{e}_r(x)=\dot{e}_r(x)+\dot{e}_{r-1}(x). \tag{2.22}$$

Lemma 2.9. *For K an arbitrary integer,*

$$\begin{aligned}\sum_{y\in\mathbb{Z}^k}&\det_{1\leqslant i,j\leqslant k}\Big(u^{y_i}t^{\frac{1}{2}Ky_i^2-(j-\frac{1}{2})y_i}\big(\dot{e}_{n-i+j-Ky_i}(x)+\dot{e}_{n+i+j-Ky_i-1}(x)\big)\Big)\\ &=\frac{1}{2}\sum_{y\in\mathbb{Z}^k}\det_{1\leqslant i,j\leqslant k}\Big(u^{y_i}t^{\frac{1}{2}Ky_i^2-(j-\frac{1}{2})y_i}\big(\ddot{e}_{n-i+j-Ky_i+1}(x)+\ddot{e}_{n+i+j-Ky_i-1}(x)\big)\Big)\end{aligned} \tag{2.23a}$$

and

$$\sum_{y\in\mathbb{Z}^k}\det_{1\leqslant i,j\leqslant k}\big(u^{y_i}t^{\frac{1}{2}Ky_i^2-jy_i}\big(\dot{e}_{n-i+j-Ky_i}(x)-\dot{e}_{n+i+j-Ky_i}(x)\big)\big)$$
$$=\sum_{y\in\mathbb{Z}^k}\det_{1\leqslant i,j\leqslant k}\big(u^{y_i}t^{\frac{1}{2}Ky_i^2-jy_i}\big(\ddot{e}_{n-i+j-Ky_i+1}(x)-\ddot{e}_{n+i+j-Ky_i}(x)\big)\big). \tag{2.23b}$$

Proof. First we consider (2.23a). Fix $y\in\mathbb{Z}^k$ such that $y_1\geqslant y_2\geqslant\cdots\geqslant y_k$. Then (2.23a) is a consequence of

$$\sum_{S_k\cdot y}\det_{1\leqslant i,j\leqslant k}\big(t^{-jy_i}\big(\dot{e}_{n-i+j-Ky_i}(x)+\dot{e}_{n+i+j-Ky_i-1}(x)\big)\big)$$
$$=\frac{1}{2}\sum_{S_k\cdot y}\det_{1\leqslant i,j\leqslant k}\big(t^{-jy_i}\big(\ddot{e}_{n-i+j-Ky_i+1}(x)+\ddot{e}_{n+i+j-Ky_i-1}(x)\big)\big), \tag{2.24}$$

where $S_k\cdot y$ denotes the S_k orbit of y. Summing over all $k!$ permutations of y instead of $S_k\cdot y$ amounts to multiplying (2.24) by the size of the stabilizer of y, and in the following it will be more convenient to consider (2.24) with y replaced by $w(y)=(y_{w_1},\dots,y_{w_k})$ and the sum over $S_k\cdot y$ replaced by a sum over $w\in S_k$. Then, after applying (2.22) to the right-hand side, the identity to be proved becomes the special case – replace $(x_i,z_i)\mapsto(t^{-y_i},Ky_i)$ followed by $E_r\mapsto\dot{e}_{n-r}(x)$ – of the formal identity

$$\sum_{w\in S_k}\det_{1\leqslant i,j\leqslant k}\big(x_{w_i}^j\big(E_{z_{w_i}+i-j}+E_{z_{w_i}-i-j+1}\big)\big)$$
$$=\frac{1}{2}\sum_{w\in S_k}\det_{1\leqslant i,j\leqslant k}\big(x_{w_i}^j\big(E_{z_{w_i}+i-j-1}+E_{z_{w_i}+i-j}+E_{z_{w_i}-i-j+1}+E_{z_{w_i}-i-j+2}\big)\big).$$

Dispensing with the determinants, this is the same as

$$\sum_{\sigma,w\in S_k}\operatorname{sgn}(\sigma)\prod_{i=1}^k x_{w_i}^{\sigma_i}F_{i,i}(\sigma,w)=\frac{1}{2}\sum_{\sigma,w\in S_k}\operatorname{sgn}(\sigma)\prod_{i=1}^k x_{w_i}^{\sigma_i}\big(F_{i,i-1}(\sigma,w)+F_{i,i}(\sigma,w)\big),$$

where

$$F_{i,j}(\sigma,w):=E_{z_{w_i}+j-\sigma_i}+E_{z_{w_i}-j-\sigma_i+1}.$$

Since $F_{i,-j}(\sigma,w)=F_{i,j+1}(\sigma,w)$ and thus $F_{1,0}(\sigma,w)=F_{1,1}(\sigma,w)$, this may be simplified to

$$\sum_{\varnothing\subset I\subseteq\{2,\dots,k\}}\sum_{\sigma,w\in S_k}\operatorname{sgn}(\sigma)\prod_{i=1}^k x_{w_i}^{\sigma_i}\prod_{i\in I}F_{i,i-1}(\sigma,w)\prod_{i\in\{1,\dots,k\}\setminus I}\prod_{i\in I}F_{i,i}(\sigma,w)=0. \tag{2.25}$$

Since $\prod_{i=1}^k x_{w_i}^{\sigma_i}=\prod_{i=1}^k x_i^{(\sigma w^{-1})_i}$, we replace $w\mapsto w\sigma$ to obtain

$$\sum_{\varnothing\subset I\subseteq\{2,\dots,k\}}\sum_{\sigma,w\in S_k}\operatorname{sgn}(\sigma)\prod_{i=1}^k x_i^{(w^{-1})_i}\prod_{i\in I}F_{i,i-1}(\sigma,w\sigma)\prod_{i\in\{1,\dots,k\}\setminus I}\prod_{i\in I}F_{i,i}(\sigma,w\sigma)=0.$$

In the following, we will show the stronger vanishing result

$$f_{I;w}:=\sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)\prod_{i\in I}F_{i,i-1}(\sigma,w\sigma)\prod_{i\in\{1,\dots,k\}\setminus I}\prod_{i\in I}F_{i,i}(\sigma,w\sigma)=0 \tag{2.26}$$

for fixed $\varnothing \subset I \subseteq \{2,\dots,k\}$ and $w \in S_k$. For $j \in \{1,\dots,k-1\}$, let $s_j \in S_k$ denote the j-th adjacent transposition, i.e., the 2-cycle $(j,j+1)$. Then, for any $\tau \in S_k$,

$$(\tau s_j)_i = \begin{cases} \tau_{j+1} & \text{if } i=j, \\ \tau_j & \text{if } i=j+1, \\ \tau_i & \text{otherwise.} \end{cases}$$

Hence

$$F_{i,i-1}(\sigma s_j, w\sigma s_j) = \begin{cases} F_{i-1,i-1}(\sigma, w\sigma) & \text{if } i=j+1, \\ F_{i,i-1}(\sigma, w\sigma) & \text{if } i \neq j, j+1 \end{cases}$$

and

$$F_{i,i}(\sigma s_j, w\sigma s_j) = \begin{cases} F_{i+1,i}(\sigma, w\sigma) & \text{if } i=j, \\ F_{i,i}(\sigma, w\sigma) & \text{if } i \neq j, j+1. \end{cases}$$

We now fix the index j of s_j as $j=\min\{I\}-1$. Then $j+1 \in I$ and $j \notin I$, and thus by replacing σ by σs_j in (2.26) and using that $\operatorname{sgn}(\sigma s_j) = -\operatorname{sgn}(\sigma)$,

$$\begin{aligned} f_{I;w} &= -\sum_{\sigma \in S_k} \bigg(\operatorname{sgn}(\sigma) F_{j+1,j}(\sigma s_j, w\sigma s_j) F_{j,j}(\sigma s_j, w\sigma s_j) \\ &\qquad\qquad \times \prod_{\substack{i \in I \\ i \neq j+1}} F_{i,i-1}(\sigma s_j, w\sigma s_j) \prod_{\substack{i \in \{1,\dots,k\} \setminus I \\ i \neq j}} \prod_{i \in I} F_{i,i}(\sigma s_j, w\sigma s_j) \bigg) \\ &= -\sum_{\sigma \in S_k} \bigg(\operatorname{sgn}(\sigma) F_{j,j}(\sigma, w\sigma) F_{j+1,j}(\sigma, w\sigma) \prod_{\substack{i \in I \\ i \neq j+1}} F_{i,i-1}(\sigma, w\sigma) \prod_{\substack{i \in \{1,\dots,k\} \setminus I \\ i \neq j}} \prod_{i \in I} F_{i,i}(\sigma, w\sigma) \bigg) \\ &= -\sum_{\sigma \in S_k} \bigg(\operatorname{sgn}(\sigma) \prod_{i \in I} F_{i,i-1}(\sigma, w\sigma) \prod_{i \in \{1,\dots,k\} \setminus I} \prod_{i \in I} F_{i,i}(\sigma, w\sigma) \bigg) \\ &= -f_{I;w}. \end{aligned}$$

Hence $f_{I;w}=0$.

Next we consider (2.23b). Proceeding as in the first proof, this time it suffices to prove the formal identity

$$\begin{aligned} &\sum_{w \in S_k} \det_{1 \leqslant i,j \leqslant k} \big(x_{w_i}^j \big(E_{z_{w_i}+i-j} - E_{z_{w_i}-i-j} \big) \big) \\ &\quad = \sum_{w \in S_k} \det_{1 \leqslant i,j \leqslant k} \big(x_{w_i}^j \big(E_{z_{w_i}+i-j-1} + E_{z_{w_i}+i-j} - E_{z_{w_i}-i-j} - E_{z_{w_i}-i-j+1} \big) \big). \end{aligned}$$

This is the same as

$$\sum_{w,\sigma \in S_k} \operatorname{sgn}(\sigma) \prod_{i=1}^k x_{w_i}^{\sigma_i} \tilde{F}_{i,i}(\sigma, w) = \sum_{w,\sigma \in S_k} \operatorname{sgn}(\sigma) \prod_{i=1}^k x_{w_i}^{\sigma_i} \big(\tilde{F}_{i,i-1}(\sigma, w) + \tilde{F}_{i,i}(\sigma, w) \big), \tag{2.27}$$

where

$$\tilde{F}_{i,j}(\sigma, w) := E_{z_{w_i}+j-\sigma_i} - E_{z_{w_i}-j-\sigma_i}.$$

Since $\tilde{F}_{i,0}=0$ and thus $F_{1,0}=0$, (2.27) can be rewritten exactly as (2.25) but with F replaced by $\tilde{F}$. The remainder of the proof if identical to the first proof. ∎

3 Proof, special cases and generalisations of Theorem 1.1

Before discussing a number of special cases and generalisations of Theorem 1.1, we provide a short proof of the theorem based on a result due to Schilling and Shimozono [68]. For the sake of brevity, throughout this section we use $Q := \{y \in \mathbb{Z}^k \mid |y| = 0\}$ for the A_{k-1} root lattice. We also mostly work in the algebra of symmetric functions writing e_r, $P_\lambda(t)$ and s_λ instead of $e_r(x)$, $P_\lambda(x;t)$ and $s_\lambda(x)$.

Proof of Theorem 1.1. We will prove the result in the form (2.5), or, equivalently, in the form

$$P_{(k^r)}(t) = \sum_{y\in Q}\sum_{\sigma\in S_k} \operatorname{sgn}(\sigma) t^{\frac{1}{2}\sum_{i=1}^k (ky_i+2i)y_i} e_{(r^k)+\sigma-\rho-ky},$$

where $\rho := (1, 2, \dots, k)$ and where the elementary symmetric function in the summand vanishes unless $(r^k) + \sigma - \rho - ky$ is a (weak) composition. By (2.7),

$$P_{(k^r)}(t) = \sum_{\lambda}\sum_{y\in Q}\sum_{\sigma\in S_k} \operatorname{sgn}(\sigma) t^{\frac{1}{2}\sum_{i=1}^k (ky_i+2i)y_i} \mathcal{R}_{\lambda,(r^k)+\sigma-\rho-ky}(t) P_\lambda(t),$$

where $\mathcal{R}_{\lambda,\alpha}(t) := 0$ if $\min\{\alpha\} < 0$. Equating coefficients of $P_\lambda(t)$ on both sides, we are left to show that

$$\sum_{y\in Q}\sum_{\sigma\in S_k} \operatorname{sgn}(\sigma) t^{\frac{1}{2}\sum_{i=1}^k (ky_i+2i)y_i} \mathcal{R}_{\lambda,(r^k)+\sigma-\rho-ky}(t) = \delta_{\lambda,(k^r)} \tag{3.1}$$

for all partitions $\lambda \vdash rk$ such that $\lambda_1 \leqslant k$. (As noted on p. 11, if $\lambda_1 > k$ or $\lambda \neq |kr|$, then $\mathcal{R}_{\lambda,(r^k)+\sigma-\rho-ky}(t) = 0$, in which case there is nothing to prove.) For s a nonnegative integer such that $s \leqslant r$, let $r - s$ be the multiplicity of parts of size k in λ, and by mild abuse of notation, write $\lambda = (k^{r-s}, \mu)$ where $\mu \vdash ks$ such that $\mu_1 < k$. Then, by (2.9), the identity (3.1) may be simplified to

$$\sum_{y\in Q}\sum_{\sigma\in S_k} \operatorname{sgn}(\sigma) t^{\frac{1}{2}\sum_{i=1}^k (ky_i+2i)y_i} \mathcal{R}_{\mu,(s^k)+\sigma-\rho-ky}(t) = \delta_{s,0}. \tag{3.2}$$

Recalling (2.10), for $k = 2$ and $\mu = (1^{2s})$ this is the well-known (see, e.g., [78, equation (2.3)])

$$\sum_{y\in\mathbb{Z}} (-1)^y t^{\binom{y}{2}} \begin{bmatrix} 2s \\ s-y \end{bmatrix}_t = \delta_{s,0}. \tag{3.3}$$

For $k = 3$, it is the $\ell = 0$ instance of [5, Proposition 5.1], where μ and s are parametrised as $\mu = (2^{2L_2-L_1}, 1^{2L_1-L_2})$ for $4L_2 \geqslant 2L_1 \geqslant L_2$ and $s = L_2$. The identity (3.2) for general k was first proposed in [78, equation (6.5)] and subsequently proved by Schilling and Shimozono [68, equation (6.6)] as an identity for $U_q(\mathrm{A}_{k-1}^{(1)})$ level-restricted path at trivial level 0. ■

In the remainder of this section, we discuss a number of special cases and generalisations of Theorem 1.1.

On [51, p. 214], Macdonald gives the following Schur expansion for the Hall–Littlewood polynomial indexed by a partition of length at most one.

Lemma 3.1. *For k a nonnegative integer,*

$$P_{(k)}(t) = \sum_{a=0}^{k-1} (-t)^a s_{(k-a,1^a)},$$

where $s_{(k-0,1^0)} := s_k$ and $s_{(1,1^{k-1})} := s_{(1^k)}$.

It is not hard to show that this follows from Theorem 1.1.

Proof of Lemma 3.1. Equation (1.3) for $r=1$ simplifies to

$$P_{(k)}(t)=\sum_{y\in Q}\det_{1\leqslant i,j\leqslant k}\bigl(t^{k\binom{y_i}{2}+iy_i}e_{1-i+j-ky_i}\bigr).$$

Since $e_r=0$ for $r<0$, all entries in the first row of the determinant are zero unless $y_1\leqslant 1$. Moreover, for $2\leqslant i\leqslant k$ all entries in row i are zero unless $y_i\leqslant 0$. Since $y\in Q$, this implies that there are exactly k choices for y such that the determinant is non-vanishing: $y=\bigl(0^k\bigr)$ and $y=\bigl(1,0^{k-a-1},-1,0^{a-1}\bigr)$ for $1\leqslant a\leqslant k-1$. For $y=\bigl(0^k\bigr)$, this yields the Schur function $s_{(k)}$ by (1.2). The contribution to the sum from $y=\bigl(1,0^{k-a-1},-1,0^{a-1}\bigr)$ is

$$D_a(t):=t^a\det_{1\leqslant i,j\leqslant k}\bigl(e_{j-i+1+k\delta_{i,k-a+1}-k\delta_{i,1}}\bigr).$$

All entries in the first row of the determinant are zero except for the last, which is 1. By a Laplace expansion along the first row,

$$D_a(t)=t^a(-1)^{k-1}\det_{1\leqslant i,j\leqslant k-1}\bigl(e_{j-i+k\delta_{i,k-a}}\bigr).$$

In the determinant, we cycle rows $1,2,\dots,k-a\mapsto k-a,1,2,\dots,k-a-1$. In other words, row $k-a$ is put as the first row, pushing down rows $1,\dots,k-a-1$, leading to

$$D_a(t)=(-t)^a\det_{1\leqslant i,j\leqslant k-1}\bigl(e_{j-i+1+a\delta_{i,1}-\chi(i>k-a)}\bigr),$$

where $\chi(\text{true})=1$ and $\chi(\text{false})=0$. By (1.2) with $k\mapsto k-1$, this yields $D_a(t)=(-t)^a s_\lambda$, where

$$\lambda'=\bigl(a+1,1^{k-a-1},0^{a-1}\bigr)=\bigl(a+1,1^{k-a-1}\bigr),$$

and thus $\lambda=(k-a,1^a)$. Hence

$$P_{(k)}(t)=s_{(k)}+\sum_{a=1}^{k-1}(-t)^a s_{(k-a,1^a)},$$

as required. ■

In [38, Theorem 7.2], Lassalle and Schlosser derived an expression for $P_\lambda(t)$ for arbitrary λ in terms of elementary symmetric functions. Their formula does not have a determinantal form, and it seems difficult to deduce Theorem 1.1 from the $\lambda=(k^r)$ case of the Lassalle–Schlosser theorem. For $k=2$, however, the two theorems are trivially the same.

Lemma 3.2 (Lassalle–Schlosser)**.** *For r a nonnegative integer,*

$$P_{(2^r)}(t)=\sum_{i=-r}^{r}(-1)^i t^{\binom{i}{2}}e_{r+i}e_{r-i}.$$

Proof. According to (1.3) for $k=2$,

$$\begin{aligned}P_{(2^r)}(t)&=\sum_{\substack{y_1,y_2\in\mathbb{Z}\\ y_1+y_2=0}}t^{2\binom{y_1}{2}+2\binom{y_2}{2}+y_1+2y_2}\bigl(e_{r-2y_1}e_{r-2y_2}-e_{r-2y_1+1}e_{r-2y_2-1}\bigr)\\&=\sum_{i\in\mathbb{Z}}t^{2i^2-i}\bigl(e_{r-2i}e_{r+2i}-e_{r-2i+1}e_{r+2i-1}\bigr)\\&=\sum_{i=-r}^{r}(-1)^i t^{\binom{i}{2}}e_{r+i}e_{r-i},\end{aligned}$$

which is the Lassalle–Schlosser expression. ■

By (2.6), the $t=1$ specialisation of Theorem 1.1 is

$$m_{(k^r)}(x;t)=\sum_{y\in Q}\det_{1\leqslant i,j\leqslant k}\big(e_{r-i+j-ky_i}(x)\big). \tag{3.4}$$

This identity is an immediate consequence of Proposition 2.1.

Proof of (3.4). For $x=(x_1,\dots,x_n)$, let $f\big(x^k\big)=f\big(x_1^k,x_2^k,\dots,x_n^k\big)$. Then

$$m_{(k^r)}(x)=e_r\big(x^k\big).$$

We thus need to show that

$$\sum_{y\in Q}\det_{1\leqslant i,j\leqslant k}\big(e_{r-i+j-ky_i}(x)\big)=e_r\big(x^k\big).$$

This follows from the $\ell=0$ cases of Proposition 2.1 and (2.3). ■

Let $\lambda\in\mathrm{Par}_{n,k}^{\ell}$. Then the above connection with cylindric Schur functions suggest that the symmetric function

$$S_{\lambda}^{k,\ell}(t)=\sum_{y\in Q}\det_{1\leqslant i,j\leqslant k}\Big(t^{(k+\ell)\binom{y_i}{2}+iy_i}e_{\lambda'_i-i+j-(k+\ell)y_i}\Big)$$

– which is a t-deformation of the right-hand side of (2.2) – warrants further study. As in the Schur case, for $\ell\geqslant l(\lambda)$ the summand on the right vanishes unless $y=\big(0^k\big)$, and for such ℓ, $S_{\lambda}^{k,\ell}(t)=s_\lambda$. Also, from (1.3), $S_{(k^r)}^{k,0}(t)=P_{(k^r)}(t)$. Using further results by Schilling and Shimozono, the Hall–Littlewood expansion of $S_{\lambda}^{k,\ell}(t)$ for all admissible partitions λ may be given. Below we will describe the simplest case $\lambda=(k^r)$ and $\ell\geqslant 1$. By abuse of notation, we denote the type A Cartan matrix and its inverse by C and C^{-1} respectively, without reference to the rank of the underlying root system which will always be assumed to be clear from the context in which these matrices are used. In the case of A_{k-1}, C and C^{-1} are $(k-1)\times(k-1)$ matrices with entries $C_{a,b}=2\delta_{a,b}-\delta_{a,b-1}-\delta_{a,b+1}$ and $C_{a,b}^{-1}=\min\{a,b\}-ab/k$ for $1\leqslant a,b\leqslant k-1$. For $\mu=\big(k^{m_k(\mu)},\dots,1^{m_1(\mu)}\big)\vdash kr$, let

$$\varphi_\ell(\mu)=\frac{1}{2\ell}\sum_{a,b=1}^{k-1}C_{a,b}^{-1}m_a(\mu)m_b(\mu)$$

and define $K_{(r^k),\mu}^{\ell}(t)$ as the coefficients in the expansion of $S_{(k^r)}^{\ell}(t)$ in terms of Hall–Littlewood polynomials:

$$S_{(k^r)}^{k,\ell}(t)=t^{-k\binom{r}{2}}\sum_{\substack{\mu\vdash kr\\ \mu_1\leqslant k}}t^{\varphi_\ell(\mu)}K_{(r^k),\mu}^{\ell}(t)P_\mu(t).$$

Proposition 3.3. *Let $\mu\vdash kr$ such that $\mu_1\leqslant k$. Then the polynomial $K_{(r^k),\mu}^{\ell}(t)$ is the level-ℓ restricted generalised Kostka polynomial $K_{\lambda R}^{\ell}(t)$ (see, e.g., [68, 69]) for $\lambda=\big(r^k\big)$ and R the sequence of single-column partitions*

$$R=\big(\underbrace{(1),\dots,(1)}_{m_1(\mu)\text{ times}},\underbrace{(1^2),\dots,(1^2)}_{m_2(\mu)\text{ times}},\dots,\underbrace{(1^k),\dots,(1^k)}_{m_k(\mu)\text{ times}}\big).$$

By [69, equation (6.7)],

$$K^{\ell}_{(k^r),\mu}(t) = t^{k\binom{r}{2}} \sum t^{\frac{1}{2}\sum_{a,b=1}^{k-1}\sum_{i,j=1}^{\ell-1} C_{a,b}C^{-1}_{i,j}\tau_i^{(a)}\tau_j^{(b)}} \prod_{a=1}^{k-1}\prod_{i=1}^{\ell-1} \begin{bmatrix} \nu_i^{(a)} + \tau_i^{(a)} \\ \tau_i^{(a)} \end{bmatrix}_t,$$

where the sum is over $\tau_i^{(a)} \in \mathbb{N}_0$ for $1 \leqslant a \leqslant k-1$ and $1 \leqslant i \leqslant \ell - 1$ such that

$$\nu_i^{(a)} := m_a(\mu)C^{-1}_{i,1} - \sum_{b=1}^{k-1}\sum_{j=1}^{\ell-1} C_{a,b}C^{-1}_{i,j}\tau_j^{(b)} \in \mathbb{N}_0.$$

In particular,

$$S^{k,1}_{(k^r)}(t) = \sum_{y\in Q} \det_{1\leqslant i,j\leqslant k}\bigl(t^{(k+1)\binom{y_i}{2}+iy_i} e_{r-i+j-(k+1)y_i}\bigr) = \sum_{\substack{\mu\vdash kr \\ \mu_1\leqslant k}} t^{\varphi_\ell(\mu)}P_\mu(t).$$

Proof of Proposition 3.3. The proof of the positive-level case is the exact same as the proof for the level-0 case, but now uses

$$\sum_{y\in Q}\sum_{\sigma\in S_k} \operatorname{sgn}(\sigma) t^{\frac{1}{2}\sum_{i=1}^{k}((k+1)y_i+2i)y_i} \mathcal{R}_{\mu,(r^k)+\sigma-\rho-(k+1)y}(t) = t^{-k\binom{r}{2}+\varphi_\ell(\mu)} K^{\ell}_{(r^k),\mu}(t)$$

for $\mu \vdash kr$ such that $\mu_1 \leqslant k$. This identity was first conjectured in [70, equation (9.2)] and proved by Schilling and Shimozono for $\ell = 1$ in [68] and for arbitrary positive level in [69, Section 6.2]. ■

We remark that the $t = \ell = 1$ case of Proposition 3.3 is

$$\sum_{y\in Q} \det_{1\leqslant i,j\leqslant k}\bigl(e_{r-i+j-(k+1)y_i}\bigr) = \sum_{\substack{\mu\vdash kr \\ \mu_1\leqslant k}} m_\mu(t),$$

which follows from the $\ell = 1$ case of Proposition 2.1 and (2.4).

We conclude this section with a straightforward corollary of Theorem 1.1 in the form of an $\mathrm{A}^{(1)}_{k-1}$ basic hypergeometric summation.

Corollary 3.4. *For $k \geqslant 1$ and $0 \leqslant r \leqslant n$ integers,*

$$\sum_{y\in Q}\prod_{1\leqslant i<j\leqslant k}\bigl(1 - t^{k(y_i-y_j)-i+j}\bigr)\prod_{i=1}^{k} t^{k(k+1)\binom{y_i}{2}-iy_i} \begin{bmatrix} n+k-1 \\ n-r-ky_i+i-1 \end{bmatrix}_t = \begin{bmatrix} n \\ r \end{bmatrix}_t \prod_{i=1}^{k}(t^{n+i};t)_{k-i}.$$

For $k = 3$, this result, which plays a key role in the A_2 Bailey lemma [5, 77], was first proved in [5, p. 692]. For general k, different proofs may be found in [37, 78, 77], see equations (3), (6.6) and (A.3) in these papers, respectively.

Proof. By [51, p. 27]

$$e_r\bigl(1,t,\dots,t^{n-1}\bigr) = t^{\binom{r}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_t$$

and [51, p. 213]

$$P_{(k^r)}\bigl(1,t\dots,t^{n-1};t\bigr) = t^{k\binom{r}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_t, \tag{3.5}$$

the principal specialisation $x_i = t^{i-1}$ of (1.3) is given by

$$\sum_{y\in Q} \det_{1\leqslant i,j\leqslant k}\left(t^{k(k+1)\binom{y_i}{2}+iy_i+(j-i)(r-i-ky_i)}\begin{bmatrix} n \\ r-i+j-ky_i\end{bmatrix}_t\right) = \begin{bmatrix} n \\ r\end{bmatrix}_t.$$

The determinant on the left evaluates by [36, p. 169]

$$\det_{1\leqslant i<j\leqslant k}\left(t^{(j-i)a_i}\begin{bmatrix} n \\ a_i+j\end{bmatrix}_t\right) = \prod_{1\leqslant i<j\leqslant k}\left(1-t^{a_i-a_j}\right)\prod_{i=1}^{k}\frac{1}{(t^{n+i};t)_{k-i}}\begin{bmatrix} n+k-1 \\ n-a_i-1\end{bmatrix}_t$$

for $a_i = r-i-ky_i$. Finally, replacing y_i by $-y_i$ for all $1\leqslant i\leqslant k$, we are done. ∎

4 Proofs of special cases of Conjectures 1.2–1.4

In this section, we prove some special cases of the B_n, C_n and BC_n Jacobi–Trudi identities.

Lemma 4.1. *Conjecture 1.2 holds for $k=1$.*

Before proving the lemma, we recall that a Bailey pair relative to q^ℓ (for ℓ a nonnegative integer) is a pair of sequences $(\alpha_n)_{n\geqslant 0}$, $(\beta_n)_{n\geqslant 0}$ such that

$$\beta_n = \frac{1}{(q^{\ell+1};q)_{2n}}\sum_{k=0}^{n}\alpha_k\begin{bmatrix} 2n+\ell \\ n-k\end{bmatrix}_q, \tag{4.1}$$

see, e.g., [3, 6, 54, 79]. Slater [73] compiled an extensive list of Bailey pairs, which when inserted into (4.1) result in identities for sums of q-binomial coefficients. The proof of Lemma 4.1 and its C_n and BC_n analogues given in Lemmas 4.2 and 4.3 below is based on the identities arising from the Bailey pairs A(1)–A(4), E(1), E(2), F(1), F(2).

Proof. For $k=1$, the conjecture simplifies to the three identities

$$P^{\mathrm{B}_n}_{(1^n)}(x;t,0) = \sum_{y\in\mathbb{Z}}(-1)^y t^{\frac{1}{2}(3y-1)y}\left(\dot{e}_{n-3y}(x)+\dot{e}_{n-3y+1}(x)\right), \tag{4.2a}$$

$$P^{\mathrm{B}_n}_{(1^n)}\big(x;t,-t^{1/2}\big) = \sum_{y\in\mathbb{Z}} t^{\frac{1}{2}(2y-1)y}\left(\dot{e}_{n-2y}(x)+\dot{e}_{n-2y+1}(x)\right), \tag{4.2b}$$

$$P^{\mathrm{B}_n}_{(1^n)}(x;t,1) = \sum_{y\in\mathbb{Z}}(-1)^y t^{y^2}\dot{e}_{n-2y}(x). \tag{4.2c}$$

First we consider (4.2a). By the $k=1$ case of the bounded Littlewood identity (2.19a) as well as Lemma 2.3, identity (4.2a) is equivalent to

$$\begin{aligned}\sum_{s,r\geqslant 0}P_{(2^r,1^s)}(x;t) &= \sum_{y\in\mathbb{Z}}(-1)^y t^{\frac{1}{2}(3y-1)y}\sum_{\substack{r,s\geqslant 0\\ s+y\text{ even}}}\begin{bmatrix} s \\ \frac{1}{2}(s-3y)\end{bmatrix}_t P_{(2^r,1^s)}(x;t)\\ &\quad+\sum_{y\in\mathbb{Z}}(-1)^y t^{\frac{1}{2}(3y-1)y}\sum_{\substack{r,s\geqslant 0\\ s+y\text{ odd}}}\begin{bmatrix} s \\ \frac{1}{2}(s-3y+1)\end{bmatrix}_t P_{(2^r,1^s)}(x;t).\end{aligned}$$

We next equate coefficients of $P_{(2^r,1^s)}(x;t)$ and make the substitution $t\mapsto q$. This leads to the q-binomial identity

$$\sum_{y\in\mathbb{Z}}(-1)^y q^{\frac{1}{2}(3y-1)y}\begin{bmatrix} s \\ \lceil\frac{1}{2}(s-3y)\rceil\end{bmatrix}_q = 1.$$

For $s=2n$, this follows from the Bailey pair A(1) in Slater's list and for $s=2n+1$ it follows from the pair A(2).

We apply analogous steps to (4.2b) and (4.2c), this time using the bounded Littlewood identities (2.19b) and (2.19c) for $k=1$. This yields

$$\sum_{y\in\mathbb{Z}} q^{\frac{1}{2}(2y-1)y}\begin{bmatrix} s \\ \lceil\frac{1}{2}(s-2y)\rceil\end{bmatrix}_q = \left(-q^{1/2};q^{1/2}\right)_s \qquad \text{and} \qquad \sum_{y\in\mathbb{Z}}(-1)^y q^{y^2}\begin{bmatrix} 2n \\ n-y\end{bmatrix} = \left(q;q^2\right)_n.$$

(In the last identity, s has been replaced by $2n$; the coefficient of $P_{(2^r,1^s)}(x;t)$ for odd s is zero on both sides of (4.2c).) For $s=2n$, the first of these identities follows from the Bailey pair F(1) while for $s=2n+$ it follows from F(2). The second identity follows from the Bailey pair E(1). ∎

Lemma 4.2. *Conjecture 1.3 holds for $k=1$.*

Proof. The identities to be proved are

$$P^{\mathrm{C}_n}_{(1^n)}(x;t,0)=\sum_{y\in\mathbb{Z}}(-1)^y t^{\binom{y}{2}}\dot{e}_{n-2y}(x) \qquad \text{and} \qquad P^{\mathrm{C}_n}_{(1^n)}(x;t,t)=\sum_{y\in\mathbb{Z}}(-1)^y t^{2\binom{y}{2}}\dot{e}_{n-2y}(x).$$

Following the approach of the previous proof, using Lemma 2.3 and the $k=1$ instances of (2.20a) and (2.20b), the underlying polynomial identities turn out to be (3.3) and

$$\sum_{y\in\mathbb{Z}}(-1)^y q^{2\binom{y}{2}}\begin{bmatrix} 2n \\ n-y\end{bmatrix}_q = q^n\left(q;q^2\right)_n,$$

where we have replaced t and s by q and n, respectively. It follows from the Bailey pair E(4). ∎

Lemma 4.3. *Conjecture 1.4 holds for $k=1$.*

Proof. The claim is that

$$P^{\mathrm{BC}_n}_{(1^n)}\left(x;t,-t^{1/2},0\right)=\sum_{y\in\mathbb{Z}}(-1)^y t^{\frac{1}{2}(3y-2)y}\big(\dot{e}_{n-3y}(x)-\dot{e}_{n-3y+2}(x)\big).$$

By Lemma 2.3 and (2.21), this leads to

$$\sum_{\substack{y\in\mathbb{Z}\\ y-s\text{ even}}} q^{\frac{1}{2}(3y-1)y}\left(\begin{bmatrix} s \\ \frac{1}{2}(s-3y)\end{bmatrix}_q - \begin{bmatrix} s \\ \frac{1}{2}(s-3y+2)\end{bmatrix}_q\right) = q^{s/2},$$

where once again we have replaced t by q. For $s=2n$, this follows from the Bailey pair A(3) and for $s=2n+1$ from the pair A(4). ∎

Lemma 4.4. *The* B_n *and* BC_n *identities* (1.6a) *and* (1.8) *hold for* $t=1$.

Proof. Throughout the proof, $K=2k+1$.

By (2.17a) and (2.18), the $t=1$ specialisation of (1.6a) is

$$\sum_{y\in\mathbb{Z}^k}\det_{1\leqslant i,j\leqslant k}\left((-1)^{y_i}\big(\dot{e}_{n-i+j-Ky_i}(x)+\dot{e}_{n+i+j-Ky_i-1}(x)\big)\right)=\prod_{i=1}^n\frac{x_i^{-k}-x_i^{k+1}}{1-x_i},$$

Similarly, by (2.18), the $t=1$ case of (1.8) is

$$\sum_{y\in\mathbb{Z}^k}\det_{1\leqslant i,j\leqslant k}\left((-1)^{y_i}\big(\dot{e}_{n-i+j-Ky_i}(x)-\dot{e}_{n+i+j-Ky_i}(x)\big)\right)=\prod_{i=1}^n\frac{x_i^{-k}-x_i^{k+1}}{1-x_i}.$$

By multilinearity and

$$\dot{e}_{n-r}(x)=(x_1\cdots x_n)^{-1}\sum_{m=0}^{n}e_m(x)e_{m+r}(x),$$

we must thus show that

$$\det_{1\leqslant i,j\leqslant k}\big(\overline{F}_{i-j,K}(x)+\overline{F}_{1-i-j,K}(x)\big)=\det_{1\leqslant i,j\leqslant k}\big(\overline{F}_{i-j,K}(x)-\overline{F}_{-i-j,K}(x)\big)=\prod_{i=1}^{n}\frac{1-x_i^K}{1-x_i},$$

where (see [27, Theorem 3.1])

$$\overline{F}_{i,N}(x):=\sum_{y,m\in\mathbb{Z}}(-1)^y e_m(x)e_{m+Ny+i}(x).$$

Easily established relations for this function are

$$\overline{F}_{i,N}(x)=\overline{F}_{-i,N}(x)=-\overline{F}_{i-N,N}(x).$$

The substitution $(i,j)\mapsto(k+1-i,k+1-j)$ in either one of the determinants thus leads to the other, so that it suffices to prove the equality of the first determinant and the product expression on the right. By the first of the above relations for $\overline{F}_{i,N}(x)$, it follows that the first determinant is exactly the right-hand side of [27, equation (3.2)] for $w=1$ and $h=k$. Hence we may replace the determinant by the $w=1$ and $h=k$ instance of the left-hand side of [27, equation (3.2)], which in our notation is

$$\sum_{\lambda\in\mathrm{Par}^1_{n,2k}}\;\sum_{T\in\mathrm{CSSYT}_{n;2k,1}(\lambda)}x^T.$$

By (2.4), this is equal to

$$\sum_{\mu\in\mathrm{Par}_{n,2k}}m_\mu(x)=\prod_{i=1}^{n}\frac{1-x_i^{2k+1}}{1-x_i}=\prod_{i=1}^{n}\frac{1-x_i^K}{1-x_i},$$

as required. ■

Out of the remaining identities (1.6b), (1.6c), (1.7a) and (1.7b), we only know how to recast the $t=1$ case of (1.7a) in terms of cylindric tableaux. As a first step, by (2.17b) and (2.18) the $t=1$ case of (1.7a) can be rewritten as

$$\sum_{y\in\mathbb{Z}^k}\det_{1\leqslant i,j\leqslant k}\big(\dot{e}_{n-Ky_i-i+j}(x)-\dot{e}_{n-Ky_i-i-j}(x)\big)=\prod_{i=1}^{n}\frac{x_i^{k+1}-x_i^{-k-1}}{x_i-x_i^{-1}},$$

where $K:=2k+2$. Following [27] and defining

$$F_{i,N}(x):=\sum_{y,m\in\mathbb{Z}}e_m(x)e_{m+Ny+i}(x), \tag{4.3}$$

this can also be stated as

$$\det_{1\leqslant i,j\leqslant k}\big(F_{i-j,K}(x)-F_{i+j,K}(x)\big)=\prod_{i=1}^{n}\frac{1-x_i^{2k+2}}{1-x_i^2}. \tag{4.4}$$

Since $F_{i,N}(x) = F_{-i,N}(x)$, the left-hand side of (4.4) is exactly the right-hand side of [27, equation (3.6)] for $w = 2$ and $h = k$. Furthermore,

$$\prod_{i=1}^{n} \frac{1 - x_i^{2k+2}}{1 - x_i^2} = \sum_{\substack{\mu \text{ even} \\ \mu_i \leqslant 2k}} m_\mu(x_1, \dots, x_n).$$

Equating this with the left-hand side of [27, equation (3.6)] yields

$$\Bigg(\sum_{\lambda \in \mathrm{Par}^2_{n,2k}}{}' - \sum_{\lambda \in \mathrm{Par}^2_{n,2k}}{}'' \Bigg) \sum_{T \in \mathrm{CSSYT}_{n;2k,2}(\lambda)} x^T = \sum_{\substack{\mu \text{ even} \\ \mu_i \leqslant 2k}} m_\mu(x_1, \dots, x_n).$$

Here the prime (resp. double prime) denotes the restriction that $\lambda \in \mathrm{Par}^2_{n,2k}$ must be of the form $((2k)^r, 2a, 2b)$ (resp. $((2k)^r, 2a+1, 2b+1)$) for $r \geqslant 0$ and $0 \leqslant b \leqslant a < k$. A proof of this identity would imply a proof of (1.7a) for $t = 1$, but so far we only managed to find a proof when $k = 1$.

5 q,t-Rogers–Ramanujan identities

In this section, we will first show how Conjectures 1.2–1.4 imply the q,t-Rogers–Ramanujan identities stated in Theorems 1.5–1.8. We will then show that after specialising the Hall–Littlewood parameter t, many well-known Rogers–Ramanujan identities follow by a new manifestation of level-rank duality. Finally, we will provide proofs of Theorems 1.5–1.8 based on Ismail's argument.

5.1 From Jacobi–Trudi to q,t-Rogers–Ramanujan identities

Because it is the most important example and includes the classical Rogers–Ramanujan identities as a special case, we will first show that (1.7a) implies (1.9). We refer to this as a 'conditional proof' since it assumes the validity of the conjectural (1.7a).

Conditional proof of Theorem 1.5. By the bounded Littlewood identity (2.20a) and the determinant identity (2.23b) for $u = 1$, (1.7a) can be rewritten as

$$\begin{aligned}(x_1 \cdots x_n)^{-k} \sum_{\substack{\lambda \text{ even} \\ \lambda_1 \leqslant 2k}} P_\lambda(x;t) &= \sum_{y \in \mathbb{Z}^k} \det_{1 \leqslant i,j \leqslant k} \Big(t^{\frac{1}{2}Ky_i^2 - jy_i} \big(\dot{e}_{n-i+j-Ky_i} - \dot{e}_{n+i+j-Ky_i} \big) \Big) \\ &= \sum_{y \in \mathbb{Z}^k} \det_{1 \leqslant i,j \leqslant k} \Big(t^{\frac{1}{2}Ky_i^2 - jy_i} \big(\ddot{e}_{n-i+j-Ky_i+1} - \ddot{e}_{n+i+j-Ky_i} \big) \Big),\end{aligned}$$

where $K = 2k+2$. We view this as two identities, one between the first two expressions and one between the first and third expression. In the first identity (second) identity we specialise $x_i = q^{i-1/2}$, $x_i = q^i$, for $1 \leqslant i \leqslant n$. By [51, p. 27]

$$e_{n-r}\big(q^{n-1/2}, q^{n-3/2}, \dots, q^{1/2-n}\big) = q^{\frac{1}{2}r^2 - \frac{1}{2}n^2} \begin{bmatrix} 2n \\ n-r \end{bmatrix}_q$$

and

$$e_{n-r}\big(q^n, q^{n-1}, \dots, q^{-n}\big) = q^{\binom{r+1}{2} - \binom{n+1}{2}} \begin{bmatrix} 2n+1 \\ n-r \end{bmatrix}_q,$$

this yields

$$\sum_{\substack{\lambda \text{ even} \\ \lambda_1 \leqslant 2k}} q^{\frac{1}{2}(\sigma+1)|\lambda|} P_\lambda\big(1,q,\dots,q^{n-1};t\big)$$
$$= \sum_{y\in\mathbb{Z}^k} \det_{1\leqslant i,j\leqslant k} \bigg(t^{\frac{1}{2}Ky_i^2-jy_i} q^{\frac{1}{2}(Ky_i+i-j)(Ky_i+i-j-\sigma)} \begin{bmatrix} 2n+\sigma \\ n+Ky_i+i-j \end{bmatrix}_q - t^{\frac{1}{2}Ky_i^2+jy_i} q^{\frac{1}{2}(Ky_i+i+j)(Ky_i+i+j-\sigma)} \begin{bmatrix} 2n+\sigma \\ n+Ky_i+i+j \end{bmatrix}_q \bigg) \tag{5.1}$$

for $\sigma\in\{0,1\}$. In the above we have also used multilinearity to replace y_i by $-y_i$ for all $1\leqslant i\leqslant n$ in the second term in the determinant. The next step is to let n tend to infinity using

$$\lim_{n\to\infty} \begin{bmatrix} 2n+a \\ n+b \end{bmatrix}_q = \frac{1}{(q;q)_\infty}.$$

Thus

$$\sum_{\substack{\lambda \text{ even} \\ \lambda_1 \leqslant 2k}} q^{\frac{1}{2}(\sigma+1)|\lambda|} P_\lambda\big(1,q,q^2,\dots;t\big)$$
$$= \frac{1}{(q;q)_\infty^k} \sum_{y\in\mathbb{Z}^k} \det_{1\leqslant i,j\leqslant k} \big(x_i^{Ky_i+i-j} p^{\frac{1}{2}Ky_i^2-jy_i} - x_i^{Ky_i+i+j} p^{\frac{1}{2}Ky_i^2+jy_i} \big),$$

where $p=tq^K$ and $x_i=q^{i-\frac{1}{2}\sigma}$. The right-hand side can be written in product form by the $\mathrm{C}_k^{(1)}$ Macdonald identity [50]

$$\sum_{y\in\mathbb{Z}^k} \det_{1\leqslant i,j\leqslant k} \big(x_i^{2(k+1)y_i+i-j} p^{(k+1)y_i^2-jy_i} - x_i^{2(k+1)y_i+i+j} p^{(k+1)y_i^2+jy_i} \big)$$
$$= (p;p)_\infty^k \prod_{i=1}^k \theta\big(x_i^2;p\big) \prod_{1\leqslant i<j\leqslant k} \theta(x_j/x_i, x_ix_j;p),$$

leading to

$$\sum_{\substack{\lambda \text{ even} \\ \lambda_1 \leqslant 2k}} q^{\frac{1}{2}(\sigma+1)|\lambda|} P_\lambda\big(1,q,q^2,\dots;t\big) = \frac{(p;p)_\infty^k}{(q;q)_\infty^k} \prod_{i=1}^k \theta\big(q^{2i-\sigma};p\big) \prod_{1\leqslant i<j\leqslant k} \theta\big(q^{j-i},q^{i+j-\sigma};p\big).$$

Since

$$\prod_{i=1}^k \theta\big(q^{2i-1};p\big) \prod_{1\leqslant i<j\leqslant k} \theta\big(q^{i+j-1};p\big) = \prod_{i=1}^k \theta\big(q^{i};p\big) \prod_{1\leqslant i<j\leqslant k} \theta\big(q^{i+j};p\big),$$

this may also be stated as in Theorem 1.5. ■

We remark that (5.1) for $k=1$ can be written in the compact form

$$\sum_{r=0}^n q^{(\sigma+1)r} P_{(2^r)}\big(1,q,\dots,q^{n-1};t\big) = \sum_{i=-\infty}^{\infty} (-1)^i t^{\binom{i}{2}} q^{i(2i-\sigma)} \begin{bmatrix} 2n+\sigma \\ n+2i \end{bmatrix}_q. \tag{5.2}$$

Since

$$P_{(2^r)}\big(1,q,\dots,q^{n-1};1\big) = \begin{bmatrix} n \\ r \end{bmatrix}_{q^2},$$

this is a t-deformation of

$$\sum_{r=0}^{N} q^{r(r+\sigma)} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} = \sum_{i=-\infty}^{\infty} (-1)^i q^{i(2i-\sigma)} \begin{bmatrix} 2n+u \\ n+2i \end{bmatrix}_q \tag{5.3}$$

due to Foda and Quano [22, Theorem 1.2, $k=2$], see also [71, equations (3.2-b) and (3.3-b)] by Sills.[2] Similarly, by (3.5), with $k=2$ and $t \mapsto q$, (5.2) is also a t-deformation of

$$\sum_{r=0}^{n} q^{r(r+\sigma)} \begin{bmatrix} n \\ r \end{bmatrix}_{q} = \sum_{i=-\infty}^{\infty} (-1)^i q^{i(5i-1)/2-i\sigma} \begin{bmatrix} 2n+\sigma \\ n+2i \end{bmatrix}_q .$$

For $\sigma = 0$, this is Bressoud's finite analogue of the first Rogers–Ramanujan identity [14, equation (1.1)] and for $\sigma = 1$ it is a companion of Bressoud's identity for the second Rogers–Ramanujan identity, given in [80, p. 249].

The conditional proofs of the remaining q,t-Rogers–Ramanujan identities all proceed in the exact same manner as the proof of Theorem 1.5, and below we only summarise the key steps in each proof.

Conditional proofs of the remaining q,t-Rogers–Ramanujan identities. *From identities* (1.6a) *and* (1.8) *to Theorem* 1.6. We rewrite the left-hand side of (1.6a) ((1.8)) using the bounded Littlewood identity (2.19a) ((2.21)), while for the right-hand side we obtain a second determinantal expression using (2.23a) ((2.23b)) for $u=-1$. Next we specialise $x_i = q^{i-1/2}$ using the first determinantal expression on the right or $x_i = q^i$ using the second determinantal expression, and take the $n \to \infty$ limit. This leads to

$$\begin{aligned} &\sum_{\substack{\lambda \\ \lambda_1 \leqslant 2k}} q^{\frac{1}{2}(\sigma+1)|\lambda|} P_\lambda\big(1,q,q^2,\dots;t\big) \\ &\qquad = \frac{1}{(\sigma+1)(q;q)_\infty^k} \sum_{y \in \mathbb{Z}^k} \det_{1 \leqslant i,j \leqslant k} \Big(x_i^{Ky_i+i-j} p^{\frac{1}{2}Ky_i^2-(j-\frac{1}{2})y_i} - x_i^{Ky_i+i+j-1} p^{\frac{1}{2}Ky_i^2+(j-\frac{1}{2})y_i} \Big) \end{aligned}$$

and

$$\begin{aligned} &\sum_{\substack{\lambda \\ \lambda_1 \leqslant 2k}} q^{\frac{1}{2}(\sigma+1)|\lambda|} t^{\frac{1}{2}l(\lambda^{\mathrm{o}})} P_\lambda\big(1,q,q^2,\dots;t\big) \\ &\qquad = \frac{1}{(q;q)_\infty^k} \sum_{y \in \mathbb{Z}^k} \det_{1 \leqslant i,j \leqslant k} \Big(x_i^{Ky_i+i-j} p^{\frac{1}{2}Ky_i^2-jy_i} - x_i^{Ky_i+i+j} p^{\frac{1}{2}Ky_i^2+jy_i} \Big), \end{aligned}$$

where $\sigma \in \{0,1\}$, $p = tq^K$, $K = 2k+1$ and $x_i = -q^{i-(\sigma+1)/2}$, $x_i = -q^{i-\sigma/2}$, in the first (second) identity. By the $\mathrm{A}_{2k}^{(2)}$ Macdonald identity [50] in the 'B_k form'

$$\begin{aligned} &\sum_{y \in \mathbb{Z}^k} \det_{1 \leqslant i,j \leqslant k} \Big(x_i^{(2k+1)y_i+i-j} p^{\frac{1}{2}(2k+1)y_i^2-(j-\frac{1}{2})y_i} - x_i^{(2k+1)y_i+i+j-1} p^{\frac{1}{2}(2k+1)y_i^2+(j-\frac{1}{2})y_i} \Big) \\ &\qquad = (p;p)_\infty^k \prod_{i=1}^{k} \theta(x_i;p)\theta\big(px_i^2;p^2\big) \prod_{1 \leqslant i<j \leqslant k} \theta(x_j/x_i, x_ix_j;p) \end{aligned} \tag{5.4}$$

[2] By the q-binomial theorem, the left-hand side of (5.3) can be simplified to $\big(-q^{\sigma+1};q^2\big)_n$.

or in the 'C_k form'

$$\sum_{y\in\mathbb{Z}^k} \det_{1\leqslant i,j\leqslant k}\big(x_i^{(2k+1)y_i+i-j}p^{\frac{1}{2}(2k+1)y_i^2-jy_i} - x_i^{(2k+1)y_i+i+j}p^{\frac{1}{2}(2k+1)y_i^2+jy_i}\big)$$
$$= (p;p)_\infty^k \prod_{i=1}^k \theta\big(p^{1/2}x_i;p\big)\theta\big(x_i^2;p^2\big) \prod_{1\leqslant i<j\leqslant k} \theta(x_j/x_i,x_ix_j;p),$$

Theorem 1.6 results.

From (1.6b) *to Theorem* 1.7. We rewrite the left-hand side of (1.6b) using the bounded Littlewood identity (2.19b) and obtain a second determinantal expression on the right using (2.23a) for $u=1$. Then specialising as in the previous two cases and taking the $n\to\infty$ limit yields

$$\sum_{\substack{\lambda\\ \lambda_1\leqslant 2k}} q^{\frac{1}{2}(\sigma+1)|\lambda|}\Bigg(\prod_{i=1}^{2k-1}\big(-t^{1/2};t^{1/2}\big)_{m_i(\lambda)}\Bigg)P_\lambda\big(1,q,q^2,\dots;t\big)$$
$$= \frac{1}{(\sigma+1)(q;q)_\infty^k}\sum_{y\in\mathbb{Z}^k}\det_{1\leqslant i,j\leqslant k}\big(x_i^{Ky_i+i-j}p^{\frac{1}{2}Ky_i^2-(j-\frac{1}{2})y_i} - x_i^{Ky_i+i+j-1}p^{\frac{1}{2}Ky_i^2+(j-\frac{1}{2})y_i}\big),$$

where $\sigma\in\{0,1\}$, $x_i=q^{i-(\sigma+1)/2}$, $p=tq^K$ and $K=2k$. By the $\mathrm{D}_{k+1}^{(2)}$ Macdonald identity [50]

$$\sum_{y\in\mathbb{Z}^k} \det_{1\leqslant i,j\leqslant k}\big(x_i^{2ky_i+i-j}p^{ky_i^2-(j-\frac{1}{2})y_i} - x_i^{2ky_i+i+j-1}p^{ky_i^2+(j-\frac{1}{2})y_i}\big)$$
$$= \big(p^{1/2};p^{1/2}\big)_\infty (p;p)_\infty^{k-1}\prod_{i=1}^k \theta\big(x_i;p^{1/2}\big)\prod_{1\leqslant i<j\leqslant k}\theta(x_j/x_i,x_ix_j;p),$$

this gives Theorem 1.7.

From (1.7b) *to Theorem* 1.8. We rewrite the left-hand side of (1.7b) using the bounded Littlewood identity (2.20b) and obtain a second determinantal form on the right using (2.23b) for $u=1$. Then specialising the x_i as before and taking the $n\to\infty$ limit results in

$$\sideset{}{'}\sum_{\substack{\lambda\\ \lambda_1\leqslant 2k}} q^{\frac{1}{2}(\sigma+1)}t^{\frac{1}{2}l(\lambda^{\mathrm{o}})}\Bigg(\prod_{i=1}^{2k-1}\big(t;t^2\big)_{\lceil m_i(\lambda)/2\rceil}\Bigg)P_\lambda\big(1,q,q^2,\dots;t\big)$$
$$= \frac{1}{(q;q)_\infty^k}\sum_{\substack{y\in\mathbb{Z}^k\\ |y|\text{ even}}}\det_{1\leqslant i,j\leqslant k}\big(x_i^{Ky_i+i-j}p^{\frac{1}{2}Ky_i^2-jy_i} - x_i^{Ky_i+i+j}p^{\frac{1}{2}Ky_i^2+jy_i}\big),$$

where $\sigma\in\{0,1\}$, $x_i=q^{i-\sigma/2}$, $p=tq^K$ and $K=2k$. By the $\mathrm{A}_{2k-1}^{(2)}$ Macdonald identity [50]

$$\sum_{\substack{y\in\mathbb{Z}^k\\ |y|\text{ even}}} \det_{1\leqslant i,j\leqslant k}\big(x_i^{2ky_i+i-j}p^{ky_i^2-jy_i} - x_i^{2ky_i+i+j}p^{ky_i^2+jy_i}\big)$$
$$= \big(p^2;p^2\big)_\infty (p;p)_\infty^{k-1}\prod_{i=1}^k \theta\big(x_i^2;p^2\big)\prod_{1\leqslant i<j\leqslant k}\theta(x_j/x_i,x_ix_j;p),$$

we obtain Theorem 1.8. ∎

5.2 From q,t-Rogers–Ramanujan to Rogers–Ramanujan identities for standard modules

Thanks to the groundbreaking work of Lepowsky together with his students and collaborators [10, 16, 17, 18, 19, 30, 31, 41, 42, 43, 44, 45, 46, 47, 48, 56, 57, 58, 59], it is well known that many Rogers–Ramanujan identities admit an interpretation as identities for principal characters of standard modules of affine Lie algebras, see also [72, Section 5]. In a beautiful manifestation of level-rank duality [23], the q,t-Rogers–Ramanujan identities of Theorems 1.5–1.8 imply many Rogers–Ramanujan identities for the characters of standard modules of affine Lie algebras, albeit typically not principal ones. Most of these were previously found, such as the GOW identities for $\mathrm{A}_{2n}^{(2)}$, $\mathrm{C}_n^{(1)}$ and $\mathrm{D}_{n+1}^{(2)}$ [26] or the identities from [63], with a total of eight new identities. Below we will show how all these Rogers–Ramanujan arise via transformations for theta functions of level-rank duality type, such as

$$\frac{(p;p)_\infty^k}{(q;q)_\infty^k}\prod_{1\leqslant i<j\leqslant k}\theta\big(q^{j-i},q^{i+j-1};p\big)=\frac{(p;p)_\infty^n}{(q;q)_\infty^n}\prod_{1\leqslant i<j\leqslant n}\theta\big(q^{j-i},q^{i+j-1};p\big), \tag{5.5}$$

for $p=q^{2k+2n-1}$.

Let $\mathfrak{g}$ be one of the affine Lie algebras shown in Figure 5.2, and let $L(\Lambda)$ denote the integrable highest weight or standard module of $\mathfrak{g}$ of highest weight Λ. The character of $L(\Lambda)$ is defined as

$$\operatorname{ch}L(\Lambda)=\sum_{\mu\in\mathfrak{h}^*}\dim(V_\mu)\,\mathrm{e}^{\mu},$$

where V_μ is the weight space of weight μ in the weight-space decomposition of $L(\Lambda)$ and $\exp(\cdot)$ is a formal exponential. Each Rogers–Ramanujan identity below is a q-series identity for

$$\chi_\Lambda(a,b;q):=\phi_{a,b;q}\big(\mathrm{e}^{-\Lambda}\operatorname{ch}L(\Lambda)\big),$$

where $\phi_{a,b;q}$ is the specialisation

$$\phi_{a,b;q}\colon\ \mathbb{Z}\big[\big[\mathrm{e}^{-\alpha_0},\dots,\mathrm{e}^{-\alpha_n}\big]\big]\to\mathbb{Z}[[a,b,q]],\qquad \mathrm{e}^{-\alpha_i}\mapsto\begin{cases}a & \text{for } i=0,\\ q & \text{for } 1\leqslant i\leqslant n-1,\\ b & \text{for } i=n.\end{cases}$$

When $a=b=q$, this is the well-known principal specialisation [40] for which Lepowsky's numerator formula [41] holds

$$\phi_{q,q;q}\big(\mathrm{e}^{-\Lambda}\operatorname{ch}L(\Lambda)\big)=\prod_{\alpha\in R^+}\bigg(\frac{1-q^{\langle\Lambda+\rho,\alpha^\vee\rangle}}{1-q^{\langle\rho,\alpha^\vee\rangle}}\bigg)^{\operatorname{mult}(\alpha)}, \tag{5.6}$$

where ρ is the Weyl vector of $\mathfrak{g}$ and, for α a root, $\alpha^\vee$ is the corresponding coroot.

After these Lie algebraic preliminaries, we are ready to give our list of Rogers–Ramanujan identities, together with an interpretation in terms of specialised characters of standard modules. In many cases, this identification needs a minor modification when n is small. For example, $\mathrm{C}_1^{(1)}\cong\mathrm{A}_1^{(1)}$, $\mathrm{A}_1^{(2)}\cong\mathrm{A}_1^{(1)}$, $\mathrm{A}_3^{(2)}\cong\mathrm{D}_3^{(2)}$, and so on. In some but not all cases, this also means that for small n the highest weight indexing the character and/or the specialisation $\phi_{a,b;q}$ are not correct as stated. For example, in Corollary 5.17 for $n=1$ the correct affine Lie algebra, standard module and specialisation are $\mathrm{A}_1^{(1)}$, $L(k\Lambda_0+k\Lambda_1)$ and $\phi_{q^3,q}$, instead of the stated $\mathrm{A}_1^{(2)}$, $L(k\Lambda_1)$ and $\phi_{q^2,q}$. Similar such modifications apply to other corollaries.

We start with three applications of Theorem 1.5.

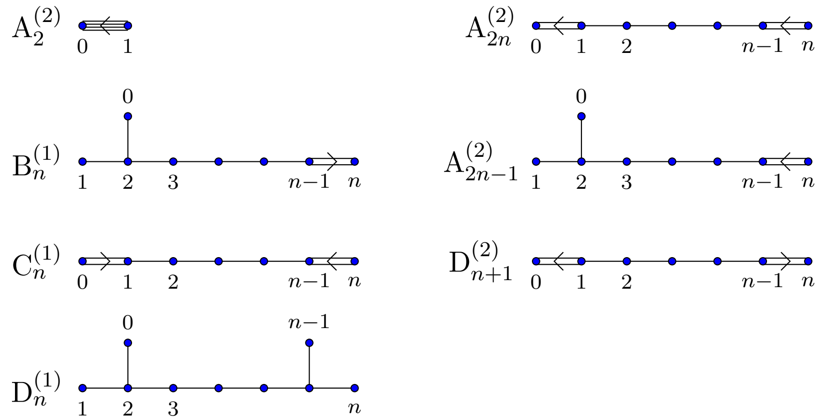


Figure 1. The Dynkin diagrams of the affine Lie algebras $\mathrm{A}_{2n}^{(2)}$ $(n \geqslant 1)$, $\mathrm{B}_n^{(1)}$ $(n \geqslant 3)$, $\mathrm{A}_{2n-1}^{(2)}$ $(n \geqslant 3)$, $\mathrm{C}_n^{(1)}$ $(n \geqslant 2)$, $\mathrm{D}_{n+1}^{(2)}$ $(n \geqslant 2)$ and $\mathrm{D}_n^{(1)}$ $(n \geqslant 4)$.

Corollary 5.1 ($\mathrm{A}_{2n}^{(2)}$ Rogers–Ramanujan identities, [26, Theorem 1.1])**.** *For k, n positive integers, let $p = q^{2k+2n+1}$. Then*

$$\chi_{k\Lambda_n}(-1,q;q) = \sum_{\substack{\lambda \text{ even} \\ \lambda_1 \leqslant 2k}} q^{|\lambda|/2} P_\lambda\big(1,q,q^2,\dots;q^{2n-1}\big)$$
$$= \frac{(p;p)_\infty^n}{(q;q)_\infty^n} \prod_{i=1}^n \theta\big(q^{i+k};p\big) \prod_{1\leqslant i<j\leqslant n} \theta\big(q^{j-i},q^{i+j-1};p\big) \tag{5.7}$$

and

$$\chi_{2k\Lambda_0}(-1,q;q) = \sum_{\substack{\lambda \text{ even} \\ \lambda_1 \leqslant 2k}} q^{|\lambda|} P_\lambda\big(1,q,q^2,\dots;q^{2n-1}\big)$$
$$= \frac{(p;p)_\infty^n}{(q;q)_\infty^n} \prod_{i=1}^n \theta\big(q^{i};p\big) \prod_{1\leqslant i<j\leqslant n} \theta\big(q^{j-i},q^{i+j};p\big), \tag{5.8}$$

By the principal specialisation formula [51, p. 213]

$$P_\lambda\big(1,q,q^2,\dots;q\big) = \prod_{i\geqslant 1} \frac{q^{\binom{\lambda_i'}{2}}}{(q;q)_{\lambda_i'-\lambda_{i+1}'}}, \tag{5.9}$$

it follows that Corollary 5.1 for $n=1$ yields the $i=k+1$ and $i=1$ instances of the modulus $2k+3$ Andrews–Gordon identities [1]

$$\sum_{n_1\geqslant\cdots\geqslant n_k\geqslant 0} \frac{q^{n_1^2+\cdots+n_k^2+n_i+\cdots+n_k}}{(q;q)_{n_1-n_2}\cdots(q;q)_{n_{k-1}-n_k}(q;q)_{n_k}} = \frac{(q^i,q^{2k-i+3},q^{2k+3};q^{2k+3})_\infty}{(q;q)_\infty}$$

for $1 \leqslant i \leqslant k+1$.

Proof. Corollary 5.1 follows after specialising $t = q^{2n-1}$ in Theorem 1.5 and using the theta function identity

$$\frac{(p;p)_\infty^k}{(q;q)_\infty^k} \prod_{i=1}^k \theta\big(q^{(2-\sigma)i};p\big) \prod_{1\leqslant i<j\leqslant k} \theta\big(q^{j-i},q^{i+j};p\big)$$

$$= \frac{(p;p)_\infty^n}{(q;q)_\infty^n} \prod_{i=1}^n \theta\big(q^{i+(1-\sigma)k};p\big) \prod_{1\leqslant i<j\leqslant n} \theta\big(q^{j-i},q^{i+j+\sigma-1};p\big),$$

for $p=q^{2k+2n+1}$ and $\sigma\in\{0,1\}$. ∎

Corollary 5.2 ($\mathrm{C}_n^{(1)}$ Rogers–Ramanujan identities, [26, Theorem 1.2])**.** *For k, n positive integers, let $p=q^{2k+2n+2}$. Then*

$$\chi_{k\Lambda_0}(q,q;q) = \sum_{\substack{\lambda \text{ even}\\ \lambda_1\leqslant 2k}} q^{|\lambda|/2} P_\lambda\big(1,q,q^2,\dots;q^{2n}\big)$$
$$= \frac{\big(p^{1/2};p^{1/2}\big)_\infty (p;p)_\infty^{n-1}}{\big(q;q^2\big)_\infty (q;q)_\infty^n} \prod_{i=1}^n \theta\big(q^i;p^{1/2}\big) \prod_{1\leqslant i<j\leqslant n} \theta\big(q^{j-i},q^{i+j};p\big). \tag{5.10}$$

Proof. The result follows by specialising $t=q^{2n}$ in the $\sigma=0$ instance of (1.9) and using

$$\frac{(p;p)_\infty^k}{(q;q)_\infty^k} \prod_{i=1}^k \theta\big(q^{2i};p\big) \prod_{1\leqslant i<j\leqslant k} \theta\big(q^{j-i},q^{i+j};p\big)$$
$$= \frac{\big(p^{1/2};p\big)_\infty (p;p)_\infty^n}{\big(q;q^2\big)_\infty (q;q)_\infty^n} \prod_{i=1}^n \theta\big(q^i;p^{1/2}\big) \prod_{1\leqslant i<j\leqslant n} \theta\big(q^{j-i},q^{i+j};p\big),$$

where $p=q^{2k+2n+2}$. ∎

Corollary 5.3 ($\mathrm{D}_{n+1}^{(2)}$ Rogers–Ramanujan identities, [26, Theorem 1.2])**.** *For k, n positive integers, let $p=q^{2k+2n}$. Then*

$$\chi_{2k\Lambda_0}(-1,-1;q) = \sum_{\substack{\lambda \text{ even}\\ \lambda_1\leqslant 2k}} q^{|\lambda|} P_\lambda\big(1,q,q^2,\dots;q^{2n-2}\big)$$
$$= \frac{(p;p)_\infty^n}{\big(q^2;q^2\big)_\infty (q;q)_\infty^{n-1}} \prod_{1\leqslant i<j\leqslant n} \theta\big(q^{j-i},q^{i+j-1};p\big). \tag{5.11}$$

Proof. This follows by specialising $t=q^{2n}$ in (1.9) for $\sigma=1$ and using

$$\frac{(p;p)_\infty^k}{(q;q)_\infty^k} \prod_{i=1}^k \theta\big(q^{i};p\big) \prod_{1\leqslant i<j\leqslant k} \theta\big(q^{j-i},q^{i+j};p\big)$$
$$= \frac{(p;p)_\infty^n}{\big(q^2;q^2\big)_\infty (q;q)_\infty^{n-1}} \prod_{1\leqslant i<j\leqslant n} \theta\big(q^{j-i},q^{i+j-1};p\big), \tag{5.12}$$

where $p=q^{2k+2n}$. ∎

The next four corollaries for even k are all consequences of (1.11a), whereas for odd k they follow from Corollaries 5.1–5.3 combined with (2.14a) for $a=1$ and $n\to\infty$.

Corollary 5.4 ($\mathrm{D}_{n+1}^{(2)}$ Rogers–Ramanujan identities, [63, Theorem 5.14])**.** *For k, n positive integers, let $p=q^{k+2n}$. Then*

$$\chi_{k\Lambda_0}\big(q^{1/2},-1;q\big) = \sum_{\substack{\lambda\\ \lambda_1\leqslant k}} q^{|\lambda|/2} P_\lambda\big(1,q,q^2,\dots;q^{2n-1}\big)$$
$$= \frac{\big(-q^{1/2};q\big)_\infty (p;p)_\infty^n}{(q;q)_\infty^n} \prod_{i=1}^n \theta\big(p^{1/2}q^{i-1/2};p\big) \prod_{1\leqslant i<j\leqslant n} \theta\big(q^{j-i},q^{i+j-1};p\big).$$

By (5.9), for $n=1$ the corollary is the $i=k+1$ case of [4, Theorem 3]

$$\sum_{n_1\geqslant\cdots\geqslant n_k\geqslant 0}\frac{q^{\frac{1}{2}(n_1^2+\cdots+n_k^2)+n_i+n_{i+2}+\cdots+n_{k-1}}}{(q;q)_{n_1-n_2}\cdots(q;q)_{n_{k-1}-n_k}(q;q)_{n_k}} = \frac{\big(-q^{1/2};q\big)_\infty\big(q^{i/2},q^{k-i/2+2},q^{k+2};q^{k+2}\big)_\infty}{(q;q)_\infty}, \tag{5.13}$$

where $1\leqslant i\leqslant k+1$ such that $i+k$ is odd, see also [11, p. 449] and [81, Theorem 4.4].

Proof. For k even, the claim follows by specialising $t=q^{2n-1}$ in (1.11a) for $\sigma=0$ and applying the $a=p^{1/2}$ case of

$$\frac{(p;p)_\infty^k}{(q;q)_\infty^k}\prod_{i=1}^k\theta\big(-p^{1/2}q^{i-1/2}/a;p\big)\theta\big(a^2q^{2i-1};p^2\big)\prod_{1\leqslant i<j\leqslant k}\theta\big(q^{j-i},q^{i+j-1};p\big) = \frac{\big(-q^{1/2};q\big)_\infty(p;p)_\infty^n}{(q;q)_\infty^n}\prod_{i=1}^n\theta\big(aq^{i-1/2};p\big)\prod_{1\leqslant i<j\leqslant n}\theta\big(q^{j-i},q^{i+j-1};p\big), \tag{5.14}$$

where $p=q^{2k+2n}$ and $a\in\{1,p^{1/2}\}$. For k odd, we combine (5.7) with the $a=1$, $n\to\infty$ and $x_i=q^{i-1/2}$ specialisation of (2.14a), and use

$$\prod_{i=1}^n\theta\big(q^{i+k};q^{2k+2n+1}\big)=\prod_{i=1}^n\theta\big(q^{i+k+n};q^{2k+2n+1}\big)\prod_{i=1}^n\theta\big(q^{(k+n+1)/2}\cdot q^{i-1/2};q^{2k+2n+1}\big)$$

to write the product of theta function in the desired form. ∎

The next result is our first new Rogers–Ramanujan identity.

Corollary 5.5 ($\mathrm{A}_{2n-1}^{(2)}$ Rogers–Ramanujan identities)**.** *For k, n positive integers, let $p=q^{k+2n}$. Then*

$$\chi_{k\Lambda_0}(q,q;q)=\sum_{\substack{\lambda\\ \lambda_1\leqslant k}}q^{|\lambda|}P_\lambda\big(1,q,q^2,\dots;q^{2n-1}\big) = \frac{(p;p)_\infty^n}{\big(q;q^2\big)_\infty(q;q)_\infty^n}\prod_{i=1}^n\theta\big(q^i;p\big)\prod_{1\leqslant i<j\leqslant n}\theta\big(q^{j-i},q^{i+j};p\big). \tag{5.15}$$

By (5.9), the $n=1$ case yields the $i=0$ instance of the following generating function identity for certain Kleshchev multipartitions [4, Theorem 3], [20, Theorem 3.5], [34, Theorem 1.4] or tight cylindric partitions of two rows [32, Theorem 13]:

$$\sum_{n_1\geqslant\cdots\geqslant n_k\geqslant 0}\frac{q^{\binom{n_1+1}{2}+\cdots+\binom{n_k+1}{2}-n_2-n_4-\cdots-n_{2i}}}{(q;q)_{n_1-n_2}\cdots(q;q)_{n_{k-1}-n_k}(q;q)_{n_k}}=\frac{\big(q^{i+1},q^{k-i+1},q^{k+2};q^{k+2}\big)_\infty}{\big(q;q^2\big)_\infty(q;q)_\infty}, \tag{5.16}$$

where $0\leqslant i\leqslant\lfloor k/2\rfloor$.

Proof. For even k, the result follows by choosing $t=q^{2n-1}$ in (1.11a) for $\sigma=1$ and applying (5.12) with k and n interchanged, recalling that $(q;q)_\infty/\big(q^2;q^2\big)_\infty=\big(q;q^2\big)_\infty$. For odd k, the result follows from (5.8) and the $a=1$, $n\to\infty$ and $x_i=q^i$ case of (2.14a). The identification of the result as $\chi_{k\Lambda_0}(q,q;q)$ for the affine Lie algebra $\mathrm{A}_{2n-1}^{(2)}$ is a direct consequence of (5.6). This formula implies that for the level-ℓ dominant integral weights of $\mathrm{A}_{2n-1}^{(2)}$ parametrised as

$$\Lambda=(\ell-\lambda_1-\lambda_2)\Lambda_0+(\lambda_1-\lambda_2)\Lambda_1+\cdots+(\lambda_{n-1}-\lambda_n)\Lambda_{n-1}+\lambda_n\Lambda_n,$$

where $(\lambda_1,\dots,\lambda_n)$ a partition such that $\lambda_1+\lambda_2\leqslant \ell$,

$$\chi_\Lambda(q,q;q)=\frac{(p;p)_\infty^n}{(q;q^2)_\infty(q;q)_\infty^n}\prod_{i=1}^n\theta\big(q^{\lambda_i+n-i+1};p\big)\prod_{1\leqslant i<j\leqslant n}\theta\big(q^{\lambda_i-\lambda_j-i+j},q^{\lambda_i+\lambda_j+2n-i-j+2};p\big).$$

Here $p=q^{\mathrm{lev}(\Lambda)+h^\vee}=q^{\ell+2n}$. For $\ell=k$ and $\lambda_1=\dots=\lambda_n=0$ this yields the right-hand side of (5.15). ■

Corollary 5.6 ($\mathrm{A}_{2n}^{(2)}$ Rogers–Ramanujan identities, [63, Theorem 5.13])**.** *For k, n positive integers, let $p=q^{k+2n+1}$. Then*

$$\begin{aligned}\chi_{k\Lambda_0}\big(q^{1/2},q;q\big)&=\sum_{\substack{\lambda\\ \lambda_1\leqslant k}}q^{|\lambda|/2}P_\lambda\big(1,q,q^2,\dots;q^{2n}\big)\\&=\frac{\big(p^{1/2};p^{1/2}\big)_\infty(p;p)_\infty^{n-1}}{\big(q^{1/2};q^{1/2}\big)_\infty(q;q)_\infty^{n-1}}\prod_{i=1}^n\theta\big(q^i;p^{1/2}\big)\prod_{1\leqslant i<j\leqslant n}\theta\big(q^{j-i},q^{i+j};p\big).\end{aligned}$$

Proof. For even k, we take $t=q^{2n}$ in the $\sigma=0$ case of (1.11a) and apply

$$\begin{aligned}&\frac{(p;p)_\infty^k}{(q;q)_\infty^k}\prod_{i=1}^k\theta\big(-q^{i-1/2};p\big)\theta\big(pq^{2i-1};p^2\big)\prod_{1\leqslant i<j\leqslant k}\theta\big(q^{j-i},q^{i+j-1};p\big)\\&\quad=\frac{\big(p^{1/2};p^{1/2}\big)_\infty(p;p)_\infty^{n-1}}{\big(q^{1/2};q^{1/2}\big)_\infty(q;q)_\infty^{n-1}}\prod_{i=1}^n\theta\big(q^i;p^{1/2}\big)\prod_{1\leqslant i<j\leqslant n}\theta\big(q^{j-i},q^{i+j};p\big),\end{aligned}$$

where $p=q^{2k+2n+1}$. The odd case arises from (5.10) and (2.14a), specialised in the same way as in the proof of Corollary 5.4. ■

A fourth and final application of (1.11a) gives our second new Rogers–Ramanujan identity.

Corollary 5.7 ($\mathrm{B}_n^{(1)}$ Rogers–Ramanujan identities)**.** *For k, n positive integers, let $p=q^{k+2n-1}$. Then*

$$\chi_{k\Lambda_0}(q,-1;q)=\sum_{\substack{\lambda\\ \lambda_1\leqslant k}}q^{|\lambda|}P_\lambda\big(1,q,q^2,\dots;q^{2n-2}\big)=\frac{(p;p)_\infty^n}{(q;q)_\infty^n}\prod_{1\leqslant i<j\leqslant n}\theta\big(q^{j-i},q^{i+j-1};p\big),\tag{5.17}$$

Proof. For the even-k case, we specialise $t=q^{2n-2}$ in (1.11a) for $\sigma=1$ and apply (5.5). The odd-k case follows from (5.11) and (2.14a) along the lines of the proof of Corollary 5.5. The identification of the identity in terms of $\mathrm{B}_n^{(1)}$ arises as follows. We parametrise the level-ℓ dominant integral weights of $\mathrm{B}_n^{(1)}$ as

$$\Lambda=(\ell-\lambda_1-\lambda_2)\Lambda_0+(\lambda_1-\lambda_2)\Lambda_1+\dots+(\lambda_{n-1}-\lambda_n)\Lambda_{n-1}+2\lambda_n\Lambda_n,$$

where $(\lambda_1,\dots,\lambda_n)$ is a partition or half-partition such that $\lambda_1+\lambda_2\leqslant\ell$. Specialising the Weyl–Kac character formula [29, equation (10.4.1)] for the $\mathrm{B}_n^{(1)}$-module $L(\Lambda)$ according to $\phi_{q,-1;q}$, and using the $\mathrm{D}_n^{(1)}$ Macdonald identity – the fact that this is the right Macdonald identity is reflected in the fact that $2n-2$ is the dual Coxeter number of $\mathrm{D}_n^{(1)}$ – yields

$$\chi_\Lambda(q,-1;q)=\begin{cases}\dfrac{(p;p)_\infty^n}{(q;q)_\infty^n}\displaystyle\prod_{1\leqslant i<j\leqslant n}\theta\big(q^{\lambda_i-\lambda_j-i+j},q^{\lambda_i+\lambda_j+2n-i-j+1};p\big) & \text{if } \lambda \text{ is a partition,}\\ 0 & \text{if } \lambda \text{ is a half-partition,}\end{cases}\tag{5.18}$$

where $p=q^{\ell+h^\vee}=q^{\ell+2n-1}$. For $\ell=k$ and $\lambda_1=\dots=\lambda_n=0$, this is the product on the right of (5.17). ■

The next four corollaries are consequences of (1.11b).

Corollary 5.8 ($\mathrm{A}_{2n-1}^{(2)}$ Rogers–Ramanujan identities, [63, Theorem 5.17])**.** *For k, n positive integers, let $p=q^{2k+2n}$. Then*

$$\begin{aligned}\chi_{k\Lambda_n}(q,q;q) &= \sum_{\substack{\lambda\\ \lambda_1\leqslant 2k}} q^{|\lambda|/2+(n-1/2)l(\lambda^{\mathrm{o}})} P_\lambda\big(1,q,q^2,\dots;q^{2n-1}\big)\\ &= \frac{(p;p)_\infty^n}{\big(q;q^2\big)_\infty (q;q)_\infty^n}\prod_{i=1}^n \theta\big(p^{1/2}q^{i-1};p\big)\prod_{1\leqslant i<j\leqslant n}\theta\big(q^{j-i},q^{i+j-2};p\big).\end{aligned}$$

This result, which for $n=1$ yields (5.16) for $(i,k)\mapsto(k,2k)$, is a companion of Corollary 5.5.

Proof. The claim follows by taking $\sigma=0$ and $t=q^{2n-1}$ in (1.11b) and applying the theta function identity

$$\begin{aligned}&\frac{(p;p)_\infty^k}{(q;q)_\infty^k}\prod_{i=1}^k\theta\big({-p^{1/2}q^i};p\big)\theta\big(q^{2i};p^2\big)\prod_{1\leqslant i<j\leqslant k}\theta\big(q^{j-i},q^{i+j};p\big)\\ &\quad=\frac{(p;p)_\infty^n}{\big(q;q^2\big)_\infty (q;q)_\infty^n}\prod_{i=1}^n\theta\big(p^{1/2}q^{i-1};p\big)\prod_{1\leqslant i<j\leqslant n}\theta\big(q^{j-i},q^{i+j-2};p\big)\end{aligned}$$

for $p=q^{2k+2n}$. ∎

The next corollary, which is a companion to Corollary 5.4 and gives (5.13) for $(k,i)\mapsto(2k,1)$, is our third new Rogers–Ramanujan identity.

Corollary 5.9 ($\mathrm{D}_{n+2}^{(2)}$ Rogers–Ramanujan identities)**.** *For k, n positive integers, let $p=q^{2k+2n}$. Then*

$$\begin{aligned}\chi_{k\Lambda_n}\big(q^{1/2},-1;q\big) &= \sum_{\substack{\lambda\\ \lambda_1\leqslant 2k}} q^{|\lambda|+(n-1/2)l(\lambda^{\mathrm{o}})} P_\lambda\big(1,q,q^2,\dots;q^{2n-1}\big)\\ &= \frac{\big({-q^{1/2}};q\big)_\infty (p;p)_\infty^n}{(q;q)_\infty^n}\prod_{i=1}^n\theta\big(q^{i-1/2};p\big)\prod_{1\leqslant i<j\leqslant n}\theta\big(q^{j-i},q^{i+j-1};p\big).\end{aligned}$$

Proof. This follows by specialising $t=q^{2n-1}$ in (1.11b) for $\sigma=1$ and using (5.14) for $a=1$. The identification of the q-series as $\chi_{k\Lambda_n}\big(q^{1/2},-1;q\big)$ for the affine Lie algebra $\mathrm{D}_{n+1}^{(2)}$ follows from the product formula for $\chi_\Lambda\big(q^{1/2},-1;q\big)$ given on [63, p. 75]. ∎

Corollary 5.10 ($\mathrm{A}_{2n}^{(2)}$ Rogers–Ramanujan identities [63, Theorem 5.13])**.** *For k, n positive integers, let $p=q^{2k+2n+1}$. Then*

$$\begin{aligned}\chi_{k\Lambda_n}\big(q^{1/2},q;q\big) &= \sum_{\substack{\lambda\\ \lambda_1\leqslant 2k}} q^{|\lambda|/2+n\,l(\lambda^{\mathrm{o}})} P_\lambda\big(1,q,q^2,\dots;q^{2n}\big)\\ &= \frac{\big(p^{1/2};p^{1/2}\big)_\infty (p;p)_\infty^{n-1}}{\big(q^{1/2};q^{1/2}\big)_\infty (q;q)_\infty^{n-1}}\prod_{i=1}^n\theta\big(q^{i-1/2};p^{1/2}\big)\prod_{1\leqslant i<j\leqslant n}\theta\big(q^{j-i},q^{i+j-1};p\big).\end{aligned}$$

Proof. The claim follows by specialising $\sigma=0$ and $t=q^{2n}$ in (1.11b) and using

$$\begin{aligned}&\frac{(p;p)_\infty^k}{(q;q)_\infty^k}\prod_{i=1}^k\theta\big(-p^{1/2}q^i;p\big)\theta\big(q^{2i};p^2\big)\prod_{1\leqslant i<j\leqslant k}\theta\big(q^{j-i},q^{i+j};p\big)\\&\quad=\frac{\big(p^{1/2};p^{1/2}\big)_\infty(p;p)_\infty^{n-1}}{\big(q^{1/2};q^{1/2}\big)_\infty(q;q)_\infty^{n-1}}\prod_{i=1}^n\theta\big(q^{i-1/2};p^{1/2}\big)\prod_{1\leqslant i<j\leqslant n}\theta\big(q^{j-i},q^{i+j-1};p\big)\end{aligned}$$

for $p=q^{2k+2n+1}$. ∎

The next identity, which is a companion of (5.17), is our fourth new Rogers–Ramanujan identity.

Corollary 5.11 ($\mathrm{B}_n^{(1)}$ Rogers–Ramanujan identities)**.** *For k, n positive integers, let $p=q^{2k+2n-1}$. Then*

$$\begin{aligned}\chi_{2k\Lambda_n}\big(q,-1;q\big)&=\sum_{\substack{\lambda\\\lambda_1\leqslant 2k}}q^{|\lambda|+(n-1)\,l(\lambda^{\mathrm{o}})}P_\lambda\big(1,q,q^2,\dots;q^{2n-2}\big)\\&=\frac{(p;p)_\infty^n}{(q;q)_\infty^n}\prod_{1\leqslant i<j\leqslant n}\theta\big(q^{j-i},q^{i+j-2};p\big).\end{aligned}$$

Proof. We take $\sigma=1$, $t=q^{2n-2}$ in (1.11b) and use

$$\begin{aligned}&\frac{(p;p)_\infty^k}{(q;q)_\infty^k}\prod_{i=1}^k\theta\big(-p^{1/2}q^{i-1/2};p\big)\theta\big(q^{2i-1};p^2\big)\prod_{1\leqslant i<j\leqslant k}\theta\big(q^{j-i},q^{i+j-1};p\big)\\&\quad=\frac{(p;p)_\infty^n}{(q;q)_\infty^n}\prod_{1\leqslant i<j\leqslant n}\theta\big(q^{j-i},q^{i+j-2};p\big)\end{aligned}$$

for $p=q^{2k+2n-1}$. The identification as $\chi_{2k\Lambda_n}\big(q,-1;q\big)$ follows from (5.18) with $k\mapsto 2k$ and $\lambda_i=k$ for all $1\leqslant i\leqslant k$. ∎

Next we consider applications of Theorem 1.7.

Corollary 5.12 ($\mathrm{B}_n^{(1)}$ Rogers–Ramanujan identities [63, Theorem 5.15 and Remark 5.16])**.** *For k, n positive integers, let $p=q^{k+2n-1}$. Then*

$$\begin{aligned}&\chi_{k\Lambda_{(1-\sigma)\Lambda_0+\sigma\Lambda_n}}\big(q^{1/2},q;q\big)\\&\quad=\sum_{\substack{\lambda\\\lambda_1\leqslant k}}q^{(\sigma+1)|\lambda|/2}\Bigg(\prod_{i=0}^{k-1}\big(-q^{n-1/2};q^{n-1/2}\big)_{m_i(\lambda)}\Bigg)P_\lambda\big(1,q,q^2,\dots;q^{2n-1}\big)\\&\quad=\frac{\big(-q^{n-1/2};q^{n-1/2}\big)_\infty(p;p)_\infty^n}{\big(q^{1/2};q^{1/2}\big)_\infty(q;q)_\infty^{n-1}}\prod_{i=1}^n\theta\big(q^{i+(1-\sigma)k/2-1/2};p\big)\prod_{1\leqslant i<j\leqslant n}\theta\big(q^{j-i},q^{i+j+\sigma-2};p\big),\end{aligned}$$

where $m_0(\lambda):=\infty$ and $\sigma\in\{0,1\}$.

For $n=1$ and $q\mapsto q^2$, the above result implies the $i=k+1$ and $i=1$ cases of Bressoud's even modulus analogues of the Andrew–Gordon identities [12, 13]:

$$\sum_{n_1\geqslant\cdots\geqslant n_k\geqslant 0}\frac{q^{n_1^2+\cdots+n_k^2+n_i+\cdots+n_k}}{(q;q)_{n_1-n_2}\cdots(q;q)_{n_{k-1}-n_k}\big(q^2;q^2\big)_{n_k}}=\frac{\big(q^i,q^{2k-i+2},q^{2k+2};q^{2k+2}\big)_\infty}{(q;q)_\infty},$$

where $1\leqslant i\leqslant k+1$.

Proof. For even k, this follows by specialising $t = q^{2n-1}$ in Theorem 1.7 and using

$$\frac{(p^{1/2};p)_\infty (p;p)_\infty^k}{(\sigma+1)(q;q)_\infty^k} \prod_{i=1}^k \theta\big(-q^{i-(\sigma+1)/2};p^{1/2}\big) \prod_{1\leqslant i<j\leqslant k} \theta\big(q^{j-i},q^{i+j-\sigma-1};p\big)$$
$$= \frac{(p;p)_\infty^n}{\big(q^{1/2};q\big)_\infty (q;q)_\infty^n} \prod_{i=1}^n \theta\big(q^{i+(1-\sigma)k-1/2};p\big) \prod_{1\leqslant i<j\leqslant n} \theta\big(q^{j-i},q^{i+j+\sigma-2};p\big)$$

for $p = q^{2k+2n-1}$. For odd k, the result follows from Corollaries 5.8 and 5.9 combined with (2.14b) for $n\to\infty$ and $x_i = q^{i+(\sigma-1)/2}$. ■

Corollary 5.13 ($\mathrm{D}_{n+1}^{(2)}$ Rogers–Ramanujan identities [63, Theorem 5.14])**.** *For k, n positive integers, let $p = q^{k+2n}$. Then*

$$\chi_{k\Lambda_0}\big(q^{1/2},q^{1/2};q\big)$$
$$= \sum_{\substack{\lambda \\ \lambda_1\leqslant k}} q^{|\lambda|/2} \Bigg(\prod_{i=0}^{k-1} (-q^n;q^n)_{m_i(\lambda)}\Bigg) P_\lambda\big(1,q,q^2,\dots;q^{2n}\big)$$
$$= \frac{\big(-q^{1/2};q\big)_\infty \big(p^{1/2};p^{1/2}\big)_\infty (p;p)_\infty^{n-1}}{\big(q^n;q^{2n}\big)_\infty \big(q^{1/2};q^{1/2}\big)_\infty (q;q)_\infty^{n-1}} \prod_{i=1}^n \theta\big(q^{i-1/2};p^{1/2}\big) \prod_{1\leqslant i<j\leqslant n} \theta\big(q^{j-i},q^{i+j-1};p\big),$$

where $m_0(\lambda) := \infty$.

By the $\mathrm{D}_{n+1}^{(2)}$ diagram automorphism $\alpha_i \mapsto \alpha_{n-i}$ for $0\leqslant i\leqslant n$, the above q-series may also be identified as $\chi_{k\Lambda_n}\big(q^{1/2},q^{1/2};q\big)$.

Proof. For even k, this follows by specialising $\sigma = 0$ and $t = q^{2n}$ in Theorem 1.7 and the application of

$$\frac{(p^{1/2};p)_\infty (p;p)_\infty^k}{(q;q)_\infty^k} \prod_{i=1}^k \theta\big(-q^{i-1/2};p^{1/2}\big) \prod_{1\leqslant i<j\leqslant k} \theta\big(q^{j-i},q^{i+j-1};p\big)$$
$$= \frac{\big(-q^{1/2};q\big)_\infty \big(p^{1/2};p^{1/2}\big)_\infty (p;p)_\infty^{n-1}}{\big(q^{1/2};q^{1/2}\big)_\infty (q;q)_\infty^{n-1}} \prod_{i=1}^n \theta\big(q^{i-1/2};p^{1/2}\big) \prod_{1\leqslant i<j\leqslant n} \theta\big(q^{j-i},q^{i+j-1};p\big)$$

for $p = q^{2k+2n}$. For odd k, the result follows from Corollary 5.10 and (2.14b) for $n\to\infty$ and $x_i = q^{i-1/2}$. ■

Our fifth new result is our first and only identity for $\mathrm{D}_n^{(1)}$.

Corollary 5.14 ($\mathrm{D}_n^{(1)}$ Rogers–Ramanujan identity)**.** *For k, n positive integers, let $p = q^{k+2n-2}$. Then*

$$\chi_{k\Lambda_0}(q,q;q) = \sum_{\substack{\lambda \\ \lambda_1\leqslant k}} q^{|\lambda|} \Bigg(\prod_{i=0}^{k-1} \big(-q^{n-1};q^{n-1}\big)_{m_i(\lambda)}\Bigg) P_\lambda\big(1,q,q^2,\dots;q^{2n-2}\big)$$
$$= \frac{(-q;q)_\infty \big(-q^{n-1};q^{n-1}\big)_\infty (p;p)_\infty^n}{(q;q)_\infty^n} \prod_{1\leqslant i<j\leqslant n} \theta\big(q^{j-i},q^{i+j-2};p\big), \tag{5.19}$$

where $m_0(\lambda) = \infty$.

Once again, by the automorphisms of the $\mathrm{D}_n^{(1)}$ Dynkin diagram, we may replace Λ_0 in the above by Λ_1, Λ_{n-1} or Λ_n.

Proof. For even k, the second equality in (5.19) follows by taking $\sigma = 1$ and $t = q^{2n-2}$ in Theorem 1.7 and the use of

$$\begin{aligned}\frac{(p^{1/2};p)_\infty (p;p)_\infty^k}{2(q;q)_\infty^k} &\prod_{i=1}^k \theta(-q^{i-1};p^{1/2}) \prod_{1\leqslant i<j\leqslant k} \theta(q^{j-i},q^{i+j-2};p) \\ &= \frac{(-q;q)_\infty (p;p)_\infty^n}{(q;q)_\infty^n} \prod_{1\leqslant i<j\leqslant n} \theta(q^{j-i},q^{i+j-2};p)\end{aligned}$$

for $p = q^{2k+2n-2}$. For odd k, the result follows from Corollary 5.11 and (2.14b) for $n \to \infty$ and $x_i = q^i$. The identification of the q-series in terms of the $\mathrm{D}_n^{(1)}$ standard module $L(k\Lambda_0)$ follows from (5.6). For $\mathrm{D}_n^{(1)}$ with parametrisation

$$\Lambda = (\ell-\lambda_1-\lambda_2)\Lambda_0 + (\lambda_1-\lambda_2)\Lambda_1 + \cdots + (\lambda_{n-1}-\lambda_n)\Lambda_{n-1} + (\lambda_{n-1}+\lambda_n)\Lambda_n,$$

where $(\lambda_1,\dots,\lambda_n)$ is a generalised[3] partition or half-partition such that $\lambda_1+\lambda_2 \leqslant \ell$, (5.6) implies that

$$\chi_\Lambda(q,q;q) = \frac{(-q;q)_\infty (-q^{n-1};q^{n-1})_\infty (p,p)_\infty^n}{(q;q)_\infty^n} \prod_{1\leqslant i<j\leqslant n} \theta(q^{\lambda_i-\lambda_j-i+j}, q^{\lambda_i+\lambda_j+2n-i-j};p).$$

Here $p = q^{\ell+h^\vee} = q^{\ell+2n-2}$. For $\ell = k$ and $\lambda_1 = \cdots = \lambda_n = 0$, this gives yields the product on the right of (5.19). ∎

Finally, we consider four applications of Theorem 1.8, all but one of which are new.

Corollary 5.15 ($\mathrm{A}_{2n}^{(2)}$ Rogers–Ramanujan identities)**.** *For k a positive integer and n a nonnegative integer, let $p = q^{2k+2n+1}$. Then*

$$\begin{aligned}\chi_{k\Lambda_n}(q,q;q) &= \sideset{}{'}\sum_{\substack{\lambda \\ \lambda_1\leqslant 2k}} q^{|\lambda|/2+(n+1/2)l(\lambda^{\mathrm{o}})} \bigg(\prod_{i=0}^{2k-1} (q^{2n+1};q^{4n+2})_{\lceil m_i(\lambda)/2\rceil}\bigg) P_\lambda(1,q,q^2,\dots;q^{2n+1}) \\ &= \frac{(q^{2n+1};q^{4n+2})_\infty (p;p)_\infty^n}{(q;q^2)_\infty (q;q)_\infty^n} \prod_{i=1}^n \theta(q^{k+i};p)\theta(q^{2i-1};p^2) \prod_{1\leqslant i<j\leqslant n} \theta(q^{j-i},q^{i+j-1};p), \end{aligned} \tag{5.20}$$

where $m_0(\lambda) := \infty$ and the prime denotes the restriction that the odd parts of λ have even multiplicity.

For $n = 0$, this is the $i = 0$ case of the somewhat unusual Rogers–Ramanujan-type identity[4]

$$\begin{aligned}\sideset{}{'}\sum_{n_1\geqslant\cdots\geqslant n_{2k}\geqslant 0} &\frac{q^{\frac{1}{2}(n_1^2+\cdots+n_{2k}^2)+\frac{1}{2}(n_1-n_2+\cdots+n_{2k-1}-n_{2k})+\frac{1}{2}(n_1+n_2+\cdots+n_{2i})}}{(q^2;q^2)_{\lfloor (n_1-n_2)/2\rfloor}\cdots(q^2;q^2)_{\lfloor (n_{2k-1}-n_{2k})/2\rfloor}(q;q)_{n_{2k}}} \\ &= \frac{(-q^{2i+1},-q^{2i+1},q^{2i+1};q^{2i+1})_\infty}{(q^2;q^2)_\infty},\end{aligned} \tag{5.21}$$

where $0 \leqslant i \leqslant k$ and the prime denotes the restriction that $n_{2j-1}-n_{2j}$ is even for all $1 \leqslant j \leqslant k$.

[3]Generalised in the sense that λ_n need not be nonnegative, as long as $|\lambda_n| \leqslant \lambda_{n-1}$.

[4]In an earlier version of this paper, only the $i = 0$ and $i = k$ cases of (5.21) were stated. Matthew Russell subsequently discovered the remaining cases, which are proved in the appendix.

Proof. We take $\sigma=0$ and $t=q^{2n+1}$ in Theorem 1.8 and apply

$$\begin{aligned}\frac{(-p;p)_\infty(p;p)_\infty^k}{(q;q)_\infty^k}&\prod_{i=1}^k\theta\big(q^{2i};p^2\big)\prod_{1\leqslant i<j\leqslant k}\theta\big(q^{j-i},q^{i+j};p\big)\\&=\frac{(-q;q)_\infty(p;p)_\infty^n}{(q;q)_\infty^n}\prod_{i=1}^n\theta\big(q^{k+i};p\big)\theta\big(q^{2i-1};p^2\big)\prod_{1\leqslant i<j\leqslant n}\theta\big(q^{j-i},q^{i+j-1};p\big)\end{aligned}$$

for $p=q^{2k+2n+1}$. Parametrising the level-ℓ dominant integral weight of $\mathrm{A}_{2n}^{(2)}$ as

$$\Lambda=(\ell-2\lambda_1)\Lambda_0+(\lambda_1-\lambda_2)\Lambda_1+\cdots+(\lambda_{n-1}-\lambda_n)\Lambda_{n-1}+\lambda_n\Lambda_n,$$

where $(\lambda_1,\dots,\lambda_n)$ is a partition such that $\lambda_1\leqslant\lfloor\ell/2\rfloor$, the $\mathrm{A}_{2n}^{(2)}$ case of (5.6) takes the form

$$\begin{aligned}\chi_\Lambda(q,q;q)&=\frac{\big(q^{2n+1};q^{4n+2}\big)_\infty(p;p)_\infty^n}{\big(q;q^2\big)_\infty(q;q)_\infty^n}\prod_{i=1}^n\theta\big(q^{\lambda_i+n-i+1};p\big)\theta\big(q^{\ell-2\lambda_i+2i-1};p^2\big)\\&\quad\times\prod_{1\leqslant i<j\leqslant n}\theta\big(q^{\lambda_i-\lambda_j-i+j},q^{\lambda_i+\lambda_j+2n-i-j+2};p\big),\end{aligned}$$

where $p=q^{\ell+h^\vee}=q^{\ell+2n+1}$. For $\ell=2k$ and $\lambda_1=\cdots=\lambda_n=k$, this is gives the product on the right of (5.20). ∎

Corollary 5.16 ($\mathrm{B}_n^{(1)}$ Rogers–Ramanujan identities)**.** *For k,n positive integers, let $p=q^{2k+2n-1}$. Then*

$$\begin{aligned}&\chi_{2k\Lambda_n}\big(q^2,-1;q\big)\\&\quad=\sum_{\substack{\lambda\\ \lambda_1\leqslant 2k}}{}^{'}\,q^{|\lambda|+(n-1/2)l(\lambda^{\mathrm{o}})}\Bigg(\prod_{i=0}^{2k-1}\big(q^{2n-1};q^{4n-2}\big)_{\lceil m_i(\lambda)/2\rceil}\Bigg)P_\lambda\big(1,q,q^2,\dots;q^{2n-1}\big)\\&\quad=\frac{\big(q^{2n-1};q^{4n-2}\big)_\infty(p;p)_\infty^n}{2\big(q^2;q^2\big)_\infty(q;q)_\infty^{n-1}}\prod_{i=1}^n\theta\big({-q^{i-1}};p\big)\prod_{1\leqslant i<j\leqslant n}\theta\big(q^{j-i},q^{i+j-2};p\big),\end{aligned}\tag{5.22}$$

where $m_0(\lambda):=\infty$ and the prime denotes the restriction that the odd parts of λ have even multiplicity.

For $n=1$, this is (5.21) for $i=k$.

Proof. We take $\sigma=1$ and $t=q^{2n+1}$ in Theorem 1.8 and apply

$$\begin{aligned}\frac{(-p;p)_\infty(p;p)_\infty^k}{(q;q)_\infty^k}&\prod_{i=1}^k\theta\big(q^{2i-1};p^2\big)\prod_{1\leqslant i<j\leqslant k}\theta\big(q^{j-i},q^{i+j-1};p\big)\\&=\frac{(p;p)_\infty^n}{2\big(q^2;q^2\big)_\infty(q;q)_\infty^{n-1}}\prod_{i=1}^n\theta\big({-q^{i-1}};p\big)\prod_{1\leqslant i<j\leqslant n}\theta\big(q^{j-i},q^{i+j};p\big)\end{aligned}$$

for $p=q^{2k+2n-1}$. The stated specialisation is somewhat unusual in that it does not lead to a product form for weights other than $\ell\Lambda_n$, for which it follows from the Weyl–Kac formula and the $\mathrm{B}_n^{(1)}$ Macdonald identity that

$$\begin{aligned}&\chi_{\ell\Lambda_n}\big(q^2,-1;q\big)\\&\quad=\begin{cases}\dfrac{\big(q^{2n-1};q^{4n-2}\big)_\infty(p;p)_\infty^n}{2\big(q^2;q^2\big)_\infty(q;q)_\infty^{n-1}}\displaystyle\prod_{i=1}^n\theta\big({-q^{i-1}};p\big)\prod_{1\leqslant i<j\leqslant n}\theta\big(q^{j-i},q^{i+j-2};p\big) & \text{if } \ell \text{ is even},\\[2mm] 0 & \text{if } \ell \text{ is odd},\end{cases}\end{aligned}$$

where $p=q^{\ell+h^\vee}=q^{\ell+2n-1}$. The product on the right of (5.22) corresponds to $\ell=2k$. ∎

Corollary 5.17 ($\mathrm{A}_{2n-1}^{(2)}$ Rogers–Ramanujan identities [63, Theorem 5.17])**.** *For k, n positive integers, let $p = q^{2k+2n}$. Then*

$$\begin{aligned}\chi_{k\Lambda_n}&\big(q^2,q;q\big)\\ &= \sideset{}{'}\sum_{\substack{\lambda\\ \lambda_1\leqslant 2k}} q^{|\lambda|/2+n\,l(\lambda^{\mathrm{o}})}\left(\prod_{i=0}^{2k-1}\big(q^{2n};q^{4n}\big)_{\lceil m_i(\lambda)/2\rceil}\right)P_\lambda\big(1,q,q^2,\dots;q^{2n}\big)\\ &= \frac{\big(q^{2n};q^{4n}\big)_\infty\big(-p^{1/2};p\big)_\infty(p;p)_\infty^n}{2(q;q)_\infty^n}\prod_{i=1}^n\theta\big(-q^{i-1},p^{1/2}q^{i-1};p\big)\prod_{1\leqslant i<j\leqslant n}\theta\big(q^{j-i},q^{i+j-2};p\big),\end{aligned}$$

where $m_0(\lambda) := \infty$ and prime denotes the restriction that the odd parts of λ have even multiplicity.

Proof. We set $\sigma = 0$ and $t = q^{2n}$ in Theorem 1.8. The claim then follows from

$$\begin{aligned}&\frac{(-p;p)_\infty(p;p)_\infty^k}{(q;q)_\infty^k}\prod_{i=1}^k\theta\big(q^{2i};p^2\big)\prod_{1\leqslant i<j\leqslant k}\theta\big(q^{j-i},q^{i+j};p\big)\\ &\quad=\frac{\big(-p^{1/2};p\big)_\infty(p;p)_\infty^n}{2(q;q)_\infty^n}\prod_{i=1}^n\theta\big(-q^{i-1},p^{1/2}q^{i-1};p\big)\prod_{1\leqslant i<j\leqslant n}\theta\big(q^{j-i},q^{i+j-2};p\big)\end{aligned}$$

for $p = q^{2k+2n}$. ∎

Corollary 5.18 ($\mathrm{D}_{n+1}^{(2)}$ Rogers–Ramanujan identities)**.** *For k, n positive integers, let $p = q^{2k+2n}$. Then*

$$\begin{aligned}\chi_{k\Lambda_0}(-1,q;q) &= \sideset{}{'}\sum_{\substack{\lambda\\ \lambda_1\leqslant 2k}} q^{|\lambda|+n\,l(\lambda^{\mathrm{o}})}\left(\prod_{i=0}^{2k-1}\big(q^{2n};q^{4n}\big)_{\lceil m_i(\lambda)/2\rceil}\right)P_\lambda\big(1,q,q^2,\dots;q^{2n}\big) \qquad (5.23)\\ &= \frac{\big(q^{2n};q^{4n}\big)_\infty\big(p^2;p^2\big)_\infty(p;p)_\infty^{n-1}}{(q;q)_\infty^n}\prod_{i=1}^n\theta\big(q^{2i-1};p^2\big)\prod_{1\leqslant i<j\leqslant n}\theta\big(q^{j-i},q^{i+j-1};p\big),\end{aligned}$$

where $m_0(\lambda) := \infty$ and the prime denotes the restriction that the odd parts of λ have even multiplicity.

Proof. Let

$$\Theta_{k,n}(q) = \frac{(p;p)_\infty^k}{(q;q)_\infty^k}\prod_{i=1}^k\theta\big(q^{2i-1};p^2\big)\prod_{1\leqslant i<j\leqslant k}\theta\big(q^{j-i},q^{i+j-1};p\big),$$

where $p = q^{2k+2n}$. Taking $\sigma = 1$ and $t = q^{2n}$ in Theorem 1.8 and applying

$$\Theta_{k,n}(q) = \Theta_{n,k}(q)$$

yields the second equality in (5.23). To obtain the first equality we parametrise the level-ℓ dominant integral weights of $\mathrm{D}_{n+1}^{(2)}$ as

$$\Lambda = (\ell - 2\lambda_1)\Lambda_0 + (\lambda_1-\lambda_2)\Lambda_1 + \cdots + (\lambda_{n-1}-\lambda_n)\Lambda_{n-1} + 2\lambda_n\Lambda_n,$$

where $(\lambda_1,\dots,\lambda_n)$ is a partition or half-partition such that $\lambda_1\leqslant\lfloor\ell/2\rfloor$. By specialising the Weyl–Kac character formula for the $\mathrm{D}_n^{(1)}$-module $L(\Lambda)$ according to $\phi_{-1,q;q}$, it follows from the $\mathrm{A}_{2n-1}^{(2)}$ Macdonald identity that

$$\chi_\Lambda(-1,q;q) = \begin{cases} \dfrac{(q^{2n};q^{4n})_\infty (p^2;p^2)_\infty (p;p)_\infty^{n-1}}{(q;q)_\infty^n}\displaystyle\prod_{i=1}^n \theta\big(q^{2\lambda_i+2n-2i+1};p^2\big) & \\ \qquad\times\displaystyle\prod_{1\leqslant i<j\leqslant n}\theta\big(q^{\lambda_i-\lambda_j-i+j},q^{\lambda_i+\lambda_j+2n-i-j+1};p\big) & \text{if } \lambda \text{ is a partition,} \\ 0 & \text{if } \lambda \text{ is a half-partition,}\end{cases}$$

where $p=q^{\ell+h^\vee}=q^{\ell+2n}$. For $k=\ell$ and $\lambda_1=\dots=\lambda_n$ this is the product on the right of (5.23). ∎

5.3 Proofs of Theorems 1.5–1.8

So far, we have only given conditional proofs of the q,t-Rogers–Ramanujan identities, based on the conjectural Jacobi–Trudi identities. For proofs that are unconditional, we apply Ismail's analytic argument [28]. Because this works in the exact same manner for all four theorems, we only present the details for Theorem 1.5.

Proof. Fix $q\in\mathbb{C}$ such that $|q|<1$ and denote the difference between the left- and right-hand sides of (1.9) by $f(t)$. Let D denote the open unit disk in $\mathbb{C}$. Then $f\colon D\to\mathbb{C}$ is analytic in an open neighbourhood $U\subset D$ of 0. By (5.7), (5.8), (5.10) and (5.11), which were previously proved in [26, Theorems 1.1–1.3], $f\big(q^N\big)=0$ for all positive integers N. Since the (geometric) sequence $\big(q^N\big)_{N\geqslant 1}$ has limit point $0\in U$, $f(t)=0$ for all $t\in D$ by the identity theorem for analytic functions. ∎

There is one problem with the above proof method in that it assumes the truth of all of the Rogers–Ramanujan identities stated in Section 5.2. Eight of these are new and therefore require independent proofs. It is not difficult however to adapt the method of proof from [26, 63] to obtain the missing proofs. In each case, the required calculations are quite lengthy and hence we will give the full details of the most complicated to prove case (Corollary 5.7) and for the remaining cases we will only briefly indicate the key steps.

Proof of Corollary 5.7. We require the special case of [63, Proposition 5.10] obtained by setting $(t_2,t_3)=(0,-1)$ and letting N tend to infinity. In this case, we may replace the nonnegative integer m in the proposition by the nonnegative integer or half-integer $k/2$, as is explained just above [63, equation (5.2.32)]. Then

$$\begin{aligned}\sum_{\substack{\lambda\\ \lambda_1\leqslant k}} t^{|\lambda|/2}P'_\lambda(x;t) &= \prod_{i=1}^n \frac{\big(-t^{1/2}x_i;t\big)_\infty}{(tx_i^2;t)_\infty}\prod_{1\leqslant i<j\leqslant n}\frac{1}{(tx_ix_j;t)_\infty} \\ &\quad\times\sum_{r_1,\dots,r_n\geqslant 0}\frac{\Delta_{\mathrm{C}}(xt^r)}{\Delta_{\mathrm{C}}(x)}\prod_{i=1}^n x_i^{(k+1)r_i}t^{\frac{1}{2}(k+1)r_i^2}\prod_{i,j=1}^n\bigg(-\frac{x_i}{x_j}\bigg)^{r_i}t^{\binom{r_i}{2}}\frac{(x_ix_j;t)_{r_i}}{(tx_i/x_j;t)_{r_i}},\end{aligned}\tag{5.24}$$

where, for $x=(x_1,\dots,x_n)$,

$$\Delta_{\mathrm{C}}(x)=\prod_{i=1}^n\big(1-x_i^2\big)\prod_{1\leqslant i<j\leqslant n}(x_i-x_j)(x_ix_j-1)=\det_{1\leqslant i,j\leqslant n}\big(x_i^{j-1}-x_i^{2n-j+1}\big).$$

We note that the simultaneous substitutions $\big(t^{1/2},x_1,\dots,x_n\big)\mapsto\big(-t^{1/2},-x_1,\dots,-x_n\big)$ leave (5.24) invariant. As our next step we observe that the series on the right of (5.24) is the function $L^{(0)}_{-,(\infty^n)}(x_1,\dots,x_n)$ defined in [7, equation (A.1b)] with $(m,q)\mapsto(k,t)$ and $b_1,\dots,b_{k+1}\to\infty$, $c_1=\dots=c_{k+1}=-t^{1/2}$. This puts us in a position to iteratively apply [7, Lemma A.1]. In particular, in (5.24) we first replace $n\mapsto 2n-2$ (for $n\geqslant 2$) and then make the substitution

$$(x_1,\dots,x_{2n-2})\mapsto(x_1,x_2,y_2,x_3,y_3,\dots,x_{n-1},y_{n-1},x_n).$$

Taking the limits $y_i\to x_i^{-1}$ for all $2\leqslant i\leqslant n-1$ using [7, Lemma A.1], and then taking the further limits $x_1\to t^{1/2}$ and $x_n\to 1$, yields

$$\sum_{\substack{\lambda\\ \lambda_1\leqslant k}} t^{|\lambda|/2}P'_\lambda\big(x_1,x_2^{\pm},\dots,x_{n-1}^{\pm},x_n;t\big)=\frac{1}{(t;t)_\infty^n}\prod_{i=1}^n\frac{1}{\theta(-x_i;t)}\prod_{1\leqslant i<j\leqslant n}\frac{1}{(-x_j)\theta\big(x_ix_j^{\pm};t\big)}$$
$$\times\sum_{\substack{r_1,r_n\geqslant 0\\ r_2,\dots,r_{n-1}\in\mathbb{Z}}}(1+\chi(r_n>0))\Delta_{\mathrm{B}}(-xt^r)\prod_{i=1}^n x_i^{(k+2n-1)r_i}t^{\frac{1}{2}(k+2n-1)r_i^2-(n-\frac{1}{2})r_i},$$

where $x_1:=t^{1/2}$, $x_n:=1$, and

$$\Delta_{\mathrm{B}}(x)=\prod_{i=1}^n(1-x_i)\prod_{1\leqslant i<j\leqslant n}(x_i-x_j)(x_ix_j-1)=\det_{1\leqslant i,j\leqslant n}\big(x_i^{j-1}-x_i^{2n-j}\big).$$

Denoting the summand on the right by S_r, we have

$$S_{(r_1,r_2,\dots,r_{n-1},r_n)}=S_{(-1-r_1,r_2,\dots,r_{n-1},r_n)}=S_{(r_1,r_2,\dots,r_{n-1},-r_n)}.$$

These symmetries imply that the above sum over $r_1,\dots,r_n$ can be simplified to

$$\sum_{\substack{r\in\mathbb{Z}^n\\ |r|\text{ even}}}\Delta_{\mathrm{B}}(-xt^r)\prod_{i=1}^n x_i^{(k+2n-1)r_i}t^{\frac{1}{2}(k+2n-1)r_i^2-(n-\frac{1}{2})r_i},$$

although in the following we find it slightly more convenient to drop the condition that $|r|$ is even and divide the series by two. We now specialise $t\mapsto q^{2n-2}$ and $x_i\mapsto q^{n-i}$ $\big($which is consistent with $x_1=t^{1/2}$ and $x_n=1\big)$ and use the homogeneity of the Hall–Littlewood polynomials, (2.15) and multilinearity, to obtain

$$\sum_{\substack{\lambda\\ \lambda_1\leqslant k}} q^{|\lambda|}P_\lambda\big(1,q,q^2,\dots;q^{2n-2}\big)=\frac{\mathcal{N}}{4(q;q)_\infty^n},\tag{5.25}$$

where

$$\mathcal{N}=\det_{1\leqslant i,j\leqslant n}\bigg(\sum_{r\in\mathbb{Z}}\big(y_j^{i-j-2(n-1)r}+y_j^{2n-i-j+2(n-1)r}\big)p^{(n-1)r^2+(n-i)r}\bigg)$$

for $y_i=q^{n-i+1/2}$ and $p=q^{k+2n-1}$. Next we swap i and j in the determinant, replace r by $-r$ in the term of the summand on the right and then again appeal to multilinearity. As a result,

$$\mathcal{N}=\sum_{r\in\mathbb{Z}^n}\prod_{i=1}^n y_i^{2(n-1)r_i-i+1}p^{2(n-1)\binom{r_i}{2}}\det_{1\leqslant i,j\leqslant n}\big((y_ip^{r_i})^{j-1}+(y_ip^{r_i})^{2n-j-1}\big)$$

$$
\begin{aligned}
&= 4(p;p)_\infty^n \prod_{1\leqslant i<j\leqslant n} \theta\big(y_i y_j^{\pm};p\big)\\
&= 4(p;p)_\infty^n \prod_{1\leqslant i<j\leqslant n} \theta\big(q^{j-i},q^{2n-i-j+1};p\big)\\
&= 4(p;p)_\infty^n \prod_{1\leqslant i<j\leqslant n} \theta\big(q^{j-i},q^{i+j-1};p\big).
\end{aligned}
$$

Here the second equality (which holds for arbitrary y_i) follows from the variant of the $\mathrm{D}_n^{(1)}$ Macdonald identity stated in [33, Appendix A.2] and the third equality uses that $y_i = q^{n-i+1/2}$. Substituting the above in (5.25) completes the proof. ∎

Sketch of proof of Corollary 5.5. As in the proof of Corollary 5.7, the staring point is (5.24), but this time we carry out the substitutions $n \mapsto 2n-1$ followed by

$$
(x_1,\dots,x_{2n-1}) \mapsto (x_1,x_2,y_2,x_3,y_3,\dots,x_n,y_n),
$$

and then take the limits $y_i \to x_i^{-1}$ for all $2 \leqslant i \leqslant n$ and $x_1 \to t^{1/2}$. Then carrying out manipulations similar (but simpler, since only the sum over r_1 instead of r_1 and r_n requires special attention) to the proof of Corollary 5.7, specialising $t \mapsto q^{2n-1}$ and $x_i \mapsto q^{n-i+1/2}$ and using the $\mathrm{B}_n^{(1)}$ Macdonald identity yields the identity (5.15). ∎

Sketch of proof of Corollary 5.9. This time the starting point is the special case of [63, Proposition 5.10] obtained by setting $(t_2,t_3) = \big(0,-t^{1/2}\big)$ and letting N tend to infinity:

$$
\begin{aligned}
\sum_{\substack{\lambda\\ \lambda_1\leqslant 2k}} t^{(|\lambda|+l(\lambda^{\mathrm{o}}))/2} P'_\lambda(x;t) &= \prod_{i=1}^n \frac{(-tx_i;t)_\infty}{(tx_i^2;t)_\infty} \prod_{1\leqslant i<j\leqslant n} \frac{1}{(tx_ix_j;t)_\infty}\\
&\times \sum_{r_1,\dots,r_n\geqslant 0} \frac{\Delta_{\mathrm{B}}(xt^r)}{\Delta_{\mathrm{B}}(x)} \prod_{i=1}^n x_i^{(2k+1)r_i} t^{\binom{r_i+1}{2}+kr_i^2} \prod_{i,j=1}^n \left(-\frac{x_i}{x_j}\right)^{r_i} t^{\binom{r_i}{2}} \frac{(x_ix_j;t)_{r_i}}{(tx_i/x_j;t)_{r_i}}.
\end{aligned}
\tag{5.26}
$$

The rest of the proof is the exact same as the proof of Corollary 5.5, including the use of the $\mathrm{B}_n^{(1)}$ Macdonald identity in the final step. ∎

Sketch of proof of Corollary 5.11. The identity follows by taking (5.26) and then following the exact same steps as in the proof of Corollary 5.7. ∎

Sketch of proof of Corollary 5.14. We begin with the special case of [63, Proposition 5.10] obtained by setting $(t_2,t_3) = \big(-1,-t^{1/2}\big)$, letting N tend to infinity and replacing m by $k/2$. Then

$$
\begin{aligned}
\sum_{\substack{\lambda\\ \lambda_1\leqslant k}} t^{|\lambda|/2} \left(\prod_{i=1}^{k-1} \big(-t^{1/2};t^{1/2}\big)_{m_i(\lambda)}\right) P'_\lambda(x;t) &= \prod_{i=1}^n \frac{1}{\big(t^{1/2}x_i;t^{1/2}\big)_\infty} \prod_{1\leqslant i<j\leqslant n} \frac{1}{(tx_ix_j;t)_\infty}\\
&\times \sum_{r_1,\dots,r_n\geqslant 0} \frac{\Delta_{\mathrm{B}}(xt^r)}{\Delta_{\mathrm{B}}(x)} \prod_{i=1}^n x_i^{kr_i} t^{\frac{1}{2}kr_i^2+\frac{1}{2}r_i} \prod_{i,j=1}^n \left(-\frac{x_i}{x_j}\right)^{r_i} t^{\binom{r_i}{2}} \frac{(x_ix_j;t)_{r_i}}{(tx_i/x_j;t)_{r_i}}.
\end{aligned}
$$

The remainder follows the exact same steps as in the proof of Corollary 5.7. ∎

Sketch of proof of Corollary 5.15. In [63, Proposition 5.10], we set $t_2 = -t_3 = -t^{1/2}$ and let N tend to infinity. Then

$$
\sum_{\substack{\lambda\\ \lambda_1\leqslant 2k}}{}' \, t^{(|\lambda|+l(\lambda^{\mathrm{o}}))/2} \left(\prod_{i=1}^{2k-1} (t;t^2)_{\lceil m_i(\lambda)/2\rceil}\right) P'_\lambda(x;t) = \prod_{i=1}^n \frac{1}{(tx_i^2;t^2)_\infty} \prod_{1\leqslant i<j\leqslant n} \frac{1}{(tx_ix_j;t)_\infty}
$$

$$\times \sum_{r_1,\dots,r_n\geqslant 0} \frac{\Delta_{\mathrm{D}}(xt^r)}{\Delta_{\mathrm{D}}(x)} \prod_{i=1}^n (-1)^{r_i} x_i^{2kr_i} t^{kr_i^2+r_i} \prod_{i,j=1}^n \left(-\frac{x_i}{x_j}\right)^{r_i} t^{\binom{r_i}{2}} \frac{(x_ix_j;t)_{r_i}}{(tx_i/x_j;t)_{r_i}}, \tag{5.27}$$

where the prime in the final sum over λ denotes the restriction that the odd parts of λ must have even multiplicity, and

$$\Delta_{\mathrm{D}}(x) = \prod_{1\leqslant i<j\leqslant n} (x_i - x_j)(x_ix_j - 1) = \frac{1}{2} \det_{1\leqslant i,j\leqslant n} \big(x_i^{j-1} + x_i^{2n-j-1}\big).$$

We now replace $n \mapsto 2n+1$ followed by

$$(x_1,\dots,x_{2n+1}) \mapsto (x_1,y_1,x_2,y_2,\dots,x_n,y_n,x_{n+1}),$$

and then take the limits $y_i \to x_i^{-1}$ for all $1 \leqslant i \leqslant n$ and $x_{n+1} \to 1$. Carrying out the specialisation $t \mapsto q^{2n+1}$ and $x_i \mapsto q^{n-i}$ and using (2.15) on the left and the $\mathrm{A}_{2n}^{(2)}$ Macdonald identity on the right yields the result. ∎

Sketch of proof of Corollary 5.16. As in the previous proof, we take (5.27) but now carry out the same steps as in the proof of Corollary 5.5, including the use of the $\mathrm{B}_n^{(1)}$ Macdonald identity to obtain the product of theta functions. ∎

Sketch of proof of Corollary 5.18. Yet again we take (5.27) but then follow the proof of Corollary 5.7, except that in the final step we use the $\mathrm{A}_{2n-1}^{(2)}$ Macdonald identity. ∎

6 Open problems

To conclude the paper, we list a number of open problems pertaining to affine Jacobi–Trudi and q,t-Rogers–Ramanujan identities, beyond the obvious problem of proving Conjectures 1.2–1.4.

One such problem is to better understand algebraically for what values of the parameters t, s in $P_{(k^n)}^{\mathrm{B}_n}(x;t,s)$ and $P_{(k^n)}^{\mathrm{C}_n}(x;t,s)$, or t, s_1, s_2 in $P_{(k^n)}^{\mathrm{C}_n}(x;t,s_1,s_2)$ one should expect further Jacobi–Trudi formulas. Our list of results is definitely not complete, and for example we conjecture that the additional $\mathrm{A}_{2k-1}^{(2)}$ identity

$$P_{(k^n)}^{\mathrm{B}_n}(x;t,t) = \frac{1}{2} \sum_{y\in\mathbb{Z}^k} \det_{1\leqslant i,j\leqslant k} \Big((-1)^{y_i} t^{\frac{1}{2}Ky_i^2-(j-1)y_i} \big(\ddot{e}_{n-i+j-Ky_i}(x) + \ddot{e}_{n+i+j-Ky_i-1}(x)\big)\Big)$$

holds, where $K = 2k$. What is unusual about this Jacobi–Trudi formula is that there does not appear to be a companion for $\dot{e}_r$. As a consequence, the q,t-Rogers–Ramanujan identity

$$\begin{aligned}\sum_{\substack{\lambda\\ \lambda_1\leqslant 2k}} q^{|\lambda|} \left(\prod_{i=1}^{2k-1} (t;t^2)_{\lceil m_i(\lambda)/2\rceil}\right) P_\lambda\big(1,q,q^2,\dots;t\big) \\ = \frac{(p^2;p^2)_\infty (p;p)_\infty^{k-1}}{(q;q)_\infty^k} \prod_{i=1}^k \theta\big(pq^{2i-1};p^2\big) \prod_{1\leqslant i<j\leqslant k} \theta\big(q^{j-i},q^{i+j-1};p\big),\end{aligned}$$

where $p = tq^{2k}$, does not appear to have a counterpart for

$$\sum_{\substack{\lambda\\ \lambda_1\leqslant 2k}} q^{|\lambda|/2} \left(\prod_{i=1}^{2k-1} (t;t^2)_{\lceil m_i(\lambda)/2\rceil}\right) P_\lambda\big(1,q,q^2,\dots;t\big).$$

It remains unclear if this is an isolated case or that further such results exist. In particular, we note that the affine Lie algebras $\mathrm{B}_k^{(1)}$ and $\mathrm{D}_k^{(1)}$ are currently missing from our list of q,t-Rogers–Ramanujan identities.

It would also be interesting to consider the determinants in Conjectures 1.2–1.4 with $K = 2k+a$ (where a is one of 0, 1, 2 depending on the conjecture) replaced by $K = 2k+2\ell+a$ for ℓ a nonnegative integer. For example, what are the $P_\lambda^{\mathrm{C}_n}(x;t,0)$ and $(x_1\cdots x_n)^{-k}P_\lambda(x;t)$ expansions of

$$\sum_{y\in\mathbb{Z}^k}\det_{1\leqslant i,j\leqslant k}\big(t^{(k+\ell+1)y_i^2-jy_i}\big(\dot{e}_{n-i+j-2(k+\ell+1)y_i}(x)-\dot{e}_{n+i+j-2(k+\ell+1)y_i}(x)\big)\big)? \tag{6.1}$$

The answers to these questions should imply further character identities. For instance, in the simplest possible case, namely $k=\ell=1$, we have

$$\begin{aligned}\sum_{y\in\mathbb{Z}}t^{3y^2-y}\big(\dot{e}_{n-6y}(x)-\dot{e}_{n-6y+2}(x)\big)&=\sum_{s=0}^{\lfloor n/2\rfloor}t^{s^2}P_{(1^{n-2s})}^{\mathrm{C}_n}(x;t,0)\\&=(x_1\cdots x_n)^{-1}\sum_{r,s\geqslant 0}t^{s^2}P_{(2^r,1^{2s})}(x;t).\end{aligned}$$

Here the equality between the expression on the left and the second expression on the right follows from

$$\sum_{y\in\mathbb{Z}}q^{3y^2-y}\left(\begin{bmatrix}2n\\n-3y\end{bmatrix}_q-\begin{bmatrix}2n\\n-3y+1\end{bmatrix}\right)=t^{n^2}$$

– an identity equivalent to the Bailey pair A(5) in Slater's list – whereas the equality between the two expressions on the right may be established using the $k=1$ case of (2.19a) and

$$P_{(1^n)}^{\mathrm{B}_n}(x;t,0)=\sum_{k=0}^{n}P_{(1^k)}^{\mathrm{C}_n}(x;t,0).$$

Specialising $x_i=q^{i-1/2}$ for $1\leqslant i\leqslant n$, or first using (2.23b) and then specialising $x_i=q^i$ for $1\leqslant i\leqslant n$, yields

$$\begin{aligned}&\sum_{r,s\geqslant 0}q^{(\sigma+1)(r+s)}t^{s^2}P_{(2^r,1^{2s})}\big(1,q,\dots,q^{n-1};t\big)\\&\quad=\sum_{y\in\mathbb{Z}}t^{3y^2-y}\left(q^{3y(6y-\sigma)}\begin{bmatrix}2n+\sigma\\n-6y+\sigma\end{bmatrix}_q-q^{(3y-1)(6y+\sigma-2)}\begin{bmatrix}2n+\sigma\\n-6y+2\end{bmatrix}_q\right)\end{aligned}$$

for $\sigma\in\{0,1\}$. Taking the large-n limit and applying Watson's quintuple product identity [83] – the $\mathrm{A}_{2k}^{(2)}$ Macdonald identity (5.4) for $k=1$ – gives the further q,t-Rogers–Ramanujan identity

$$\sum_{r,s\geqslant 0}q^{(\sigma+1)(r+s)}t^{s^2}P_{(2^r,1^{2s})}\big(1,q,q^2,\dots;t\big)=\frac{(p;p)_\infty}{(q;q)_\infty}\,\theta\big(q^{2-\sigma};p\big)\theta\big(pq^{4-2\sigma};p^2\big),$$

where $p=t^2q^{12}$. For $t=q$, this is

$$\sum_{r,s\geqslant 0}\frac{q^{(r+s)^2+2s^2+\sigma(r+s)}}{(q;q)_r(q;q)_{2s}}=\frac{(q^{14};q^{14})_\infty}{(q;q)_\infty}\,\theta\big(q^{2-\sigma};q^{14}\big)\theta\big(q^{10+2\sigma};q^{28}\big),$$

which is a well known formula for the (normalised) character $\chi_{1,2-\sigma}^{(3,7)}(q)$ of the Virasoro algebra of central charge $c=-25/7$ and conformal dimension $-5(1-\sigma)/28$, see, e.g., [9].

Recalling the notation (4.3), (6.1) for $t=1$ is

$$(x_1\cdots x_n)^{-k}\det_{1\leqslant i,j\leqslant k}\big(F_{i-j,2(k+\ell+1)}(x)-F_{i+j,2(k+\ell+1)}(x)\big),$$

for which Huh et al. [27, Theorem 3.3] obtained a combinatorial expression in terms of cylindric tableaux as follows:

$$(x_1\cdots x_n)^{-k}\Bigg(\sum_{\lambda\in\mathrm{Par}_{n,2k}^{2\ell+2}}{}^{'}-\sum_{\lambda\in\mathrm{Par}_{n,2k}^{2\ell+2}}{}^{''}\Bigg)\sum_{T\in\mathrm{CSSYT}_{n;2k,2\ell+2}(\lambda)}x^T.$$

Here the prime in the first sum denotes the restriction that λ must be even and the double prime in the second sum denotes the restrictions that $\lambda'_1-\lambda'_{2k}=2\ell+2$, the last $2\ell+2$ parts of λ are odd and the other parts are even. Is it possible to extend this result to (6.1) by introducing an additional statistic on cylindric tableaux?

A final question is to find partition theoretic interpretations for the Rogers–Ramanujan identities listed in Section 5.2. From the recent work in [15, 19, 21, 33, 61, 62, 67] on the Capparelli–Meurman–Primc–Primc (CMPP) conjectures, we know that, at least conjecturally, the GOW identities of Corollaries 5.1–5.3 admit interpretations as identities for the generating function of restricted sets of coloured partitions. It is natural to suspect that such an interpretation is not limited to the first three corollaries, and it would be extremely interesting to find a partition theoretic interpretation in terms of coloured partitions for the remaining 14 corollaries.

A Proof of (5.21)

In this appendix, we prove the full set of Rogers–Ramanujan identities (5.21), which for $1\leqslant i\leqslant k-1$ were conjectured by Matthew Russell after reading an earlier version of this paper.

Theorem A.1. *For k a positive integer and i an integer such that $0\leqslant i\leqslant k$,*

$$\begin{aligned}&\sum_{n_1\geqslant\cdots\geqslant n_{2k}\geqslant 0}{}^{'}\frac{q^{\frac12(n_1^2+\cdots+n_{2k}^2)+\frac12(n_1-n_2+\cdots+n_{2k-1}-n_{2k})+\frac12(n_1+n_2+\cdots+n_{2i})}}{\left(q^2;q^2\right)_{\lfloor (n_1-n_2)/2\rfloor}\cdots\left(q^2;q^2\right)_{\lfloor (n_{2k-1}-n_{2k})/2\rfloor}(q;q)_{n_{2k}}}\\&\qquad=\frac{\left(-q^{2i+1},-q^{2i+1},q^{2i+1};q^{2i+1}\right)_\infty}{\left(q^2;q^2\right)_\infty},\end{aligned}\tag{A.1}$$

where the prime denotes the restriction that $n_{2j-1}-n_{2j}$ is even for all $1\leqslant j\leqslant k$.

Before proving this theorem, we prepare some preliminary results. For nonnegative integers r and s, let ${}_r\phi_s$ denote the basic hypergeometric series

$${}_r\phi_s\left[\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};q,z\right]=\sum_{k=0}^\infty\frac{(a_1,\dots,a_r;q)_k}{(q,b_1,\dots,b_s;q)_k}\left((-1)^kq^{\binom{k}{2}}\right)^{s-r+1}z^k.$$

Lemma A.2. *For n a nonnegative integer,*

$$\sum_{r=0}^n w^r q^{\binom{r+1}{2}}H_{n-r}(z;q)\begin{bmatrix}n\\ r\end{bmatrix}_q={}_2\phi_0\left[\begin{matrix}-wq/z,q^{-n}\\ -\end{matrix};q,zq^n\right],\tag{A.2}$$

where H_m is a Rogers–Szegő polynomial, see (2.12).

Proof. Denote the expression on the left of (A.2) by $f_n(w,z;q)$. Then

$$f_n(w,z;q)=\sum_{r=0}^{n}\sum_{k=0}^{n-r} w^r z^k q^{\binom{r+1}{2}} \begin{bmatrix} n-r \\ k \end{bmatrix}_q \begin{bmatrix} n \\ r \end{bmatrix}_q = \sum_{k=0}^{n} z^k \begin{bmatrix} n \\ k \end{bmatrix}_q \sum_{r=0}^{n-k} w^r q^{\binom{r+1}{2}} \begin{bmatrix} n-k \\ r \end{bmatrix}_q.$$

The sum over r evaluates to $(-wq;q)_{n-k}$ by the q-binomial theorem [25, equation (II.4)]

$$\sum_{k=0}^{n} (-z)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q = (z;q)_n.$$

Thus

$$f_n(w,z;q)=\sum_{k=0}^{n} z^k (-wq;q)_{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q = (-wq;q)_n \, {}_2\phi_1\left[\begin{matrix} 0, q^{-n} \\ -q^{-n}/w \end{matrix}; q, -\frac{z}{w}\right].$$

Taking the $b\to 0$ limit in the transformation formula [25, equation (III.8)] yields

$${}_2\phi_1\left[\begin{matrix} 0, q^{-n} \\ c \end{matrix}; q, z\right] = \frac{1}{\left(q^{1-n}/c;q\right)_n} \, {}_2\phi_0\left[\begin{matrix} q/z, q^{-n} \\ - \end{matrix}; q, \frac{z}{c}\right].$$

Applying this to the above expression for $f_n(w,z;q)$, we obtain the right-hand side of (A.2). ∎

Corollary A.3. *For n a nonnegative integer,*

$$\sum_{r=0}^{n} w^r q^{\binom{r+1}{2}} H_{n-r}(-wq;q) \begin{bmatrix} n \\ r \end{bmatrix}_q = 1.$$

Proof. We specialise $z=-wq$ in (A.2). Since the ${}_2\phi_0$ series has a numerator parameter equal to 1, the series trivialises to 1, resulting in the claim. ∎

Corollary A.4. *For n a nonnegative integer,*

$$g_n(q) := \sum_{\substack{0\leqslant s\leqslant r\leqslant n \\ r-s \text{ even}}} \frac{q^{\binom{r+1}{2}+\binom{s}{2}}(q;q)_n}{\left(q^2;q^2\right)_{\lfloor (n-r)/2\rfloor}\left(q^2;q^2\right)_{(r-s)/2}(q;q)_s} = 1.$$

Before proving this result, we note that in the $n\to\infty$ limit it yields (A.1) for $i=0$ and $k=1$.

Proof. We begin by noting that

$$\frac{1}{\left(q^2;q^2\right)_{\lfloor m/2\rfloor}} = \frac{\left(q;q^2\right)_{\lceil m/2\rceil}}{(q;q)_m} = \frac{H_m(-q;q)}{(q;q)_m},$$

where the final equality follows from [82, equation (1.10c)]. Using the above identity and replacing the summation index $s\mapsto r-2s$, it follows that

$$\begin{aligned} g_n(q) &= \sum_{r=0}^{n}\sum_{s=0}^{\lfloor r/2\rfloor} \frac{q^{r^2-2rs+s(2s+1)}(q;q)_n H_{n-r}(-q;q)}{(q;q)_{n-r}\left(q^2;q^2\right)_s (q;q)_{r-2s}} \\ &= \sum_{r=0}^{n} q^{r^2} H_{n-r}(-q;q) \begin{bmatrix} n \\ r \end{bmatrix}_q {}_2\phi_1\left[\begin{matrix} q^{-r}, q^{1-r} \\ 0 \end{matrix}; q^2, q^2\right]. \end{aligned}$$

According to the $c \to 0$ limit of the q-Chu–Vandermonde summation [25, equation (II.6)]

$$ {}_2\phi_1\left[\begin{matrix} a, q^{-n} \\ 0 \end{matrix}; q, q\right] = a^n. $$

Applying this for $(q,a,n) \mapsto \big(q^2, q^{1-2\lceil r/2\rceil}, \lfloor r/2\rfloor\big)$, we find

$$ g_n(q) = \sum_{r=0}^{n} q^{\binom{r+1}{2}} H_{n-r}(-q;q) \begin{bmatrix} n \\ r \end{bmatrix}_q. $$

By Corollary A.3 for $w=1$, we are done. ■

We are now ready to give a proof of Theorem A.1 assuming the truth of the result for $i=k$. This case is the $n=1$ instance of Corollary 5.16.

Proof of Theorem A.1. As remarked above, we may assume the theorem holds for $i=k$. Combining this with the remark immediately following Corollary A.4, there is nothing to prove for $k=1$. In the following we thus assume that $k \geqslant 2$ and $0 \leqslant i < k$. Now denote the sum on the left-hand side of (A.1) by $S_{i,k}(q)$ and its summand by $S_{i,k;n_1,\dots,n_{2k}}(q)$. Then

$$ \begin{aligned} S_{i,k}(q) &= \sideset{}{'}\sum_{n_1\geqslant\dots\geqslant n_{2k}\geqslant 0} S_{i,k;n_1,\dots,n_{2k}}(q) = \sideset{}{'}\sum_{n_1\geqslant\dots\geqslant n_{2k-2}\geqslant 0} S_{i,k-2;n_1,\dots,n_{2k-2}}(q) g_{n_{2k-2}(q)} \\ &= \sideset{}{'}\sum_{n_1\geqslant\dots\geqslant n_{2k-2}\geqslant 0} S_{i,k-2;n_1,\dots,n_{2k-2}}(q) = S_{i,k-1}(q), \end{aligned} $$

where the second equality uses the definition of $g_n(q)$ and the third equality follows from Corollary A.4. If $i \geqslant 1$, this may be iterated to give

$$ S_{i,k}(q) = S_{i,i}(q) = \frac{\big(-q^{2i+1}, -q^{2i+1}, q^{2i+1}; q^{2i+1}\big)_\infty}{\big(q^2;q^2\big)_\infty}. $$

If, on the other hand, $i=0$, iteration leads to

$$ S_{0,k}(q) = S_{0,1}(q) = \frac{(-q,-q,q;q)_\infty}{\big(q^2;q^2\big)_\infty}. $$

This completes the proof. ■

Acknowledgements

I am grateful to Anne Schilling for helpful discussions, to Matthew Russell for pointing out the full set of identities (5.21), and to both referees for their careful reading of the paper. This work was supported by the Australian Research Council Discovery Project DP200102316.

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