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Intermediate Macdonald Polynomials and Their Vector Versions

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Abstract. Intermediate Macdonald polynomials for an affine root system S with fixed origin and finite Weyl group W_0 are orthogonal polynomials invariant under a parabolic subgroup $W_J \leq W_0$. The extreme cases of $W_J = 1$ and $W_J = W_0$ correspond to the non-symmetric and symmetric Macdonald polynomials, respectively. In this paper, we use double-affine Hecke algebras to study their basic properties, including that they form an orthogonal basis and that they diagonalise a commutative algebra of difference-reflection operators, and calculate their norms. Finally, we provide two interpretations of intermediate Macdonald polynomials as vector-valued polynomials and connect them to the literature.

Key words: intermediate Macdonald polynomials; double-affine Hecke algebras; vector-valued orthogonal polynomials

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1 Introduction

Let R be a (finite) root system and L a lattice associated with R, such as the weight lattice, the coweight lattice, or the coroot lattice. Let $A := K[L]$ be the group algebra of L over a suitable field K. Orthogonal polynomials associated with R, such as the ones detailed in [6, 17, 18], are typically elements of A that are either symmetric under the Weyl group W_0 of R (or transform according to a given 1-dimensional representation) or possess no symmetry at all.

In this paper, we study the intermediate cases: polynomials that transform under a character ϵ of a parabolic subgroup $W_J \leq W_0$, in the context of Macdonald polynomials.

In the context of Heckman–Opdam polynomials, the W_J-invariant polynomials have been considered in [23]. Our approach is different and leads to the notion of *intermediate Macdonald polynomials.*

In the case where R is of type A_n and $\epsilon = 1$, these polynomials have been studied extensively, e.g., in [1, 2, 8, 14] under the name "Macdonald polynomials with prescribed symmetry", and connected to the equivariant K-theory of parabolic flag Hilbert schemes in [9] as well as to the theory of parahoric Whittaker functions in [4]. In a preprint [7] that appeared shortly after this one, it is furthermore shown that certain specialisations of intermediate Macdonald polynomials can be obtained as characters of representations of parahoric subalgebras of affine Kac–Moody algebras.

For general Macdonald data and characters ϵ of W_J, we prove basic facts about the intermediate Macdonald polynomials, such as leading terms, orthogonality, and that they diagonalise a commutative algebra of difference-reflection operators, and compute their norms.

Finally, we provide two vector-valued interpretations for $\epsilon = 1$, and connect them to the literature. For the first interpretation as vector-valued polynomials, we can artificially increase the symmetry of elements of $A_J := A^{W_J}$ by tensoring with the finite-dimensional vector space $K[W_0/W_J]$. In fact,

$$A_J \cong (A \otimes K[W_0/W_J])^{W_0}$$

as modules over $A_0 := A^{W_0}$ by mapping $A_J \ni f \mapsto f \otimes e(W_J)$ and then symmetrising ($e(W_J)$ is the basis vector of $K[W_0/W_J]$ corresponding to the coset of the neutral element). However, in the context of Macdonald polynomials, the group W_0 and its subgroups are rather artificial, and it makes much more sense to phrase our constructions in terms of the double-affine Hecke algebra $\tilde{\mathfrak{H}}$ from [17]. Luckily, the authors of [19] show that it is possible to establish a $\tilde{\mathfrak{H}}$-module structure (even more than one) on $A \otimes K[W_0/W_J]$, which they call $\mathbb{M}_J(\phi)$. We show that the "spherical" vectors of $\mathbb{M}_J(\phi)$ are indeed in A_0-linear equivalence with A_J. This result has been proven independently in [24].

This vector-valued interpretation is especially valuable if we seek to connect our theory to representation theory. In the classical theory (i.e., Lie groups), $(A \otimes K[W_0/W_J])^{W_0}$ is exactly the form that suitable matrix-spherical functions take. In fact, besides the well-known correspondence between symmetric Heckman–Opdam polynomials ($W_J = W_0$) and zonal spherical functions of symmetric pairs from, e.g., [10], a concrete connection between intermediate Heckman–Opdam polynomials and matrix spherical functions of symmetric pairs has been shown in [22, 23] for some examples. For quantum groups, a similar connection between symmetric Macdonald polynomials and zonal spherical functions of quantum symmetric pairs was proven in [15], so it stands to reason to look for matrix spherical functions in the Macdonald version of $(A \otimes K[W_0/W_J])^{W_0}$, i.e., the spherical elements of $\mathbb{M}_J(\phi)$.

For a second interpretation of A_J as a ring of vector-valued polynomials, recall that by a well-known result by Chevalley, A_0 is a free (commutative) K-algebra, i.e., a polynomial ring. The same, however, cannot be said about A_J (except when $W_J = W_0$) or even A. Thus, we are confronted with the pedantic philosophical question of how exactly intermediate and even non-symmetric Macdonald polynomials can even be thought of as polynomials in the first place. A possible solution to this conundrum comes from noting that A_J is a free A_0-module (as is detailed in [21]), so that we can view elements of A_J as column vectors of elements of A_0, i.e., as vector-valued polynomials. The inner product on A_J can then be rewritten in terms of a matrix weight with symmetric entries, which allows for an interpretation in the theory of vector-valued orthogonal polynomials.

In Section 2, we begin by introducing some notions about double-affine Hecke algebras for arbitrary affine root systems S, as well as setting up the necessary notions of duality, labels, and the monomial order on the ring where our orthogonal polynomials live. The notation (with exception of the τ's, as explained in Remark 2.29 (ii)) and ordering of topics have been taken from [17], which we will refer to for most of the proofs.

In Section 3, we introduce various tools that are specifically related to parabolic subgroups, and that are not taken wholesale from [17]. They include an adaption of the symmetrisers from [17, Section 5.5] as well as a study of W_J-orbits of monomials.

Section 4 is the central section of this paper and contains its protagonist: the (ϵ-symmetric) intermediate Macdonald polynomials. We introduce them in analogy with the ϵ-symmetric Macdonald polynomials from [17, Section 5.7] by applying a suitable symmetriser to non-symmetric Macdonald polynomials and then normalising. We show that the symmetric versions form a basis of the W_J-symmetric polynomials, and that they are orthogonal. Then we proceed to relate their norms to those of the non-symmetric Macdonald polynomials.

In Section 5, we consider the first vector-valued interpretation of A_J. We introduce the Y-parabolically induced modules $\mathbb{M}_J(\phi)$ from [19, Section 3.2] and then prove that their spaces of spherical vectors is in A_0-linear equivalence with A_J.

Finally, in Section 6, we consider the second vector-valued interpretation and show the existence of a symmetric matrix weight. To illustrate this general theory, we then consider two examples: the case of non-symmetric Askey–Wilson polynomials $\big($type $\big(C_1^\vee, C_1\big)$ with $W_J = 1$, so that $A^{W_J} = A\big)$ from [12], and the case of type A_2 with a nontrivial parabolic subgroup $W_J \leq W_0$ that was also considered in the classical case in [23].

2 Double-affine Hecke algebras

For the convenience of the reader, we now recap the theory from [17] that we will rely on in the following sections. Readers wishing to skip the recap are advised to consult Remark 2.29 for notational differences to Macdonald, and then proceed to Section 3.

Let E be a finite-dimensional Euclidean space with translation space V, and let F be the real vector space of affine functions $E \to \mathbb{R}$. Let $S \subseteq F$ be an affine root system (see [16]) and W_S its affine Weyl group.

2.1 Duality and labels

To each possible choice of S (potentially up to embedding in a larger root system, cf. the remark immediately proceeding [17, Section 1.4]), we are going to associate another affine root system S', two (linear) root systems R, R', and two lattices L, L'.

Lemma 2.1. *Let S be irreducible, then after choosing an appropriate origin to identify $E = V$ and after appropriate re-scaling, one of the following is true:*

(*i*) *There is a reduced linear root system $R \subseteq V$ such that $S = S(R)$.*

(*ii*) *There is a reduced linear root system $R \subseteq V$ such that $S = S(R)^\vee$.*

(*iii*) *There is an affine root system $\tilde{S}$ of type $\big(C_n^\vee, C_n\big)$ of which S forms a $W_{\tilde{S}}$-invariant subsystem.*

Here, $S(R)$ is as in [17, Section 1.2.1].

Proof. This is the content of [17, Section 1.3]. For reference, we mention for (iii), which orbits of $\tilde{S}$ make up S. We adopt the notation of [17, Section 1.3] and define $O_1, \dots, O_5$ to be the orbits satisfying $a_n \in O_1$, $2a_n \in O_2$, $a_0 \in O_3$, $2a_0 \in O_4$, $a_1, \dots, a_{n-1} \in O_5$ ($a_0, \dots, a_n$ as in [17, Section 1.3.18]). Then S consists of the following $\tilde{S}$-orbits:

$$BC_n\colon\ O_1,\ O_4,\ O_5, \qquad (BC_n, C_n)\colon\ O_1,\ O_2,\ O_4,\ O_5, \qquad \big(C_n^\vee, BC_n\big)\colon\ O_1,\ O_2,\ O_3,\ O_5,$$
$$\big(B_n, B_n^\vee\big)\colon\ O_1,\ O_2,\ O_5, \qquad \big(C_n^\vee, C_n\big)\colon\ O_1,\ O_2,\ O_3,\ O_4,\ O_5. \qquad \blacksquare$$

For the remainder of this paper, we replace S by $\tilde{S}$ if Lemma 2.1 (c) is satisfied.

Definition 2.2. If S is irreducible, and

(i) $S = S(R)$ for a reduced (linear) root system R, let $S' := S\big(R^\vee\big)$, $R' := R^\vee$, $L := P(R)$, $L' := P\big(R^\vee\big)$. Furthermore, for $\alpha \in R$ write $\alpha' := \alpha^\vee \in R'$.

(ii) $S = S(R)^\vee$ for a reduced (linear) root system R, let $S' := S$, $R' := R$, $L := L' := P\big(R^\vee\big)$. Furthermore, for $\alpha \in R$ write $\alpha' := \alpha \in R'$.

(iii) S is of type $\big(C_n^\vee, C_n\big)$, let R be the linear root system of type C_n such that $S = S(R) \cup S(R)^\vee$. Then let $S' := S$, $R' := R$, $L := L' := \mathbb{Z}R^\vee$. Furthermore, for $\alpha \in R$ write $\alpha' := \alpha \in R'$.

If $S = S_1 \cup \cdots \cup S_n$ is reducible, assume without loss of generality that every S_i is either $S(R_i)$ (R_i reduced), $S(R_i)^\vee$ (R_i reduced), or that $S(R_i) \cup S(R_i)^\vee$ (R_i of type C_{n_i}). This has the side-effect that all constant affine functions contained in S_i and S_i', are half-integer-valued. Write c for the constant function that is 1 everywhere. Then define S_i', R_i, R_i', L_i, L_i' as above and take

$$\begin{aligned} S' &:= S_1' \cup \cdots \cup S_n', & R &:= R_1 \cup \cdots \cup R_n, & R' &:= R_1' \cup \cdots \cup R_n', \\ L &:= L_1 \oplus \cdots \oplus L_n, & L' &:= L_1' \oplus \cdots \oplus L_n'. \end{aligned}$$

We interpret R, R', L, L' as subsets of V and S, S' as sets of affine-linear functions on V, and we call (S, S', R, R', L, L') a *set of duality data.*

Remark 2.3.

(i) If (S, S', R, R', L, L') is a set of duality data, then so is (S', S, R', R, L', L).

(ii) Let $\langle \cdot, \cdot \rangle$ be the inner product of the translation space V, and use it to identify $V \cong V^*$. If $D\colon F \to V$ is the gradient map, i.e., $f(p + v) = f(p) + \langle Df, v\rangle$ for $f \in F$, $p \in E$, $v \in V$, we then have $DS \subseteq L$, $DS' \subseteq L'$. Furthermore, we have have $R^\vee \subseteq S'$ and $R'^\vee \subseteq S$.

(iii) Furthermore, $\langle R, L'\rangle, \langle R', L\rangle \subseteq \mathbb{Z}$. In particular, unless S contains components of type $\left(C_n^\vee, C_n\right)$, the pairings of $R \times L'$, $R' \times L$ are perfect pairings. If S does contain a component of type $\left(C_n^\vee, C_n\right)$ and if $a \in S$ lies in the orbit of the highest root of R, we have $\langle a, L'\rangle = \langle a', L\rangle = 2\mathbb{Z}$.

(iv) It is possible to choose systems of simple roots $(a_i)_{i\in I}$, $(a_i')_{i\in I}$, and $(\alpha_i)_{i\in I_0}$ of S, S' and R, respectively, such that $\alpha_i^\vee = a_i'$ ($i \in I_0$). Here, $I_0 \subseteq I$ indexes the simple linear roots and leaves out only one (non-linear) affine root per irreducible component. For each irreducible component R_i, depending on whether $R_i' = R_i$ or $R_i' = R_i^\vee$, write $\alpha' := \alpha$ or $\alpha' := \alpha^\vee$ for $\alpha \in R_i$.

Definition 2.4. Let (S, S', R, R', L, L') be a set of duality data, write W_0 for the (finite) Weyl group of R and R' (they have the same). Then define the *extended affine Weyl groups*

$$W := W(R, L') := W_R \ltimes t(L'), \qquad W' := W(R', L).$$

Here, $t(\lambda')\colon V \to V$ is the translation $x \mapsto x + \lambda'$ (for λ'). Since W acts on V by affine transformations, it also acts on F, the space of affine functions. In particular, W acts on L' and S; similarly, W' acts on L and S'. Furthermore, we have $W_S \leq W$ and $W_{S'} \leq W'$, where W_S (resp. $W_{S'}$) is the reflection group generated by $s_i = s_{a_i}$ (resp. $s_{a_i'}$) for $i \in I$ (s_a is the affine reflection in a, see [17, Section 1.1.6]).

Write $W_0 := W_R = W_{R'}$ for the reflection group generated by $s_i = s_{\alpha_i}$ for $i \in I_0$. It acts on S, S', R, R', L, L' and is the parabolic subgroup of W_S and $W_{S'}$ generated by the index set I_0.

Definition 2.5. A function $k\colon S \to \mathbb{R}$ is called a *W-labelling* if k is constant on W-orbits.

Remark 2.6. Let S be irreducible.

(i) If $S = S(R)$ for R simply-laced, there is only one W-orbit, and hence any labelling k is given by one number.

(ii) If $S = S(R)$ or $S(R)^\vee$ for R not simply-laced, there are two W-orbits, so any labelling k is given by two numbers k_s, k_l for short and long roots, respectively.

(iii) If S is of type $(C_n, C_n^\vee)$, we have $W = W_S$ and therefore we have the four ($n = 1$) or five orbits $O_1, \ldots, O_5$ from the proof of Lemma 2.1. In that case, a labelling k is given by five numbers $k_1, \ldots, k_5$.

Definition 2.7. The *dual labelling* of a W-labelling k is a W'-labelling $k'\colon S' \to \mathbb{R}$ given by

(i) $S = S(R)$: $k'(\alpha^\vee + c) := k(\alpha + rc)$ for $\alpha \in R, r \in \frac{1}{2}\mathbb{Z}$;

(ii) $S = S(R)^\vee$: $k' := k$;

(iii) Type $\big(C_n^\vee, C_n\big)$: $k'_5 := k_5$ and

$$\begin{pmatrix} k'_1 \\ k'_2 \\ k'_3 \\ k'_4 \end{pmatrix} := \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix}$$

on irreducible components.

Evidently, if k' is dual to k, then k is dual to k'.

Lemma 2.8. *Let $S_k \subseteq S$ be an irreducible component and let $i \in I_0$ such that $a_i \in S_k$. Then*

$$k(a_i) + k(2a_i) = k'(a'_i) + k'(2a'_i).$$

Proof. We differentiate between the three cases:

(i) If $S_k = S(R_k)$, then $a_i = \alpha_i$ and $a'_i = \alpha_i^\vee$, so that $k'(a'_i) = k'\big(\alpha_i^\vee\big) = k(\alpha_i) = k(a_i)$ and $k'(2a'_i) = 0 = k(2a_i)$.

(ii) If $S_k = S(R_k)^\vee$, then $a'_i = a_i$ and $k = k'$, so that $k'(a'_i) = k(a_i)$ and $k'(2a'_i) = 0 = k(2a_i)$.

(iii) If S_k is of type $\big(C_n^\vee, C_n\big)$, then a_i is indivisible and linear. Hence a_i belongs either to O_1 or to O_5. In the first case we have $2a_i \in O_2$ and hence

$$k'(a'_i) + k'(2a'_i) = k'_1 + k'_2 = k_1 + k_2 = k(a_i) + k(2a_i).$$

In the latter case, $2a_i \notin S$ and hence

$$k'(a'_i) + k'(2a'_i) = k'_5 = k_5 = k(a_i) + k(2a_i). \qquad \blacksquare$$

Lemma 2.9. *Let $S_k \subseteq S$ be an irreducible component and let $i \in I \setminus I_0$ be such that $a_i \in S_k$ (in particular, a_i is not linear). Let $j \in I_0$ be such that $W_0\alpha_j$ contains the highest root ϕ of R_k. Then $k(a_i) + k(2a_i) = k'(a'_j) - k'(2a'_j)$.*

Proof. We distinguish between the three cases.

(i) If $S_k = S(R_k)$, then $Da_i = -\phi$, so that $k(a_i) = k(-\phi) = k(\phi) = k'\big(\phi^\vee\big)$. Since $\alpha_j \in W_0\phi$, we have $\alpha_j^\vee = a'_j \in W_0\phi^\vee$ and hence $k(a_i) = k'(a'_j)$. Since S_k is reduced, the elements $2a_i$, $2a'_j$ are not in S.

(ii) If $S_k = S(R_k)^\vee$, then $Da_i = -\phi^\vee$, so that

$$k(a_i) = k\big(-\phi^\vee\big) = k\big(\phi^\vee\big) = k'\big(\phi^\vee\big).$$

Since $\alpha_j \in W_0\phi$, we again have $a'_j \in W_0\phi^\vee$, so that $k(a_i) = k'(a'_j)$.

(iii) If S_k is of type $\big(C_n^\vee, C_n\big)$, we have $a_i \in O_3$, so that

$$k(a_i) + k(2a_i) = k_3 + k_4 = k'_1 - k'_2.$$

Since α_j is long, its corresponding simple a'_j (of S_k) is short (see [17, Section 1.4.3], where $j = n$) and lies in O_1. Therefore,

$$k(a_i) + k(2a_i) = k'_1 - k'_2 = k'(a'_j) - k'(2a'_j). \qquad \blacksquare$$

2.2 Extended affine Weyl group

Write $S_1 := \{a \in S \mid 2a \notin S\}$ for the subsystem of inmultipliable roots. We shall now have a closer look at the extended affine Weyl groups W, W', in particular at their length functions.

Definition 2.10. For $w \in W$, define

$$S_1(w) := S^+ \cap w^{-1}S^- = \left\{a \in S^+ \;\middle|\; wa \in S^-\right\},$$

which we can also interpret as the set of hyperplanes lying between the fundamental alcove C and $w^{-1}C$. Define the *length* of W to be $\ell(w) := \#S_1(w)$. Write $\Omega := \{w \in W \mid \ell(w) = 0\}$.

Lemma 2.11 ([17, below equation (2.2.3)])**.** *The group Ω is a finite abelian group, and we have $W = W_S \rtimes \Omega$.*

Lemma 2.12 ([17, Section 2.2.4])**.** *If $w, v \in W$, then $\ell(vw) \leq \ell(v) + \ell(w)$ with equality iff $S_1(w) \subseteq S_1(vw)$.*

Lemma 2.13 ([17, Section 2.2.8])**.** *Define $\sigma\colon S \to \mathbb{R}$ by $\sigma(a) := \pm 1$ for $a \in S^{\pm}$. For all $w \in W$ and $i \in I$, we have*

$$\ell(s_i w) = \ell(w) + \sigma\big(w^{-1}a_i\big), \qquad \ell(ws_i) = \ell(w) + \sigma(wa_i).$$

Lemma 2.14 ([17, Section 2.2.9])**.** *Every element $w \in W$ can be expanded as a reduced expression of the form $w = us_{i_1} \cdots s_{i_p}$, where $p = \ell(w)$ and $u \in \Omega$. For such a w, we have $S_1(w) = \{b_1, \ldots, b_p\}$, where $b_r := s_{i_p} \cdots s_{i_{r+1}}(a_{i_r})$.*

Definition 2.15. W_S as a Coxeter group comes equipped with a Bruhat order. We extend it to W as follows: $uw \leq vw'$ for $u, v \in \Omega$ and $w, w' \in W_S$ iff $u = v$ and $w \leq w'$.

Lemma 2.16. *Let $W_J \leq W_S$ be a parabolic subgroup. Define*

$$W^J := \{v \in W \mid \forall w \in W_J \colon \ell(vw) \geq \ell(v)\},$$

the set of shortest coset representatives. Then every element $w \in W$ has a unique decomposition vw' with $v \in W^J$, $w' \in W_J$ and $\ell(w) = \ell(v) + \ell(w')$.

Proof. If W is a Coxeter group, the claim is a well-known result that can be found, e.g., in [11, Section 5.12, Proposition 1.12]. Since we are here also allowing for the semidirect product with the finite automorphism group Ω, there is something left to prove.

Write $w = u\tilde{w}$ for $u \in \Omega, \tilde{w} \in W_S$. By [11, Section 5.12, Proposition 1.12], $\tilde{w} = \tilde{v}\tilde{w}'$ for $\tilde{w}' \in W_J$ and $\tilde{v}$ a shortest coset representative. Then picking $w' := \tilde{w}'$ and $v := u\tilde{v}$ does the trick: evidently, $w' = \tilde{w}' \in W_J$, and the map $W_S \to W, x \mapsto ux$ is an order-isomorphism onto uW_S, both with respect to the quasi-order induced by the length function and the Bruhat order. ■

2.3 ρ, u, v, and r

We shall now decompose translations in terms of shortest coset representatives with respect to W_0 and define a few auxiliary objects that make use of the W-labelling k.

Definition 2.17. Define

$$\rho_{k'} := \frac{1}{2}\sum_{\alpha \in R} k'\big(\alpha^\vee\big)\alpha \qquad \rho'_k := \frac{1}{2}\sum_{\alpha' \in R'} k\big(\alpha'^\vee\big)\alpha' \in V.$$

Lemma 2.18. *We have $\langle\rho_{k'},\alpha_i^\vee\rangle = k'(\alpha_i^\vee)$ for $i\in I_0$. In particular, if $k'(a')\neq 0$ for all $a'\in S'$, then $\rho_{k'}$ is regular under the action of W_0 (viewed as an element of $E\cong V$). If $k'(a')>0$ for all $a'\in S'$, it is even strictly dominant.*

Proof. For $i\in I_0$, we have

$$s_i\rho_{k'} = \frac{1}{2}\sum_{\alpha\in R^+} k'(\alpha^\vee)s_i\alpha = \frac{1}{2}\sum_{\alpha\in s_iR^+} k'(\alpha^\vee)\alpha.$$

By [3, Chapitre VI, Section 1.6, Corollaire 1], the simple reflection s_i permutes the positive roots that are not α_i. Therefore, we have $s_iR^+ = R^+\setminus\{\alpha_i\}\cup\{-\alpha_i\}$, hence

$$s_i\rho_{k'} = \rho_{k'} - k'(\alpha_i^\vee)\alpha_i = \rho_{k'} - \langle\rho_{k'},\alpha_i^\vee\rangle\alpha_i.$$

Consequently, $k'(\alpha_i^\vee) = \langle\rho_{k'},\alpha_i^\vee\rangle$. So if k' is never zero, $\rho_{k'}$ lies in the interior of a Weyl chamber. If k' is in particular positive, $\rho_{k'}$ lies in the interior of the fundamental Weyl chamber. ∎

Definition 2.19. For $\lambda'\in L'$, let $t(\lambda') =: u(\lambda')v(\lambda')\in W^0W_0$ be the decomposition from Lemma 2.16. Similarly, decompose $t(\lambda) =: u'(\lambda)v(\lambda)\in (W')^0W_0$ for $\lambda\in L$.

Define

$$L'_+ := \left\{\lambda'\in L' \;\middle|\; \forall i\in I_0\colon \langle\lambda',a_i\rangle\geq 0\right\}$$

and analogously, L_+, and call the elements of L'_+, L_+ *dominant*. The dominant elements form a fundamental domain for the action of W_0 on L', L, respectively. Elements of $-L'_+$, $-L_+$ are called *antidominant*. For $\lambda'\in L'$, write λ'_+, λ'_- for the unique dominant (resp. antidominant) element in $W_0\lambda'$.

Lemma 2.20. *Let $\lambda'\in L'$ (resp. $\lambda\in L$), then $v(\lambda')\lambda'\in -L'_+$ (resp. $v(\lambda)\lambda\in -L_+$).*

Proof. In [17, Section 2.4], this follows from the definition of $v(\lambda')$. That the definition used there is equivalent to ours can be seen from [17, Section 2.4.9]. ∎

Corollary 2.21. *We have $u(\lambda') = v(\lambda')^{-1}t(\lambda'_-)$ for $\lambda'\in L'$, where λ'_- is the antidominant element in its W_0-orbit.*

Proof. By Lemma 2.20, we have $v(\lambda')\lambda' = \lambda'_-$, so that

$$v(\lambda')^{-1}t(\lambda'_-) = v(\lambda')^{-1}v(\lambda)t(\lambda')v(\lambda)^{-1} = u(\lambda')v(\lambda')v(\lambda')^{-1} = u(\lambda'). \qquad ∎$$

Definition 2.22. For $\lambda'\in L'$, $\lambda\in L$, define

$$\begin{aligned} r'_k(\lambda') &:= u(\lambda')(-\rho'_k) = v(\lambda')^{-1}(\lambda'_- - \rho'_k),\\ r_{k'}(\lambda) &:= u'(\lambda)(-\rho_{k'}) = v(\lambda)^{-1}(\lambda_- - \rho_{k'}).\end{aligned}$$

Lemma 2.23 ([17, Section 2.8.4])**.**

(i) *If $u\in\Omega$ and $\lambda'\in L'$, then $r'_k(u\lambda') = ur'_k(\lambda')$.*

(ii) *If $i\in I_0$ with $s_i\lambda'\neq\lambda'$, then $r'_k(s_i\lambda') = s_ir'_k(\lambda')$.*

2.4 Partial order on the lattices

We shall now define partial orders on L, L' that have finite down-sets and thus allow for the application of the Gram–Schmidt algorithm. These partial orders will turn out to be intimately related to the Bruhat ordering and the elements $(u(\lambda'))_{\lambda' \in L'}$ (resp. $(u'(\lambda))_{\lambda \in L}$).

Definition 2.24. For $\lambda' \in L'$ and $\lambda \in L$ define $\overline{v}(\lambda')$ and $\overline{v}(\lambda)$ to be the shortest elements $w \in W_0$ mapping λ'_+ to λ' and λ_+ to λ, respectively.

Definition 2.25. Let $\lambda', \mu' \in L'$, then define $\lambda' \leq \mu'$ if

(i) $\mu' - \lambda' \in \mathbb{N}_0 R^{\vee,+} \setminus \{0\}$ or

(ii) $\mu' = \lambda'$ and $\overline{v}(\lambda') \leq \overline{v}(\mu')$.

Analogously on L.

Remark 2.26. We later refer to results from [19] although the authors of that paper use a different definition for the order on L': they define $\lambda' \leq \mu'$ iff $u(\lambda') \leq u(\mu')$, i.e., they identify L' with W/W_0 and take the Bruhat order on the shortest coset representatives. This definition is actually equivalent to our definition by [19, Proposition 5.21] and [17, Section 2.7.13].

Lemma 2.27. *Let $\lambda' \in L'_+$ and let $\mu' \in W_0\lambda'$. Let $\overline{v}(\mu') = s_{i_1} \cdots s_{i_p}$ be a reduced expression. Define $\mu'_r := s_{i_{r+1}} \cdots s_{i_p}\lambda'$, then $\mu' = \mu'_0 > \cdots > \mu'_p = \lambda'$.*

Proof. Follows from [17, Section 2.7.10 (ii)] after some minor adjustments of notation. ■

2.5 Double-affine Hecke algebra

For this subsection, we immediately give definitions for the (double-)affine Hecke algebra in terms of generators and relations and thereby forego any mentions of the (double-)affine braid groups from [17, Section 3]. In order to still be able to use Macdonald's results, we prove that our definitions are equivalent.

We fix a field extension $K|\mathbb{Q}$ and a group homomorphism $q\colon \mathbb{R} \to K^\times$ such that $q(x)$ is transcendental over $\mathbb{Q}$ for all $x \neq 0$ (e.g., $K = \mathbb{Q}(\mathbb{R})$, the quotient field of the group algebra of $\mathbb{R}$ over $\mathbb{Q}$). Write $A := K[L]$ and $A' := K[L']$ for the group algebras of L, L', whose monomials shall be written $e(\lambda)$, $e(\lambda')$. Furthermore, define $e(rc) := q(r)$, which provides a definition for $e(a)$ for any $a \in S$.

Definition 2.28. For a W-labelling $k\colon S \to \mathbb{R}$ and a root $a \in S$, define

$$\tau_a := \tau_{a,k} := q\left(\frac{k(a) + k(2a)}{2}\right), \qquad \tilde{\tau}_a := \tilde{\tau}_{a,k} := q\left(\frac{k(a) - k(2a)}{2}\right),$$

where we take $k(2a) = 0$ if $2a \notin S$. In case $a = a_i$ for $i \in I$, we shall write $\tau_i := \tau_{a_i}$, $\tilde{\tau}_i := \tilde{\tau}_{a_i}$ as a shorthand.

Remark 2.29.

(i) In the notation of Lemma 2.8, we can conclude $\tau_i = \tau_{a'_i,k'}$.

(ii) In the notation of Lemma 2.9, we can conclude $\tau_i = \tilde{\tau}_{a'_j,k'}$.

(iii) In [17], the notation τ'_i is used instead of $\tilde{\tau}_i$, which can be easily confused as referring to the root system S' and the labelling k'. To prevent such confusion, and in order to allow us to more easily talk about the τ's derived from k', we adopted the notation used here.

Definition 2.30. Define the rational functions

$$\boldsymbol{b}(t,u;x) := \frac{t-t^{-1}+\big(u-u^{-1}\big)x}{1-x^2}, \qquad \boldsymbol{c}(t,u;x) := \frac{tx-t^{-1}x^{-1}+u-u^{-1}}{x-x^{-1}}.$$

Note that $\boldsymbol{c}\big(t^{-1},u^{-1};x^{-1}\big)=\boldsymbol{c}(t,u;x)$. Define furthermore the following functions:

$$\boldsymbol{b}_a := \boldsymbol{b}_{a,k,S} := \boldsymbol{b}(\tau_{a,k},\tilde{\tau}_{a,k},e(a)) \in \operatorname{Frac}(A), \qquad \boldsymbol{b}'_{a'} := \boldsymbol{b}_{a',k',S'} \in \operatorname{Frac}(A')$$

for $a\in S$, $a'\in S'$.

Definition 2.31. The *affine Hecke algebra* $\mathfrak{H}$ is the associative K-algebra generated by $T(w)$ for $w\in W$ subject to $T(v)T(w)=T(vw)$ if $\ell(vw)=\ell(v)+\ell(w)$ and

$$(T(s_i)-\tau_i)\big(T(s_i)+\tau_i^{-1}\big)=0$$

for $i\in I$. For $i\in I$, write $T_i := T(s_i)$. Then $\mathfrak{H}$ is in particular generated by $(T_i)_{i\in I}$ and $(T(u))_{u\in\Omega}$. Since all elements of Ω have length 0, the latter family is a subgroup, and the T_i satisfy the braid relations of the corresponding Coxeter system.

Lemma 2.32. *There exists a group homomorphism* $Y\colon L'\to\mathfrak{H}^\times$ *with* $Y^{\lambda'}=T(t(\lambda'))$ *for* $\lambda'\in L'_+$. *Furthermore,* $\mathfrak{H}$ *is generated by* $Y^{L'}$ *and* $(T_i)_{i\in I_0}$ *subject to the following Bernstein–Lusztig–Zelevinsky relation:*[1]

$$Y^{\lambda'}T_i - T_iY^{s_i\lambda'} = \boldsymbol{b}'_{a'_i}\big(Y^{-1}\big)\big(Y^{\lambda'}-Y^{s_i\lambda'}\big) \tag{2.1}$$

for $\lambda'\in L'$, $i\in I_0$, *where* $\boldsymbol{b}'_{a'_i}\big(Y^{-1}\big)$ *means that we extend the group homomorphism* $Y^{-1}\colon\lambda'\mapsto Y^{-\lambda'}$, *extend it to* $\operatorname{Frac}(A')$, *and then apply it to* $\boldsymbol{b}'_{a'_i}$.

Proof. The existence of Y follows from [17, equation (3.2.2)], the relation, from [17, Section 4.2.4], and that $\mathfrak{H}$ is generated as claimed, from [17, Section 4.2.7]. ■

Lemma 2.33 ([17, Section 3.1.8])**.** *If* $i,j\in I$ *and* $w\in W$ *such that* $ws_iw^{-1}=s_j$, *then* $T(w)T_iT(w)^{-1}=T_j$.

Definition 2.34. The *double-affine Hecke algebra* $\tilde{\mathfrak{H}}$ is the associative K-algebra generated by $\mathfrak{H}$ and elements X^λ ($\lambda\in L$) such that

(i) $\mathfrak{H}$ is a subalgebra and $X\colon L\to\tilde{\mathfrak{H}}^\times$ is a group homomorphism. We extend X to affine maps on E whose gradient lies in L by defining $X^c := q(1)$.

(ii) The following Bernstein–Lusztig–Zelevinsky relation holds: Let $i\in I$ and $f\in F$ with $Df\in L$. Then

$$T_iX^f - X^{s_if}T_i = \boldsymbol{b}_{a_i}(X)\big(X^f-X^{s_if}\big). \tag{2.2}$$

(iii) $T(u)X^fT\big(u^{-1}\big)=X^{uf}$ for $u\in\Omega$.

Remark 2.35. In [17, Section 4.7], as well as other sources like [5, Definition 2.1], $\tilde{\mathfrak{H}}$ is defined with other relations, most of which are immediately a special case of the Bernstein presentation. Macdonald's use of potentially non-reduced affine root systems necessitates the imposition of another set of quadratic Hecke relations (that turn out to be only necessary for components

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[1] Also referred to as Bernstein relation, Bernstein–Zelevinsky cross relation, or Lusztig's relation.

of type $\left(C_n^\vee, C_n\right)$, i.e., for exactly these non-reduced root systems): we define $\tilde{T}_i := X^{-a_i}T_i^{-1}$ for $i \in I$. Then

$$\begin{aligned}\tilde{T}_i^{-1} - \tilde{\tau}_i^{-1} &= T_iX^{a_i} - \tilde{\tau}_i = X^{-a_i}T_i + \boldsymbol{b}_{a_i}(X)\left(X^{a_i} - X^{-a_i}\right) - \tilde{\tau}_i^{-1}\\ &= X^{-a_i}T_i + \frac{\tau_i - \tau_i^{-1} + \left(\tilde{\tau}_i - \tilde{\tau}_i^{-1}\right)X^{a_i}}{1 - X^{2a_i}}\left(X^{a_i} - X^{-a_i}\right) - \tilde{\tau}_i^{-1}\\ &= X^{-a_i}T_i - X^{-a_i}\left(\tau_i - \tau_i^{-1}\right) - \tilde{\tau}_i\\ &= X^{-a_i}\left(T_i - \tau_i + \tau_i^{-1}\right) - \tilde{\tau}_i = X^{-a_i}T_i^{-1} - \tilde{\tau}_i = \tilde{T}_i - \tilde{\tau}_i.\end{aligned}$$

This shows that the quadratic Hecke relations also hold for $\tilde{T}_i$, and therefore, that our definition for the double-affine Hecke algebra is indeed compatible with [5, 17].

Lemma 2.36 ([17, Section 4.7.5])**.** *The families*

$$\left(Y^{\lambda'}T(w)X^{\mu}\right)_{\lambda'\in L', w\in W_0, \mu\in L}, \qquad \left(X^{\mu}T(w)Y^{\lambda'}\right)_{\mu\in L, w\in W_0, \lambda'\in L'}$$

are K-bases of $\tilde{\mathfrak{H}}$ and hence give rise to Poincaré–Birkhoff–Witt-like decompositions

$$\tilde{\mathfrak{H}} \cong A(X) \otimes \mathfrak{H}_0 \otimes A'(Y) \cong A'(Y) \otimes \mathfrak{H}_0 \otimes A(X),$$

where $\mathfrak{H}_0 \leq \mathfrak{H}$ is the (finite-dimensional) subalgebra generated by $(T_i)_{i\in I_0}$. These decompositions are as vector spaces and as left (resp. right) A-modules and as right (resp. left) A'-modules. In particular, we have $\tilde{\mathfrak{H}} \cong A(X) \otimes \mathfrak{H} \cong \mathfrak{H} \otimes A(X)$ as left (resp. right) A-modules and as right (resp. left) $\mathfrak{H}$-modules.

Definition 2.37. Fix an involutive $\mathbb{Q}$-automorphism $*\colon K \to K$ mapping $q(x)$ to $q(-x)$. This exists because we required all $q(x)$ $(x \neq 0)$ to be transcendent.

Lemma 2.38. *$*$ extends to an involutive anti-automorphism of $\tilde{\mathfrak{H}}$ satisfying*

$$(X^{\mu})^* = X^{-\mu}, \qquad \left(Y^{\lambda'}\right)^* = Y^{-\lambda'}, \qquad (T(w))^* = T(w)^{-1}$$

for $\mu \in L$, $\lambda' \in L'$, $w \in W_0$.

Proof. Write $*$ also for the involutive automorphisms of A, A' that are derived from the inverse map of L, L'. If $t^* = t^{-1}$, $u^* = u^{-1}$, $x^* = x^{-1}$, we have

$$\boldsymbol{b}(t,u;x)^* = \frac{t^{-1} - t + \left(u^{-1} - u\right)x^{-1}}{1 - x^{-2}} = \frac{x^2\left(t - t^{-1}\right) + x\left(u - u^{-1}\right)}{1 - x^2} = \boldsymbol{b}(t,u;x) - t + t^{-1}.$$

Consequently, it remains to show that the two Bernstein–Lusztig–Zelevinsky relations as well as the relations between Ω and X^L are preserved by $*$.

For $i \in I$ and $Df \in L$, we have

$$\begin{aligned}\left(T_iX^f - X^{s_if}T_i\right)^* &= X^{-f}T_i^{-1} - T_i^{-1}X^{-s_if}\\ &= X^{-f}\left(T_iX^{s_if} - X^fT_i - \left(\tau_i - \tau_i^{-1}\right)\left(X^{s_if} - X^f\right)\right)X^{-s_if}\\ &= X^{-f}\left(\left(\boldsymbol{b}_{a_i}(X) - \tau_i + \tau_i^{-1}\right)\left(X^{s_if} - X^f\right)\right)X^{-s_if}\\ &= \left(\boldsymbol{b}_{a_i} - \tau_i + \tau_i^{-1}\right)(X)\left(X^{-f} - X^{-s_if}\right)\\ &= \boldsymbol{b}_{a_i}(X)^*\left(X^f - X^{s_if}\right)^*.\end{aligned}$$

A similar calculation shows the correct relations between $Y^{\lambda'}$ and $(T_i)_{i\in I_0}$, and for Ω note that $T(u)^* = T(u)^{-1}$ for $u \in \Omega$, so that

$$\left(T(u)X^fT(u)^{-1}\right)^* = T(u)X^{-f}T(u)^{-1} = X^{u(-f)} = \left(X^{u(f)}\right)^*$$

since Ω acts linearly on F. ∎

2.6 Basic representation and inner product

We shall now describe the basic representation of $\tilde{\mathfrak{H}}$. Contrary to the approach of [17] but in anticipation of Section 5, we shall describe the basic representation as an induced module.

Lemma 2.39. *There is a unique ring homomorphism* $\phi\colon \mathfrak{H} \to K$ *satisfying* $\phi(T_i) = \tau_{a'_i,k'} = \tau_i$ *for* $i \in I_0$ *and* $\phi\big(Y^{\lambda'}\big) = q(\langle \lambda', \rho_{k'}\rangle)$.

Proof. By Lemma 2.32, $\mathfrak{H}$ is generated by $(T_i)_{i\in I_0}$ and $Y^{L'}$ satisfying (2.1). Thus, any ring homomorphism is uniquely defined by what it maps $Y^{L'}$ and the T_i to. This shows uniqueness.

For existence, we need to show that the given data for ϕ satisfies the relations of $\mathfrak{H}$. The relations among the T_i are the Hecke relations (which are trivially satisfied) and the braid relations of the corresponding Coxeter system. Let $(m_{ij})_{i,j\in I}$ be the Coxeter matrix and let $i, j \in I_0$ with $i \neq j$. If m_{ij} is even, then the braid relation is satisfied because τ_i and τ_j commute with each other. If m_{ij} is odd, the roots a'_i and a'_j lie in the same $W_{S'}$-orbit and therefore, $\tau_i = \tau_j$. This shows that the braid relations are also satisfied for this case.

It thus remains to show the Bernstein–Lusztig–Zelevinsky relation, so let $\lambda' \in L'$ and $i \in I_0$. From Lemma 2.18, we know that $\langle \alpha_i^\vee, \rho_{k'}\rangle = k'(a'_i)$. Furthermore, $\tilde{\tau}_{a'_i,k'} q(-k'(a'_i)) = \tau^{-1}_{a'_i,k'}$, so that

$$\begin{aligned}\frac{\tau_{a'_i,k'} - \tau^{-1}_{a'_i,k'} + \big(\tilde{\tau}_{a'_i,k'} - \tilde{\tau}^{-1}_{a'_i,k'}\big) q(-k'(a'_i))}{1 - q(-2k'(a'_i))} &= \frac{\tau_{a'_i,k'} - \tau^{-1}_{a'_i,k'} + \tau^{-1}_{a'_i,k'} - \tau_{a'_i,k'} q(-2k'(a'_i))}{1 - q(-2k'(a'_i))} \\ &= \tau_{a'_i,k'} = \tau_i\end{aligned}$$

by Remark 2.29. As a consequence, we have

$$\begin{aligned}&\frac{\tau_{a'_i,k'} - \tau^{-1}_{a'_i,k'} + \big(\tilde{\tau}_{a'_i,k'} - \tilde{\tau}^{-1}_{a'_i,k'}\big) q(-k'(a'_i))}{1 - q(-2k'(a'_i))} (q(\langle \lambda', \rho_{k'}\rangle) - q(\langle s_i\lambda', \rho_{k'}\rangle)) \\ &\quad = q(\langle \lambda', \rho_{k'}\rangle)\tau_i - \tau_i q(\langle s_i\lambda', \rho_{k'}\rangle),\end{aligned}$$

which shows that our data satisfies (2.1), and hence gives rise to a ring homomorphism $\mathfrak{H} \to K$. ■

Via the ring homomorphism ϕ from Lemma 2.39, we can view K as a $\mathfrak{H}$-module and then consider the $\tilde{\mathfrak{H}}$-module $M := \tilde{\mathfrak{H}} \otimes_{\mathfrak{H}} \otimes K$.

Lemma 2.40. *For* $i \in I$, *we have* $\phi(T_i) = \tau_i$ *(in particular, also for* $i \notin I_0$*), and for* $u \in \Omega$, *we have* $\phi(T(u)) = 1$.

Proof. For $i \in I_0$, we have $\phi(T_i) = \tau_i$ by Remark 2.29 (i). Otherwise, by [17, equation (3.3.5)], if $w \in W_0$ and $w\alpha_j \in R$ is the highest root in the same irreducible component as a_i, we have

$$T_i = T(w) Y^{\alpha_j^\vee} T_j^{-1} T(w)^{-1}.$$

Applying ϕ, we obtain

$$\phi(T_i) = \tau_j^{-1} q\big(\langle \alpha_j^\vee, \rho_{k'}\rangle\big) = \tau^{-1}_{a'_j,k'} q(k'(a'_j))$$

by Lemma 2.18. Observe that by the proof of Lemma 2.39, this equals $\tilde{\tau}_{a'_j,k'}$, which equals τ_i by Remark 2.29 (i).

For u, note that ϕ restricted to $T(\Omega)$ is a character of Ω. Since we can write $T(u)$ as a product of T_i's and elements of $Y^{L'}$, and since ϕ is defined in terms of q, we have $\phi(T(u)) = q(x)$ for some $x \in \mathbb{R}$. Since Ω is a finite group, u has finite order, say n. Then $\phi(T(u)^n) = \phi(T(u^n)) = 1 = q(nx)$. As q is injective, we have $nx = 0$ and hence $x = 0$. ■

Remark 2.41. Evidently, it would have been easier to define ϕ using the relations from Lemma 2.40. The approach chosen here makes a clearer connection to the content of Sections 5.1 and 5.2.

Lemma 2.42. *The map* $e(\mu) \mapsto X^\mu \otimes 1$ *defines an isomorphism* $A \cong M$ *of* K*-vector spaces and of left* A*-modules.*

Proof. By Lemma 2.36, the elements $(X^\mu)_{\mu \in L}$ are a $\mathfrak{H}$-basis (with respect to the action from the right), and thus $(X^\mu \otimes 1)_{\mu \in L} \subseteq M$ are K-linearly independent and a K-generating system. Since our described map maps a K-basis of A to a K-basis of M, it is a K-linear isomorphism $A \cong M$. Since it also respects the A-module structures on A and M, it is even an A-linear isomorphism. ■

Definition 2.43. A, viewed as an $\tilde{\mathfrak{H}}$-module using the isomorphism from Lemma 2.42 is called the *basic representation.*

Lemma 2.44. *Let* $f \in A$*, then we have* $T_i f = \tau_i s_i f + \boldsymbol{b}_{a_i}(f - s_i f)$*,* $T(u) f = uf$*,* $X^\mu f = e(\mu) f$ *for* $i \in I$*,* $u \in \Omega$*,* $\mu \in L$*.*

Proof. By (2.2) and linear extension, we have

$$T_i f(X) \otimes 1 = (s_i f)(X) \otimes \phi(T_i) + \boldsymbol{b}_{a_i}(X)(f - s_i f)(X) \otimes 1.$$

By Lemma 2.40, we have $\phi(T_i) = \tau_i$, whence we have the first claim. For the second claim, let $u \in \Omega$, then

$$T(u) f(X) \otimes 1 = (uf)(X) \otimes \phi(u) = (uf)(X) \otimes 1$$

by Lemma 2.40. Lastly, the third claim follows from the fact that our identification is A-linear. ■

We can equip the basic representation with an inner product and make it unitary.

Definition 2.45. Let $\Delta = \Delta_{k,S}$ be an appropriately defined infinite product

$$\prod_{a \in S_1^+} \tau_a^{-1} \boldsymbol{c}(\tau_a, \tilde{\tau}_a, e(a))^{-1}$$

as in [17, Section 5.1.7 ff]. Define furthermore

$$\Delta_0 := \Delta_{0,S} := \prod_{\substack{a \in S_1^+ \\ Da = 0}} \tau_a^{-1} \boldsymbol{c}(\tau_a, \tilde{\tau}_a, e(-a))^{-1}, \qquad \nabla := \nabla_S := \Delta \Delta_0.$$

Lemma 2.46.

(*i*) *There are distributions* $\Delta_1, \nabla_1 \in K[[L]]$ *that are proportional to* Δ*,* ∇ *and have* 1 *as their constant term;*

(*ii*) ∇ *is* W_0*-symmetric;*

(*iii*) *we have* $\sum_{w \in W_0} w \Delta_0^{-1} = W_0(\tau^2)$ (*already using notation from Example* 3.4 (i), (ii)).

Proof. (i) Follows by [17, Section 5.1.10] and the fact that $\Delta_0 \in K(L)$. (ii) Follows by [17, Section 5.1.27]. (iii) Follows by [17, Section 5.1.36]. ■

Recall that we can use any distribution $h \in K[[L]]$ to define a Hermitean bilinear form on A as follows $(f,g) \mapsto \mathrm{ct}(fg^*h)$ and a symmetric bilinear form on A as follows $(f,g) \mapsto \mathrm{ct}(f\bar{g}h)$. Here $\mathrm{ct}\colon K[[L]] \to K$ maps a distribution to its constant term, and $*$ is the involution of A extending $*$ on K (i.e., mapping $q(a) \mapsto q(-a)$ for $a \in \mathbb{Q}$) and mapping $e(\mu) \mapsto e(-\mu)$ for $\mu \in L$. Furthermore, $\bar{\cdot}$ is the K-linear involution mapping $e(\mu) \mapsto e(-\mu)$ for $\mu \in L$ (but preserving $q(a)$ for $a \in \mathbb{Q}$).

We thus define our inner products as follows:

Definition 2.47. For $f, g \in A$, define $(f,g) := (f,g)_{k,S} := \mathrm{ct}(fg^*\Delta_{k,S})$. Furthermore, define $(f,g)_1 := \frac{(f,g)}{(1,1)} = \mathrm{ct}(fg^*\Delta_{k,S,1}) \in K$.

By [17, Section 5.1.20], the equation $(f,f) = 0$ implies $f = 0$. Consequently, we can apply the Gram–Schmidt procedure to $(e(\mu))_{\mu \in L}$, with the monomial order from Section 2.4.

Theorem 2.48 ([17, Section 5.2.1ff])**.** *There is a unique family $(E_\lambda)_{\lambda \in L} \subseteq A$ satisfying*

(i) $E_\lambda = e(\lambda) + \text{l.o.t.}$ *(here, "lower" refers to the order $\leq$ on L);*

(ii) $f(Y)E_\lambda = f(-r_{k'}(\lambda))E_\lambda$*;*

(iii) $(E_\lambda, E_\mu) = 0$ *for $\lambda \neq \mu$,*

called the non-symmetric Macdonald polynomials. In particular, the span of $(E_\mu)_{\mu \in W_0\lambda}$ is an $\mathfrak{H}_0$-module.

That last remark of the $(E_\mu)_{\mu \in W_0\lambda}$ being an $\mathfrak{H}_0$-module can be made more precise:

Lemma 2.49 ([17, Section 5.4.3f])**.** *Let $\lambda \in I$ and $i \in I_0$.*

(i) If $\langle \lambda, a_i' \rangle = 0$, so that $s_i\lambda = \lambda$, we have $T_iE_\lambda = \tau_iE_\lambda$.

(ii) If $\langle \lambda, a_i' \rangle > 0$, so that $s_i\lambda > \lambda$, we have $T_iE_\lambda = \tau_i^{-1}E_{s_i\lambda} + \boldsymbol{b}'_{a_i'}(r_{k'}(\lambda))E_\lambda$.

3 Adapting Macdonald's formalism to parabolic subgroups

Fix a subset $J \subseteq I_0$ and write $W_J \leq W_0$ and $\mathfrak{H}_J \leq \mathfrak{H}_0$ for the subgroup/-algebra generated by s_j and T_j ($j \in J$), respectively. Let $A_J := A^{W_J}$, $A'_J := (A')^{W_J}$, and let W_0^J be the set of shortest representatives of elements of W_0/W_J. Let w_J, w_0 be the longest elements of W_J, W_0, respectively. We shall now see how the formalism from [17] can be used in a W_J-invariant context.

3.1 Poincaré polynomials

We begin by recalling some facts about (much generalised) Poincaré polynomials and series, as it turns out, much of our theory requires them. Throughout this subsection, let $(\mathcal{W}, \mathcal{S})$ be a finite Coxeter group and $\mathcal{A}$ a ring.

Definition 3.1. A function $\tau\colon \mathcal{W} \to \mathcal{A}^\times$ is called an ($\mathcal{A}$-valued) *multiplicative labelling* of $\mathcal{W}$ if $\ell(vw) = \ell(v) + \ell(w)$ implies that $\tau_{vw} = \tau_v\tau_w$.

For any multiplicative labelling τ and a subset $X \subseteq \mathcal{W}$, define $X(\tau) := \sum_{w \in X} \tau_w$. This is the *Poincaré polynomial* of the subset X with respect to τ.

Remark 3.2. The usual meaning of "Poincaré polynomial" can be recovered by considering $A = \mathbb{Z}[t]$ and $\tau_w := t^{\ell(w)}$.

Lemma 3.3. *Let $(\tau_s)_{s \in \mathcal{S}} \subseteq \mathcal{A}^\times$ satisfy the braid relations of $(\mathcal{W}, \mathcal{S})$. Then the map $\tau\colon \mathcal{S} \to \mathcal{A}^\times$, $s \mapsto \tau_s$ can be extended uniquely to a multiplicative labelling of $\mathcal{W}$.*

Proof. In analogy with [17, Section 3.1], the Artin braid group $\mathfrak{B}$ of $\mathcal{W}$ is generated by $(T(w))_{w\in\mathcal{W}}$ subject to $\ell(vw)=\ell(v)+\ell(w)\Rightarrow T(v)T(w)=T(vw)$. It follows therefore that the multiplicative labellings (with values in $\mathcal{A}$) of $\mathcal{W}$ are exactly the representations of $\mathfrak{B}$ (in $\mathcal{A}^\times$). By [17, Section 3.1.6], the braid group can also be presented in terms of the generators $(T_s)_{s\in\mathcal{S}}$ and the braid relations. Thus a representation can be specified by providing elements $(\tau_s)_{s\in\mathcal{S}}$ satisfying the braid relations. ■

Example 3.4.

(i) If τ is a multiplicative labelling that maps to a commutative algebra,then any integer power of τ is a multiplicative labelling as well.

(ii) Let k be a W-labelling of S. For $j\in J$, take $\tau_j:=\tau_{a_j,k}$ as in Definition 2.28. These elements satisfy the braid relations, so that $\tau=\tau_k$ is a multiplicative labelling of W_J (after appropriate restriction).

(iii) Let in addition $\epsilon\colon W_J\to\mathbb{C}^\times$ be a multiplicative character. Then define

$$\tau_j^{(\epsilon)}:=\tau_{j,k}^{(\epsilon)}:=\begin{cases}\tau_{j,k}, & \epsilon(s_j)=1,\\ -\tau_{j,k}^{-1}, & \epsilon(s_j)=-1,\end{cases}$$

the *q-deformation of* ϵ. This also gives rise to a multiplicative labelling $\tau_k^{(\epsilon)}$ of W_J, and even to a ring homomorphism $\mathfrak{H}_J\to K$. The name q-deformation comes from the fact that $\mathfrak{H}_J$ is a deformation of $K[W_J]$, and that ϵ can be extended to a ring homomorphism $K[W_J]\to K$ that can be obtained from $\tau_k^{(\epsilon)}$ as a limit case.

(iv) Take $A=\mathfrak{H}_J$, then the $(T_j)_{j\in J}$ satisfy the braid relations, giving rise to a multiplicative labelling T.

(v) Similarly, the $(\tau_{j,k}T_j)_{j\in J}$ also satisfy the braid relations, as do $\big(\tau_{j,k}^{(\epsilon)}T_j\big)_{j\in J}$, yielding multiplicative labellings τ_kT and $\tau_k^{(\epsilon)}T$.

Lemma 3.5. *Let $\mathcal{J}\subseteq\mathcal{S}$ a subset. Let τ be a multiplicative labelling on $\mathcal{W}$, then we have the following identity of Poincaré polynomials: $\mathcal{W}(\tau)=\mathcal{W}^{\mathcal{J}}(\tau)\mathcal{W}_{\mathcal{J}}(\tau)$, where $\mathcal{W}_{\mathcal{J}}$ is the subgroup generated by $\mathcal{J}$, and $\mathcal{W}^{\mathcal{J}}$ is the set of shortest coset representatives of $\mathcal{W}/\mathcal{W}_{\mathcal{J}}$.*

Proof. For the case $\tau_w=t^{\ell(w)}$ or more generally, $\mathcal{A}$ commutative, this result is well known, e.g., [11, Section 5.12], but it even holds for $\mathcal{A}$ non-commutative.

For every $w\in\mathcal{W}$, there is a unique decomposition vw' with $v\in\mathcal{W}^{\mathcal{J}}$ and $w'\in\mathcal{W}_{\mathcal{J}}$ such that $\ell(w)=\ell(v)+\ell(w')$. Therefore,

$$\mathcal{W}(\tau)=\sum_{v\in\mathcal{W}^{\mathcal{J}},\,w\in\mathcal{W}_{\mathcal{J}}}\tau_{vw}=\sum_{v\in\mathcal{W}^{\mathcal{J}},\,w\in\mathcal{W}_{\mathcal{J}}}\tau_v\tau_w=\sum_{v\in\mathcal{W}^{\mathcal{J}}}\tau_v\sum_{w\in\mathcal{W}_{\mathcal{J}}}\tau_w=\mathcal{W}^{\mathcal{J}}(\tau)\mathcal{W}_{\mathcal{J}}(\tau).$$ ■

Lemma 3.6. *For any multiplicative labelling $\tau\colon\mathcal{W}\to\mathcal{A}^\times$, the map $\tau^{-1}\colon w\mapsto\tau_w^{-1}$ is an "anti-multiplicative" labelling: we have $\tau_{vw}^{-1}=\tau_w^{-1}\tau_v^{-1}$ whenever $\ell(vw)=\ell(v)+\ell(w)$. We then have $\mathcal{W}(\tau^{-1})=\tau_{w_0}^{-1}\mathcal{W}(\tau)$ (where $\mathcal{W}(\tau^{-1})$ is defined as if τ^{-1} were a multiplicative labelling, and where w_0 is the longest element of $\mathcal{W}$).*

Proof. Let $w\in\mathcal{W}$. By [11, Corollary 1.8], we have $\ell(w_0w)=\ell(w_0)-\ell(w)$, i.e., $\ell(w_0w)+\ell(w)=\ell(w_0)$. Consequently, we have $\tau_{w_0w}\tau_w=\tau_{w_0}$, or equivalently $\tau_w^{-1}=\tau_{w_0}^{-1}\tau_{w_0w}$. Thus,

$$\mathcal{W}(\tau^{-1})=\sum_{w\in\mathcal{W}}\tau_w^{-1}=\tau_{w_0}^{-1}\sum_{w\in\mathcal{W}}\tau_{w_0w}=\tau_{w_0}^{-1}\sum_{w\in\mathcal{W}}\tau_w=\tau_{w_0}^{-1}\mathcal{W}(\tau).$$ ■

Remark 3.7. All results of this subsection except for Lemma 3.6 also work in case $(\mathcal{W},\mathcal{S})$ is an infinite Coxeter group, provided sufficient convergence of the series, e.g., $\mathcal{A}$ being filtered and τ mapping to $\mathcal{A}^+$. In this case, we speak not of Poincaré polynomials, but of *Poincaré series*.

3.2 Symmetrisers

From this general finite Coxeter group interlude, we now return to our main programme of affine root systems S, S', their extended affine Weyl groups W, W', and the parabolic subgroup $W_J \leq W_0$. Using the language of (generalised) Poincaré polynomials, we now define symmetrisers in a way generally inspired by [17, Section 5.5].

Definition 3.8. Let $\epsilon\colon W_J \to \mathbb{C}^\times$ be a multiplicative character of W_J. Define

$$U_J^{(\epsilon)} := U_{J,k}^{(\epsilon)} := \big(\tau_{w_J,k}^{(\epsilon)}\big)^{-1} W_J\big(\tau_k^{(\epsilon)} T\big),$$

the ϵ-*symmetriser* for W_J. Here, we use Example 3.4 (v).

We furthermore adopt the convention that for all notation that has an exponent (ϵ), we interpret the absence of that exponent to mean the trivial character.

Lemma 3.9. *Let $j \in J$, then*

$$\big(T_j - \tau_j^{(\epsilon)}\big) U_J^{(\epsilon)} = U_J^{(\epsilon)} \big(T_j - \tau_j^{(\epsilon)}\big) = 0.$$

Proof. Follows from [17, Section 5.5.9] applied to $\mathfrak{H}_J$. ■

This shows that for any representation of $\mathfrak{H}_J$, the application of $U_J^{(\epsilon)}$ produces elements that transform according to the character $\tau^{(\epsilon)}$. In fact, $U_J^{(\epsilon)}$ is (a scalar multiple of) the projection onto the corresponding isotypic component:

Corollary 3.10. *Let V be a representation of $\mathfrak{H}_J$, then*

$$\big\{v \in V \mid \forall j \in J\colon T_j v = \tau_j^{(\epsilon)} v\big\} = U_J^{(\epsilon)} V.$$

Proof. "$\supseteq$": Follows from Lemma 3.9.

"$\subseteq$": Let $v \in V$ have the desired transformation behaviour, then we have $\tau_w^{(\epsilon)} T(w) v = \tau_w^{(\epsilon)2} v$ for all w, hence

$$U_J^{(\epsilon)} v = \tau_{w_J}^{-1} W_J\big(\tau^{(\epsilon)} T\big) v = \frac{W_J\big(\tau^{(\epsilon)2}\big)}{\tau_{w_J}} v.$$

Multiplying by an appropriate scalar shows that v lies in $U_J^{(\epsilon)} V$. ■

An immediate consequence of Lemma 3.5 is the following statement about symmetrisers.

Corollary 3.11. *Let $J' \subseteq J$ be a subset, then*

$$U_J^{(\epsilon)} = \big(\tau_{w_J w_{J'}}^{(\epsilon)}\big)^{-1} \sum_{v \in W_J^{J'}} \tau_v^{(\epsilon)} T(v) U_{J'}^{(\epsilon)} = \big(\tau_{w_J w_{J'}}^{(\epsilon)}\big)^{-1} W_J^{J'}\big(\tau^{(\epsilon)} T\big) U_{J'}^{(\epsilon)}.$$

We conclude with some further properties of symmetrisers.

Proposition 3.12.

(i) $\big(U_J^{(\epsilon)}\big)^2 = \frac{W_J(\tau_k^{(\epsilon)2})}{\tau_{w_J}^{(\epsilon)}} U_J^{(\epsilon)}$.

(ii) $\big(U_J^{(\epsilon)}\big)^* = U_J^{(\epsilon)}$.

(iii) *For $f, g \in A$, we have*

$$\big(U_J^{(\epsilon)} f, U_J^{(\epsilon)} g\big) = \frac{W_J\big(\tau_k^{(\epsilon)2}\big)}{\tau_{w_J}^{(\epsilon)}} \big(U_J^{(\epsilon)} f, g\big).$$

Proof. [17, Section 5.5.17 (ii)] applied to $\mathfrak{H}_J$. [17, Section 5.5.17 (iii)] applied to $\mathfrak{H}_J$. Using the first two statements, we get

$$\big(U_J^{(\epsilon)} f, U_J^{(\epsilon)} g\big) = \big(\big(U_J^{(\epsilon)}\big)^2 f, g\big) = \frac{W_J\big(\tau_k^{(\epsilon)2}\big)}{\tau_{w_J}^{(\epsilon)}} \big(U_J^{(\epsilon)} f, g\big).$$ ■

3.3 W_J-orbits

In this subsection, we set out for a suitable way of labelling W_J-orbits within L, and examine what $\leq$ looks like when restricted to a W_J-orbit.

Definition 3.13. Let

$$L_{+,J} := \left\{\lambda \in L \,\middle|\, \forall j \in J\colon \langle \lambda, a'_j \rangle \geq 0\right\} \subseteq L$$

be the set of J-*dominant* elements.

A useful property of L_+ is that all of its stabilisers are parabolic. It turns out that $L_{+,J}$ possesses the same property.

Lemma 3.14. *Let $\lambda \in L_{+,J}$, then the group*

$$W_{J,\lambda} := \{w \in W_J \mid w\lambda = \lambda\} = W_\lambda \cap W_J$$

is a parabolic subgroup of W_J.

Proof. Let W' be the subgroup of $W_{J,\lambda}$ generated by its simple reflections. We now show that $W_{J,\lambda} \subseteq W'$. Let $w \in W_{J,\lambda}$; we proceed by induction in $p = \ell(w)$:

"$p = 0$": Then $w = 1$, so $w \in W'$.

"$p - 1 \to p$": Let $\ell(w) = p$. Since $1 \neq w$, there is $j \in J$ such that $wa'_j < 0$. Then we have $\ell(ws_j) < \ell(w)$. Furthermore, we have

$$0 \leq \langle \lambda, a'_j \rangle = \langle w\lambda, wa'_j \rangle = \langle \lambda, wa'_j \rangle \leq 0$$

since $\lambda \in L_{+,J}$ and since $wa'_j < 0$ is a negative linear combination of a_i $(i \in J)$. Thus, $s_j\lambda = \lambda$. As a consequence, $ws_j\lambda = w\lambda = \lambda$. Since $\ell(ws_j) = p-1$, we can apply the induction hypothesis to find that $ws_j \in W'$, as is s_j, and hence also $w \in W'$. ■

An immediate corollary to Lemma 3.14 concerns the decomposition of W_J-symmetrisers from Corollary 3.11.

Corollary 3.15. *Let $\lambda \in L_{+,J}$ such that $\epsilon(W_{J,\lambda}) = \{1\}$, then*

$$U_J^{(\epsilon)} = \left(\tau^{(\epsilon)}_{w_J w_{J,\lambda}}\right)^{-1} \sum_{v \in W_J^\lambda} \tau_v^{(\epsilon)} T(v) U_{J'}$$

where $J' = \{j \in J \mid \langle \lambda, a'_j \rangle = 0\}$; in other words, where $W_{J'} = W_{J,\lambda}$.

In order to prove that $L_{+,J}$ is a fundamental domain for W_J, it is useful to know how to produce (the) elements of $L_{+,J}$.

Lemma 3.16. *Let $v \in W_0^J$ and $\lambda \in L_+$, then $v^{-1}\lambda \in L_{+,J}$.*

Proof. Let $j \in J$. Since v is the shortest element of its right coset, we have $\ell(vs_j) > \ell(v)$ and hence $va'_j > 0$ by Lemma 2.13. Since λ is dominant, we therefore have

$$\left\langle v^{-1}\lambda, a'_j \right\rangle = \langle \lambda, va'_j \rangle \geq 0. \qquad ■$$

Proposition 3.17. *The set $L_{+,J}$ is a fundamental domain for the action of W_J on L.*

Proof. Let $\lambda \in L$, we show that there is a unique element in $L_{+,J} \cap W_J\lambda$.

Uniqueness. Let $\lambda \in L_{+,J}$, we show that $\{w \in W_J \mid w\lambda \in L_{+,J}\} \subseteq W_{J,\lambda}$, which we do by induction on length p.

"$p = 0$": Evidently true.

"$p-1 \to p$": Let $\ell(w) = p$ and $w\lambda \in L_{+,J}$. Then there is $j \in J$ such that $\ell(s_jw) = p-1$. By Lemma 2.13, this implies that $w^{-1}a'_j < 0$ is a negative linear combination of a'_i $(i \in J)$, and hence that $\langle \lambda, w^{-1}a'_j \rangle \le 0$. Thus we have $0 \ge \langle \lambda, w^{-1}a'_j \rangle = \langle w\lambda, a'_j \rangle \ge 0$, which implies that $s_jw\lambda = w\lambda \in L_{+,J}$. Since $\ell(s_jw) = p-1$, we can apply the induction hypothesis to find that $s_jw \in W_{J,\lambda}$ and in particular $s_jw\lambda = \lambda$, which then also shows that $w\lambda = \lambda$. As a consequence, every W_J-orbit contains at most one element in $L_{+,J}$.

Existence. We know that L_+ is a fundamental domain for W_0, so let $w \in W_0$ such that $\mu = w\lambda \in L_+$. Decompose $w = vw'$ for $v \in W_0^J$, $w' \in W_J$. Then $w'\lambda = v^{-1}\mu \in L_{+,J}$ by Lemma 3.16. ■

Remark 3.18. The reader might be tempted to prove Proposition 3.17 using the fact that W_J is the Weyl group of a finite root system whose dominant weights are always a fundamental domain for W_J's action. However, this does not work here since we're considering W_J's action on a much larger lattice L.

We now want to understand the order restricted to a W_J-orbit. This requires a more specialised version of $\overline{v}$.

Definition 3.19. Let $\lambda \in L_{+,J}$ and let $\mu \in W_J\lambda$. Write $\overline{v}_J(\mu)$ for the shortest element $w \in W_J$ satisfying $w\lambda = \mu$.

It turns out that this notion is related to $\overline{v}$ from Section 2.4 via the shortest coset representatives decomposition.

Lemma 3.20. *Let $\lambda \in L_{+,J}$ and $\mu \in W_J\lambda$, then $\overline{v}(\mu) = \overline{v}_J(\mu)\overline{v}(\lambda)$ where the lengths add up. In particular, $\overline{v}(\lambda)^{-1} \in W_0^J$.*

Proof. Let $\lambda_+ \in L_+$ be the unique element in whose W_0-orbit λ, μ lie.

"$\le$": We have $\overline{v}_J(\mu)\overline{v}(\lambda)\lambda_+ = \overline{v}_J(\mu)\lambda = \mu$, so we can write $\overline{v}_J(\mu)\overline{v}(\lambda)$ as $\overline{v}(\mu)w$ where w fixes λ_+. Since the stabiliser of λ_+ is a parabolic subgroup of W_0, and since $\overline{v}(\mu)$ has minimal length, in the above decomposition the lengths add up. Therefore, $\overline{v}(\mu) \le \overline{v}_J(\mu)\overline{v}(\lambda)$.

"$\ge$": Let $v \in W_0^J, w \in W_J$ such that $\overline{v}(\mu)^{-1} = vw$, i.e., $w^{-1}v^{-1} = \overline{v}(\mu)$. Note that $v^{-1}\lambda_+ \in L_{+,J}$ by Lemma 3.16. Since $w^{-1}v^{-1}\lambda_+ = \mu$, the element $v^{-1}\lambda_+$ lies in $W_J\mu$. Since it is J-dominant, it equals λ by Proposition 3.17.

Thus, $w^{-1} \in W_J$ maps λ to μ, hence $\overline{v}_J(\mu) \le w^{-1}$ (same argument as for "$\le$"). Furthermore, v^{-1} maps λ_+ to λ, whence $\overline{v}(\lambda) \le v^{-1}$. Since the lengths of v^{-1} and w^{-1} add up, we can conclude

$$\overline{v}_J(\mu)\overline{v}(\lambda) \le w^{-1}v^{-1} = \overline{v}(\mu),$$

and hence the desired equality.

To show that $v^{-1} = \overline{v}_J(\mu)$, note that modulo W_J we have $vW_J = \overline{v}(\lambda)^{-1}W_J$. Since v is the shortest representative of this coset, we have $v \le \overline{v}(\lambda)^{-1}$. Together with $v^{-1} \ge \overline{v}(\lambda)$, we obtain $v = \overline{v}(\lambda)^{-1} \in W_0^J$. ■

In order to later determine the leading coefficients of a W_J-invariant polynomial, we now need to find out which elements of a given W_J-orbit are the highest and lowest.

Corollary 3.21. *Let $\lambda \in L_{+,J}$, then λ is the smallest element of its W_J-orbit and $w_J\lambda$ is the largest.*

Proof. As in Corollary 3.15, write W_J^λ for the set of shortest representatives of $W_{I,\lambda}$-cosets of W_J and let $w_{J,\lambda}$ be the longest element of $W_{J,\lambda}$. As in the proof of Lemma 3.6, the lengths of $w_J w_{J,\lambda}$ and $w_{J,\lambda}$ add up so that $w_J w_{J,\lambda}$ is the shortest representative of $w_J W_{J,\lambda}$. Consequently, $\overline{v}_J(w_J\lambda) = w_J w_{J,\lambda}$. Let $v \in W_J^\lambda$, then $v \leq w_J$, so we can choose reduced expressions for $w_J w_{J,\lambda}$ and $w_{J,\lambda}$ and obtain a reduced expression for v by deleting reflections. Since v is the shortest element in its right $W_{J,\lambda}$-coset, we need to delete all reflections belonging to $w_{J,\lambda}$, so that we have $v \leq w_J w_{J,\lambda}$.

Let now $\mu \in W_J\lambda$, then $\overline{v}_J(\mu) \in W_J^\lambda$ because it is the shortest element in its right $W_{J,\lambda}$-coset. Then we have $1 \leq \overline{v}_J(\mu) \leq w_J w_{J,\lambda}$, i.e., $\overline{v}_J(\lambda) \leq \overline{v}_J(\mu) \leq \overline{v}_J(w_J\lambda)$. We can multiply everything by $\overline{v}(\lambda) \in (W_0^J)^{-1}$ on the right since the lengths will always add up and obtain $\overline{v}(\lambda) \leq \overline{v}(\mu) \leq \overline{v}(w_J\lambda)$ by Lemma 3.20. By definition of the partial ordering, this implies that $\lambda \leq \mu \leq w_J\lambda$. ■

4 Intermediate Macdonald polynomials

We now begin by defining a class of polynomials possessing the desired transformation behaviour under $\mathfrak{H}_J$.

4.1 General

Definition 4.1. For $\lambda \in L$, define $F_{J,\lambda}^{(\epsilon)} := U_J^{(\epsilon)} E_\lambda$, where E_λ is the non-symmetric Macdonald polynomial from Theorem 2.48. Later, these will be suitably normalised to yield the intermediate Macdonald polynomials.

Corollary 4.2. *Let $\lambda \in L$, $i \in J$, then $T_i F_{J,\lambda}^{(\epsilon)} = \tau_i^{(\epsilon)} F_{J,\lambda}^{(\epsilon)}$. In particular, if ϵ is the trivial character, the polynomial $F_{J,\lambda}^{(\epsilon)} = F_{J,\lambda}$ is W_J-invariant.*

Proof. Follows from applying Corollary 3.10 to the basic representation. If ϵ is trivial, we have

$$0 = (T_i - \tau_i) F_{J,\lambda}^{(\epsilon)} = (\boldsymbol{b}_{a_i} - \tau_i)(1 - s_i) F_{J,\lambda}^{(\epsilon)}$$

for all $i \in J$, and thus that $F_{J,\lambda}^{(\epsilon)}$ is W_J-symmetric. ■

Corollary 4.3. *Let $\lambda \in L$, $f \in A'_J$, then*

$$f(Y) F_{J,\lambda}^{(\epsilon)} = f(-r_{k'}(\lambda)) F_{J,\lambda}^{(\epsilon)},$$

i.e., the commutative algebra $A'_J(Y)$ of difference-reflection operators diagonalises $F_{J,\lambda}^{(\epsilon)}$.

Proof. $f(Y)$ commutes with $\mathfrak{H}_J$, hence also with $U_J^{(\epsilon)}$. Then the claim follows from Theorem 2.48 (ii). ■

In some cases, the $F_{J,\lambda}^{(\epsilon)}$ turns out to be zero ("when symmetrising something anti-symmetric"). We shall later see that the following lemma indeed covers all instances where this happens.

Lemma 4.4. *If there is $j \in J$ with $\epsilon(s_j) = -1$ and $s_j\lambda = \lambda$, then $F_{J,\lambda}^{(\epsilon)} = 0$.*

Proof. Analogously to [17, Section 5.7.1], using Lemma 3.9. ■

Next, we will see that up to scalar multiples, $F_{J,\lambda}^{(\epsilon)}$ only depends on λ's W_J-orbit. For that we define the labelling $\epsilon k \colon S_J \to K, a \mapsto \epsilon(s_a)k(a)$ of the root system S_J spanned by $(a_j)_{j\in J}$, analogously $\epsilon k'$.

Lemma 4.5. *Let $j \in J$ with $\langle \lambda, \alpha'_j \rangle > 0$, then*

$$F^{(\epsilon)}_{J,s_j\lambda} = \epsilon(s_j)\tau_j \boldsymbol{c}'_{a'_j}(\epsilon(s_j) r_{k'}(\lambda)) F^{(\epsilon)}_{J,\lambda} = \epsilon(s_j)\tau_j \boldsymbol{c}_{a'_j,\epsilon k'}(r_{k'}(\lambda)) F^{(\epsilon)}_{J,\lambda}.$$

Proof. Analogously to [17, Section 5.7.2]. ■

This statement can be iterated.

Lemma 4.6. *Let $\lambda \in L_{+,J}$ and $\mu \in W_J\lambda$, then*

$$F^{(\epsilon)}_{J,\mu} = \epsilon(\overline{v}_J(\mu))\tau_{\overline{v}_J(\mu)} \boldsymbol{c}_{\epsilon k',S'}(\overline{v}_J(\mu))(r_{k'}(\lambda)) F^{(\epsilon)}_{J,\lambda},$$

where as in [17, equation (4.4.6)] *we have $\boldsymbol{c}_{k,S}(w) := \prod_{a\in S_1(w)} \boldsymbol{c}_{a,k}$ for $w \in W$ (analogously, for $k', S', w \in W'$).*

Proof. Write $\overline{v}_J(\mu) = s_{i_1} \cdots s_{i_p}$ and define $\mu_r := s_{i_{r+1}} \cdots s_{i_p}\lambda$, as well as $b'_r := s_{i_p} \cdots s_{i_{r+1}} a'_{i_r}$. By Lemma 2.14, we have $S'_1(\overline{v}_I(\mu)) = \{b'_1, \dots, b'_p\}$. From Lemma 2.27, we furthermore know that $\mu = \mu_0 > \cdots > \mu_p = \lambda$, so in particular the μ_r are all different and we have $r_{k'}(\mu_r) = s_{i_{r+1}} \cdots s_{i_p} r_{k'}(\lambda)$ by iterating Lemma 2.23. This then shows that

$$\boldsymbol{c}_{a'_{i_r},\epsilon k'}(r_{k'}(\lambda_r)) = \boldsymbol{c}_{a'_{i_r},\epsilon k'}(s_{i_{r+1}} \cdots s_{i_p} r_{k'}(\lambda)) = \boldsymbol{c}_{s_{i_p}\cdots s_{i_{r+1}} a'_{i_r},\epsilon k'}(r_{k'}(\lambda)) = \boldsymbol{c}_{b'_r,\epsilon k'}(r_{k'}(\lambda)).$$

We can now apply Lemma 4.5 recursively to find

$$F^{(\epsilon)}_{J,\mu} = \prod_{r=1}^{p} (\epsilon(s_{i_r})\tau_{i_r} \boldsymbol{c}_{b'_r,\epsilon k'}(r_{k'}(\lambda))) F^{(\epsilon)}_{J,\lambda} = \epsilon(\overline{v}_J(\mu))\tau_{\overline{v}_J(\mu)} \boldsymbol{c}_{\epsilon k',S'}(\overline{v}_J(\mu))(r_{k'}(\lambda)) F^{(\epsilon)}_{J,\lambda}.$$ ■

4.2 Leading terms

Using results from the last section, we can now prove that the $F^{(\epsilon)}_{J,\lambda}$ have non-vanishing leading coefficients unless the conditions of Lemma 4.4 are met.

Lemma 4.7. *Let $\lambda \in L_{+,J}$ be such that $\epsilon(W_{J,\lambda}) = 1$. Then*

$$F^{(\epsilon)}_{J,\lambda} = \frac{W_{J,\lambda}(\tau_k^2)}{\tau_{w_J}} e(w_J\lambda) + l.o.t.$$

In particular, $(F_{J,\lambda})_{\lambda\in L_{+,J}}$ is a basis of A_J.

Proof. By Lemma 2.49 (i), we have $T(w)E_\lambda = \tau_w E_\lambda$ for $w \in W_{J,\lambda}$, which shows that

$$U_{J'} E_\lambda = \frac{W_{J,\lambda}(\tau_k^2)}{\tau_{w_{J,\lambda}}} E_\lambda,$$

where $J' \subseteq J$ is the set of $i \in J$ such that $s_i\lambda = \lambda$. Using the decomposition from Corollary 3.15, we find

$$F^{(\epsilon)}_{J,\lambda} = U^{(\epsilon)}_J E_\lambda = \frac{W_{J,\lambda}(\tau_k^2)}{\tau_{w_{J,\lambda}} \tau^{(\epsilon)}_{w_J w_{J,\lambda}}} \sum_{v\in W_J^\lambda} \tau^{(\epsilon)}_v T(v) E_\lambda.$$

From Lemma 2.49 and Theorem 2.48, we conclude recursively that for all $v \in W_J^\lambda$ we have

$$T(v)E_\lambda = \sum_{w\leq v} c_{v,w} e(w\lambda) E_{w\lambda} = \sum_{w\leq v} \sum_{\mu\leq w\lambda} c_{v,w} d_{w\lambda,\mu} e(\mu)$$

with $c_{v,v} = \tau_v^{-1}$ and $d_{\mu,\mu} = 1$. Since for $w \in W_J$ we have $\overline{v}_J(w\lambda) \le w$, we have $w\lambda \le v\lambda$, and hence $\mu \le w\lambda \le v\lambda$ for every term μ, w occurring in this sum. Consequently, the highest term in $T(v)E_\lambda$ is $\tau_v^{-1}e(v\lambda)$, so that the leading coefficient of $F^{(\epsilon)}_{J,\lambda}$ comes only from $v = w_J w_{J,\lambda}$, and equals

$$\frac{W_{J,\lambda}\big(\tau_k^2\big)}{\tau_{w_{J,\lambda}}\tau^{(\epsilon)}_{w_J w_{J,\lambda}}}\tau^{(\epsilon)}_{w_J w_{J,\lambda}}\tau^{-1}_{w_J w_{J,\lambda}} = \frac{W_{J,\lambda}\big(\tau_k^2\big)}{\tau_{w_J}},$$

which is nonzero.

To see that the $F^{(\epsilon)}_{J,\lambda}$ form a basis, note that for $\lambda \in L_{+,J}$ we can define

$$m_{J,\lambda} := \sum_{\mu \in W_J\lambda} e(\mu),$$

which is evidently a basis of A_J. In this basis, we have

$$F_{J,\lambda} = \sum_{\substack{\mu \in L_{+,J} \\ \mu \le \lambda}} c_{\lambda,\mu} m_{J,\mu}$$

(note that for $\mu \le \lambda$ a nonzero term occurring in $F_{J,\lambda}$, the J-dominant element of μ's W_J-orbit is lower than μ, hence also $\le \lambda$) with $c_{\lambda,\lambda} = \frac{1}{\tau_{w_J}} W_{J,\lambda}\big(\tau_k^2\big)$. This gives rise to a triangular matrix with invertible diagonals, so we can invert it (locally) and therefore find relations for the $m_{J,\lambda}$ in terms of $(F_{J,\mu})_{\mu \le \lambda}$. ■

With this in hand we are ready to define the protagonist of this work.

Definition 4.8. Let

$$P^{(\epsilon)}_{J,\lambda} := \frac{\tau_{w_J}}{W_{J,\lambda}\big(\tau_k^2\big)} F^{(\epsilon)}_{J,\lambda}.$$

These are the *intermediate Macdonald polynomials* associated to the parabolic subgroup $W_J \le W_0$ and its character ϵ.

By Lemma 4.7 and Definition 4.8, we obtain the following leading term.

Corollary 4.9. *If $\epsilon(W_{J,\lambda}) = 1$, we have $P^{(\epsilon)}_{J,\lambda} = e(w_J\lambda) +$ l.o.t. In particular, $P_{J,\lambda} = m_{J,\lambda} +$ l.o.t. Otherwise, we have $P^{(\epsilon)}_{J,\lambda} = 0$.*

4.3 Orthogonality

As a consequence of the definition of the polynomials $F^{(\epsilon)}_{J,\lambda}$ in terms of symmetrisers, we have easy access to their $A'_J(Y)$-eigenvalues. This allows us to prove their orthogonality.

Theorem 4.10. *For any ϵ, the polynomials $\big(P^{(\epsilon)}_{J,\lambda}\big)_{\lambda \in L_{+,J}}$ from Definition 4.8 form a family of orthogonal polynomials with respect to the inner product from Definition 2.47, with the caveat that $P^{(\epsilon)}_{J,\lambda} = 0$ in case $-1 \in \epsilon(W_{0,\lambda})$. In particular, the nonzero elements are an orthogonal basis of their span. For $\epsilon = 1$, we thus obtain an orthogonal basis of A_J.*

Proof. Let $\mu \ne \lambda \in L_{+,J}$. We will show that the corresponding inner product of F's is zero for all choices of k where all $k'\big(\alpha_i^\vee\big) > 0$ for all $i \in I$.

The weights λ, μ lie on different W_J-orbits. We first show that their $r_{k'}$s also lie on different W_J-orbits. Assume that there is $w \in W_J$ such that $wr_{k'}(\lambda) = r_{k'}(\mu)$. By definition, this means that

$$wv(\lambda)^{-1}(\lambda_- - \rho_{k'}) = v(\mu)^{-1}(\mu_- - \rho_{k'}).$$

Since $\rho_{k'}$ is strictly dominant by Lemma 2.18, the vectors $\lambda_- - \rho_{k'}$, $\mu_- - \rho_{k'}$ are both strictly antidominant. Since they lie in the same W_0-orbit, they must be equal, hence $\lambda_- = \mu_-$ and hence μ, λ lie in the same W_0-orbit. Furthermore, their being strictly antidominant implies that they are regular, whence $wv(\lambda)^{-1} = v(\mu)^{-1}$. Consequently, we have

$$w\lambda = wv(\lambda)^{-1}\lambda_- = v(\mu)^{-1}\mu_- = \mu,$$

so that μ, λ are W_J-related. By Proposition 3.17, we thus have $\mu = \lambda$, which is a contradiction. Therefore, $r_{k'}(\lambda)$, $r_{k'}(\mu)$ lie on different W_J-orbits. Hence the same is true for $-r_{k'}(\lambda)$, $-r_{k'}(\mu)$.

Since A'_J separates W_J-orbits of V, there exists $f \in A'_J$ such that $f(-r_{k'}(\lambda)) \neq f(-r_{k'}(\mu))$. Then by (2.1), we have $f(Y)U_J^{(\epsilon)} = U_J^{(\epsilon)}$, so that

$$\begin{aligned} f(-r_{k'}(\lambda))\big(F_{J,\lambda}^{(\epsilon)}, F_{J,\mu}^{(\epsilon)}\big) &= \big(f(-r_{k'}(\lambda))F_{J,\lambda}^{(\epsilon)}, F_{J,\mu}^{(\epsilon)}\big) = \big(f(Y)F_{J,\lambda}^{(\epsilon)}, F_{J,\mu}^{(\epsilon)}\big) = \big(F_{J,\lambda}^{(\epsilon)}, f^*(Y)F_{J,\mu}^{(\epsilon)}\big) \\ &= \big(F_{J,\lambda}^{(\epsilon)}, f^*(-r_{k'}(\mu))F_{J,\mu}^{(\epsilon)}\big) = f(-r_{k'}(\mu))\big(F_{J,\lambda}^{(\epsilon)}, F_{J,\mu}^{(\epsilon)}\big), \end{aligned}$$

whence $\big(F_{J,\lambda}^{(\epsilon)}, F_{J,\mu}^{(\epsilon)}\big) = 0$. Consequently, we also have $\big(F_{J,\lambda}^{(\epsilon)}, F_{J,\mu}^{(\epsilon)}\big) = 0$.

By definition, the element $\big(P_{J,\lambda}^{(\epsilon)}, P_{J,\mu}^{(\epsilon)}\big)_1$ is a rational function in finitely many values of q, and since the space of these values for positive k is Zariski-dense, the equality $\big(P_{J,\lambda}^{(\epsilon)}, P_{J,\mu}^{(\epsilon)}\big) = 0$ holds identically. ■

Example 4.11. For $\epsilon = 1$ and $J = I_0$, the leading coefficient from Corollary 4.9 together with the orthogonality from Theorem 4.10 implies that

- $\forall \mu \in L_+ \colon P_{J,\mu} = m_\mu + \text{l.o.t.}$
- $\forall \lambda, \mu \in L_+ \colon \lambda < \mu \Rightarrow (P_{J,\mu}, m_\lambda) = 0.$

Together with [17, Sections 5.1.35 and 5.3.1f], this implies that our intermediate Macdonald polynomials are just the symmetric Macdonald polynomials from [17, Section 5.3].

For $\epsilon = 1$ and $J = \varnothing$, our definitions show that $P_{J,\lambda} = E_\lambda$, i.e., that the intermediate Macdonald polynomials are the nonsymmetric Macdonald polynomials.

These two examples illustrate the choice of the name "intermediate Macdonald polynomials".

4.4 Norms

Using an approach similar to [17, Section 5.7.12], we can now also express the norm of $P_{J,\lambda}^{(\epsilon)}$ in terms of the norm of $E_{w_J\lambda}$.

Theorem 4.12. *If* $\epsilon(W_{J,\lambda}) = 1$, *we have*

$$\big(P_{J,\lambda}^{(\epsilon)}, P_{J,\lambda}^{(\epsilon)}\big) = \epsilon(w_J)\frac{\boldsymbol{c}_{S',-\epsilon k'}(w_J w_{J,\lambda})(r_{k'}(\lambda))W_J^\lambda\big(\tau_k^{(\epsilon)2}\big)}{\tau_{w_J w_{J,\lambda}}^2 \tau_{w_J w_{J,\lambda}}^{(\epsilon)}}(E_{w_J\lambda}, E_{w_J\lambda}).$$

Proof. We have

$$\big(P_{J,\lambda}^{(\epsilon)}, P_{J,\lambda}^{(\epsilon)}\big) = \frac{\tau_{w_{J,\lambda}}^2}{W_{J,\lambda}\big(\tau_k^2\big)^2}\big(F_{J,\lambda}^{(\epsilon)}, F_{J,\lambda}^{(\epsilon)}\big).$$

By Lemma 4.6, this equals

$$\epsilon(w_J w_{J,\lambda})\frac{\tau_{w_{J,\lambda}}^2 \tau_{w_J w_{J,\lambda}}}{W_{J,\lambda}\big(\tau_k^2\big)^2 \boldsymbol{c}_{-\epsilon k',S'}(w_J w_{J,\lambda})(r_{k'}(\lambda))}\big(F_{J,\lambda}^{(\epsilon)}, F_{J,w_J\lambda}^{(\epsilon)}\big).$$

Note that $\epsilon(w_{J,\lambda}) = 1$, so that $\epsilon(w_J w_{J,\lambda}) = \epsilon(w_J)$. Applying Proposition 3.12 (iii), we get

$$= \epsilon(w_J) \frac{W_J\bigl(\tau_k^{(\epsilon)2}\bigr)\tau_{w_{J,\lambda}}^2 \tau_{w_J w_{J,\lambda}}}{W_{J,\lambda}\bigl(\tau_k^2\bigr)^2 \tau_{w_J}^{(\epsilon)} \boldsymbol{c}_{-\epsilon k', S'}(w_J w_{J,\lambda})(r_{k'}(\lambda))} \bigl(F_{J,\lambda}^{(\epsilon)}, E_{w_J\lambda}\bigr).$$

Since the $e(w_J\lambda)$-coefficient (the leading coefficient) in $F_{J,\lambda}^{(\epsilon)}$ is $\frac{W_{J,\lambda}(\tau_k^2)}{\tau_{w_J}}$, we have

$$F_{J,\lambda}^{(\epsilon)} = \sum_{\mu \in W_J\lambda} c_\mu E_\mu$$

with $c_{w_J\lambda} = \frac{W_{J,\lambda}(\tau_k^2)}{\tau_{w_J}}$. Since all other E_μ are orthogonal to $E_{w_J\lambda}$, we have

$$\bigl(F_{J,\lambda}^{(\epsilon)}, E_{w_J\lambda}\bigr) = \frac{W_{J,\lambda}\bigl(\tau_k^2\bigr)}{\tau_{w_J}} (E_{w_J\lambda}, E_{w_J\lambda}),$$

so that

$$\bigl(P_{J,\lambda}^{(\epsilon)}, P_{J,\lambda}^{(\epsilon)}\bigr) = \epsilon(w_J) \frac{W_J\bigl(\tau_k^{(\epsilon)2}\bigr)\tau_{w_{J,\lambda}} \tau_{w_J w_{J,\lambda}}}{W_{J,\lambda}\bigl(\tau_k^2\bigr) \tau_{w_J}^{(\epsilon)} \boldsymbol{c}_{-\epsilon k', S'}(w_J w_{J,\lambda})(r_{k'}(\lambda))} (E_{w_J\lambda}, E_{w_J\lambda}).$$

Next, note that $\tau_{w_{J,\lambda}} = \tau_{w_{J,\lambda}}^{(\epsilon)}$, and that $W_{J,\lambda}\bigl(\tau_k^2\bigr) = W_{J,\lambda}\bigl(\tau_k^{(\epsilon)2}\bigr)$, so that

$$\bigl(P_{J,\lambda}^{(\epsilon)}, P_{J,\lambda}^{(\epsilon)}\bigr) = \epsilon(w_J) \frac{W_J^\lambda\bigl(\tau_k^{(\epsilon)2}\bigr)\tau_{w_J w_{J,\lambda}}}{\tau_{w_J w_{J,\lambda}}^{(\epsilon)} \boldsymbol{c}_{-\epsilon k', S'}(w_J w_{J,\lambda})(r_{k'}(\lambda))} (E_{w_J\lambda}, E_{w_J\lambda}). \qquad \blacksquare$$

5 Invariant vector-valued polynomials

A method of obtaining a representation of $\tilde{\mathfrak{H}}$ on vector-valued polynomials is presented in [5]. Given a $\mathfrak{H}$-module V, we can consider the $\tilde{\mathfrak{H}}$-module $\tilde{\mathfrak{H}} \otimes_{\mathfrak{H}} V$. But which module V should we start with? Taking inspiration from [19, Section 3], we consider principal series modules.

5.1 Y-parabolically induced modules of $\mathfrak{H}$

Lemma 5.1. *The set $\mathfrak{H}_J A'(Y)$ is a subalgebra of $\mathfrak{H}$.*

Proof. Obviously $\mathfrak{H}_J$ and $A'(Y)$ are subalgebras of their own, and the Bernstein–Lusztig–Zelevinsky presentation (2.1) guarantees that their product forms an algebra. $\blacksquare$

Definition 5.2. Let $\phi\colon \mathfrak{H}_J A'(Y) \to K$ be a ring homomorphism. We can define the following induced modules with it:

(i) The *principal series representation* is the $\mathfrak{H}$-module $M_J(\phi) {:=} \mathfrak{H} \otimes_{\mathfrak{H}_J A'(Y)} K$ (where $\mathfrak{H}_J A'(Y)$ acts on K via ϕ);

(ii) the *standard Y-parabolically induced $\tilde{\mathfrak{H}}$-module* is the $\tilde{\mathfrak{H}}$-module

$$\mathbb{M}_J(\phi) := \tilde{\mathfrak{H}} \otimes_{\mathfrak{H}} M_J(\phi) = \tilde{\mathfrak{H}} \otimes_{\mathfrak{H}_J A'(Y)} K.$$

Moreover, an element $v \in \mathbb{M}_J(\phi)$ is called *spherical* if $(T_i - \tau_i)v = 0$ for all $i \in I_0$. Write $\mathbb{M}_J(\phi)^{\mathfrak{H}_0}$ for the vector space of spherical vectors.

Lemma 5.3. *A K-basis of $\mathbb{M}_J(\phi)$ is given by $(X^\mu T(v))_{\mu \in L, v \in W_0^J}$. In particular, $\mathbb{M}_J(\phi)$ is a free A-module with basis $(T(v))_{v \in W_0^J}$.*

Proof. By Lemma 2.36, we can decompose $\tilde{\mathfrak{H}} \cong A(X) \otimes \mathfrak{H}_0 \otimes A'(Y)$ as right $A'(Y)$-modules. Note that using the shortest coset representatives, we can also decompose

$$\mathfrak{H}_0 = \bigoplus_{v \in W_0^J} T(v)\mathfrak{H}_J$$

as right $\mathfrak{H}_J$-modules. Consequently,

$$\tilde{\mathfrak{H}} \cong A(X) \otimes \bigoplus_{v \in W_0^J} KT(v) \otimes \mathfrak{H}_J A'(Y)$$

as right $\mathfrak{H}_J A'(Y)$-modules. Consequently, the family $(X^\mu T(v))_{\mu \in L, v \in W_0^J}$ is a basis of the right $\mathfrak{H}_J A'(Y)$-module $\tilde{\mathfrak{K}}$, and hence a K-basis of $\mathbb{M}_J(\phi)$. ■

Purely on a vector space level, we can therefore establish a bijection $A \otimes K[W_0/W_J] \cong \mathbb{M}_J(\phi)$ by mapping $f \otimes \delta_{vW_J} \mapsto f(X)T(v)$ where $v \in W_0^J$.

Example 5.4. For $J = I_0$, the Y-parabolic subalgebra $\mathfrak{H}_J A'(Y)$ is just $\mathfrak{H}$. For the map ϕ from Lemma 2.39, the standard Y-parabolically induced $\tilde{\mathfrak{H}}$-module $\mathbb{M}_J(\phi)$ is how we defined the basic representation on A.

5.2 Spherical vectors

We shall assume now that our character ϕ restricts to the trivial character of $\mathfrak{H}_J$. Then the induced module $\mathbb{M}_J(\phi)$ corresponds to $(\mathbb{C}[P] \otimes \mathbb{C}[W/W_J])^W$ from [23] in the classical limit. We will see that the result [23, Lemma 5.1] carries over to the q-setting and the spherical vectors of $\mathbb{M}_J(\phi)$ correspond to elements of A_J.

Definition 5.5. We define $\Gamma\colon A_J \to \mathbb{M}_J(\phi)$ by mapping $f \mapsto U_0 f(X)$, where U_0 refers to the symmetriser for the whole group W_0.

Lemma 5.6. *The map Γ is a well-defined A_0-linear map $A_J \to (\mathbb{M}_J(\phi))^{\mathfrak{H}_0}$. In particular, we have*

$$\Gamma(f) = \sum_{v \in W_0^J} f_v(X)T(v), \qquad f_{w_0 w_J} = \frac{W_J\left(\tau_k^2\right)}{\tau_{w_J}} w_0 f.$$

Proof. By Lemma 3.9, we have $(T_i - \tau_i)U_0 = 0$ for all $i \in I_0$. Consequently, by associativity also $0 = (T_i - \tau_i)U_0 f(X) = (T_i - \tau_i)\Gamma(f)$. For A_0-linearity, note that $A_0(X)$ commutes with $\mathfrak{H}_0$, so that it also commutes with U_0.

For the second claim, note that by Corollary 3.11 we have

$$\Gamma(f) = \tau_{w_0 w_J}^{-1} \sum_{v \in W_0^J} \tau_v T(v) U_J f(X).$$

Since $f \in A_J$, the elements $f(X)$ and U_J commute, and we have

$$U_J = \frac{W_J\left(\tau_k^2\right)}{\tau_{w_J}}$$

in $\mathbb{M}_J(\phi)$. This shows that

$$\Gamma(f) = \frac{W_J\left(\tau_k^2\right)}{\tau_{w_0}} \sum_{v \in W_0^J} \tau_v T(v) f(X) = \frac{W_J\left(\tau_k^2\right)}{\tau_{w_0}} \sum_{v \in W_0^J} \tau_v \sum_{w \leq v} f_{v,w}(X) T(w),$$

where $f_{v,v} = vf$ (follows inductively from (2.2)). Since w_0w_J is the smallest element of its right W_J-coset, the only summand contributing to the $T(w_0w_J)$ term is where $v = w = w_0w_J$. As a consequence, when we expand $\Gamma(f)$ in terms of our basis from Lemma 5.3, say

$$\Gamma(f) = \sum_{v \in W_0^J} f_v(X)T(v)$$

with

$$f_{w_0w_J} = \frac{W_J\big(\tau_k^2\big)}{\tau_{w_0}} \tau_{w_0w_J} w_0w_J f = \frac{W_J\big(\tau_k^2\big)}{\tau_{w_J}} w_0 f. \qquad \blacksquare$$

Remark 5.7. It is tempting to view Γ as the concatenation of the $\tilde{\mathfrak{H}}$-linear map $\iota\colon A \to \mathbb{M}_J(\phi)$ mapping $1 \mapsto 1$ and multiplication with U_0, rendering Γ in fact not only $A(X)$-linear but indeed $Z_{\tilde{\mathfrak{H}}}(U_0)$-linear $\big($the centraliser of U_0 in $\tilde{\mathfrak{H}}\big)$. This, however, falls flat when we try to investigate if the map ι even exists in the first place. Since we defined both $\tilde{\mathfrak{H}}$-module structures in terms of induced representations, we can map $1 \mapsto 1$ if $\mathrm{Ann}_A(1) \subseteq \mathrm{Ann}_{\mathbb{M}_J(\phi)}$. Since 1 lies in the centre of $\tilde{\mathfrak{H}}$, the annihilators are the kernels of $\tilde{\phi}$, ϕ, respectively (here, we write $\tilde{\phi}$ for the trivial character ϕ from Lemma 2.39). However, unless $J = I_0$, the kernel of $\tilde{\phi}$ is bigger than that of ϕ.

Corollary 5.8. Γ *is injective.*

Proof. Since $\frac{W_J(\tau_k^2)}{\tau_{w_J}}$ is nonzero and w_0 is an automorphism of A, Lemma 5.6 implies that we can read off f from $\Gamma(f)$'s expansion in the basis from Lemma 5.3. $\blacksquare$

If we want to show that Γ is also surjective (onto the spherical vectors of $\mathbb{M}_J(\phi)$), we need to find out more about the spherical vectors, especially how the coefficients are related.

Proposition 5.9. *Let $h \in \mathbb{M}_J(\phi)$ be spherical, say*

$$h = \sum_{v \in W_0^J} f_v(X)T(v),$$

then

(i) Then h is uniquely determined by any f_v.

(ii) $f_{w_0w_J}$ is $w_0W_Jw_0$-symmetric.

(iii) $w_0 f_{w_0w_J} \in A_J$.

Proof. We begin by describing the action of $\mathfrak{H}_0$ on $M_J(\phi)$. If $v \in W_0^J$ and $i \in I_0$, we can decompose $s_iv \in W_0$ according to $W_0^JW_J$, say as $s_iv =: (s_i \bullet v)m_i(v)$. Then we have $s_ivW_J = (s_i \bullet v)W_J$, so that $\bullet$ describes the W_0-action on W_0/W_J on shortest coset representatives, and the m's provide the cocycles. If $\ell(s_iv) > \ell(v)$, we have

$$T_iT(v) = T(s_iv) = T((s_i \bullet v)m_i(v)) = T(s_i \bullet v)T(m_i(v)) = \tau_{m_i(v)}T(s_i \bullet v)$$

in $M_J(\phi)$. Otherwise, we have

$$T_iT(v) = T(s_iv) + \big(\tau_i - \tau_i^{-1}\big)T(v) = \tau_{m_i(v)}T(s_i \bullet v) + \big(\tau_i - \tau_i^{-1}\big)T(v).$$

In the special case of $v = w_0w_J$, and $s_i \in w_0W_Jw_0$, we have $w_0s_iw_0 \in W_J$, and since the longest element of a Coxeter group always permutes the simple reflections (by conjugation), we have that $w_Jw_0s_iw_0w_J = s_j$ is a simple reflection in W_J. This implies $s_iw_0w_J = w_0w_Js_j$, which by Lemma 2.33 implies that

$$T_iT(w_0w_J) = T(w_0w_J)T_j = \tau_jT(w_0w_J) = \tau_iT(w_0w_J).$$

(i) We now consider T_i's action on h. By the Bernstein–Lusztig–Zelevinsky presentation (2.2) and the action of T_i on $M_I(\phi)$, we obtain

$$\begin{aligned} T_i h &= \sum_{v\in W_0^J} (s_i f_v)(X) T_i T(v) + \boldsymbol{b}_i(X) \sum_{v\in W_0^J} (f_v - s_i f_v)(X) T(v) \\ &= \sum_{v\in W_0^J} (s_i f_v)(X) \tau_{m_i(v)} T(s_i \bullet v) + \boldsymbol{b}_i(X) \sum_{v\in W_0^J} (f_v - s_i f_v)(X) T(v) \\ &\quad + \big(\tau_i - \tau_i^{-1}\big) \sum_{\substack{v\in W_0^J \\ \ell(s_i v)<\ell(v)}} f_v(X) T(v) \\ &= \sum_{v\in W_0^J} \big(\tau_{m_i(s_i\bullet v)} s_i f_{s_i\bullet v} + \boldsymbol{b}_i (f_v - s_i f_v)\big)(X) T(v) \\ &\quad + \big(\tau_i - \tau_i^{-1}\big) \sum_{\substack{v\in W_0^J \\ \ell(s_i v)<\ell(v)}} (s_i f_v)(X) T(v). \end{aligned}$$

Since the $(T(v))_{v\in W_0^J}$ are $A(X)$-linearly independent, we have

$$\tau_i f_v = \tau_{m_i(s_i\bullet v)} s_i f_{s_i\bullet v} + \boldsymbol{b}_i (f_v - s_i f_v) + \begin{cases} 0, & \ell(s_i v) > \ell(v), \\ \big(\tau_i - \tau_i^{-1}\big)(s_i f_v), & \ell(s_i v) < \ell(v), \end{cases}$$

We can solve for $f_{s_i\bullet v}$ and thus obtain that $f_{s_i\bullet v}$ is uniquely determined by f_v. Inductively, we thus obtain that $f_{w\bullet v}$ is uniquely determined by f_v for all $w \in W_0$. Since W_0 acts transitively on W_0/W_J and hence also on W_0^J, this shows that all coefficient polynomials $(f_v)_{v\in W_0^J}$ are determined by any one f_v.

(ii) If we consider the case of $s_i \in w_0 W_J w_0$ and $v = w_0 w_J$, we find that $s_i \bullet v = v$, $m_i(v) = s_i$, and $\ell(s_i v) > \ell(v)$, so that $\tau_i f_v = \tau_i (s_i f_v) + \boldsymbol{b}_i (f_v - s_i f_v)$, in other words, $0 = (\boldsymbol{b}_i - \tau_i)(f_v - s_i f_v)$, which implies that $f_v = s_i f_v$. In other words, $f_{w_0 w_J}$ is $w_0 W_J w_0$-symmetric.

(iii) Let $w \in W_J$, then $w w_0 f_{w_0 w_J} = w_0 w_0 w w_0 f_{w_0 w_J} = w_0 f_{w_0 w_J}$ as $w_0 w w_0 \in w_0 W_J w_0$. ■

Theorem 5.10. *$\Gamma\colon A_J \to \mathbb{M}_J(\phi)^{\mathfrak{H}_0}$ is an A_0-linear isomorphism.*

Proof. From Lemma 5.6, we know that Γ is well-defined and A_0-linear. With Corollary 5.8, we conclude that Γ is injective. For surjectivity, let

$$h = \sum_{v\in W_0^J} f_v(X) T(v) \in \mathbb{M}_J(\phi)^{\mathfrak{H}_0}.$$

From Proposition 5.9 (iii), we know that $w_0 f_{w_0 w_J} \in A_J$. Consider now

$$h' := \Gamma\left(\frac{\tau_{w_J}}{W_J\big(\tau_k^2\big)} w_0 f_{w_0 w_J} \right) = \sum_{v\in W_0^J} f'_v(X) T(v).$$

By Lemma 5.6, we know that

$$f'_{w_0 w_J} = \frac{W_J\big(\tau_k^2\big)}{\tau_{w_J}} \frac{\tau_{w_J}}{W_J\big(\tau_k^2\big)} w_0 w_0 f_{w_0 w_J} = f_{w_0 w_J}.$$

By Proposition 5.9 (i) therefore, all other coefficient polynomials of h, h' also have to be equal, whence $h = h'$. ■

Using this isomorphism, we can now push $(\cdot,\cdot)_1$ to $\mathbb{M}_J(\phi)^{\mathfrak{H}_0}$ and obtain the basis of vector-valued orthogonal polynomials $(\Gamma(P_{J,\lambda}))_{\lambda\in L_{+,J}}$.

In [19, Section 6.6], the authors introduce a basis of W_0-symmetric orthogonal elements of $\mathbb{M}_J(\phi)$ that diagonalise the action of $A_0'(Y)$. Since our map Γ is not expected to preserve any of the Y-actions, we do not expect this basis to coincide with the intermediate Macdonald polynomials. However, in [24, Theorem 7.6] it is shown that they coincide in the $q\to\infty$ limit.

6 Vector-valued invariant polynomials

Classically, there exists an isomorphism

$$(A\otimes K[W_0/W_J])^{W_0}\cong A^{W_0}\otimes K[W_0/W_J]=A_0^{\#W_0^J}.$$

The reason for this is that as is shown in [21], A_J is a free A_0-module. This is proved by constructing an appropriate basis $(e_v)_{v\in W_0^J}$, where $e_1=1$. Note that this particular choice of basis is by no means unique or canonical, and we shouldn't restrict ourselves to it.

Lemma 6.1. *Let $(e_v)_{v\in W_0^J}$ be an A_0-basis of A_J. Via this basis, we identify elements $f\in A_J$ with column vectors $\underline{f}$ (indexed by W_0^J). Define*

$$m_{v,v'}:=\frac{1}{\#W_0}\sum_{w\in W_0}w\frac{e_ve_{v'}^*}{\Delta_0}\in K(L)^{W_0}$$

and write $M=(m_{v,v'})_{v,v'\in W_0^J}$ for the matrix thus defined. Then, $(f,g)=\mathrm{ct}\left(\underline{f}^{\mathsf{T}}M\underline{g}^*\nabla\right)$ for all $f,g\in A_J$. Here, ∇ is the W_0-invariant weight function defined in Definition 2.45.

Proof. If we expand

$$f=\sum_{v\in W_0^J}f_ve_v,\qquad g=\sum_{v\in W_0^J}g_ve_v,$$

then the claim states that

$$(f,g)=\sum_{v,v'\in W_0^J}\mathrm{ct}\left(f_vm_{v,v'}g_{v'}^*\nabla\right).$$

To prove this, we substitute the expansions of f, g

$$(f,g)=\sum_{v,v'\in W_0^J}\mathrm{ct}\left(f_vg_{v'}^*\nabla\frac{e_ve_{v'}^*}{\Delta_0}\right).$$

Recall that we can write $\Delta=\nabla\Delta_0^{-1}$ where ∇ is W_0-invariant by Lemma 2.46 (ii). Next, note that the notion of the constant term is W_0-invariant, as is $f_vg_{v'}^*\nabla$, so the summands are not changed by also symmetrising over W_0

$$(f,g)=\sum_{v,v'\in W_0^J}\mathrm{ct}\left(f_vg_{v'}^*\nabla\frac{1}{\#W_0}\sum_{w\in W_0}w\frac{e_ve_{v'}^*}{\Delta_0}\right)=\sum_{v,v'\in W_0^J}\mathrm{ct}\left(f_vm_{v,v'}g_{v'}^*\nabla\right).\qquad\blacksquare$$

Corollary 6.2. *For any fixed basis $(e_v)_{v\in W_0^J}$, the family $(\underline{P_{J,\lambda}})_{\lambda\in L_{+,J}}$ is a family of vector-valued polynomials that is orthogonal with respect to the matrix weight ∇M.*

We close this discussion of general theory with some observations about matrix weights.

Definition 6.3. A matrix weight $M \in (K(L)^{W_0})^{n\times n}$ is called *reducible* if there is $R \in \mathrm{GL}_n(K)$ such that $RMR^{*\mathsf{T}}$ is a block matrix. (Here, $*$ refers to entry-wise application of K's $*$ involution.)

Corollary 6.4. *If M is a matrix weight whose entries are polynomials, say $M = \sum_{\mu\in L_+} M_\mu m_\mu$, then M is reducible if all M_μ can be brought into the same block shape by a similarity transformation with the same matrix R. In particular, M can be made diagonal iff all the M_μ can be made diagonal with the same matrix R.*

Proof. If $D = RMR^{*\mathsf{T}}$ is a block matrix, we can expand D as a polynomial matrix as

$$D = \sum_{\mu\in L_+} RM_\mu R^{*\mathsf{T}} m_\mu.$$

Since the m_μ are K-linearly independent, all $RM_\mu R^{*\mathsf{T}}$ have to have the same block structure as D. ■

We now consider two examples, where we are able to transform the basis constructed in [21] by a unipotent triangular matrix (over A_0) to obtain a maximally reduced matrix weight.

6.1 Example: $(C_1^\vee, C_1)$ with $J = \varnothing$

This example was taken (with slightly different conventions) from [12] and [13, Section 6]. The $q \to 1$ limit can be compared to [22]. For an affine root system of type $(C_1^\vee, C_1)$, the double-affine Hecke algebra $\tilde{\mathfrak{H}}$ is generated by T_1, X, Y (in the notation of [17, Section 6.4]) and depends on the four parameters k_1, k_2, k_3, k_4. As is customary, we use these four parameters to define the more convenient Askey–Wilson parameters

$$(a,b,c,d) = \left(q^{k_1}, -q^{k_2}, q^{1/2+k_3}, -q^{1/2+k_4}\right).$$

We pick $J = \varnothing$ and thus represent A as a free A_0-module. Steinberg's proof provides the basis $\tilde{e}_1 = 1$, $\tilde{e}_{s_1} = x^{-1}$ (writing $x := e(a_1)$). We shall first consider this basis and compare our findings to [22].

Lemma 6.5. *In the basis $(\tilde{e}_1, \tilde{e}_{s_1})$, we obtain (as in Lemma 6.1) the following weight matrix:*

$$\tilde{M} = \frac{1}{2}\begin{pmatrix} 1-ab & x+x^{-1}-a-b \\ -ab\left(x+x^{-1}\right)+a+b & 1-ab \end{pmatrix}.$$

We can now do a similarity transform with the constant matrix

$$U := \begin{pmatrix} -a & 1 \\ -b & 1 \end{pmatrix}$$

to find

$$U\tilde{M}U^{*\mathsf{T}} = \frac{a-b}{2}\begin{pmatrix} -\left(x+x^{-1}\right)+a+a^{-1} & 0 \\ 0 & x+x^{-1}-b-b^{-1} \end{pmatrix}.$$

Proof. Note that

$$\Delta_0 = \frac{x-x^{-1}}{(x-a)\left(1-bx^{-1}\right)} = \frac{\delta}{F},$$

where $\delta = x - x^{-1}$ is the Weyl-denominator, and $F = x\bigl(1 - ax^{-1}\bigr)\bigl(1 - bx^{-1}\bigr)$. Then

$$\tilde{m}_{v,v'} = \frac{1}{2\delta}(1 - s_1)(F\tilde{e}_v\tilde{e}^*_{v'})$$

for all $v, v' \in W_0$ since $s_1\delta^{-1} = -\delta^{-1}s_1$. This means that

$$\begin{aligned}
\tilde{m}_{1,1} = \tilde{m}_{s_1,s_1} &= \frac{1}{2\delta}(1 - s_1)F \\
&= \frac{1}{2\delta}\bigl(x - a - b + abx^{-1} - x + a + b - abx\bigr) = \frac{1 - ab}{2}, \\
\tilde{m}_{1,s_1} &= \frac{1}{2\delta}(1 - s_1)(Fx) = \frac{1}{2\delta}\bigl(x^2 - ax - bx + ab - x^{-2} + ax^{-1} + bx^{-1} - ab\bigr) \\
&= \frac{x + x^{-1} - a - b}{2}, \\
\tilde{m}_{s_1,1} &= \frac{1}{2\delta}(1 - s_1)(Fx^{-1}) = \frac{1}{2\delta}\bigl(1 - ax^{-1} - bx^{-1} + abx^{-2} - 1 + ax + bx - abx^2\bigr) \\
&= \frac{-ab\bigl(x + x^{-1}\bigr) + a + b}{2}.
\end{aligned}$$

Lastly, to see the similarity transform, note that

$$2\tilde{M} = \begin{pmatrix} 1 - ab & -a - b \\ a + b & 1 - ab \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -ab & 0 \end{pmatrix}\bigl(x + x^{-1}\bigr) = 2\tilde{M}_0 m_0 + 2\tilde{M}_{a_1} m_{\alpha_1}$$

(using notation from Corollary 6.4). We have

$$\begin{aligned}
U2\tilde{M}_0U^{*\mathsf{T}} &= \begin{pmatrix} -a & 1 \\ -b & 1 \end{pmatrix}\begin{pmatrix} 1 - ab & -a - b \\ a + b & 1 - ab \end{pmatrix}\begin{pmatrix} -a^{-1} & -b^{-1} \\ 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} (a - b)\bigl(a + a^{-1}\bigr) & 0 \\ 0 & -(a - b)(b + b^{-1}) \end{pmatrix} \\
&= (a - b)\begin{pmatrix} a + a^{-1} & 0 \\ 0 & -b - b^{-1} \end{pmatrix}, \\
U2\tilde{M}_{a_1}U^{*\mathsf{T}} &= \begin{pmatrix} -a & 1 \\ -b & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -ab & 0 \end{pmatrix}\begin{pmatrix} -a^{-1} & -b^{-1} \\ 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} b - a & 0 \\ 0 & a - b \end{pmatrix} = (a - b)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},
\end{aligned}$$

so that $UMU^{*\mathsf{T}}$ has the shape that was claimed. ■

Remark 6.6. Note that in the $q \to 1$ limit we have

$$\begin{aligned}
\nabla_k &\to \bigl((1 - x)\bigl(1 - x^{-1}\bigr)\bigr)^{k_1 + k_3}\bigl((1 + x)\bigl(1 + x^{-1}\bigr)\bigr)^{k_2 + k_4} \\
&= \bigl(2 - x - x^{-1}\bigr)^{k_1 + k_3}\bigl(2 + x + x^{-1}\bigr)^{k_2 + k_4}
\end{aligned}$$

and

$$\tilde{M} \to \begin{pmatrix} 1 & \frac{x + x^{-1}}{2} \\ \frac{x + x^{-1}}{2} & 1 \end{pmatrix}, \qquad U \to \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

as well as

$$U\tilde{M}U^{*\mathsf{T}} \to \begin{pmatrix} 1 - \frac{x + x^{-1}}{2} & 0 \\ 0 & 1 + \frac{x + x^{-1}}{2} \end{pmatrix}.$$

This yields the overall matrix weight

$$U\tilde{M}\nabla U^{*\mathsf{T}} \to 2^{k_1+k_2+k_3+k_4}\begin{pmatrix}(1-z)^{k_1+k_3+1}(1+z)^{k_2+k_4} & 0\\ 0 & (1-z)^{k_1+k_3}(1+z)^{k_2+k_4+1}\end{pmatrix}$$

$\big($writing $2z = x + x^{-1}\big)$, which corresponds to [22, equation (13)]. Note that by Lemma 2.1, we need to set $k_2 = k_3 = 0$ to emulate an affine root system of type BC_1, which leaves the two constants $\alpha = k_1$, $\beta = k_4$.

Besides the Steinberg basis there happens to be another A_0-basis of A that appears naturally.

Lemma 6.7. *The family* $e_1 := 1, e_{s_1} := x^{-1}(1-ax)(1-bx)$ *is an* A_0*-basis of* A_J *(*$= A$*), which diagonalises* T_1*.*

Proof. We have

$$\begin{aligned}
&T_1\tilde{e}_1 = \tau_1\tilde{e}_1,\\
&T_1\tilde{e}_{s_1} = \tau_1 x + (\tau_1 - \tau_1^{-1})x^{-1} + \tilde{\tau}_1 - \tilde{\tau}_1^{-1} = \tau_1\left(\frac{1}{ab}x^{-1} + x + x^{-1} - \frac{1}{b} - \frac{1}{a}\right)\\
&\phantom{T_1\tilde{e}_{s_1}} = \tau_1\left(\left(x + x^{-1} - \frac{a+b}{ab}\right)\tilde{e}_1 + \frac{1}{ab}\tilde{e}_{s_1}\right).
\end{aligned}$$

In other words, as a matrix we have

$$T_1 = \tau_1\begin{pmatrix}1 & x + x^{-1} - \frac{a+b}{ab}\\ 0 & \frac{1}{ab}\end{pmatrix}.$$

This is upper triangular with distinct eigenvalues 1, $a^{-1}b^{-1}$. Therefore, it can be diagonalised with another upper triangular matrix, which transforms $\tilde{e}_1$, $\tilde{e}_{s_1}$ to 1, $x^{-1}(1-ax)(1-bx)$. ■

Corollary 6.8. *Let* $f, g \in A_0$*, then* $(fe_1, ge_{s_1}) = 0$*.*

Proof. Since T_1 acts on A_J by means of an A_0-endomorphism, its eigenspaces are A_0-modules, and hence by Lemma 6.7, fe_1 is an eigenvector with eigenvalue 1 and ge_{s_1} is an eigenvector with eigenvalue $a^{-1}b^{-1}$. Since these are distinct and since T_1 is unitary with respect to $(\cdot,\cdot)$, the corresponding eigenspaces are orthogonal. ■

Corollary 6.9. *In the expression from Lemma 6.1 of* $(\cdot,\cdot)$ *in terms of a matrix weight for the basis* e_1*,* e_{s_1}*, the matrix weight* M *is diagonal.*

Lemma 6.10. *Write* $M = \left(\begin{smallmatrix}d_1 & 0\\ 0 & d_2\end{smallmatrix}\right)$ *for the matrix weight from Corollary 6.9. Then*

$$d_1 = \frac{1-ab}{2}, \qquad d_2 = \frac{1-a^{-1}b^{-1}}{2}\frac{\nabla_{k+l}}{\nabla_k},$$

where $l = (1,1,0,0)$*, so that the labelling* $k+l$ *corresponds to the coefficients* aq*,* bq*,* c*,* d*.*

Proof. Just like in the proof of Lemma 6.5, we have

$$d_1 = \frac{1}{2\delta}(1-s_1)F = \frac{1-ab}{2}, \qquad d_2 = \frac{1}{2\delta}(1-s_1)(Fhh^*).$$

For the second diagonal entry, we get

$$d_2 = \frac{1}{2\delta}(1-s_1)x\big(1-ax^{-1}\big)\big(1-bx^{-1}\big)(1-ax)(1-bx)\big(1-a^{-1}x^{-1}\big)\big(1-b^{-1}x^{-1}\big)$$

$$
\begin{aligned}
&= \frac{1}{2\delta}\big(x\big(1-ax^{-1}\big)\big(1-bx^{-1}\big)(1-ax)(1-bx)\big(1-a^{-1}x^{-1}\big)\big(1-b^{-1}x^{-1}\big) \\
&\quad - x^{-1}(1-ax)(1-bx)\big(1-ax^{-1}\big)\big(1-bx^{-1}\big)\big(1-a^{-1}x\big)\big(1-b^{-1}x\big)\big) \\
&= \frac{(1-ax)\big(1-ax^{-1}\big)(1-bx)\big(1-bx^{-1}\big)}{2\delta}\big(x\big(1-a^{-1}x^{-1}\big)\big(1-b^{-1}x^{-1}\big) \\
&\quad - x^{-1}\big(1-a^{-1}x\big)\big(1-b^{-1}x\big)\big) \\
&= \frac{1-a^{-1}b^{-1}}{2}(1-ax)\big(1-ax^{-1}\big)(1-bx)\big(1-bx^{-1}\big) \\
&= \frac{1-a^{-1}b^{-1}}{2}\frac{\nabla_{k+l}}{\nabla_k}. \qquad \blacksquare
\end{aligned}
$$

Corollary 6.11. *The polynomials* $(\underline{E_n})_{n\in\mathbb{Z}}$ *(i.e., the non-symmetric Askey–Wilson polynomials expressed in* e_1*,* e_{s_1}*) form an orthogonal basis with respect to the matrix weight*

$$
\frac{1}{2}\begin{pmatrix} (1-ab)\nabla_k & 0 \\ 0 & \big(1-a^{-1}b^{-1}\big)\nabla_{k+l} \end{pmatrix}.
$$

Remark 6.12. In fact, what happened in this subsection is by no means surprising, when we consider the contents of [17, Section 5.8]. Let ϵ be the sign character of W_0, then according to [17, Section 5.8.8], every ϵ-symmetric polynomial lies in $\delta_{\epsilon,k}A_0$ for a particular $\delta_{\epsilon,k}\in A$. It just so happens that $\delta_{\epsilon,k}$ is a scalar multiple of what we call e_{s_1} here, so that we can infer from [17, Section 5.8.6] that $(fe_{s_1}, ge_{s_1})_k \sim (f,g)_{k+l}$ for $f,g\in A_0$.

In addition, from [17, Sections 5.8.10f] (and by comparing leading coefficients), we can conclude that $P_{m,k+l}e_{s_1} = P^{(\epsilon)}_{m+1,k}$, so that writing a polynomial as a two-component vector amounts to determining its symmetric and anti-symmetric part (with respect to T_1). If we expand the components of $\underline{E_n}$ in terms of appropriate symmetric Macdonald polynomials, say as

$$
\begin{pmatrix} \displaystyle\sum_{m\in\mathbb{N}_0} a_{n,m}P_{n,k} \\ \displaystyle\sum_{m\in\mathbb{N}_0} b_{n,m}P_{n,k+l} \end{pmatrix},
$$

the $Y+Y^{-1}$-eigenvalues dictate that $a_{n,m}=a_n\delta_{|n|,m}$ and $b_{n,m}=b_n\delta_{|n|,m+1}$. Consequently, (for $n\neq 0$), the two polynomials $\underline{E_n}$ and $\underline{E_{-n}}$ lie in the space spanned by

$$
\begin{pmatrix} P_{n,k} \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ P_{n-1,k+l} \end{pmatrix}.
$$

6.2 Example: A_2 with $J=\{2\}$

This example is the q-version of [23, Section 7] (which is linked to matrix spherical functions of symmetric pairs with restricted root system A_2, see [20]). Here, our affine root system has type A_2, so that $\mathfrak{H}_0$ is generated by T_1, T_2, and the lattice L by ω_1, ω_2, the fundamental weights corresponding to α_1, α_2. There is only one parameter, $\tau = q(k/2)$.

We consider $J=\{2\}$, leading to the parabolic subgroup W_J generated by s_2 and the subalgebra $\mathfrak{H}_J$ generated by T_2. We now present A_J, the s_2-invariant polynomials, as a free A_0-module. The construction from [21] gives the following basis $\tilde{e}_1 = 1$, $\tilde{e}_{s_1} = e(\omega_2-\omega_1)+e(-\omega_2)$, $\tilde{e}_{s_2s_1} = e(-\omega_1)$.

Like in the previous subsection, we attempt to get a decomposition by looking at the action of elements of $\mathfrak{H}_0$ on A_J. Those that let A_J invariant, act as A_0-endomorphisms, and therefore can be expressed as matrices in our existing Steinberg basis.

Lemma 6.13. *Let* $x := T_1T_2 + T_2T_1 - \big(\tau - \tau^{-1}\big)(T_1 + T_2)$*. Then* $x \in Z_{\mathfrak{H}_0}(\mathfrak{H}_J)$ *and* $x^* = x$*.*

Proof. Write $\alpha := \tau - \tau^{-1}$, then we have

$$\begin{aligned} T_2x &= T_1T_2T_1 + T_1 + \alpha T_2T_1 - \alpha(T_2T_1 + 1 + \alpha T_2) = T_1T_2T_1 + T_1 - \alpha^2T_2 - \alpha, \\ xT_2 &= T_1 + \alpha T_1T_2 + T_1T_2T_1 - \alpha(T_1T_2 + 1 + \alpha T_2) = T_1 + T_1T_2T_1 - \alpha^2T_2 - \alpha = T_2x, \end{aligned}$$

which shows that x centralises $\mathfrak{H}_J$, the algebra generated by T_2. For self-adjointness, note that

$$\begin{aligned} x^* &= T_2^{-1}T_1^{-1} + T_1^{-1}T_2^{-1} + \alpha(T_1^* + T_2^*) \\ &= (T_2 - \alpha)(T_1 - \alpha) + (T_1 - \alpha)(T_2 - \alpha) + \alpha(T_1 - \alpha + T_2 - \alpha) \\ &= T_2T_1 + T_1T_2 - 2\alpha T_1 - 2\alpha T_2 + 2\alpha^2 + \alpha T_1 + \alpha T_2 - 2\alpha^2 = x. \end{aligned}$$

■

Lemma 6.14. *In the basis* $\tilde{e}_1$, $\tilde{e}_{s_1}$, $\tilde{e}_{s_2s_1}$*, the element* x *from Lemma 6.13 has the following matrix representation:*

$$\begin{pmatrix} 2 & \big(\tau^2+1\big)m_{\omega_1} & \tau^2 m_{\omega_2} \\ 0 & 1-\tau^2-\tau^{-2} & 0 \\ 0 & 0 & 1-\tau^2-\tau^{-2} \end{pmatrix}.$$

Lemma 6.15. *The family* $e_1 = 1$, $e_{s_1} = h$, $e_{s_2s_1} = h^*$ *with*

$$h = \big(\tau^2 + 1\big)e(\omega_1) - \tau^{-2}\big(e(\omega_2 - \omega_1) + e(-\omega_2)\big)$$

is an A_0*-basis of* A_J *that diagonalises* x*.*

Proof. Note that

$$h = \big(\tau^2 + 1\big)m_{\omega_1}\tilde{e}_1 - \big(\tau^2 + 1 + \tau^{-2}\big)\tilde{e}_{s_1}, \qquad h^* = -\tau^2 m_{\omega_2}\tilde{e}_1 + \big(\tau^2 + 1 + \tau^{-2}\big)\tilde{e}_{s_2s_1},$$

so the claimed new basis arises from $\tilde{e}_1$, $\tilde{e}_{s_1}$, $\tilde{e}_{s_2s_1}$ by a triangular matrix whose diagonal entries 1 and $\pm\big(\tau^2 + 1 + \tau^{-2}\big)$ are units of A_0. Consequently, e_1, e_{s_1}, $e_{s_2s_1}$ is itself an A_0-basis of A_J.

Next, by Lemma 6.14 $\ker(x - 2)$ is the null space of

$$\begin{pmatrix} 0 & \big(\tau^2+1\big)m_{\omega_1} & \tau^2 m_{\omega_2} \\ 0 & -1-\tau^2-\tau^{-2} & 0 \\ 0 & 0 & -1-\tau^2-\tau^{-2} \end{pmatrix}$$

and since it is at most 2-dimensional, is spanned by $\tilde{e}_1 = e_1$. Moreover, the kernel of $x - 1 + \tau^2 + \tau^{-2}$ is the null space of

$$\begin{pmatrix} \tau^2+1+\tau^{-2} & \big(\tau^2+1\big)m_{\omega_1} & \tau^2 m_{\omega_2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which we can readily see to be spanned by h, h^*. ■

Corollary 6.16. *Let* $f, g \in A_0$*, then* $(fe_1, ge_{s_1}) = (fe_1, ge_{s_2s_1}) = 0$*.*

Corollary 6.17. *By Lemmas 6.15 and 6.13, we have* $xfe_1 = fxe_1 = 2fe_1$ *and*

$$xge_{s_1} = gxe_{s_1} = \big(1 - \tau^2 - \tau^{-2}\big)ge_{s_1}$$

and analogously for $e_{s_2s_1}$*. We therefore see that* fe_1, ge_{s_1} *and* fe_1, $ge_{s_2s_1}$ *are pairs of eigenvectors of the self-adjoint operator* x *with distinct eigenvalues. Therefore, they are orthogonal.*

We therefore see that our matrix weight from Lemma 6.1 has the following block shape:

$$\begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

It turns out, there are some relations between these entries that can easily be determined.

Lemma 6.18. *We have*

$$m_{1,1} = \frac{W_0(\tau^2)}{6}, \qquad m_{s_1,s_1} = m_{s_2s_1,s_2s_1}, \qquad m_{s_1,s_2s_1} = \tau^6 m^*_{s_2s_1,s_1}.$$

Proof. (i) The first claim follows from Lemma 2.46 (iii). (ii) This follows from the definition of M, since $e^*_{s_1} = e_{s_2s_1}$. (iii) By definition, we have $\Delta_0^{-1} = \tau^3 \boldsymbol{c}_{-\alpha_1}\boldsymbol{c}_{-\alpha_2}\boldsymbol{c}_{-\alpha_1-\alpha_2}$. Since $\boldsymbol{c}$ is invariant under $*$, we have $\Delta_0^{-*} = \tau^{-6}\Delta_0^{-1}$, and hence

$$m^*_{s_2s_1,s_1} = \frac{1}{6}\sum_{w\in W_0} w\frac{h^2}{\Delta_0^*} = \tau^{-6}\frac{1}{6}\sum_{w\in W_0} w\frac{h^2}{\Delta_0} = \tau^{-6}m_{s_1,s_2s_1}. \qquad \blacksquare$$

This leaves two entries to be determined: m_{s_1,s_1} and m_{s_1,s_1s_2}. Our first observation is that they are both polynomials.

Lemma 6.19. *All $m_{v,v'}$ $\big(v, v' \in W_0^J\big)$ are polynomials.*

Proof. We proceed similarly to the proof of Lemma 6.10: we decompose Δ_0^{-1} in a useful way

$$\Delta_0^{-1} = \tau^3 \prod_{\alpha\in R^+} \boldsymbol{c}_{-\alpha,k} = \tau^3 \prod_{\alpha\in R^+} \frac{\tau^{-1}e(\alpha/2) - \tau e(-\alpha/2)}{e(\alpha/2) - e(-\alpha/2)} =: \frac{F}{\delta},$$

where

$$\begin{aligned} \delta &= \prod_{\alpha\in R^+}(e(\alpha/2) - e(-\alpha/2)) = \sum_{w\in W_0}(-1)^{\ell(w)}e(w\rho), \\ F &= \tau^3\prod_{\alpha\in R^+}\big(\tau^{-1}e(\alpha/2) - \tau e(-\alpha/2)\big) = \sum_{w\in W_0}\big(-\tau^2\big)^{\ell(w)}e(w\rho). \end{aligned}$$

δ is the Weyl denominator and every anti-symmetric polynomial (i.e., polynomial that transforms under the sign representation of W_0) is divisible by it. Consequently, we have

$$m_{v,v'} = \frac{1}{6}\sum_{w\in W_0} w\frac{e_v e^*_{v'}F}{\delta} = \frac{1}{6\delta}\sum_{w\in W_0}(-1)^{\ell(w)}w(e_v e^*_{v'}F).$$

Since $e_v e^*_{v'}F \in A$, its anti-symmetrisation is an anti-symmetric polynomial and hence divisible by δ. As a consequence, $m_{v,v'} \in A_0$. $\blacksquare$

Lemma 6.20. *The matrix weight M we are considering in this subsection cannot be further reduced.*

Proof. Assume that it can. By Corollary 6.4, all M_μ can be made diagonal by the same matrix. In particular $M_{2\omega_1}$. We will show that this is not possible.

For that we first find out, which entries of $M_{2\omega_1}$ are definitely zero by determining the supports of the coefficients $m_{v,v'}$ $(v, v' \in M_0^J)$, i.e., the W_0-orbits of L whose coefficient within $m_{v,v'}$ may be nonzero. We have the following supports:

$$e_{s_1}e^*_{s_1} = hh^*\colon\ W_0\omega_1 + W_0\omega_2 = W_0\rho \sqcup 0,$$

$$\begin{aligned}
e_{s_1}e^*_{s_1}F = Fhh^* &\colon\ W_0\rho + (W_0\rho \sqcup 0) = W_0 2\rho \sqcup W_0 3\omega_1 \sqcup W_0 3\omega_2 \sqcup W_0\rho \sqcup 0,\\
e_{s_1}e^*_{s_2s_1} = h^2 &\colon\ W_0\omega_1 + W_0\omega_1 = W_0 2\omega_1 \sqcup W_0\omega_2,\\
e_{s_1}e^*_{s_2s_1}F = Fh^2 &\colon\ W_0\omega_1 + (W_0 2\omega_1 \sqcup W_0\omega_2)\\
&\qquad = W_0(3\omega_1+\omega_2) \sqcup W_0(\omega_1+2\omega_2) \sqcup W_0 2\omega_1 \sqcup W_0\omega_2
\end{aligned}$$

(where we use the polynomial F from the proof of Lemma 6.19). Since m_{s_1,s_1} and m_{s_1,s_2s_1} are the anti-symmetrisations of the polynomials listed above, divided by δ, we can remove all singular orbits since they vanish under anti-symmetrisation, leaving $W_0 2\rho \sqcup W_0\rho$ in the first case, and $W_0(3\omega_1+\omega_2) \sqcup W_0(\omega_1+2\omega_2)$ in the second case. Dividing such polynomials by δ yields the following supports $m_{s_1,s_1}\colon W_0\rho \sqcup 0$, $m_{s_1,s_2s_1}\colon W_0 2\omega_1 \sqcup W_0\omega_2$. This shows that

$$M = \begin{pmatrix} \frac{W_0(\tau^2)}{6} & 0 & 0\\ 0 & a + bm_\rho & \tau^3(cm_{\omega_2} + dm_{2\omega_1})\\ 0 & \tau^3(c^*m_{\omega_1} + d^*m_{2\omega_2}) & a + bm_\rho \end{pmatrix}$$

for constants $a, b, c, d \in K$. Therefore,

$$M_{2\omega_1} = d\tau^3 \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0 \end{pmatrix},$$

which can only be brought into diagonal form if $d = 0$. To see that $d \neq 0$, note that of the orbits on which Fh^2 is supported, only $W_0(3\omega_1+\omega_2)$ contributes to d, i.e., $d\tau^3$ is the coefficient of $e(3\omega_1+\omega_2)$ in the anti-symmetrisation of Fh^2. If we expand $h^2 = \sum_\mu a_\mu e(\mu)$, we can use the sum expansion of F

$$\frac{1}{6}\sum_{w\in W_0}(-1)^{\ell(w)} w\big(Fh^2\big) = \frac{1}{6}\sum_{w,w'\in W_0,\mu}(-1)^{\ell(w)}\big(-\tau^2\big)^{\ell(w')} a_\mu e(w(\mu + w'\rho)).$$

The only way of obtaining $w(\mu + w'\rho) = 3\omega_1 + \omega_2$ is if $w' = w^{-1}$ and $\mu = w' 2\omega_1$, thus,

$$d\tau^3 = \frac{1}{6}\sum_{w\in W_0}\tau^{2\ell(w)} a_{w^{-1}2\omega_1} = \frac{1}{6}\sum_{w\in W_0}\tau^{2\ell(w)} a_{w2\omega_1} = \frac{\tau^2+1}{6}\sum_{w\in W_0^I}\tau^{2\ell(w)} a_{w2\omega_1}.$$

Now, the only way of obtaining $2\omega_1$ within $W_0\omega_1 + W_0\omega_1$ is as $\omega_1 + \omega_1$, so that

$$a_{2\omega_1} = \big(\tau^2+1\big)^2, \qquad a_{2\omega_2 - 2\omega_1} = a_{-2\omega_2} = \tau^{-4}.$$

Hence,

$$\begin{aligned}
d\tau^3 &= \frac{\tau^2+1}{6}\big(\big(\tau^2+1\big)^2 + \tau^{-4}\big(\tau^2+\tau^4\big)\big) = \frac{1+\tau^{-2}}{6}\big(\tau^6 + 2\tau^4 + 2\tau^2 + 1\big)\\
&= \big(1+\tau^{-2}\big)\frac{W_0\big(\tau^2\big)}{6} \neq 0.
\end{aligned}$$

■

Corollary 6.21. *If we use the notation from Lemma 6.1, the polynomials* $(P_{J,\lambda})_{\lambda\in L_{+,J}}$ *are an orthogonal basis of* A_0^3 *with respect to the inner product with matrix weight* $\overline{\nabla M}$*. Here,* M *is maximally reduced, so we have also found an irreducible* 2×2 *matrix weight on* A_0*.*

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