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Linear Independence for $A_1^{(1)}$ by Using $C_2^{(1)}$

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Abstract. In the previous paper, the authors proved linear independence of the combinatorial spanning set for standard $C_\ell^{(1)}$-module $L(k\Lambda_0)$ by establishing a connection with the combinatorial basis of Feigin–Stoyanovsky's type subspace $W(k\Lambda_0)$ of $C_{2\ell}^{(1)}$-module $L(k\Lambda_0)$. In this note we extend this argument for $C_1^{(1)} \cong A_1^{(1)}$ to all standard $A_1^{(1)}$-modules $L(\Lambda)$. In the proof we use a coefficient of an intertwining operator of the type $\binom{L(\Lambda_2)}{L(\Lambda_1)\ L(\Lambda_1)}$ for standard $C_2^{(1)}$-modules.

Key words: affine Lie algebras; standard modules; Feigin–Stoyanovsky's type subspace; combinatorial basis

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1 Introduction

In [7], the authors proved linear independence of the combinatorial spanning set for standard $C_\ell^{(1)}$-module $L(k\Lambda_0)$ by establishing a connection with the combinatorial basis of Feigin–Stoyanovsky's type subspace $W(k\Lambda_0)$ of $C_{2\ell}^{(1)}$-module $L(k\Lambda_0)$ constructed in [1]. For $\ell = 1$ we have $C_1^{(1)} \cong A_1^{(1)}$ and the combinatorial basis of $L_{A_1^{(1)}}(k\Lambda_0)$ is a part of the general construction of combinatorial bases of all standard $A_1^{(1)}$-modules $L_{A_1^{(1)}}(k_0\Lambda_0 + k_1\Lambda_1)$, $k_0 + k_1 = k$, obtained independently in [5] and [3]. In this note we extend the argument from [7] to all standard $A_1^{(1)}$-modules by using a coefficient of an intertwining operator of the type $\binom{L(\Lambda_2)}{L(\Lambda_1)\ L(\Lambda_1)}$ for standard $C_2^{(1)}$-modules. This gives a new proof of linear independence of combinatorial bases of standard $A_1^{(1)}$-modules and, hopefully, this approach may lead to a proof of linear independence of combinatorial bases of all standard $C_\ell^{(1)}$-modules conjectured in [2].

As in [7], the key idea for the proof of linear independence of the spanning set B_1 of monomial vectors $x(\pi)v_{\bar\Lambda}$ in $L_{A_1^{(1)}}(\bar\Lambda)$ is to embed Lie algebra $\mathfrak{l}$ of type A_1 into $\mathfrak{g}$ of type C_2 (see Figure 1) together with its standard module

$$L_{A_1^{(1)}}(\bar\Lambda) \subset L_{C_2^{(1)}}(\Lambda) \supset W_{C_2^{(1)}}(\Lambda),$$

and then, by using the inner derivations T of $\mathfrak{g}$, connect the set B_1 and the basis B_2^1 of the Feigin–Stoyanovsky subspace $W_{C_2^{(1)}}(\Lambda)$ consisting of monomial vectors $\overline{x}(\pi)v_\Lambda$.

Monomials $x(\pi)$ and $\overline{x}(\pi)$ in the universal enveloping algebra $U(\hat{\mathfrak{g}})$, given by (2.2) and (2.8), are parameterized with colored partitions π in three colors, determined by frequencies $\{a_j, b_j, c_j \mid j \geq 0\}$ satisfying *the same* difference conditions (2.3)–(2.5). The case when $\bar\Lambda = k\Lambda_0$ and $\Lambda = k\Lambda_0$ is relatively simple because π in monomials $x(\pi)$ and $\overline{x}(\pi)$ satisfy the *the same* initial conditions $a_0 = b_0 = c_0 = 0$ and for the proper power $T^{N'}$ of T we have

$$T^{N'}\colon\ x(\pi) \to \overline{x}(\pi).$$

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However, in general the initial conditions for π in monomials $x(\pi)$ and $\overline{x}(\pi)$ *are not* the same, see (2.6)–(2.7) compared to (2.9)–(2.10), and, together with T, a coefficient w of an intertwining operator of the type $\binom{L(\Lambda_2)}{L(\Lambda_1)\ L(\Lambda_1)}$ for standard $C_2^{(1)}$-modules is used to circumvent this difficulty.

2 Affine Lie algebras of type $A_1^{(1)} \subset C_2^{(1)}$ and their standard modules

2.1 Affine Lie algebras and standard modules

Let $\mathfrak{g}$ be a simple Lie algebra with a Cartan decomposition $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha\in R} \mathfrak{g}_\alpha$, where R is a root system of $\mathfrak{g}$. Let $\alpha_1, \dots, \alpha_n$ be a basis of R, θ the maximal root, and $\omega_1, \dots, \omega_n$ the corresponding fundamental weights. For each root α fix a root vector x_α in $\mathfrak{g}_\alpha$. Let $\langle\ ,\ \rangle$ be the normalized Killing form so that $\langle\theta, \theta\rangle = 2$, through which we identify $\mathfrak{h}$ and $\mathfrak{h}^*$. Let B be a basis of $\mathfrak{g}$ consisting of root vectors and elements of $\mathfrak{h}$, with a linear order $\succ$.

Let $\tilde{\mathfrak{g}}$ be the affine Lie algebra associated to $\mathfrak{g}$,

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}\left[t, t^{-1}\right] + \mathbb{C}c, \qquad \tilde{\mathfrak{g}} = \hat{\mathfrak{g}} + \mathbb{C}d,$$

with commutation relations

$$[x(i), y(j)] = [x, y](i + j) + i\langle x, y\rangle\delta_{i+j,0}c, \qquad [c, \tilde{\mathfrak{g}}] = 0, \qquad [d, x(j)] = jx(j),$$

where $x(n) = x\otimes t^n$, for $x \in \mathfrak{g}$ and $n \in \mathbb{Z}$. Identify $\mathfrak{g} = \mathfrak{g}\otimes 1 \subset \hat{\mathfrak{g}}$. Set $\overline{B} = \{b(n) \mid b \in B,\ n \in \mathbb{Z}\}$; so that $\overline{B} \cup \{c\}$ is a basis of $\hat{\mathfrak{g}}$. Extend the order on B to $\overline{B}$: $b(n) \succ b'(n')$ if $n > n'$ or $n = n'$, $b \succ b'$.

Denote by $\Lambda_0, \dots, \Lambda_n$ the fundamental weights of $\tilde{\mathfrak{g}}$. For a given $\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + \cdots + k_n\Lambda_n$, let $L(\Lambda) = L_{\tilde{\mathfrak{g}}}(\Lambda)$ be a standard (i.e., integrable highest weight) $\tilde{\mathfrak{g}}$-module, v_Λ a fixed highest weight vector, and $k = \Lambda(c)$ the level of $L(\Lambda)$ (cf. [4]).

2.2 Bases of standard modules for affine Lie algebra of type $A_1^{(1)} \subset C_2^{(1)}$

Let $\mathfrak{g}$ be a simple Lie algebra of the type C_2. Let

$$R = \{2\epsilon_1, \epsilon_1 + \epsilon_2, 2\epsilon_2, \epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_1, -2\epsilon_2, -\epsilon_1 - \epsilon_2, -2\epsilon_1\} \subset \mathbb{R}^2 \tag{2.1}$$

be a root system of $\mathfrak{g}$. Let $\alpha_1 = \epsilon_1 - \epsilon_2$, $\alpha_2 = 2\epsilon_2$ be a root basis, $\theta = 2\epsilon_1$ the maximal root, and $\omega_1 = \epsilon_1$, $\omega_2 = \epsilon_1 + \epsilon_2$ the corresponding fundamental weights.

Fix root vectors x_{11}, x_{12}, x_{22}, $x_{1\underline{2}}$, $x_{2\underline{1}}$, $x_{\underline{2}\underline{2}}$, $x_{\underline{2}\underline{1}}$, $x_{\underline{1}\underline{1}}$ corresponding respectively to the roots in (2.1) and let $x_{1\underline{1}}$, $x_{2\underline{2}}$ be the simple coroots in $\mathfrak{h}$ corresponding to positive roots $2\epsilon_1$ and $2\epsilon_2$. These vectors form a weight basis B of $\mathfrak{g}$. Define an order on B in the following way: set $1 \succ 2 \succ \underline{2} \succ \underline{1}$ and define a lexicographic order $x_{ab} \succ x_{a'b'}$ if $a \succ a'$ or $a = a'$, $b \succ b'$.

Denote by Λ_0, Λ_1, Λ_2 the fundamental weights of $\tilde{\mathfrak{g}}$. For $\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2$, let $L(\Lambda) = L_{C_2^{(1)}}(\Lambda)$ be a standard $\tilde{\mathfrak{g}}$-module of the level $k = k_0 + k_1 + k_2$ with a fixed highest weight vector v_Λ. The standard module $L(\Lambda)$ can be realised in the tensor product of level 1 standard modules, $L(\Lambda) \subset L(\Lambda_0)^{\otimes k_0} \otimes L(\Lambda_1)^{\otimes k_1} \otimes L(\Lambda_2)^{\otimes k_2}$.

On the top of the standard module $L(\Lambda_0)$ there is the trivial 1-dimensional module for $\mathfrak{g}$; denote by v_0 its weight vector. On the top of $L(\Lambda_1)$ there is the 4-dimensional irreducible module $V(\omega_1)$ for $\mathfrak{g}$ with weights ϵ_1, ϵ_2, $-\epsilon_1$, $-\epsilon_2$ and corresponding weight vectors v_1, v_2, $v_{\underline{1}}$, $v_{\underline{2}}$. On the top of $L(\Lambda_2)$ there is the 5-dimensional irreducible module $V(\omega_2)$ for $\mathfrak{g}$ with weights $\epsilon_1 + \epsilon_2$, $-\epsilon_1 + \epsilon_2$, 0, $\epsilon_1 - \epsilon_2$, $-\epsilon_1 - \epsilon_2$ and corresponding weight vectors v_{12}, $v_{\underline{1}2}$, v_{00}, $v_{1\underline{2}}$, $v_{\underline{1}\underline{2}}$. Note that $v_0 = v_{\Lambda_0}$, $v_1 = v_{\Lambda_1}$, $v_{12} = v_{\Lambda_2}$.

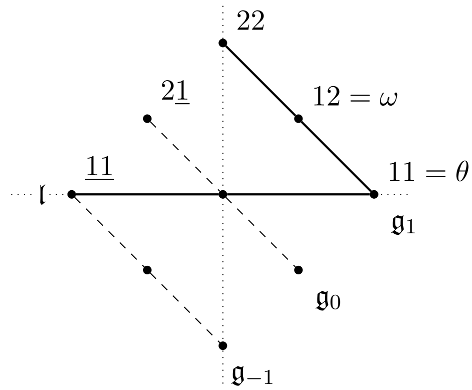

Figure 1. The root system of type C_2.

The subalgebra $\mathfrak{l} = \operatorname{span}\{x_{11}, x_{1\underline{1}}, x_{\underline{11}}\} \subset \mathfrak{g}$ is a simple algebra of type A_1 with the simple root $\theta = 2\epsilon_1$.

The inclusion $\mathfrak{l} \subset \mathfrak{g}$ induces an inclusion of affine Lie algebras $\tilde{\mathfrak{l}} \subset \tilde{\mathfrak{g}}$; the subalgebra $\tilde{\mathfrak{l}}$ is of type $A_1^{(1)}$. Denote by $\bar{\Lambda}_0$ and $\bar{\Lambda}_1$ fundamental weights of $\tilde{\mathfrak{l}}$. Standard $\tilde{\mathfrak{l}}$-modules can be found as $\tilde{\mathfrak{l}}$-submodules of standard $\tilde{\mathfrak{g}}$-modules

$$L_{A_1^{(1)}}(\bar{\Lambda}_0) \cong U(\tilde{\mathfrak{l}})v_{\Lambda_0} \subset L_{C_2^{(1)}}(\Lambda_0),$$
$$L_{A_1^{(1)}}(\bar{\Lambda}_1) \cong U(\tilde{\mathfrak{l}})v_{\Lambda_1} \subset L_{C_2^{(1)}}(\Lambda_1) \cong U(\tilde{\mathfrak{l}})v_{\Lambda_2} \subset L_{C_2^{(1)}}(\Lambda_2),$$
$$L_{A_1^{(1)}}(\bar{\Lambda}) \cong U(\tilde{\mathfrak{l}})v_{\Lambda} \subset L_{C_2^{(1)}}(\Lambda),$$

for $\bar{\Lambda} = \bar{k}_0\bar{\Lambda}_0 + \bar{k}_1\bar{\Lambda}_1$, $\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2$, where $\bar{k}_0 = k_0$, $\bar{k}_1 = k_1 + k_2$. For this reason we will use the notation Λ_0, Λ_1, Λ_2 also for $\tilde{\mathfrak{l}}$.

The PBW spanning set of $L_{A_1^{(1)}}(\Lambda)$, $\Lambda = k_0\Lambda_0 + k_1\Lambda_1$, can be reduced to a monomial basis consisting of monomial vectors

$$\prod_{j\geq 0} x_{\underline{11}}(-j)^{c_j} x_{1\underline{1}}(-j)^{b_j} x_{11}(-j)^{a_j} v_\Lambda \tag{2.2}$$

satisfying *difference conditions*

$$a_i + b_i + a_{i+1} \leq k, \qquad c_i + b_i + a_{i+1} \leq k, \tag{2.3}$$
$$c_i + b_{i+1} + a_{i+1} \leq k, \tag{2.4}$$
$$c_i + b_{i+1} + c_{i+1} \leq k \tag{2.5}$$

and *initial conditions*

$$a_0 = b_0 = 0, \qquad c_0 \leq k_1, \tag{2.6}$$
$$a_1 \leq k_0 \tag{2.7}$$

for $L_{A_1^{(1)}}(\Lambda)$. This was proved by different methods in [5] and [3]. In this note we give a new proof of linear independence by transforming its elements to elements of a monomial basis of a Feigin–Stoyanovsky subspace for $\tilde{\mathfrak{g}}$.

2.3 Feigin–Stoyanovsky subspace

For a simple Lie algebra $\mathfrak{g}$, let ω be a minuscule coweight of $\mathfrak{g}$, $\langle\omega, R\rangle = \{-1, 0, 1\}$. The minuscule coweight ω induces a $\mathbb{Z}$-gradation on $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1, \qquad \mathfrak{g}_0 = \mathfrak{h} + \sum_{\langle\omega,\alpha\rangle=0} \mathfrak{g}_\alpha, \qquad \mathfrak{g}_{\pm 1} = \sum_{\langle\omega,\alpha\rangle=\pm 1} \mathfrak{g}_\alpha.$$

The set $\Gamma = \{\alpha \in R \mid \langle \omega, \alpha \rangle = 1\}$ is called the set of colors. The subalgebras $\mathfrak{g}_{\pm 1} \subset \mathfrak{g}$ are commutative, while $\mathfrak{g}_0$ is reductive.

The $\mathbb{Z}$-gradation of $\mathfrak{g}$ induces a $\mathbb{Z}$-gradation of $\tilde{\mathfrak{g}}$, $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_{-1} + \tilde{\mathfrak{g}}_0 + \tilde{\mathfrak{g}}_1$, where

$$\tilde{\mathfrak{g}}_0 = \mathfrak{g}_0 \otimes \mathbb{C}\big[t, t^{-1}\big] + \mathbb{C}c + \mathbb{C}d, \qquad \tilde{\mathfrak{g}}_{\pm 1} = \mathfrak{g}_{\pm 1} \otimes \mathbb{C}\big[t, t^{-1}\big].$$

Again, the subalgebras $\tilde{\mathfrak{g}}_{\pm 1} \subset \tilde{\mathfrak{g}}$ are commutative.

Feigin–Stoyanovsky subspace $W(\Lambda) = W_{\tilde{\mathfrak{g}}}(\Lambda)$ of a standard module $L_{\tilde{\mathfrak{g}}}(\Lambda)$ is a $\tilde{\mathfrak{g}}_1$-submodule generated by the highest weight vector $W(\Lambda) = U(\tilde{\mathfrak{g}}_1)v_\Lambda$.

2.4 Bases of Feigin–Stoyanovsky subspaces for affine Lie algebra of type $C_2^{(1)}$

Let $\mathfrak{g}$ be a simple Lie algebra of type C_2, as before. The minuscule coweight $\omega = \omega_2$ induces a $\mathbb{Z}$-gradation on $\mathfrak{g}$ with the set of colors $\Gamma = \{2\epsilon_1, \epsilon_1 + \epsilon_2, 2\epsilon_2\}$ (see Figure 1).

The Feigin–Stoyanovsky subspace $W_{C_2^{(1)}}(\Lambda)$ for $\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2$, $k = k_0 + k_1 + k_2$, has a monomial basis

$$\prod_{j \geq 0} x_{22}(-j)^{c_j} x_{12}(-j)^{b_j} x_{11}(-j)^{a_j} v_\Lambda \tag{2.8}$$

satisfying *difference conditions* (2.3)–(2.5) and *initial conditions*

$$a_0 = b_0 = c_0 = 0, \qquad a_1 \leq k_0, \qquad a_1 + b_1 \leq k - k_2, \tag{2.9}$$

$$b_1 + c_1 \leq k - k_2 \tag{2.10}$$

for $W_{C_2^{(1)}}(\Lambda)$ (cf. [1] and [6][1]).

3 Proof of linear independence for $A_1^{(1)}$

3.1 Translation $L_{A_1^{(1)}}(\Lambda) \to W_{C_2^{(1)}}(\Lambda')$

Let $\Lambda = k_0\Lambda_0 + k_1\Lambda_1$. Let

$$\pi = \prod_{j \geq 0} (-j)^{c_j} (-j)^{b_j} (-j)^{a_j}$$

be a colored partition with parts $-j$ of colors a, b, c appearing with frequencies a_j, b_j, c_j, which satisfy difference conditions (2.3)–(2.5). Denote by $x(\pi)$ and $\overline{x}(\pi)$ monomials (2.2) and (2.8), respectively. Note that $x(\pi)$ is noncommutative and $\overline{x}(\pi)$ is commutative.

Denote by $T = \mathrm{ad} x_{12}$ a derivative on $\hat{\mathfrak{g}}$ and $U(\hat{\mathfrak{g}})$. Then, up to a scalar, $Tx_{1\underline{1}}(j) = x_{12}(j)$, $Tx_{\underline{1}\underline{1}}(j) = x_{2\underline{1}}(j)$, $Tx_{2\underline{1}}(j) = x_{22}(j)$. Note that $Tx_{11}(j) = Tx_{12}(j) = 0$.

For a monomial $x(\pi)$ satisfying difference and initial conditions for $L_{A_1^{(1)}}(\Lambda)$ set

$$N' = \sum_{j \geq 0} b_j + 2 \sum_{j \geq 0} c_j.$$

The action by $T^{N'}$ transforms $x(\pi)$ to $\overline{x}(\pi)$: $T^{N'}x(\pi) = \overline{x}(\pi)$. Furthermore, $T^{N'+1}x(\pi) = 0$.

Let $v_\Lambda = v_0^{\otimes k_0} \otimes v_1^{\otimes k_1}$ be a highest weight vector of $L_{C_2^{(1)}}(\Lambda)$. Then

$$\begin{aligned} x_{12}(0)^{N'} x(\pi) v_\Lambda &= x_{12}(0)^{N'-1} (T x(\pi)) v_\Lambda + x_{12}(0)^{N'-1} x(\pi) x_{12}(0) v_\Lambda \\ &= \cdots = \big(T^{N'} x(\pi)\big) v_\Lambda = \overline{x}(\pi) v_\Lambda. \end{aligned}$$

[1] When interchanging $C_2^{(1)} \leftrightarrow B_2^{(1)}$ we should interchange $\Lambda_1 \leftrightarrow \Lambda_2$.

Note that $x_{12}(0)$ annihilates v_Λ since the corresponding root is positive. Furthermore, initial conditions for $W_{C_2^{(1)}}(\Lambda)$ imply that $\overline{x}(\pi)v_\Lambda \neq 0 \Leftrightarrow c_0 = 0$.

Hence for a monomial $x(\pi)$ set

$$N = N(\pi) = \sum_{j\geq 0} b_j + 2\sum_{j\geq 0} c_j - c_0.$$

Let $x(\pi) = x(\pi_1)x_{\underline{1}1}(0)^{c_0}$. Note that $N(\pi) = N(\pi_1) + c_0$. Then

$$\begin{aligned} x_{12}(0)^N x(\pi)v_\Lambda &= \big(T^N x(\pi)\big)v_\Lambda = \big(T^N\big(x(\pi_1)x_{\underline{1}1}(0)^{c_0}\big)\big)v_\Lambda \\ &= \overline{x}(\pi_1)x_{2\underline{1}}(0)^{c_0}v_\Lambda + \sum \cdots x_{22}(0)v_\Lambda \qquad (3.1)\\ &= \overline{x}(\pi_1)x_{2\underline{1}}(0)^{c_0}v_\Lambda. \qquad (3.2)\end{aligned}$$

In (3.1) the sum goes over all the other possibilities of action of T^N on factors in $x(\pi)$. In all of these at least two T's act on the same $x_{\underline{1}1}(0)$ factor. Hence one gets at least one $x_{22}(0)$ factor, which commutes with $x_{\underline{1}1}(0)$ and $x_{2\underline{1}}(0)$, and annihilates v_Λ.

Notice that $x_{2\underline{1}}(0)$ acts on the v_1's in the tensor product $v_0^{\otimes k_0} \otimes v_1^{\otimes k_1}$. Since $c_0 \leq k_1$, then $x_{2\underline{1}}(0)^{c_0}v_\Lambda \neq 0$. Moreover, $x_{2\underline{1}}(0)v_1 = v_2$ and $x_{2\underline{1}}(0)v_2 = 0$.

Hence,

$$x_{12}(0)^N x(\pi)v_\Lambda = \overline{x}(\pi_1)x_{2\underline{1}}(0)^{c_0}v_\Lambda. \qquad (3.3)$$

Furthermore, $x_{12}(0)^{N+1}x(\pi)v_\Lambda = 0$.

Different possibilities of distribution of $x_{2\underline{1}}(0)$'s on the tensor product $v_0^{\otimes k_0} \otimes v_1^{\otimes k_1}$ in (3.2) will be handled by certain coefficients of intertwining operators.

3.2 Intertwining operators

For $L_{C_2^{(1)}}(\Lambda_1)$ there is an operator (a coefficient of an intertwining operator) $w\colon L_{C_2^{(1)}}(\Lambda_1) \to L_{C_2^{(1)}}(\Lambda_2)$ such that $v_1 \xrightarrow{w} 0$, $v_2 \xrightarrow{w} v_{12}$, where v_{12} is a highest weight vector of $L_{C_2^{(1)}}(\Lambda_2)$, and w commutes with the action of $\tilde{\mathfrak{g}}_1$ (see [1, Proposition 7] or [6, Remark 6.3]).

On $L(\Lambda)$ use tensor products of these operators

$$w_{k_1,s} = \underbrace{1\otimes\cdots\otimes 1}_{k_0} \otimes \underbrace{1\otimes\cdots\otimes 1}_{k_1-s} \otimes \underbrace{w\otimes\cdots\otimes w}_{s},$$

where $s \leq k_1$

Act on (3.2) by w_{k_1,c_0}

$$\begin{aligned} &w_{k_1,c_0}\overline{x}(\pi_1)x_{2\underline{1}}(0)^{c_0}v_\Lambda \\ &\quad= \overline{x}(\pi_1)w_{k_1,c_0}x_{2\underline{1}}(0)^{c_0}v_\Lambda \\ &\quad= \overline{x}(\pi_1)w_{k_1,c_0}v_0^{\otimes k_0}\otimes v_1^{\otimes(k_1-c_0)}\otimes v_2^{\otimes c_0} + \sum \overline{x}(\pi_1)w_{k_1,c_0}v_0^{\otimes k_0}\otimes\cdots \qquad (3.4)\\ &\quad= \overline{x}(\pi_1)\underbrace{v_0^{\otimes k_0}\otimes v_1^{\otimes(k_1-c_0)}\otimes v_{12}^{\otimes c_0}}_{v_{\Lambda'}},\end{aligned}$$

where $\Lambda' = k_0\Lambda_0 + (k_1 - c_0)\Lambda_1 + c_0\Lambda_2$. In (3.4), the sum goes over all other distributions of $x_{2\underline{1}}(0)$'s on tensor factors. These have at least one v_1 among the last c_0 tensor factors and hence are annihilated by w_{k_1,c_0}.

Since $x(\pi)$ satisfies initial conditions (2.6)–(2.7) for $L_{A_1^{(1)}}(\Lambda)$, then, by (2.4) and (2.5) for $i = 0$, $\overline{x}(\pi_1)$ satisfies initial conditions (2.9)–(2.10) for $W_{C_2^{(1)}}(\Lambda')$, with $k_0' = k_0$, $k_1' = k_1 - c_0$, $k_2' = c_0$.

Hence $\overline{x}(\pi_1)x_{2\underline{1}}(0)^{c_0}v_\Lambda \xrightarrow{w_{k_1,c_0}} \overline{x}(\pi_1)v_{\Lambda'}$ and $\overline{x}(\pi_1^2)$ satisfies initial conditions for $W_{C_2^{(1)}}(\Lambda')$. Note also that $\overline{x}(\pi_1)x_{2\underline{1}}(0)^{c_0}v_\Lambda \xrightarrow{w_{k_1,s}} 0$, for $c_0 < s \leq k_1$.

3.3 Proof of linear independence

Let

$$\sum_{\pi} C_\pi x(\pi) v_\Lambda = 0, \tag{3.5}$$

be a relation of linear dependence, where all monomials in (3.5) satisfy difference and initial conditions for $L_{A_1^{(1)}}(\Lambda)$. Let $N = \max_\pi N(\pi)$. Proceed inductively on N; act on (3.5) by $x_{12}(0)^N$

$$\begin{aligned} 0 &= \sum_{N(\pi)<N} C_\pi x_{12}(0)^N x(\pi) v_\Lambda + \sum_{N(\pi)=N} C_\pi x_{12}(0)^N x(\pi) v_\Lambda = \sum_{N(\pi)=N} C_\pi x_{12}(0)^N x(\pi) v_\Lambda \\ &= \sum_{\substack{N(\pi)=N \\ c_0=0}} C_\pi \overline{x}(\pi_1) v_\Lambda + \sum_{\substack{N(\pi)=N \\ c_0=1}} C_\pi \overline{x}(\pi) x_{2\underline{1}}(0) v_\Lambda + \cdots + \sum_{\substack{N(\pi)=N \\ c_0=s}} C_\pi \overline{x}(\pi_1) x_{2\underline{1}}(0)^s v_\Lambda, \end{aligned} \tag{3.6}$$

for some $s \le k_1$. The second equality follows since $x_{12}(0)^N x(\pi) = 0$ if $N(\pi) < N$, while the third follows from (3.3). Proceed inductively on s; act on (3.6) by $w_{k_1,s}$. Then all sums except the last one are annihilated. One gets

$$\sum_{\substack{N(\pi)=N \\ c_0=s}} C_\pi \overline{x}(\pi_1) v_{\Lambda'} = 0, \tag{3.7}$$

for $\Lambda' = k_0\Lambda_0 + (k_1 - s)\Lambda_1 + s\Lambda_2$. This is a relation of linear dependence in $W_{C_2^{(1)}}(\Lambda')$. Note that all monomials in (3.7) satisfy difference and initial conditions for $W_{C_2^{(1)}}(\Lambda')$. Since these are linearly independent, all C_π in (3.7) are zero.

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