Symmetry, Integrability and Geometry: Methods and Applications SIGMA **21** (2025), 068, 25 pages

From Toda Hierarchy to KP Hierarchy

Di YANG [a] *and Jian ZHOU* [b]

a) *School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026, P.R. China*
E-mail: *diyang@ustc.edu.cn*

b) *Department of Mathematical Sciences, Tsinghua University, Beijing 100084, P.R. China*
E-mail: *jianzhou@mail.tsinghua.edu.cn*

Received October 08, 2024, in final form July 27, 2025; Published online August 09, 2025
https://doi.org/10.3842/SIGMA.2025.068

Abstract. Using the matrix-resolvent method and a formula of the second-named author on the n-point function for a KP tau-function, we show that the tau-function of an arbitrary solution to the Toda lattice hierarchy is a KP tau-function. We then generalize this result to tau-functions for the extended Toda hierarchy (ETH) by developing the matrix-resolvent method for the ETH. As an example the partition function of Gromov–Witten invariants of the complex projective line is a KP tau-function, and an application on irreducible representations of the symmetric group is obtained.

Key words: Toda hierarchy; KP hierarchy; matrix-resolvent method; complex projective line; Gromov–Witten invariant

2020 Mathematics Subject Classification: 37K10; 05E05; 14N35; 53D45; 05E10

1 Introduction

The Kadomtsev–Petviashvili (KP) hierarchy and the Toda lattice hierarchy are two important integrable hierarchies. In this paper, we will show that the tau-function of an arbitrary solution to the Toda lattice hierarchy gives an infinite family of tau-functions of the KP hierarchy. This is achieved by combining two different results: a formula [50] of the second-named author on the n-point function for a KP tau-function in the big cell, and a work [44] of the first-named author on the n-point function for a Toda tau-function.

The KP hierarchy is an infinite family of equations with infinitely many unknown functions, which can be written using the Lax pair formalism as follows:

$$\frac{\partial L_{\mathrm{KP}}}{\partial T_k} = \big[\big(L_{\mathrm{KP}}^k\big)_+, L_{\mathrm{KP}}\big], \qquad k \geq 1, \tag{1.1}$$

where

$$L_{\mathrm{KP}} = \partial + \sum_{j\geq 1} u_j \partial^{-j}, \qquad \partial = \partial/\partial X,$$

is a pseudodifferential operator, called the *Lax operator*. For details about pseudodifferential operators and their operations see for example [12]. The independent variables $T_1, T_2, T_3, \dots$ are called *times*. Since $\partial L_{\mathrm{KP}}/\partial T_1 = \partial L_{\mathrm{KP}}/\partial X$, we identify T_1 with X.

Let $\mathbf{T} = (T_1, T_2, \dots)$ denote the infinite vector of KP times. It is well known (see, e.g., [12]) that an arbitrary solution $(u_1(\mathbf{T}), u_2(\mathbf{T}), \dots)$ to the KP hierarchy (1.1) can be compactly represented by a single function $\tau = \tau(\mathbf{T})$ called the *tau-function* as

$$u_1(\mathbf{T}) = \frac{\partial^2 \log \tau}{\partial T_1^2}, \qquad u_2(\mathbf{T}) = \frac{1}{2}\frac{\partial^2 \log \tau}{\partial T_1 \partial T_2} - \frac{1}{2}\frac{\partial^3 \log \tau}{\partial T_1^3}, \qquad \dots,$$

and the tau-function τ satisfies the *Hirota bilinear identities* given by

$$\mathrm{res}_{\lambda=\infty}\tau\big(\mathbf{T}-\big[\lambda^{-1}\big]\big)\tau\big(\mathbf{T}'+\big[\lambda^{-1}\big]\big)\mathrm{e}^{\sum_{k\geq 1}(T_k-T_k')\lambda^k}\mathrm{d}\lambda=0. \tag{1.2}$$

Here $\big[\lambda^{-1}\big]:=\big(\frac{1}{\lambda},\frac{1}{2\lambda^2},\frac{1}{3\lambda^3},\dots\big)$. Equivalence between (1.2) and (1.1) is a standard result in the theory of integrable systems; see, for example, [12]. Equation (1.2) itself gives the defining equations for a KP tau-function (namely, we do not need to start with a solution $(u_1(\mathbf{T}),u_2(\mathbf{T}),\dots)$ to the KP hierarchy (1.1)).

Let $\Lambda\colon f(x)\mapsto f(x+\epsilon)$ be the shift operator and

$$L:=\Lambda+v(x)+w(x)\Lambda^{-1}$$

the *Lax operator*. Here ϵ is a parameter. The *Toda lattice hierarchy* (also known as the 1d Toda chain or 1-Toda hierarchy) can be defined using the Lax pair formalism as

$$\epsilon\frac{\partial L}{\partial m_i}=\frac{1}{(i+1)!}\big[\big(L^{i+1}\big)_+,L\big],\qquad i\geq 0. \tag{1.3}$$

Here, for a difference operator P written in the form $P=\sum_{k\in\mathbb{Z}}P_k\Lambda^k$, P_+ is defined as $\sum_{k\geq 0}P_k\Lambda^k$. We also denote $\mathbf{m}=(m_0,m_1,m_2,\dots)$.

It is known [43] that an arbitrary tau-function $\tau(N,\mathbf{x},\mathbf{y})$ of the 2-Toda hierarchy [43] is a KP tau-function with respect to either $\mathbf{x}$ or $\mathbf{y}$ as KP times (fixing the others as parameters). The following Theorem 1.1, which could essentially be deduced from [43] (see Remark 1.2 below), gives a similar statement for the 1-Toda hierarchy. Let $(v(x,\mathbf{m};\epsilon),w(x,\mathbf{m};\epsilon))$ be an arbitrary power-series-in-$\mathbf{m}$ solution to the Toda lattice hierarchy with coefficients being in a ring V of functions of x closed under shifting x by $\pm\epsilon$ (e.g., V could be $\mathbb{C}[x,\epsilon]$), and $\tau(x,\mathbf{m};\epsilon)$ the tau-function [18] of this solution.

Theorem 1.1. *The tau-function $\tau(x,\mathbf{m};\epsilon)$ for the Toda lattice hierarchy is a tau-function of the KP hierarchy for any x and ϵ, where $\mathbf{m}$ and the KP times $\mathbf{T}$ are related by*

$$m_i=(i+1)!\,\epsilon\,T_{i+1},\qquad i\geq 0. \tag{1.4}$$

Remark 1.2. Theorem 1.1 should be known to experts, and could be deduced from a claim [43, p. 30] (cf. [31]) that the Lax representation of the Toda lattice hierarchy can be obtained from a reduction of the 2-Toda hierarchy with the reduction condition

$$\partial/\partial x_i=\partial/\partial y_i,\qquad i\geq 1,$$

up to a quadratic function of $\mathbf{x}$, $\mathbf{y}$. Here, $\mathbf{x}=(x_1,x_2,\dots)$, $\mathbf{y}=(y_1,y_2,\dots)$. (Our formula (3.2) below should shed light on the Toda chain discussions in [31].) We would like to thank A. Alexandrov for bringing our attention to the reduction arguments of [43]. We also point out the following well-known fact: the first nontrivial Hirota bilinear equation of the 2-Toda hierarchy under the reduction condition coincides with that of the 1-Toda hierarchy. Thus, an alternative way to show the equivalence between the 1-Toda hierarchy and the reduction of the 2-Toda hierarchy is to compare the Hirota bilinear equations of the two hierarchies [35, 43], whose detail might deserve a further study. In this work we take a different approach. For the KP hierarchy, a formula for the connected n-point functions was derived by the second-named author using the fermionic approach in [50], whereas for the Toda lattice hierarchy a formula of the same form was derived by the second-named author [44], but using a totally different method: the matrix-resolvent method. So we present a different proof combining these two approaches. An advantage of this proof is that it can be generalized to the extended Toda hierarchy [9] whose fermionic approach has not yet appeared. See our Theorem 1.3 below. The main content of Section 4 is to generalize the matrix-resolvent method to the case of the extended Toda hierarchy.

Let us now recall some earlier results in the literature that motivate us to obtain this result. It is well known in matrix model theory [1, 25] that using the theory of orthogonal polynomials, the GUE partition function $\{Z_N(\mathbf{T};\epsilon)\}_{N\geq 1}$ is a tau-function for the Toda lattice hierarchy. Here N is the size of the Hermitian matrix used to define the partition function, and $x = N\epsilon$. A perhaps less well-known result is that for each N, $Z_N(\mathbf{T};\epsilon)$ is also a tau-function of the KP hierarchy [41] (see also [31]). Both of these results have been revisited recently and some new perspectives naturally arise.

First of all, in [18, Definition 1.2.2], a notion of matrix resolvent for the Toda lattice hierarchy was introduced. Based on this, a definition of the tau-function of Dubrovin–Zhang type of the Toda lattice hierarchy was given in [18, Definition 1.2.4]. Usually a normalization constant is chosen so that the partition function of Gaussian unitary ensemble (GUE) is equal to 1 when all the coupling constants are set to be equal to zero. In [18, Appendix A], it was shown that after multiplying by a suitable correction factor, the GUE partition functions give us a tau-function of Dubrovin–Zhang type. In [18], it was also shown that the correction factor can be obtained by using the theory of Toda tau-functions, so this method is applicable to other examples.

Note the major goal of [18] is to develop a method for computing n-point function associated with the tau-function of the Toda lattice hierarchy based on the matrix-resolvent method, by generalizing the earlier results developed by Bertola, Dubrovin and the first-named author in the cases of KdV hierarchy [5] and Drinfeld–Sokolov hierarchies associated with simple Lie algebras [6]. Earlier the second-named author proved an explicit formula [49] for the Schur expansion of the Witten–Kontsevich tau-function of the KdV hierarchy. Balogh and the first-named author interpreted this formula in terms of the affine coordinates of the Witten–Kontsevich tau-function in [4]. Inspired by [5], the second-named author proved a formula [50] of n-point function associated with an arbitrary tau-function of the KP hierarchy, based on the affine coordinates of the element in the Sato Grassmannian corresponding to the tau-function. The formula in [18] has a difference from the formula in [50]: In the former matrices of power series are used whereas ordinary power series are used in the latter. To remedy the difference, the first-named author proved a formula for n-point function using only power series in [44] for tau-functions of Toda lattice hierarchy (see also [20]).

Secondly, it was shown in [51] that the normalized GUE partition function $\frac{Z_N(\mathbf{T};\epsilon)}{Z_N(\mathbf{0};\epsilon)}$ gives rise to a family of KP tau-functions in the big cell parameterized by the t'Hooft coupling constant $x = N\epsilon$ (see also [29, 31, 36, 41]). Furthermore, an explicit formula for the affine coordinates for this family of KP tau-function was derived in [51]. As an application, the formulas for the corresponding n-point functions were obtained by applying the formula in [50].

Now the GUE partition functions can be studied from two different perspectives: either as a tau-function of Dubrovin–Zhang type of the Toda lattice hierarchy, or as a family of tau-functions of the KP hierarchy. The belief that this is just a special case of general phenomenon leads us to Theorem 1.1.

There are earlier results that also lead to Theorem 1.1. It is well known that by introducing the t'Hooft coupling constant $x = N\epsilon$, $\log \frac{Z_N(\mathbf{T};\epsilon)}{Z_N(\mathbf{0};\epsilon)}$ has an expansion as weighted sum of ribbon graphs [18, 28, 51]. In other words, the enumeration of ribbon graphs gives rise to a special family of tau-functions of the KP hierarchy which become a special tau-function for the Toda lattice hierarchy by multiplying by a suitable correction factor. Recall that ribbon graphs can be regarded as *clean* dessins. As a generalization, the weighted sum of Grothendieck's dessins d'enfants has been shown by Zograf [54] (see also Kazarian and Zograf [30]) to be a tau-function of the KP hierarchy, that is referred to as the dessin partition function. Based on this fact, the second-named author had found the affine coordinates of the tau-function of the KP hierarchy associated with the dessin counting [52], and so explicit formula for the n-point functions of the dessin partition function can be written down using the general formula in [50]. The dessin partition function is a family of KP tau-functions parameterized by two parameters. It reduces to some other well-known tau-function by suitably specifying these parameters [52]. A proposal to study the dualities among different models based on the theory of KP hierarchy was then proposed in [52] and was further elaborated in [53].

The study of KP tau-functions and the study of Toda tau-functions were merged in our earlier work [46] in our study of the dessin tau-function and its role in unifying various theories. Since counting dessins is similar to counting ribbon graphs and GUE provides a matrix model for enumerating the ribbon graphs, one naturally expects a matrix model for enumerating dessins. The problem was addressed by Ambjørn and Chekhov [3], where they proposed a matrix model in the class of generalized Kontsevich models and an equivalent Hermitian 1-matrix model. In [46], we found that the Laguerre unitary ensemble (LUE) gives a simpler and rigorous matrix model for the dessins counting in the sense that the dessin partition function is equal to the *normalized* LUE partition function. Furthermore, by multiplying by a suitable correction factor, the dessin partition function gives rise to a Dubrovin–Zhang type tau-function of the Toda lattice hierarchy. It was proposed in [46] to study the duality between dessin partition function with partition functions of other theories from the viewpoint of Toda lattice hierarchy. In this new approach, one can apply the theory of normal forms of integrable hierarchies and the extended Toda hierarchy [9] as developed by Dubrovin and Zhang [21, 22]. In fact, the relevant Frobenius manifold is the Frobenius manifold associated with the Gromov–Witten invariants of $\mathbb{P}^1$. It has already been used in [16] (cf. [18, 45]) to calculate the GUE correlators.

Based on some earlier results of Dijkgraaf and Witten [13], Eguchi and Yang [24] proposed a matrix model for the Gromov–Witten invariants of $\mathbb{P}^1$ (see also [23]). Besides the Toda lattice hierarchy, the $\mathbb{P}^1$-partition function satisfies an extra family of flows introduced in [24, 26, 48] (see also [8, 9, 14, 24, 35]). These flows are called *extended flows* by Getzler [26] and Zhang [48]. They can be defined using the Lax pair formalism [9] as follows:

$$\epsilon\frac{\partial L}{\partial b_i} = \frac{2}{i!}\big[\big(L^i(\log L - c_i)\big)_+, L\big], \qquad i \geq 0, \tag{1.5}$$

where $c_i := \sum_{j=1}^{i} \frac{1}{j}$ are harmonic numbers and for the definition of $\log L$ see [9, 35]. The flows (1.5) commute with the traditional flows (1.3) of the Toda lattice hierarchy, and they also pairwise commute. All-together, (1.3), (1.5) form the *extended Toda hierarchy* (ETH) [9, 26, 48].

Let $(v(\mathbf{b}+x\mathbf{1},\mathbf{m};\epsilon), w(\mathbf{b}+x\mathbf{1},\mathbf{m};\epsilon))$ be an arbitrary solution to the ETH, and $\tau(\mathbf{b}+x\mathbf{1},\mathbf{m};\epsilon)$ the tau-function [9] of this solution (although it is uniquely determined up to multiplying by the exponential of an affine-linear function of $\mathbf{b},\mathbf{m}$). Here $\mathbf{b}=(b_0,b_1,b_2,\dots)$ and $\mathbf{1}=(1,0,0,\dots)$. The following theorem generalizes Theorem 1.1.

Theorem 1.3. *The tau-function $\tau(\mathbf{b}+x\mathbf{1},\mathbf{m};\epsilon)$ for the ETH is a KP tau-function for any x and $\mathbf{b}$, where $\mathbf{m}$ and $\mathbf{T}$ are related by* (1.4).

The proof is in Section 4.

Remark 1.4. For our intended applications to Gromov–Witten theory, we are content with tau-functions as formal power series. The results of Theorems 1.1 and 1.3 raise the problem to characterize the KP tau-functions obtained from the tau-functions of Toda lattice hierarchy or the extended Toda hierarchy. This problem is beyond the scope of this work and will be pursued in the future investigation. In viewpoint of Sato Grassmannian we think that our formula (3.2) below sheds light on this discussion, which we also mentioned briefly in Remark 1.2. We thank an anonymous referee for posing this problem to us.

It was conjectured in [14, 15, 24, 26], and was proved by Dubrovin and Zhang [22] that assuming the validity of the so-called Virasoro constraints (which is proved, e.g., in [39]) the partition function $Z^{\mathbb{P}^1}(\mathbf{b}+x\mathbf{1},\mathbf{m};\epsilon)$ of Gromov–Witten (GW) invariants of $\mathbb{P}^1$ (see Section 5 for the definition) is the tau-function of a particular solution to the ETH. Independently, Okounkov and Pandharipande [37, 38] proved that $Z^{\mathbb{P}^1}(x\mathbf{1},\mathbf{m};\epsilon)$ satisfies the bilinear equations for the 1-Toda hierarchy. Based on the above-mentioned results and on Theorem 1.3 we immediately obtain the following corollary.

Corollary 1.5. *The partition function $Z^{\mathbb{P}^1}(\mathbf{b}+x\mathbf{1},\mathbf{m};\epsilon)$ of GW invariants of $\mathbb{P}^1$ is a KP tau-function for any x, ϵ and $\mathbf{b}$, where $\mathbf{m}$ is related to $\mathbf{T}$ by (1.4). In particular, $Z^{\mathbb{P}^1}(x\mathbf{1},\mathbf{m};\epsilon)$ is a KP tau-function for any x and ϵ.*

The stationary sector of the $\mathbb{P}^1$-partition function is $Z^{\mathbb{P}^1}(0\mathbf{1},\mathbf{m};\epsilon)$. We compute in Section 5 its affine coordinates by two different methods: One by the results of [20], the other by the results of [38]. The former gives us an explicit closed formula, and the latter gives an expression in terms of characters of the symmetric groups. We then get a closed formula for some Plancherel averages in Corollary 5.8.

The rest of the paper is organized as follows. In Section 2, we recall the formula of n-point function for a KP tau-function. In Section 3, we show that an arbitrary tau-function for the Toda lattice hierarchy is a KP tau-function with the ground ring being the ring of functions of the space variable of Toda. The generalization to ETH is given in Section 4. In Section 5, we give an application of Corollary 1.5.

2 The formula for the n-point function for a KP tau-function

In this section, we first review the explicit formula (see Theorem 2.1 below) obtained by the second-named author for the n-point function for an arbitrary KP tau-function in the big cell, and then consider the converse statement that gives a criterion for a KP tau-function.

Before entering into the details let us introduce some *notations*. Denote by $\mathcal{R}$ a suitable ground ring. For a formal power series $F(\mathbf{p})\in\mathcal{R}[[\mathbf{p}]]$, with $\mathbf{p}=(p_1,p_2,\dots)$, define the n-point function associated with $F(\mathbf{p})$ by

$$G_n(\xi_1,\dots,\xi_n)=\sum_{k_1,\dots,k_n\geq 1}\prod_{i=1}^{n}\frac{k_i}{\xi_i^{k_i+1}}\cdot\frac{\partial^n F(\mathbf{p})}{\partial p_{k_1}\cdots\partial p_{k_n}}\bigg|_{\mathbf{p}=\mathbf{0}},\qquad n\geq 0.$$

By a *partition* $\mu=(\mu_1,\mu_2,\dots)$, we mean a sequence of weakly decreasing non-negative integers with $\mu_k=0$ for sufficiently large k. The length $\ell(\mu)$ is the number of the non-zero parts of μ, the weight $|\mu|:=\mu_1+\mu_2+\cdots$, and the multiplicity of i in μ is denoted by $m_i(\mu)$. The set of all partitions will be denoted by $\mathcal{P}$, and the set of partitions of weight d is denoted by $\mathcal{P}_d$. The Schur polynomial $s_\mu(\mathbf{p})$ associated to $\mu\in\mathcal{P}$ is a polynomial in the variables $\mathbf{p}=(p_1,p_2,\dots)$, defined by

$$s_\mu(\mathbf{p}):=\det_{1\leq i,j\leq\ell(\mu)}\left(h_{\mu_i-i+j}(\mathbf{p})\right),\tag{2.1}$$

where $h_j(\mathbf{p})$ are polynomials defined by the generating function

$$\sum_{j=0}^{\infty}h_j(\mathbf{p})z^j:=\mathrm{e}^{\sum_{k=1}^{\infty}\frac{p_k}{k}z^k}.$$

One constructive way to describe tau-functions of the KP hierarchy is to use Sato's Grassmannian. Let $z^{1/2}\mathcal{R}\big[\big[z,z^{-1}\big]\big]$ be the space of formal series in z of half integral powers. An element V in the big cell Gr_0 of Sato's Grassmannian is specified by a sequence of series

$$\Psi_j(z)=z^{j+1/2}+\sum_{i=0}^{\infty}A_{i,j}z^{-i-1/2},\qquad j\geq 0,$$

where the coefficients $A_{i,j}$ are called the *affine coordinates* of V. To any element $V \in Gr_0$, there corresponds a particular KP tau-function Z_V defined by

$$Z_V := \sum_{\lambda \in \mathcal{P}} c_\lambda s_\lambda(\mathbf{p}), \tag{2.2}$$

where $T_k = p_k/k$, and for a partition λ written in terms of the Frobenius notation [32, Section I.1, p. 3] $\lambda = (m_1, \ldots, m_k | n_1, \ldots, n_k)$,

$$c_\lambda := (-1)^{n_1+\cdots+n_k} \det_{1\le i,j\le k} \big(A_{m_i,n_j}\big).$$

In particular, for $\lambda = \big(i+1, 1^j\big)$ we have $c_\lambda = (-1)^j A_{i,j}$. So one can easily read off the affine coordinates from the expansion (2.2).

The following theorem was obtained in [50].

Theorem 2.1 ([50])**.** *The n-point function associated with $\log Z_V$ is given by the following formula: For $n = 1$,*

$$G_1(\xi) = \sum_{i,j\ge 0} A_{i,j} \xi^{-i-j-2}, \tag{2.3}$$

and for $n \ge 2$,

$$G_n(\xi_1, \ldots, \xi_n) = (-1)^{n-1} \sum_{n\text{-cycles}} \prod_{i=1}^{n} B\big(\xi_{\sigma(i)}, \xi_{\sigma(i+1)}\big) - \frac{\delta_{n,2}}{(\xi_1 - \xi_2)^2}, \tag{2.4}$$

where $\sigma(n+1)$ is understood as $\sigma(1)$, and

$$B(\xi_i, \xi_j) = \begin{cases} \dfrac{1}{\xi_i - \xi_j} + A(\xi_i, \xi_j), & i \ne j, \\ A(\xi_i, \xi_i), & i = j, \end{cases}$$

and

$$A(\xi, \eta) = \sum_{i,j\ge 0} A_{i,j} \xi^{-j-1} \eta^{-i-1}.$$

Theorem 2.1 enables one to easily compute the n-point function once we have found the affine coordinates $A_{i,j}$. For example, there is only one 2-cycle, so the two-point function is given by

$$G_2(\xi_1, \xi_2) = \frac{A(\xi_1, \xi_2) - A(\xi_2, \xi_1)}{\xi_1 - \xi_2} - A(\xi_1, \xi_2) \cdot A(\xi_2, \xi_1).$$

Let us consider the converse of the above Theorem 2.1. Recall the simple fact, which can be verified directly using (1.2), that multiplying a KP tau-function $\tau(\mathbf{T}(\mathbf{p}))$ by the exponential of an arbitrary affine linear function of $\mathbf{p}$

$$\tau(\mathbf{T}(\mathbf{p})) \cdot \mathrm{e}^{C_0 + \sum_{k\ge 1} C_k p_k}, \qquad C_k \in \mathcal{R}, \quad k \ge 0, \tag{2.5}$$

produces again a KP tau-function. We recall here that $T_k = p_k/k$. The renormalized variables p_k are also sometimes called KP times.

Corollary 2.2. *Let F be a formal power series in $\mathbf{p}$ and $G_n(\xi_1, \ldots, \xi_n)$ its n-point function. If there are $(A_{i,j})_{i,j\ge 0}$ such that for all $n \ge 2$ G_n are given by (2.4), then $Z = \mathrm{e}^F$ is a tau-function of the KP hierarchy.*

Proof. By Theorem 2.1, the tau-function corresponds to the point in the big cell with $A_{i,j}$ being the affine coordinates can only differ from Z by multiplying by the exponential of some affine-linear function. ■

Remark 2.3. From the above proof, we see that Corollary 2.2 directly follows from Theorem 2.1. In [2], Theorem 2.1 together with Corollary 2.2 is regarded as *Zhou's theorem*.

The next proposition describes how the affine coordinates change under (2.5).

Proposition 2.4. *Let V, V' be two elements in the big cell having affine coordinates $A_{i,j}$, $A'_{i,j}$, respectively, and let Z_V, $Z_{V'}$ be the KP tau-functions associated to V, V'. If there exist $C_1, C_2, \dots$ such that $Z_{V'} = \mathrm{e}^{\sum_{k\geq 1} C_k p_k} Z_V$, then we have the following identity:*

$$\sum_{i,j\geq 0} A'_{i,j}\xi^{-j-1}\eta^{-i-1} + \frac{1}{\xi-\eta} = \left(\sum_{i,j\geq 0} A_{i,j}\xi^{-j-1}\eta^{-i-1} + \frac{1}{\xi-\eta}\right)\frac{\mathrm{e}^{-\sum_{\ell\geq 1} C_\ell \xi^{-\ell}}}{\mathrm{e}^{-\sum_{\ell\geq 1} C_\ell \eta^{-\ell}}}.$$

Proof. By direct verifications using (2.3). ■

3 From the Toda lattice hierarchy to the KP hierarchy

In this section, we show that an arbitrary tau-function for the Toda lattice hierarchy is a KP tau-function.

Let $(v(x,\mathbf{m};\epsilon), w(x,\mathbf{m};\epsilon))$ be an arbitrary solution in $W[[\mathbf{m}]]^2$ to the Toda lattice hierarchy (1.3), and $\tau(x,\mathbf{m};\epsilon)$ the tau-function [9, 18, 22] of this solution. Here, W is a certain ring of functions of x closed under shifting x by $\pm\epsilon$, and $\tau(x,\mathbf{m};\epsilon)$ lives in $\widetilde{W}[[\mathbf{m}]]$ with $\widetilde{W}$ being some extension of W. The solution $(v(x,\mathbf{m};\epsilon), w(x,\mathbf{m};\epsilon))$ is in one-to-one correspondence with its initial value

$$v(x,\mathbf{0};\epsilon) =: f(x,\epsilon), \qquad w(x,\mathbf{0};\epsilon) =: g(x,\epsilon).$$

Denote by L_{ini} the initial Lax operator

$$L_{\mathrm{ini}} = \Lambda + f(x,\epsilon) + g(x,\epsilon)\Lambda^{-1},$$

and let

$$s(x,\epsilon) = -\left(1-\Lambda^{-1}\right)^{-1}(\log g(x,\epsilon)),$$

which lives in the extended ring $\widetilde{W}$. Recall from [44] that two elements

$$\psi_1(\lambda,x;\epsilon) = \left(1+\mathrm{O}\left(\lambda^{-1}\right)\right)\lambda^{x/\epsilon}, \qquad \psi_2(\lambda,x;\epsilon) = \left(1+\mathrm{O}\left(\lambda^{-1}\right)\right)\mathrm{e}^{-s(x,\epsilon)}\lambda^{-x/\epsilon}$$

are called forming *a pair of wave functions* of L_{ini} if

$$L_{\mathrm{ini}}(\psi_1) = \lambda\psi_1, \qquad L_{\mathrm{ini}}(\psi_2) = \lambda\psi_2,$$
$$d(\lambda,x;\epsilon) := \psi_1\,\Lambda^{-1}(\psi_2) - \psi_2\,\Lambda^{-1}(\psi_1) = \lambda\mathrm{e}^{-s(x-\epsilon,\epsilon)}.$$

The following formula is proved in [44]:

$$\epsilon^n \sum_{i_1,\dots,i_n\geq 0} \frac{(i_1+1)!\cdots(i_n+1)!}{\lambda_1^{i_1+2}\cdots\lambda_n^{i_n+2}} \frac{\partial^n \log\tau(x,\mathbf{m};\epsilon)}{\partial m_{i_1}\cdots\partial m_{i_n}}\bigg|_{\mathbf{m}=\mathbf{0}}$$
$$= (-1)^{n-1}\frac{\mathrm{e}^{ns(x-\epsilon,\epsilon)}}{\prod_{j=1}^n \lambda_j} \sum_{n\text{-cycles}} \prod_{j=1}^n D\left(\lambda_{\sigma(j)},\lambda_{\sigma(j+1)};x;\epsilon\right) - \frac{\delta_{n,2}}{(\lambda-\mu)^2}, \tag{3.1}$$

where $n\geq 2$, and

$$D(\lambda,\mu;x;\epsilon) := \frac{\psi_1(\lambda,x;\epsilon)\psi_2(\mu,x-\epsilon;\epsilon) - \psi_1(\lambda,x-\epsilon;\epsilon)\psi_2(\mu,x;\epsilon)}{\lambda-\mu}. \tag{3.2}$$

Remark 3.1. The function $D(\lambda,\mu;x;\epsilon)$ looks similar to the Christoffel–Darboux kernel in matrix models [11, 34]. We hope to study their relations in a future work.

It was observed in [44, (99)] that $\frac{e^{s(x-\epsilon,\epsilon)}}{\mu}D(\lambda,\mu;x;\epsilon)\frac{\mu^{x/\epsilon}}{\lambda^{x/\epsilon}}-\frac{1}{\lambda-\mu}$ is a power series of λ^{-1}, μ^{-1}. By a more careful analysis we can show that

$$\frac{e^{s(x-\epsilon,\epsilon)}}{\mu}D(\lambda,\mu;x;\epsilon)\frac{\mu^{\frac{x}{\epsilon}}}{\lambda^{\frac{x}{\epsilon}}}=\frac{1}{\lambda-\mu}+\sum_{i,j\geq 0}\frac{A_{i,j}(x,\epsilon)}{\lambda^{j+1}\mu^{i+1}}=:B(\lambda,\mu;x;\epsilon) \tag{3.3}$$

for some coefficients $A_{i,j}(x,\epsilon)$. In the next section, we will give a complete proof of a generalized version of (3.3).

Using (3.3) and (3.1) we find

$$\begin{aligned}\epsilon^n\sum_{i_1,\dots,i_n\geq 0}&\frac{(i_1+1)!\cdots(i_n+1)!}{\lambda_1^{i_1+2}\cdots\lambda_n^{i_n+2}}\frac{\partial^n\log\tau(x,\mathbf{m};\epsilon)}{\partial m_{i_1}\cdots\partial m_{i_n}}\bigg|_{\mathbf{m}=\mathbf{0}}\\&=(-1)^{n-1}\sum_{n\text{-cycles}}\prod_{j=1}^{n}B\big(\lambda_{\sigma(j)},\lambda_{\sigma(j+1)};\epsilon\big)-\frac{\delta_{n,2}}{(\lambda-\mu)^2},\qquad n\geq 2.\end{aligned} \tag{3.4}$$

(See also [44, Corollary 1].)

We are ready to prove Theorem 1.1.

Proof of Theorem 1.1. By using (3.3), (3.4) and (1.4), we know that the n-point function for $n\geq 2$ associated to $\log\tau(x,\mathbf{m};\epsilon)$ has the form (2.4). The theorem is then proved by applying Corollary 2.2. ■

If we write

$$\tau(x,\mathbf{m};\epsilon)=\tau_{\mathrm{corr}}(x,\epsilon)\tau_1(x,\mathbf{m};\epsilon),\qquad \tau_1(x,\mathbf{m}=\mathbf{0};\epsilon)\equiv 1,$$

then by Theorem 1.1 we know that $\tau_1(x,\mathbf{m};\epsilon)$ is a KP tau-function in the big cell. The factor $\tau_{\mathrm{corr}}(x,\epsilon)$ can [9, 44] be determined by

$$\frac{\tau_{\mathrm{corr}}(x+\epsilon,\epsilon)\tau_{\mathrm{corr}}(x-\epsilon,\epsilon)}{\tau_{\mathrm{corr}}(x,\epsilon)^2}=g(x,\epsilon)=e^{s(x-\epsilon,\epsilon)-s(x,\epsilon)}. \tag{3.5}$$

This factor can be identified with the *correction factor* in [18, 44, 46].

Let $\tilde{\tau}_1(x,\mathbf{m};\epsilon)$ be the KP tau-function associated to the point in Sato's Grassmannian having the affine coordinates $A_{i,j}(x,\epsilon)$, with $A_{i,j}(x,\epsilon)$ given in (3.3). Then there exist $a_i(x,\epsilon)$, $i\geq 0$, such that

$$\tau_1(x,\mathbf{m};\epsilon)=e^{\sum_{i\geq 0}a_i(x,\epsilon)m_i}\tilde{\tau}_1(x,\mathbf{m};\epsilon).$$

By further applying Proposition 2.4, we get the affine coordinates for $\tau_1(x,\mathbf{m};\epsilon)$.

4 Extended Toda flows in the KP hierarchy

The goal of this section is to prove Theorem 1.3.

Before entering into the main construction for this section, we briefly recall here the definition of $\log L$. It is shown in [9] that there exist dressing operators

$$P=\sum_{k\geq 0}P_k\Lambda^{-k},\qquad P_0=1,\qquad Q=\sum_{k\geq 0}Q_k\Lambda^{k},$$

such that

$$L = P \circ \Lambda \circ P^{-1} = Q \circ \Lambda^{-1} \circ Q^{-1}.$$

Here, P_k, Q_k belong to a certain extension of the differential polynomial ring. The logarithm of the Lax operator is then defined by [9]

$$\log L = \frac{1}{2} P \circ \epsilon\partial_x \circ P^{-1} - \frac{1}{2} Q \circ \epsilon\partial_x \circ Q^{-1}.$$

As in the introduction, for an arbitrary solution $(v(\mathbf{b}+x\mathbf{1}, \mathbf{m}; \epsilon), w(\mathbf{b}+x\mathbf{1}, \mathbf{m}; \epsilon))$ to the ETH, let $\tau(\mathbf{b}+x\mathbf{1}, \mathbf{m}; \epsilon)$ be the tau-function (again in the sense of [9, 22]) of the solution. Let us also denote

$$\sigma(x, \mathbf{b}, \mathbf{m}; \epsilon) = \log \frac{\tau(\mathbf{b}+x\mathbf{1}, \mathbf{m}; \epsilon)}{\tau(\mathbf{b}+(x+\epsilon)\mathbf{1}, \mathbf{m}; \epsilon)}. \tag{4.1}$$

4.1 The matrix-resolvent method

The matrix-resolvent method for studying tau-functions for the Toda lattice hierarchy was developed in [18] (see also [44]). By the locality nature of this method, the same formulation as for the Toda lattice hierarchy applies to the ETH.

Denote by

$$\mathcal{A} = \mathbb{Z}[v_0, w_0, v_{\pm1}, w_{\pm1}, v_{\pm2}, w_{\pm2}, \dots]$$

the polynomial ring. Recall that the *basic matrix resolvent* $R(\lambda)$ is defined [18] as the unique element in $\mathrm{Mat}\big(2, \mathcal{A}\big[\big[\lambda^{-1}\big]\big]\big)$ satisfying

$$\Lambda(R(\lambda))U(\lambda) - U(\lambda)R(\lambda) = 0,$$
$$\operatorname{tr} R(\lambda) = 1, \qquad \det R(\lambda) = 0, \qquad R(\lambda) - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{Mat}\big(2, \mathcal{A}\big[\big[\lambda^{-1}\big]\big]\lambda^{-1}\big),$$

where

$$U(\lambda) := \begin{pmatrix} v_0 - \lambda & w_0 \\ -1 & 0 \end{pmatrix}.$$

Define $S_i = \frac{1}{(i+1)!}\mathrm{Coef}\big(\Lambda(R(\lambda)_{21}), \lambda^{-i-2}\big) \in \mathcal{A}$, $i \geq 0$, and a sequence of elements $\Omega_{i,j} \in \mathcal{A}$ by

$$\epsilon^2 \sum_{i,j\geq 0} \frac{(i+1)!(j+1)!}{\lambda^{i+2}\mu^{j+2}} \Omega_{i,j} = \frac{\operatorname{tr} R(\lambda)R(\mu)}{(\lambda-\mu)^2} - \frac{1}{(\lambda-\mu)^2}.$$

According to [18], the above-defined $(\Omega_{i,j}, S_i)$ gives rise to part of the canonical tau-structure for the ETH in [9, 22]. Indeed, it is shown in [18] that $(\Omega_{i,j}, S_i)$ is associated to the tau-symmetric hamiltonian densities $h_{\alpha,p}$, $\alpha = 1, 2$, $p \geq -1$, [9, 22] for the Toda lattice hierarchy. More precisely, $S_i = h_{2,i-1}$ and the following identities hold for an arbitrary solution $(v(\mathbf{b} + x\mathbf{1}, \mathbf{m}; \epsilon), w(\mathbf{b}+x\mathbf{1}, \mathbf{m}; \epsilon))$ to the ETH:

$$\epsilon^2 \frac{\partial^2 \log \tau(\mathbf{b}+x\mathbf{1}, \mathbf{m}; \epsilon)}{\partial m_i \partial m_j} = \Omega_{i,j}(\mathbf{b}+x\mathbf{1}, \mathbf{m}; \epsilon),$$

$$\epsilon \frac{\partial}{\partial m_i}\left(\log \frac{\tau(\mathbf{b}+(x+\epsilon)\mathbf{1}, \mathbf{m}; \epsilon)}{\tau(\mathbf{b}+x\mathbf{1}, \mathbf{m}; \epsilon)}\right) = S_i(\mathbf{b}+x\mathbf{1}, \mathbf{m}; \epsilon), \tag{4.2}$$

$$\frac{\tau(\mathbf{b}+(x+\epsilon)\mathbf{1}, \mathbf{m}; \epsilon)\tau(\mathbf{b}+(x-\epsilon)\mathbf{1}, \mathbf{m}; \epsilon)}{\tau(\mathbf{b}+x\mathbf{1}, \mathbf{m}; \epsilon)^2} = w(\mathbf{b}+x\mathbf{1}, \mathbf{m}; \epsilon). \tag{4.3}$$

Here, $\Omega_{i,j}(\mathbf{b}+x\mathbf{1},\mathbf{m};\epsilon)$ and $S_i(\mathbf{b}+x\mathbf{1},\mathbf{m};\epsilon)$ are obtained by replacing v_k, w_k by $v(\mathbf{b}+x\mathbf{1}+k\epsilon,\mathbf{m};\epsilon)$ and $w(\mathbf{b}+x\mathbf{1}+k\epsilon,\mathbf{m};\epsilon)$, respectively. According to [18], for any $n\geq 2$,

$$\epsilon^n \sum_{i_1,\dots,i_n\geq 0}\prod_{j=1}^n \frac{(i_j+1)!}{\lambda_j^{i_j+2}}\frac{\partial^n \log\tau(\mathbf{b}+x\mathbf{1},\mathbf{m};\epsilon)}{\partial m_{i_1}\cdots\partial m_{i_n}}$$
$$= -\sum_{n\text{-cycles}}\frac{\mathrm{tr}\,\prod_{j=1}^n R(\lambda_{\sigma(j)})}{\prod_{j=1}^n(\lambda_{\sigma(j)}-\lambda_{\sigma(j+1)})}-\frac{\delta_{n,2}}{(\lambda-\mu)^2}. \tag{4.4}$$

Note that by (4.1) and (4.2), (4.3), we have

$$-\epsilon\frac{\partial\sigma(x,\mathbf{b},\mathbf{m};\epsilon)}{\partial m_i}=S_i(\mathbf{b}+x\mathbf{1},\mathbf{m};\epsilon), \tag{4.5}$$
$$\mathrm{e}^{\sigma(x-\epsilon,\mathbf{b},\mathbf{m};\epsilon)-\sigma(x,\mathbf{b},\mathbf{m};\epsilon)}=w(\mathbf{b}+x\mathbf{1},\mathbf{m};\epsilon). \tag{4.6}$$

It is also clear from (3.5) that we can choose $\sigma(x,b_0=0,b_1=0,\dots,\mathbf{m}=\mathbf{0};\epsilon)=s(x,\epsilon)$.

4.2 Wave functions for ETH and the KP hierarchy

It is shown in [8] (see also [10, 35]) that there exist dressing operators P, Q of the forms

$$P=\sum_{k\geq 0}P_k(x,\mathbf{b},\mathbf{m};\epsilon)\Lambda^{-k},\qquad P_0(x,\mathbf{b},\mathbf{m};\epsilon)\equiv 1, \tag{4.7}$$
$$Q=\sum_{k\geq 0}Q_k(x,\mathbf{b},\mathbf{m};\epsilon)\Lambda^{k}, \tag{4.8}$$

such that

$$L=P\circ\Lambda\circ P^{-1}=Q\circ\Lambda^{-1}\circ Q^{-1},$$
$$\epsilon\frac{\partial P}{\partial m_i}=-\frac{1}{(i+1)!}\left(L^{i+1}\right)_-\circ P,$$
$$\epsilon\frac{\partial P}{\partial b_i}=-\frac{2}{i!}\left(L^i(\log L-c_i)\right)_-\circ P,$$
$$\epsilon\frac{\partial Q}{\partial m_i}=\frac{1}{(i+1)!}\left(L^{i+1}\right)_+\circ Q,$$
$$\epsilon\frac{\partial Q}{\partial b_i}=\frac{2}{i!}\left(L^i(\log L-c_i)\right)_+\circ Q.$$

Define

$$\psi_1(\lambda;x,\mathbf{b},\mathbf{m};\epsilon):=P\left(\lambda^{\frac{x}{\epsilon}}\mathrm{e}^{\sum_{i\geq 0}\frac{2}{i!}\frac{b_i}{\epsilon}\lambda^i(\log\lambda-c_i)+\sum_{i\geq 0}\frac{m_i}{(i+1)!\epsilon}\lambda^{i+1}}\right),$$
$$\psi_2(\lambda;x,\mathbf{b},\mathbf{m};\epsilon):=Q\left(\lambda^{-\frac{x}{\epsilon}}\mathrm{e}^{-\sum_{i\geq 0}\frac{2}{i!}\frac{b_i}{\epsilon}\lambda^i(\log\lambda-c_i)-\sum_{i\geq 0}\frac{m_i}{(i+1)!\epsilon}\lambda^{i+1}}\right).$$

Then $\psi_a=\psi_a(\lambda;x,\mathbf{b},\mathbf{m};\epsilon)$, $a=1,2$, satisfy the following equations:

$$L\psi_a=\lambda\psi_a,\qquad a=1,2, \tag{4.9}$$
$$\epsilon\frac{\partial\psi_1}{\partial m_i}=\frac{1}{(i+1)!}\left(L^{i+1}\right)_+\psi_1,$$
$$\epsilon\frac{\partial\psi_1}{\partial b_i}=\frac{2}{i!}L^i(\log L-c_i)_+\psi_1, \tag{4.10}$$
$$\epsilon\frac{\partial\psi_2}{\partial m_i}=-\frac{1}{(i+1)!}\left(L^{i+1}\right)_-\psi_2,$$
$$\epsilon\frac{\partial\psi_2}{\partial b_i}=-\frac{2}{i!}L^i(\log L-c_i)_-\psi_2. \tag{4.11}$$

Moreover, ψ_1, ψ_2 have the form

$$\psi_1 = \phi_1(\lambda; x, \mathbf{b}, \mathbf{m}; \epsilon)\lambda^{\frac{x}{\epsilon}} \mathrm{e}^{\sum_{i\geq 0} \frac{2}{i!}\frac{b_i}{\epsilon}\lambda^i(\log\lambda - c_i) + \sum_{i\geq 0}\frac{m_i}{(i+1)!\epsilon}\lambda^{i+1}}, \tag{4.12}$$

$$\psi_2 = \phi_2(\lambda; x, \mathbf{b}, \mathbf{m}; \epsilon)\mathrm{e}^{-\sigma(x,\mathbf{b},\mathbf{m};\epsilon)}\lambda^{-\frac{x}{\epsilon}} \mathrm{e}^{-\sum_{i\geq 0} \frac{2}{i!}\frac{b_i}{\epsilon}\lambda^i(\log\lambda - c_i) - \sum_{i\geq 0}\frac{m_i}{(i+1)!\epsilon}\lambda^{i+1}}, \tag{4.13}$$

where

$$\phi_1(\lambda; x, \mathbf{b}, \mathbf{m}; \epsilon) = \sum_{k\geq 0} P_k(x, \mathbf{b}, \mathbf{m}; \epsilon)\lambda^{-k}, \tag{4.14}$$

$$\phi_2(\lambda; x, \mathbf{b}, \mathbf{m}; \epsilon) = \sum_{k\geq 0} B_k(x, \mathbf{b}, \mathbf{m}; \epsilon)\lambda^{-k}. \tag{4.15}$$

We recall that $P_k(x, \mathbf{b}, \mathbf{m}; \epsilon)$ are given in (4.7), and we also note using (4.8) that $B_k(x, \mathbf{b}, \mathbf{m}; \epsilon) = Q_k(x, \mathbf{b}, \mathbf{m}; \epsilon)\mathrm{e}^{\sigma(x,\mathbf{b},\mathbf{m};\epsilon)}$, $k \geq 0$, and $B_0(x, \mathbf{b}, \mathbf{m}; \epsilon) \equiv 1$. We call ψ_1 *the wave function of type A* and ψ_2 *the wave function of type B, associated to the solution* $(v(x, \mathbf{b}, \mathbf{m}; \epsilon), w(x, \mathbf{b}, \mathbf{m}; \epsilon))$.

Denote

$$\begin{aligned} &d(\lambda; x, \mathbf{b}, \mathbf{m}; \epsilon) \\ &\quad := \psi_1(\lambda; x, \mathbf{b}, \mathbf{m}; \epsilon)\psi_2(\lambda; x-\epsilon, \mathbf{b}, \mathbf{m}; \epsilon) - \psi_1(\lambda; x-\epsilon, \mathbf{b}, \mathbf{m}; \epsilon)\psi_2(\lambda; x, \mathbf{b}, \mathbf{m}; \epsilon). \end{aligned}$$

The following lemma is important.

Lemma 4.1. *For any $i \geq 0$, we have the identities*

$$\frac{\partial\big(\mathrm{e}^{\Lambda^{-1}(\sigma(x,\mathbf{b},\mathbf{m};\epsilon))}d(\lambda; x, \mathbf{b}, \mathbf{m}; \epsilon)\big)}{\partial b_i} = 0, \tag{4.16}$$

$$\frac{\partial\big(\mathrm{e}^{\Lambda^{-1}(\sigma(x,\mathbf{b},\mathbf{m};\epsilon))}d(\lambda; x, \mathbf{b}, \mathbf{m}; \epsilon)\big)}{\partial m_i} = 0. \tag{4.17}$$

Proof. We first introduce some notations,

$$\begin{aligned} &R_j := \mathrm{res} L^j, \qquad N_j = \mathrm{Coef}\big(L^j, \Lambda^{-1}\big), \\ &r_j := \mathrm{res}\log L\circ L^j, \qquad n_j = \mathrm{Coef}\big(\log L \circ L^j, \Lambda^{-1}\big). \end{aligned}$$

(*Warning: avoid from confusing the notation R_j with the notation for the matrix resolvent.*) For example (see [8]),

$$\begin{aligned} &R_0 = 1, \qquad R_1 = v, \qquad N_0 = 0, \qquad N_1 = w, \\ &r_0 = \frac{1}{2}\Lambda\circ\frac{\epsilon\partial}{\Lambda - 1}(v), \qquad n_0 = \frac{1}{2}\frac{\epsilon\partial}{\Lambda-1}(\log w). \end{aligned}$$

Here we omitted the arguments in R_j, r_j, N_j, n_j, v, w. Below, when no ambiguity will occur, we often do this type of omission.

As it was given in [9], the tau-symmetric hamiltonian densities $h_{\alpha,p}$ for the ETH are related to R_j, r_j by

$$\begin{aligned} h_{2,p} &= \frac{1}{(p+2)!}R_{p+2}, \\ h_{1,p} &= \frac{2}{(p+1)!}r_{p+1} - \frac{2c_{p+1}}{(p+1)!}R_{p+1}, \qquad p \geq -1. \end{aligned} \tag{4.18}$$

We also recall from [9] the following formula:

$$h_{\alpha,i-1} = \epsilon(\Lambda-1)\frac{\partial \log\tau}{\partial t^{\alpha,i}} = -\epsilon\frac{\partial\sigma}{\partial t^{\alpha,i}}, \qquad i\geq 0.$$

Let us now prove (4.17). (Although (4.17) was already proved in [44], our proof here will be slightly different from [44] and can be generalized to proving (4.16).) We have

$$\mathrm{e}^{-\Lambda^{-1}(\sigma)}\frac{\partial\big(\mathrm{e}^{\Lambda^{-1}(\sigma)}d\big)}{\partial m_i} = \frac{\partial\Lambda^{-1}(\sigma)}{\partial m_i}d + \frac{\partial d}{\partial m_i}, \qquad i\geq 0. \tag{4.19}$$

Denote

$$B_i := -\big(L^{i+1}\big)_-, \qquad i\geq -1.$$

Then for all $i\geq 0$, we have

$$\begin{aligned} B_i &= -\big(L^i\circ L\big)_- = -\big(\big(L^i\big)_+ + \big(L^i\big)_-\big)\circ L_- = -(R_iL)_- - \big(\big(L^i\big)_-\circ L\big)_- \\ &= B_{i-1}\circ L + N_i - R_iw\Lambda^{-1} = \cdots = \sum_{j=0}^{i}\big(N_j - R_jw\Lambda^{-1}\big)\circ L^{i-j}. \end{aligned}$$

(The resulting equality is also valid for $i=-1$ as $B_{-1}=0$.) Thus,

$$(i+1)!\epsilon\frac{\partial\psi_2}{\partial m_i} = B_i(\psi_2) = \sum_{j=0}^{i}\lambda^{i-j}\big(N_j - R_jw\Lambda^{-1}\big)(\psi_2), \tag{4.20}$$

and

$$(i+1)!\epsilon\frac{\partial\psi_1}{\partial m_i} = \lambda^{i+1}\psi_1 + \sum_{j=0}^{i}\lambda^{i-j}\big(N_j - R_jw\Lambda^{-1}\big)(\psi_1). \tag{4.21}$$

By substituting (4.21) and (4.20) in (4.19) and by a lengthy but straightforward calculation using also (4.9) and (4.5), we obtain

$$\begin{aligned} \mathrm{e}^{-\Lambda^{-1}(\sigma)}\epsilon\frac{\partial\big(\mathrm{e}^{\Lambda^{-1}(\sigma)}d\big)}{\partial m_i} &= -\Lambda^{-1}(S_i)d + \frac{\Lambda^{-1}(R_{i+1})}{(i+1)!}d \\ &\quad + \frac{d}{(i+1)!}\sum_{j=0}^{i}\lambda^{i-j}\big(N_j + \Lambda^{-1}(N_j) + \Lambda^{-1}(vR_j - R_{j+1})\big) = 0. \end{aligned}$$

Note that in the last equality we used (4.2), (4.18) and the relation $L^{k+1} = L\circ L^k = L^k\circ L$ (which implies the vanishing of each summand in the above $\sum_j$).

We proceed to prove (4.16). Like in [44], we note that

$$\begin{aligned} \Lambda\big(\mathrm{e}^{\Lambda^{-1}(\sigma(x,\mathbf{b},\mathbf{m};\epsilon))}d(\lambda;x,\mathbf{b},\mathbf{m};\epsilon)\big) &= \mathrm{e}^{\sigma}\left(\Lambda(\psi_1)\psi_2 - \psi_1\Lambda(\psi_2)\right) \\ &= \mathrm{e}^{\sigma}\big\{\big((\lambda-v)\psi_1 - w\Lambda^{-1}(\psi_1)\big)\psi_2 - \psi_1\big((\lambda-v)\psi_2 - w\Lambda^{-1}(\psi_2)\big)\big\} \\ &= \mathrm{e}^{\Lambda^{-1}(\sigma(x,\mathbf{b},\mathbf{m};\epsilon))}d(\lambda;x,\mathbf{b},\mathbf{m};\epsilon) \end{aligned}$$

(the last equality used (4.6)). It follows that

$$\epsilon\partial_x\big(\mathrm{e}^{\Lambda^{-1}(\sigma(x,\mathbf{b},\mathbf{m};\epsilon))}d(\lambda;x,\mathbf{b},\mathbf{m};\epsilon)\big) = \frac{\epsilon\partial_x}{\Lambda-1}\circ(\Lambda-1)\big(\mathrm{e}^{\Lambda^{-1}(\sigma(x,\mathbf{b},\mathbf{m};\epsilon))}d(\lambda;x,\mathbf{b},\mathbf{m};\epsilon)\big) = 0.$$

Thus,

$$0 = e^{-\Lambda^{-1}(\sigma(x,\mathbf{b},\mathbf{m};\epsilon))}\partial_x\big(e^{\Lambda^{-1}(\sigma(x,\mathbf{b},\mathbf{m};\epsilon))}d(\lambda;x,\mathbf{b},\mathbf{m};\epsilon)\big)$$
$$= -\Lambda^{-1}(h_{1,-1}) + \partial_x(d(\lambda;x,\mathbf{b},\mathbf{m};\epsilon)) = -2\Lambda^{-1}(r_0)d + \partial_x(d(\lambda;x,\mathbf{b},\mathbf{m};\epsilon)).$$

Denote

$$Q_i = -2\big(L^i(\log L - c_i)\big)_- = -2\big(L^i\log L\big)_0 - 2c_iB_{i-1}, \qquad i\geq 0.$$

In the way similar to the derivation for B_i, we find

$$Q_i = \sum_{j=0}^{i-1}\big(2m_j - 2r_jw\Lambda^{-1}\big)\circ L^{i-j} - 2(\log L)_-\circ L^i - 2c_iB_{i-1}.$$

Therefore,

$$i!\frac{\partial\psi_2}{\partial b_i} = Q_i(\psi_2) = \sum_{j=0}^{i-1}\lambda^{i-j}\big(2m_j - 2r_jw\Lambda^{-1}\big)(\psi_2)$$
$$- 2(\log L)_-(\psi_2) - 2c_i\sum_{j=0}^{i-1}\lambda^{i-1-j}\big(N_j - R_jw\Lambda^{-1}\big)(\psi_2),$$

and

$$i!\frac{\partial\psi_1}{\partial b_i} = 2\lambda^i(\log\lambda - c_i)\psi_1 + \sum_{j=0}^{i-1}\lambda^{i-j}\big(2m_j - 2r_jw\Lambda^{-1}\big)(\psi_1)$$
$$- 2(\log L)_-(\psi_1) - 2c_i\sum_{j=0}^{i-1}\lambda^{i-1-j}\big(N_j - R_jw\Lambda^{-1}\big)(\psi_1).$$

Using these relations and using (4.10) and (4.11) with $i=0$, and again by a lengthy calculation, we find

$$e^{-\Lambda^{-1}(\sigma)}\frac{\partial\big(e^{\Lambda^{-1}(\sigma)}d\big)}{\partial b_i} = -2\Lambda^{-1}(r_0)d + \partial_x(d) = 0.$$

The lemma is proved. ■

It follows from Lemma 4.1 and (4.12) and (4.13) that

$$d(\lambda;x,\mathbf{b},\mathbf{m};\epsilon) = \lambda e^{-\Lambda^{-1}(\sigma(x,\mathbf{b},\mathbf{m};\epsilon))}\sum_{k\geq 0}d_k\lambda^{-k},$$

where d_k, $k\geq 0$, are constants with $d_0 = 1$. Therefore, for any choice ψ_1 of the wave functions of type A associated to $(v(x,\mathbf{b},\mathbf{m};\epsilon), w(x,\mathbf{b},\mathbf{m};\epsilon))$, there exists a choice, ψ_2, of the wave functions of type B associated to $(v(x,\mathbf{b},\mathbf{m};\epsilon), w(x,\mathbf{b},\mathbf{m};\epsilon))$, such that

$$d(\lambda;x,\mathbf{b},\mathbf{m};\epsilon) = \lambda e^{-\Lambda^{-1}(\sigma(x,\mathbf{b},\mathbf{m};\epsilon))}. \tag{4.22}$$

As a generalization of the terminology in [44], we say that ψ_1, ψ_2 form *a pair of wave functions* associated to $(v(\mathbf{b}+x\mathbf{1},\mathbf{m};\epsilon), w(\mathbf{b}+x\mathbf{1},\mathbf{m};\epsilon))$ if (4.22) holds.

We now define

$$D(\lambda,\mu;x,\mathbf{b},\mathbf{m};\epsilon)$$
$$:= \frac{\psi_1(\lambda;x,\mathbf{b},\mathbf{m};\epsilon)\psi_2(\mu;x-\epsilon,\mathbf{b},\mathbf{m};\epsilon) - \psi_1(\lambda;x-\epsilon,\mathbf{b},\mathbf{m};\epsilon)\psi_2(\mu;x,\mathbf{b},\mathbf{m};\epsilon)}{\lambda-\mu},$$

which coincides with the one introduced in [44] when restricted to $b_2 = b_3 = \cdots = 0$.

We arrive at the following theorem.

Theorem 4.2. *The following identity holds:*

$$\epsilon^n \sum_{i_1,\dots,i_n\geq 0} \prod_{j=1}^n \frac{(i_j+1)!}{\lambda_j^{i_j+2}} \frac{\partial^n \log\tau(\mathbf{b}+x\mathbf{1},\mathbf{m};\epsilon)}{\partial m_{i_1}\cdots\partial m_{i_n}}$$
$$= (-1)^{n-1} \sum_{n\text{-cycles}} \prod_{j=1}^n B\big(\lambda_{\sigma(j)},\lambda_{\sigma(j+1)};x,\mathbf{b},\mathbf{m};\epsilon\big) - \frac{\delta_{n,2}}{(\lambda-\mu)^2}, \tag{4.23}$$

where $n\geq 2$ *and*

$$B(\lambda,\mu;x,\mathbf{b},\mathbf{m};\epsilon)$$
$$:= \frac{e^{\sigma(x-\epsilon,\mathbf{b},\mathbf{m};\epsilon)}}{\mu} D(\lambda,\mu;x,\mathbf{b},\mathbf{m};\epsilon) \frac{e^{\sum_{i\geq 0}\frac{2}{i!}\frac{b_i}{\epsilon}\mu^i(\log\mu-c_i)+\sum_{i\geq 0}\frac{m_i}{(i+1)!\epsilon}\mu^{i+1}}}{e^{\sum_{i\geq 0}\frac{2}{i!}\frac{b_i}{\epsilon}\lambda^i(\log\lambda-c_i)+\sum_{i\geq 0}\frac{m_i}{(i+1)!\epsilon}\lambda^{i+1}}} \frac{\mu^{\frac{x}{\epsilon}}}{\lambda^{\frac{x}{\epsilon}}}.$$

Proof. Let

$$\Psi_{\rm pair}(\lambda;x,\mathbf{b},\mathbf{m};\epsilon) := \begin{pmatrix} \psi_1(\lambda;x,\mathbf{b},\mathbf{m};\epsilon) & \psi_2(\lambda;x,\mathbf{b},\mathbf{m};\epsilon) \\ \psi_1(\lambda;x-\epsilon,\mathbf{b},\mathbf{m};\epsilon) & \psi_2(\lambda;x-\epsilon,\mathbf{b},\mathbf{m};\epsilon)\end{pmatrix}.$$

By using the arguments the same as in [44], one can show that

$$R(\lambda;x,\mathbf{b},\mathbf{m};\epsilon) = \Psi_{\rm pair}(\lambda;x,\mathbf{b},\mathbf{m};\epsilon)\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\Psi_{\rm pair}(\lambda;x,\mathbf{b},\mathbf{m};\epsilon)^{-1} \tag{4.24}$$

(see [44, the proof of Proposition 3]). It follows from (4.4) and (4.24) that for $n\geq 2$,

$$\epsilon^n \sum_{i_1,\dots,i_n\geq 0} \prod_{j=1}^n \frac{(i_j+1)!}{\lambda_j^{i_j+2}} \frac{\partial^n \log\tau(\mathbf{b}+x\mathbf{1},\mathbf{m})}{\partial m_{i_1}\cdots\partial m_{i_n}}$$
$$= (-1)^{n-1}\frac{e^{n\sigma(x-\epsilon,\mathbf{b},\mathbf{m};\epsilon)}}{\prod_{j=1}^n \lambda_j} \sum_{n\text{-cycles}} \prod_{j=1}^n D\big(\lambda_{\sigma(j)},\lambda_{\sigma(j+1)};x,\mathbf{b},\mathbf{m};\epsilon\big) - \frac{\delta_{n,2}}{(\lambda-\mu)^2}.$$

This yields identity (4.23). ∎

By taking $\mathbf{m}=\mathbf{0}$, we immediately obtain the following corollary.

Corollary 4.3. *We have*

$$\epsilon^n \sum_{i_1,\dots,i_n\geq 0} \prod_{j=1}^n \frac{(i_j+1)!}{\lambda_j^{i_j+2}} \frac{\partial^n \log\tau(\mathbf{b}+x\mathbf{1},\mathbf{m};\epsilon)}{\partial m_{i_1}\cdots\partial m_{i_n}}\bigg|_{\mathbf{m}=\mathbf{0}}$$
$$= (-1)^{n-1} \sum_{n\text{-cycles}} \prod_{j=1}^n B\big(\lambda_{\sigma(j)},\lambda_{\sigma(j+1)};x,\mathbf{b};\epsilon\big) - \frac{\delta_{n,2}}{(\lambda-\mu)^2}, \tag{4.25}$$

where $B(\lambda,\mu;x,\mathbf{b};\epsilon) := B(\lambda,\mu;x,\mathbf{b},\mathbf{m}=\mathbf{0};\epsilon)$.

In terms of the functions ϕ_1 and ϕ_2, the pair condition (4.22) reads

$$\phi_1(\lambda;x,\mathbf{b},\mathbf{m};\epsilon)\phi_2(\lambda;x-\epsilon,\mathbf{b},\mathbf{m};\epsilon)$$
$$- w(x,\mathbf{b},\mathbf{m};\epsilon)\lambda^{-2}\phi_2(\lambda;x,\mathbf{b},\mathbf{m};\epsilon)\phi_1(\lambda;x-\epsilon,\mathbf{b},\mathbf{m};\epsilon) \equiv 1, \tag{4.26}$$

and the function $B(\lambda,\mu;\mathbf{b},\mathbf{m};\epsilon)$ reads

$$B(\lambda,\mu;x,\mathbf{b},\mathbf{m};\epsilon) \tag{4.27}$$
$$:= \frac{\phi_1(\lambda;x,\mathbf{b},\mathbf{m};\epsilon)\phi_2(\mu;x-\epsilon,\mathbf{b},\mathbf{m};\epsilon) - \frac{w(x,\mathbf{b},\mathbf{m};\epsilon)}{\lambda\mu}\phi_1(\lambda;x-\epsilon,\mathbf{b},\mathbf{m};\epsilon)\phi_2(\mu;x,\mathbf{b},\mathbf{m};\epsilon)}{\lambda-\mu}.$$

Lemma 4.4. *The function $B(\lambda,\mu;x,\mathbf{b},\mathbf{m};\epsilon)$ admits the following expansion:*

$$B(\lambda,\mu;x,\mathbf{b},\mathbf{m};\epsilon)=\frac{1}{\lambda-\mu}+\sum_{i,j\geq 0}\frac{A_{i,j}(x,\mathbf{b},\mathbf{m};\epsilon)}{\lambda^{j+1}\mu^{i+1}}. \tag{4.28}$$

In particular,

$$B(\lambda,\mu;x,\mathbf{b};\epsilon)=\frac{1}{\lambda-\mu}+\sum_{i,j\geq 0}\frac{A_{i,j}(x,\mathbf{b};\epsilon)}{\lambda^{j+1}\mu^{i+1}}, \tag{4.29}$$

where $A_{i,j}(x,\mathbf{b};\epsilon)=A_{i,j}(x,\mathbf{b},\mathbf{m}=\mathbf{0};\epsilon)$.

Proof. Recall the following identity:

$$\phi_2(\mu)=\phi_2(\lambda)+\phi_2'(\lambda)(\mu-\lambda)+(\mu-\lambda)^2\partial_\lambda\left(\frac{\phi_2(\lambda)-\phi_2(\mu)}{\lambda-\mu}\right),$$

where we omit the arguments x, $\mathbf{b}$, $\mathbf{m}$ from $\phi_2(\lambda;x,\mathbf{b},\mathbf{m};\epsilon)$. Similarly,

$$\frac{\phi_2(\mu)}{\mu}=\frac{\phi_2(\lambda)}{\lambda}+\partial_\lambda\left(\frac{\phi_2(\lambda)}{\lambda}\right)(\mu-\lambda)+(\mu-\lambda)^2\partial_\lambda\left(\frac{\phi_2(\lambda)/\lambda-\phi_2(\mu)/\mu}{\lambda-\mu}\right).$$

Substituting these identities in (4.27) and using (4.26), (4.14) and (4.15), we find the validity of the expansion (4.28). ∎

Proof of Theorem 1.3. By using (4.25) and (4.29) and Corollary 2.2. ∎

Like in the previous section, if we write

$$\tau(\mathbf{b}+x\mathbf{1},\mathbf{m};\epsilon)=\tau_0(\mathbf{b}+x\mathbf{1};\epsilon)\tau_1(\mathbf{b}+x\mathbf{1},\mathbf{m};\epsilon),\qquad \tau_1(\mathbf{b}+x\mathbf{1},\mathbf{0};\epsilon)\equiv 1,$$

then by Theorem 1.3 we know that $\tau_1(\mathbf{b}+x\mathbf{1},\mathbf{m};\epsilon)$ is a KP tau-function in the big cell. The factor $\tau_0(\mathbf{b}+x\mathbf{1};\epsilon)$ can be determined from the definition of the tau-structure [9]. Let $\tilde\tau_1(x,\mathbf{b},\mathbf{m};\epsilon)$ be the KP tau-function associated to the point in the infinite Grassmannian with affine coordinates $A_{i,j}(x,\mathbf{b};\epsilon)$, with $A_{i,j}(x,\mathbf{b};\epsilon)$ given in (4.29). Then there exists $a_i(x,\mathbf{b};\epsilon)$, $i\geq 0$, such that

$$\tau_1(\mathbf{b}+x\mathbf{1},\mathbf{m};\epsilon)=\mathrm{e}^{\sum_{i\geq 0}a_i(x,\mathbf{b};\epsilon)m_i}\tilde\tau_1(x,\mathbf{b},\mathbf{m};\epsilon).$$

By further applying Proposition 2.4, we can find a formula for the affine coordinates for $\tau_1(\mathbf{b}+x\mathbf{1},\mathbf{m};\epsilon)$, whose dependence on b_0,x is through b_0+x.

5 Application to Gromov–Witten invariants of $\mathbb{P}^1$

In this section, we give an application of the above results regarding the topological solution to the ETH.

Recall that the *free energy* $\mathcal{F}^{\mathbb{P}^1}=\mathcal{F}^{\mathbb{P}^1}(\mathbf{b},\mathbf{m};\epsilon,q)$ *of GW invariants of* $\mathbb{P}^1$ is defined by

$$\mathcal{F}^{\mathbb{P}^1}=\sum_{k\geq 1}\frac{1}{k!}\sum_{\substack{1\leq\alpha_1,\dots,\alpha_k\leq 2\\ i_1,\dots,i_k\geq 0}}\prod_{j=1}^{k}t^{\alpha_j,i_j}\sum_{g\geq 0}\epsilon^{2g-2}\sum_{d\geq 0}q^d\langle\tau_{i_1}(\alpha_1)\cdots\tau_{i_k}(\alpha_k)\rangle_{g,d}^{\mathbb{P}^1}$$

where $b_i=t^{1,i}$, $m_i=t^{2,i}$, $i\geq 0$, and $\langle\tau_{i_1}(\alpha_1)\cdots\tau_{i_k}(\alpha_k)\rangle_{g,d}^{\mathbb{P}^1}$ are genus g and degree d GW invariants of $\mathbb{P}^1$ [20, 22, 37, 38]. When $\alpha_1=\cdots=\alpha_k=2$ the GW invariants $\langle\tau_{i_1}(\alpha_1)\cdots\tau_{i_k}(\alpha_k)\rangle_{g,d}^{\mathbb{P}^1}$

are called in the *stationary sector* [37, 38], and are denoted as $\langle\prod_{j=1}^n \tau_{i_j}(H)\rangle_{g,d}^{\mathbb{P}^1}$ in some literatures. The exponential

$$\exp \mathcal{F}^{\mathbb{P}^1}(\mathbf{b},\mathbf{m};\epsilon,q) =: Z^{\mathbb{P}^1}(\mathbf{b},\mathbf{m};\epsilon,q)$$

is called the *partition function of GW invariants of* $\mathbb{P}^1$. The restrictions

$$\mathcal{F}^{\mathbb{P}^1}(\mathbf{b}=\mathbf{0},\mathbf{m};\epsilon,q) =: \mathcal{F}^{\mathbb{P}^1}(\mathbf{m};\epsilon,q), \qquad Z^{\mathbb{P}^1}(\mathbf{b}=\mathbf{0},\mathbf{m};\epsilon,q) =: Z^{\mathbb{P}^1}(\mathbf{m};\epsilon,q)$$

are called the free energy and respectively the partition function of stationary GW invariants of $\mathbb{P}^1$, which are of particular interest due to their closed formulas [20] and their relations to the representations of the symmetric group [38].

It is proved by Dubrovin and Zhang [22] that assuming Virasoro constraints the partition function $Z^{\mathbb{P}^1}(\mathbf{b}+x\mathbf{1},\mathbf{m};\epsilon,q=1) =: Z^{\mathbb{P}^1}(\mathbf{b}+x\mathbf{1},\mathbf{m};\epsilon)$ of GW invariants of $\mathbb{P}^1$ is a particular tau-function for the ETH. (Note that by degree-dimension counting the Novikov variable q can be recovered via a rescaling $\epsilon\to q^{-1/2}\epsilon$, $b_i\to q^{(i-1)/2}b_i$, and $m_i\to q^{i/2}m_i$.) Then, as given in Introduction, the result of Dubrovin–Zhang together with Theorem 1.3 implies Corollary 1.5. As before, we can write the partition function $Z^{\mathbb{P}^1}(\mathbf{b}+x\mathbf{1},\mathbf{m};\epsilon)$ as follows:

$$Z^{\mathbb{P}^1}(\mathbf{b}+x\mathbf{1},\mathbf{m};\epsilon) = Z^{\mathbb{P}^1}_{\rm corr}(\mathbf{b}+x\mathbf{1};\epsilon)\cdot Z^{\mathbb{P}^1}_1(\mathbf{b}+x\mathbf{1},\mathbf{m};\epsilon),$$

where the factor $Z^{\mathbb{P}^1}_{\rm corr}(\mathbf{b}+x\mathbf{1};\epsilon)$ can be determined by using the definition of tau-function for the ETH [22], and the factor $Z^{\mathbb{P}^1}_1(\mathbf{b}+x\mathbf{1},\mathbf{m};\epsilon)$, which satisfies $Z^{\mathbb{P}^1}_1(\mathbf{b}+x\mathbf{1},\mathbf{0};\epsilon)\equiv 1$, is a particular KP tau-function in the big cell.

Denote $Z^{\mathbb{P}^1}_1(\mathbf{b}+x\mathbf{1},\mathbf{m};\epsilon,q)$ the normalized factor with q recovered, and by $A^{\mathbb{P}^1}_{i,j}(\mathbf{b}+x\mathbf{1};\epsilon,q)$ the affine coordinates of $Z^{\mathbb{P}^1}_1(\mathbf{b}+x\mathbf{1},\mathbf{m};\epsilon,q)$. The restrictions of these affine coordinates to $\mathbf{b}=\mathbf{0}$, $x=0$ are of particular interest, and will be denoted by $A^{\mathbb{P}^1}_{i,j}(\epsilon,q)$. In the next two subsections, we will derive two formulas for $A^{\mathbb{P}^1}_{i,j}(q) := A^{\mathbb{P}^1}_{i,j}(1,q)$.

5.1 Explicit generating series for $A^{\mathbb{P}^1}_{i,j}(q)$

It was proved in [20, Theorems 1–5 and equation (93)] (see also [19, 33, 44]) that, for each $n\ge 2$,

$$\sum_{i_1,\dots,i_n\ge 0}\prod_{j=1}^n \frac{(i_j+1)!}{\xi_j^{i_j+2}}\sum_{g\ge0}\sum_{d\ge0} q^d\left\langle\prod_{j=1}^n \tau_{i_j}(2)\right\rangle_{g,d}^{\mathbb{P}^1}$$
$$= (-1)^{n-1}\sum_{n\text{-cycles}}\prod_{j=1}^n B\big(\xi_{\sigma(j)},\xi_{\sigma(j+1)};q\big) - \frac{\delta_{n,2}}{(\xi_1-\xi_2)^2},$$

where $B(\lambda,\mu;q)$ is the asymptotic expansion of the following analytic function

$$-\frac{1}{\mu-\lambda}\,{}_2F_3\left(\frac{\lambda-\mu}{2},\frac{\lambda-\mu+1}{2};\,\frac12-\mu\,,\,\frac12+\lambda\,,\lambda-\mu+1;\,-4q\right) \tag{5.1}$$

for $\lambda,\mu\notin\mathbb{Z}+\frac12$ as $\lambda,\mu\to\infty$, and is explicitly given by

$$B(\lambda,\mu;q) = \frac{1}{\lambda-\mu} \tag{5.2}$$
$$-\sum_{i,j\ge0}\frac{(-1)^{j+1}}{\lambda^{j+1}\mu^{i+1}}\sum_{k\ge1}\frac{q^k}{k!}\sum_{1\le i_1,j_1\le k}(-1)^{i_1+j_1}\frac{(i_1+j_1-2k)_{k-1}\big(i_1-\frac12\big)^i\big(j_1-\frac12\big)^j}{(i_1-1)!(j_1-1)!(k-i_1)!(k-j_1)!}.$$

If we define $A^{\mathbb{P}^1,+}_{i,j}(q)$ via

$$\sum_{i,j\geq 0} A^{\mathbb{P}^1,+}_{i,j}(q)\lambda^{-j-1}\mu^{-i-1} := B(\lambda,\mu;q) - \frac{1}{\lambda-\mu} \tag{5.3}$$

and let $Z^{\mathbb{P}^1,+}(\mathbf{m},q)$ be the KP tau-function associated to the point in Gr_0 with the affine coordinates $A^{\mathbb{P}^1,+}_{i,j}(q)$ (i.e., using formula (2.2)), then we know that $Z^{\mathbb{P}^1}_1(\mathbf{0},\mathbf{m};1,q) = Z^{\mathbb{P}^1}(\mathbf{m};1,q)$ and $Z^{\mathbb{P}^1,+}(\mathbf{m},q)$ can only differ by multiplication by the exponential of a particular linear function of $\mathbf{m}$.

Proposition 5.1. *We have*

$$\log Z^{\mathbb{P}^1,+}(\mathbf{m},q) = \sum_{n\geq 1}\frac{1}{n!}\sum_{i_1,\dots,i_n\geq 0}\prod_{j=1}^n m_{i_j}\sum_{g\geq 0}\sum_{d>0} q^d\left\langle \prod_{j=1}^n \tau_{i_j}(2)\right\rangle^{\mathbb{P}^1}_{g,d}. \tag{5.4}$$

Proof. We already know that $\log Z^{\mathbb{P}^1}(\mathbf{m};1,q) - \log Z^{\mathbb{P}^1,+}(\mathbf{m},q)$ is a linear function of $\mathbf{m}$. Below, we will often omit the arguments "1, q", "q" for simplifying the notations. According to, e.g., [38, p. 529], $\langle\prod_{j=1}^n \tau_{i_j}(2)\rangle^{\mathbb{P}^1}_{g,0}$ vanish whenever $n\geq 2$, thus we get the validity of the nonlinear part of (5.4). By using [20, equations (31), (32) and (34)] and the asymptotic expansion of the digamma function

$$\psi\left(\xi+\frac{1}{2}\right) \sim \log\xi + \sum_{m\geq 2}(-1)^{m-1}\left(2^{1-m}-1\right)\frac{B_m}{m}\xi^{-m}$$

as ξ being large with $\arg\xi < \pi-\epsilon$, we calculate out the linear function

$$\log Z^{\mathbb{P}^1}(\mathbf{m}) - \log Z^{\mathbb{P}^1,+}(\mathbf{m}) = -\sum_{k\geq 1}\left(1-2^{-k}\right)\zeta(-k)\frac{m_{k-1}}{(k-1)!}, \tag{5.5}$$

with $\zeta(-k)$ given by

$$\zeta(-k) = (-1)^k\frac{B_{k+1}}{k+1}, \qquad k\geq 1.$$

Here B_j denotes the j-th Bernoulli number. The proposition is then proved by identifying this linear function with the degree zero part of stationary $\mathbb{P}^1$ free energy (see [20, p. 163] or [38, equation (0.26)]). ■

Proposition 5.1, formula (5.5) and Proposition 2.4 imply the following theorem.

Theorem 5.2. *The generating series of the affine coordinates $A^{\mathbb{P}^1}_{i,j}(q)$ is given by*

$$\sum_{i,j\geq 0} A^{\mathbb{P}^1}_{i,j}(q)\xi^{-j-1}\eta^{-i-1} = B(\xi,\eta;q)\,\frac{\mathrm{e}^{-\left(\log\frac{\Gamma(\xi+\frac{1}{2})}{\sqrt{2\pi}}-\xi\log\xi+\xi\right)}}{\mathrm{e}^{-\left(\log\frac{\Gamma(\eta+\frac{1}{2})}{\sqrt{2\pi}}-\eta\log\eta+\eta\right)}} - \frac{1}{\xi-\eta},$$

where $B(\xi,\eta;q)$ is given by (5.1) (cf. (5.2)), and $\Gamma\left(\xi+\frac{1}{2}\right)$ is understood as its asymptotic expansion for ξ large with $\arg\xi<\pi-\epsilon$.

Remark 5.3. We consider the computations of $\log Z^{\mathbb{P}^1}_0(\mathbf{b})$ and $A^{\mathbb{P}^1}_{i,j}(\mathbf{b})$ by using Virasoro constraints [22] and the ETH in a future publication.

5.2 Affine coordinates from the Okounkov–Pandharipande formula

We proceed with recalling the Okounkov–Pandharipande formula [38] for $Z^{\mathbb{P}^1}(\mathbf{m})$, which is derived from the GW/H (Gromov–Witten/Hurwitz) correspondence established by Okounkov and Pandharipande [38].

To consider the GW invariants of $\mathbb{P}^1$, one has to consider all the moduli spaces $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)$ of stable maps, where g is the genus of the domain curve, n is the number of marked point on the domain curve, d is the degree of the map. For the purpose of GW/H correspondence, it is necessary to consider the $n = 0$ case. The expected dimension of $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d)$ is $2g - 2 + 2d$. When

$$2g - 2 + 2d = 0,$$

we have a contribution of q^d to the free energy. This happens only when $(g, d) = (0, 1)$ and $(g, d) = (1, 0)$, The first case corresponds to $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, 1)$, and the second case is impossible.

With the contribution of zero point correlators, the free energy of the stationary GW invariants of $\mathbb{P}^1$ is then

$$\tilde{\mathcal{F}}^{\mathbb{P}^1}(\mathbf{m}) = q + \mathcal{F}^{\mathbb{P}^1}(\mathbf{m}).$$

Let $\tilde{Z}^{\mathbb{P}^1}(\mathbf{m}) = \exp \tilde{F}^{\mathbb{P}^1}(\mathbf{m})$ be the corrected partition function. Then

$$\tilde{Z}^{\mathbb{P}^1}(\mathbf{m}) = \mathrm{e}^q \cdot Z^{\mathbb{P}^1}(\mathbf{m}). \tag{5.6}$$

As a corollary to Corollary 1.5, we get the following.

Corollary 5.4. *The corrected partition function $\tilde{Z}^{\mathbb{P}^1}(\mathbf{m})$ is a KP tau-function.*

The corrected partition function $\tilde{Z}^{\mathbb{P}^1}$ has the following expansion:

$$\tilde{Z}^{\mathbb{P}^1}(\mathbf{m}) = \sum_{d\geq 0} q^d \sum_{n\geq 0} \sum_{i_1,\dots,i_n\geq 0} \frac{m_{i_1}\cdots m_{i_n}}{n!} \left\langle \prod_{j=1}^n \tau_{i_j}(2) \right\rangle_d^{\bullet\mathbb{P}^1},$$

where $\langle \prod_{j=1}^n \tau_{i_j}(2)\rangle_d^{\bullet\mathbb{P}^1}$ denotes the *not-necessarily connected* GW invariants of $\mathbb{P}^1$ of degree d in the stationary sector. By using the GW/H correspondence Okounkov and Pandharipande obtained [38] the following formula:

$$\left\langle \prod_{j=1}^n \tau_{i_j}(2) \right\rangle_d^{\bullet\mathbb{P}^1} = \sum_{\lambda\in\mathcal{P}_d} \left(\frac{\dim V^\lambda}{d!}\right)^2 \prod_{j=1}^n \frac{\mathbf{p}_{i_j+1}(\lambda)}{(i_j+1)!}, \tag{5.7}$$

where V^λ is the irreducible representation of S_d indexed by λ, and $\mathbf{p}_k(\lambda)$ is defined by

$$\mathbf{p}_k(\lambda) = \sum_{i\geq 1} \left[\left(\lambda_i - i + \frac{1}{2}\right)^k - \left(-i + \frac{1}{2}\right)^k\right] + \left(1 - 2^{-k}\right)\zeta(-k),$$

where $k \geq 1$. Dubrovin [17] gave a different proof of formula (5.7) using symplectic field theory. (We would like to mention that $\mathbf{p}_k(\lambda)$ are generators of shifted symmetric functions that play a key role in the Bloch–Okounkov theorem [7], and are explained in detail in, e.g., [7, 47].)

We now have

$$\begin{aligned}\tilde{Z}^{\mathbb{P}^1}(\mathbf{m}) &= \sum_{d\geq 0} q^d \sum_{n\geq 0} \sum_{i_1,\dots,i_n\geq 0} \frac{m_{i_1}\cdots m_{i_n}}{n!} \sum_{\lambda\in\mathcal{P}_d} \left(\frac{\dim V^\lambda}{d!}\right)^2 \prod_{j=1}^n \frac{\mathbf{p}_{i_j+1}(\lambda)}{(i_j+1)!} \\ &= \sum_{d\geq 0} \frac{q^d}{d!} \sum_{\lambda\in\mathcal{P}_d} \frac{\left(\dim V^\lambda\right)^2}{d!} \exp \sum_{i\geq 0} \frac{m_i}{(i+1)!}\mathbf{p}_{i+1}(\lambda).\end{aligned}$$

When all m_i's are taken to be 0, we get the vacuum expectation value

$$\tilde{Z}^{\mathbb{P}^1}(\mathbf{m}=\mathbf{0}) = \sum_{d\geq 0} \frac{q^d}{d!} \sum_{\lambda\in\mathcal{P}_d} \frac{\left(\dim V^\lambda\right)^2}{d!} = \mathrm{e}^q.$$

By the definition (5.6), we have

$$\begin{aligned} Z^{\mathbb{P}^1}(\mathbf{m}) &= \mathrm{e}^{-q}\tilde{Z}^{\mathbb{P}^1}(\mathbf{m}) \\ &= \sum_{d\geq 0} \frac{q^d}{d!}\mathrm{e}^{-q} \sum_{\lambda\in\mathcal{P}_d} \frac{\left(\dim V^\lambda\right)^2}{d!} \exp\sum_{i\geq 0}\frac{m_i}{(i+1)!}\mathbf{p}_{i+1}(\lambda). \end{aligned}$$

Now note that $\big\{\big\{\frac{q^d}{d!}\mathrm{e}^{-q}\frac{(\dim V^\lambda)^2}{d!}\big\}_{\lambda\in\mathcal{P}_d}\big\}_{d\geq 0}$ is the Poissonized Plancherel measure on the set of partitions and $\exp\sum_{i\geq 0}\frac{m_i}{(i+1)!}\mathbf{p}_{i+1}$ is a random variable on the set of partitions, so the partition function $Z^{\mathbb{P}^1}(\mathbf{m})$ is just the expectation value,

$$Z^{\mathbb{P}^1}(\mathbf{m}) = \left\langle \exp\sum_{i\geq 0}\frac{m_i}{(i+1)!}\mathbf{p}_{i+1}(\lambda)\right\rangle_{\text{Plancherel}}.$$

Now we notice that $p_k = \frac{m_{k-1}}{(k-1)!}$, $k\geq 1$, so we get

$$Z^{\mathbb{P}^1}(\mathbf{m}) = \sum_{d\geq 0} \frac{q^d}{d!}\mathrm{e}^{-q} \sum_{\lambda\in\mathcal{P}_d} \frac{\left(\dim V^\lambda\right)^2}{d!} \exp\sum_{k\geq 1}\frac{p_k}{k}\mathbf{p}_k(\lambda).$$

Recall

$$\exp\sum_{k\geq 1}\frac{1}{k}p_k p'_k = \sum_{\mu\in\mathcal{P}} s_\mu(\mathbf{p})s_\mu(\mathbf{p}'),$$

where s_μ is the Schur polynomial (cf. (2.1)) which has the expression

$$s_\mu(\mathbf{p}) = \sum_{\nu\in\mathcal{P}} \frac{\chi^\mu_\nu}{z_\nu}p_\nu, \qquad \mu\in\mathcal{P} \tag{5.8}$$

(see Macdonald [32] for the details). Here, χ^μ_ν denotes the character table, $z_\nu = \prod_{i=1}^\infty i^{m_i(\nu)}m_i(\nu)!$, and $p_\nu = p_{\nu_1}\cdots p_{\nu_{\ell(\nu)}}$. It follows that

$$Z^{\mathbb{P}^1}(\mathbf{m}) = \sum_{d\geq 0} \frac{q^d}{d!}\mathrm{e}^{-q} \sum_{\lambda\in\mathcal{P}_d} \frac{\left(\dim V^\lambda\right)^2}{d!} \sum_{\mu\in\mathcal{P}} s_\mu(\mathbf{p})\mathbf{s}_\mu(\lambda) = \sum_{\mu\in\mathcal{P}} \langle\mathbf{s}_\mu\rangle_{\text{Plancherel}}\cdot s_\mu(\mathbf{p}).$$

Together with Corollary 1.5, we have the following.

Theorem 5.5. *The affine coordinates $A^{\mathbb{P}^1}_{i,j}$ of $Z^{\mathbb{P}^1}(\mathbf{m})$ are given by the Plancherel expectation value of $\mathbf{s}_{(i+1,1^j)}$:*

$$A^{\mathbb{P}^1}_{i,j} = (-1)^j\left\langle\mathbf{s}_{(i+1,1^j)}\right\rangle_{\text{Plancherel}},$$

or explicitly,

$$A^{\mathbb{P}^1}_{i,j} = (-1)^j \sum_{d\geq 0} \frac{q^d}{d!}\mathrm{e}^{-q} \sum_{\lambda\in\mathcal{P}_d} \frac{\left(\dim V^\lambda\right)^2}{d!}\mathbf{s}_{(i+1,1^j)}(\lambda).$$

Combining Theorems 5.2 and 5.5, we have the following.

Corollary 5.6. *The following identity holds:*

$$\begin{aligned}\frac{1}{\xi-\eta}&+\sum_{i,j\geq 0}(-1)^j\sum_{d\geq 0}\frac{q^d}{d!}\mathrm{e}^{-q}\sum_{\lambda\in\mathcal{P}_d}\frac{\left(\dim V^\lambda\right)^2}{d!}\frac{\mathbf{s}_{(i+1,1^j)}(\lambda)}{\xi^{j+1}\eta^{i+1}}\\&=B(\xi,\eta;q)\,\frac{\mathrm{e}^{-\left(\log\frac{\Gamma(\xi+\frac12)}{\sqrt{2\pi}}-\xi\log\xi+\xi\right)}}{\mathrm{e}^{-\left(\log\frac{\Gamma(\eta+\frac12)}{\sqrt{2\pi}}-\eta\log\eta+\eta\right)}},\end{aligned}\tag{5.9}$$

where $\Gamma\big(\xi+\frac12\big)$ *is understood as its asymptotic expansion for* ξ *large with* $\arg\xi<\pi-\epsilon$, *and* $B(\xi,\eta;q)$ *is given by* (5.2) (*cf.* (5.1)).

For example,

$$\begin{aligned}A^{\mathbb{P}^1}_{0,0}&=\sum_{d\geq 0}\frac{q^d}{d!}\mathrm{e}^{-q}\sum_{\lambda\in\mathcal{P}_d}\frac{\left(\dim V^\lambda\right)^2}{d!}\mathbf{s}_{(1)}(\lambda)\\&=\mathrm{e}^{-q}\biggl[-\frac{1}{24}+\frac{23}{24}q+\left(\left(\frac{1}{2!}\right)^2\frac{47}{24}+\left(\frac{1}{2!}\right)^2\frac{47}{24}\right)q^2\\&\qquad+\left(\left(\frac{1}{3!}\right)^2\frac{71}{24}+\left(\frac{2}{3!}\right)^2\frac{71}{24}+\left(\frac{1}{3!}\right)^2\frac{71}{24}\right)q^3+\cdots\biggr]\\&=-\frac{1}{24}+q,\\A^{\mathbb{P}^1}_{1,0}&=\sum_{d\geq 0}\frac{q^d}{d!}\mathrm{e}^{-q}\sum_{\lambda\in\mathcal{P}_d}\frac{\left(\dim V^\lambda\right)^2}{d!}\mathbf{s}_{(2)}(\lambda)\\&=\sum_{d\geq 0}\frac{q^d}{d!}\mathrm{e}^{-q}\sum_{\lambda\in\mathcal{P}_d}\frac{\left(\dim V^\lambda\right)^2}{d!}\left(\frac12\mathbf{p}_{(2)}(\lambda)+\frac12\mathbf{p}_1(\lambda)^2\right)\\&=\frac12\mathrm{e}^{-q}\biggl[0+0\cdot q+\left(\left(\frac{1}{2!}\right)^2\cdot 2+\left(\frac{1}{2!}\right)^2\cdot(-2)\right)q^2\\&\qquad+\left(\left(\frac{1}{3!}\right)^2\cdot 6+\left(\frac{2}{3!}\right)^2\cdot 0+\left(\frac{1}{3!}\right)^2\cdot(-6)\right)q^3+\cdots\\&\qquad+\left(-\frac{1}{24}\right)^2+\left(\frac{23}{24}\right)^2q+\left(\left(\frac{1}{2!}\right)^2\left(\frac{47}{24}\right)^2+\left(\frac{1}{2!}\right)^2\left(\frac{47}{24}\right)^2\right)q^2\\&\qquad+\left(\left(\frac{1}{3!}\right)^2\left(\frac{71}{24}\right)^2+\left(\frac{2}{3!}\right)^2\left(\frac{71}{24}\right)^2+\left(\frac{1}{3!}\right)^2\left(\frac{71}{24}\right)^2\right)q^3+\cdots\biggr]\\&=\frac{1}{1152}+\frac{11q}{24}+\frac{q^2}{2}.\end{aligned}$$

These match with the right-hand side of (5.9).

Inspired by Han's conjecture [27], it was proved by Stanley [42] that

$$\langle\mathbf{p}_\mu\rangle_d:=\sum_{\lambda\in\mathcal{P}_d}\frac{\left(\dim V^\lambda\right)^2}{d!}\mathbf{p}_\mu(\lambda)$$

is a polynomial in d of degree $|\mu|$, where $\mathbf{p}_\mu(\lambda):=\prod_{j=1}^{\ell(\mu)}\mathbf{p}_{\mu_j}(\lambda)$. Note that Han's conjecture is on the polynomiality of $\langle\mathbf{p}_k\rangle_d$ (confirmed also in [40]). It follows that

$$\langle\mathbf{s}_\mu\rangle_d:=\sum_{\lambda\in\mathcal{P}_d}\frac{\left(\dim V^\lambda\right)^2}{d!}\mathbf{s}_\mu(\lambda)$$

is a polynomial in d of degree $|\mu|$, where $\mathbf{s}_\mu(\lambda) := s_\mu(\mathbf{p}(\lambda))$ with s_μ denoting Schur functions as in (5.8) or (2.1). For example,

$$\begin{aligned}
&\langle \mathbf{p}_{(1)} \rangle_d = d - \frac{1}{24},\\
&\langle \mathbf{p}_{(2)} \rangle_d = 0,\\
&\langle \mathbf{p}_{(1^2)} \rangle_d = \left(d - \frac{1}{24}\right)^2,\\
&\langle \mathbf{p}_{(3)} \rangle_d = \frac{3d^2}{2} - \frac{5d}{4} + \frac{7}{960},\\
&\langle \mathbf{p}_{(21)} \rangle_d = 0,\\
&\langle \mathbf{p}_{(1^3)} \rangle_d = d^3 - \frac{1}{8}d^2 + \frac{1}{192}d - \frac{1}{13824},
\end{aligned}$$

and from these explicit expressions we get

$$\begin{aligned}
&\langle \mathbf{s}_{(1)} \rangle_d = d - \frac{1}{24},\\
&\langle \mathbf{s}_{(2)} \rangle_d = \langle \mathbf{s}_{(1^2)} \rangle_d = \left(d - \frac{1}{24}\right)^2,\\
&\langle \mathbf{s}_{(3)} \rangle_d = \langle \mathbf{s}_{(1^3)} \rangle = \frac{d^3}{6} + \frac{23}{48}d^2 - \frac{479}{1152}d + \frac{1003}{414720},\\
&\langle \mathbf{s}_{(21)} \rangle_d = \frac{d^3}{3} - \frac{13}{24}d^2 + \frac{241}{576}d - \frac{509}{207360}.
\end{aligned}$$

When μ are hook partitions, Corollary 5.6 not only confirms polynomiality of $\langle \mathbf{s}_\mu \rangle_d$ (so of the original Han's conjecture as well) but also leads to elementary formulas for them. The explicit expressions, as well as general ones using results in [18, 19, 20, 33], will be given in a subsequent publication.

Note that by (5.5), we have

$$Z^{\mathbb{P}^1}(\mathbf{m}) = Z^{+,\mathbb{P}^1}(\mathbf{m}) \cdot \exp \sum_{k\geq 1} (1 - 2^{-k})\zeta(-k) \frac{m_{k-1}}{(k-1)!},$$

so we can also define

$$\hat{\mathbf{p}}_k(\lambda) = \sum_{i\geq 1} \left[\left(\lambda_i - i + \frac{1}{2}\right)^k - \left(-i + \frac{1}{2}\right)^k \right].$$

Then we repeat everything we have done in this subsection by replacing every $\mathbf{p}_k$ with $\hat{\mathbf{p}}_k$ to get

$$Z^{\mathbb{P}^1,+}(\mathbf{m}) = \sum_{d\geq 0} \frac{q^d}{d!} \mathrm{e}^{-q} \sum_{\lambda\in\mathcal{P}_d} \frac{(\dim V^\lambda)^2}{d!} \exp \sum_{i\geq 0} \frac{m_i}{(i+1)!} \hat{\mathbf{p}}_{i+1}(\lambda),$$

and

$$Z^{\mathbb{P}^1,+}(\mathbf{m}) = \left\langle \exp \sum_{i\geq 0} \frac{m_i}{(i+1)!} \hat{\mathbf{p}}_{i+1}(\lambda) \right\rangle_{\text{Plancherel}}.$$

And if we set

$$\hat{\mathbf{s}}_\mu = \sum_{\nu\in\mathcal{P}} \frac{\chi^\mu_\nu}{z_\nu} \hat{\mathbf{p}}_\nu,$$

then we have

$$Z^{\mathbb{P}^1,+}(\mathbf{m}) = \sum_{d\geq 0} \frac{q^d}{d!} e^{-q} \sum_{\lambda\in\mathcal{P}_d} \frac{\left(\dim V^\lambda\right)^2}{d!} \sum_{\mu\in\mathcal{P}} s_\mu(\mathbf{p})\hat{\mathbf{s}}_\mu(\lambda) = \sum_{\mu\in\mathcal{P}} \langle \hat{\mathbf{s}}_\mu\rangle_{\text{Plancherel}} \cdot s_\mu(\mathbf{p}).$$

Theorem 5.7. *The affine coordinates $A_{i,j}^{\mathbb{P}^1,+}$ of $Z^{\mathbb{P}^1,+}(\mathbf{m})$ are given by the Plancherel expectation value of $\hat{\mathbf{s}}_{(i+1,1^j)}$,*

$$A_{i,j}^{\mathbb{P}^1,+} = (-1)^j \langle \hat{\mathbf{s}}_{(i+1,1^j)}\rangle_{\text{Plancherel}},$$

or explicitly,

$$A_{i,j}^{\mathbb{P}^1,+} = (-1)^j \sum_{d\geq 0} \frac{q^d}{d!} e^{-q} \sum_{\lambda\in\mathcal{P}_d} \frac{\left(\dim V^\lambda\right)^2}{d!} \hat{\mathbf{s}}_{(i+1,1^j)}(\lambda).$$

Combining (5.3) and Theorem 5.7, we have the following.

Corollary 5.8. *The following identity holds:*

$$\sum_{i,j} (-1)^j \sum_{d\geq 0} \frac{q^d}{d!} e^{-q} \sum_{\lambda\in\mathcal{P}_d} \frac{\left(\dim V^\lambda\right)^2}{d!} \frac{\hat{\mathbf{s}}_{(i+1,1^j)}(\lambda)}{\xi^{j+1}\eta^{i+1}} = B(\xi,\eta;q), \tag{5.10}$$

where $B(\xi,\eta;q)$ is given by (5.2).

For example,

$$\begin{aligned}
A_{0,0}^{\mathbb{P}^1,+} &= \sum_{d\geq 0} \frac{q^d}{d!} e^{-q} \sum_{\lambda\in\mathcal{P}_d} \frac{\left(\dim V^\lambda\right)^2}{d!} \hat{\mathbf{s}}_{(1)}(\lambda) \\
&= e^{-q}\Bigg[0 + 1\cdot q + \left(\left(\frac{1}{2!}\right)^2 \cdot 2 + \left(\frac{1}{2!}\right)^2 \cdot 2\right) q^2 \\
&\qquad + \left(\left(\frac{1}{3!}\right)^2 \cdot 3 + \left(\frac{2}{3!}\right)^2 \cdot 3 + \left(\frac{1}{3!}\right)^2 \cdot 3\right) q^3 + \cdots \Bigg] = q, \\
A_{1,0}^{\mathbb{P}^1,+} &= \sum_{d\geq 0} \frac{q^d}{d!} e^{-q} \sum_{\lambda\in\mathcal{P}_d} \frac{\left(\dim V^\lambda\right)^2}{d!} \hat{\mathbf{s}}_{(2)}(\lambda) \\
&= \sum_{d\geq 0} \frac{q^d}{d!} e^{-q} \sum_{\lambda\in\mathcal{P}_d} \frac{\left(\dim V^\lambda\right)^2}{d!} \left(\frac{1}{2}\hat{\mathbf{p}}_{(2)}(\lambda) + \frac{1}{2}\hat{\mathbf{p}}_{(1)}(\lambda)^2\right) \\
&= \frac{1}{2} e^{-q}\Bigg[0 + 0\cdot q + \left(\left(\frac{1}{2!}\right)^2 \cdot 2 + \left(\frac{1}{2!}\right)^2 \cdot (-2)\right) q^2 \\
&\qquad + \left(\left(\frac{1}{3!}\right)^2 \cdot 6 + \left(\frac{2}{3!}\right)^2 \cdot 0 + \left(\frac{1}{3!}\right)^2 \cdot (-6)\right) q^3 + \cdots \\
&\qquad + 0^2 q + 1^2 q + \left(\left(\frac{1}{2!}\right)^2 \cdot 2^2 + \left(\frac{1}{2!}\right)^2 \cdot 2^2\right) q^2 \\
&\qquad + \left(\left(\frac{1}{3!}\right)^2 \cdot 3^2 + \left(\frac{2}{3!}\right)^2 \cdot 3^2 + \left(\frac{1}{3!}\right)^2 \cdot 3^2\right) q^3 + \cdots \Bigg] = \frac{q}{2} + \frac{q^2}{2}.
\end{aligned}$$

These match with the right-hand side of (5.10). The Plancherel averages of $\hat{\mathbf{p}}_\mu$ and $\hat{\mathbf{s}}_\mu$ are again polynomials, but they seem to be simpler than the averages of $\mathbf{p}_\mu$ and $\mathbf{s}_\mu$. For example,

$$\langle \hat{\mathbf{p}}_1 \rangle_d = d, \qquad \langle \hat{\mathbf{p}}_2 \rangle_d = 0, \qquad \langle \hat{\mathbf{p}} \rangle_1^2 = d^2,$$
$$\langle \hat{\mathbf{p}}_3 \rangle_d = \frac{3}{2}d^2 - \frac{5d}{4}, \qquad \langle \hat{\mathbf{p}}_2\hat{\mathbf{p}}_1 \rangle_d = 0, \qquad \langle \hat{\mathbf{p}}_1^3 \rangle_d = d^3,$$

and from these we get

$$\langle \hat{\mathbf{s}}_{(1)} \rangle_d = d,$$
$$\langle \hat{\mathbf{s}}_{(2)} \rangle_d = \langle \hat{\mathbf{s}}_{(1^2)} \rangle = \frac{d^2}{2},$$
$$\langle \hat{\mathbf{s}}_{(3)} \rangle_d = \langle \hat{\mathbf{s}}_{(1^3)} \rangle = \frac{d^3}{6} + \frac{1}{2}d^2 - \frac{5}{12}d,$$
$$\langle \hat{\mathbf{s}}_{(21)} \rangle_d = \frac{d^3}{3} - \frac{1}{2}d^2 + \frac{5}{12}d.$$

Acknowledgements

One of the authors D.Y. is grateful to Marco Bertola, Boris Dubrovin and Youjin Zhang for their advice. We thank Don Zagier for several insightful and helpful comments. We also thank the anonymous referees for constructive comments that help to improve the presentation. The work is supported by NSFC (No. 12371254, No. 11890662) and CAS No. YSBR-032.

References

[1] Adler M., van Moerbeke P., Matrix integrals, Toda symmetries, Virasoro constraints, and orthogonal polynomials, *Duke Math. J.* **80** (1995), 863–911, arXiv:solv-int/9706010.

[2] Alexandrov A., Bychkov B., Dunin-Barkowski P., Kazarian M., Shadrin S., KP integrability through the $x - y$ swap relation, *Selecta Math. (N.S.)* **31** (2025), 42, 37 pages, arXiv:2309.12176.

[3] Ambjørn J., Chekhov L., The matrix model for dessins d'enfants, *Ann. Inst. Henri Poincaré D* **1** (2014), 337–361, arXiv:1404.4240.

[4] Balogh F., Yang D., Geometric interpretation of Zhou's explicit formula for the Witten–Kontsevich tau function, *Lett. Math. Phys.* **107** (2017), 1837–1857, arXiv:1412.4419.

[5] Bertola M., Dubrovin B., Yang D., Correlation functions of the KdV hierarchy and applications to intersection numbers over $\overline{\mathcal{M}}_{g,n}$, *Phys. D* **327** (2016), 30–57, arXiv:1504.06452.

[6] Bertola M., Dubrovin B., Yang D., Simple Lie algebras, Drinfeld–Sokolov hierarchies, and multi-point correlation functions, *Mosc. Math. J.* **21** (2021), 233–270, arXiv:1610.07534.

[7] Bloch S., Okounkov A., The character of the infinite wedge representation, *Adv. Math.* **149** (2000), 1–60, arXiv:alg-geom/9712009.

[8] Carlet G., Extended Toda hierarchy and its Hamiltonian structure, Ph.D. Thesis, International School for Advanced Studies, 2003, available at `https://hdl.handle.net/20.500.11767/4024`.

[9] Carlet G., Dubrovin B., Zhang Y., The extended Toda hierarchy, *Mosc. Math. J.* **4** (2004), 313–332, arXiv:nlin.SI/0306060.

[10] Carlet G., van de Leur J., Hirota equations for the extended bigraded Toda hierarchy and the total descendent potential of $\mathbb{CP}^1$ orbifolds, *J. Phys. A* **46** (2013), 405205, 16 pages, arXiv:1304.1632.

[11] Deift P.A., Orthogonal polynomials and random matrices: a Riemann–Hilbert approach, *Courant Lect. Notes Math.*, Vol. 3, American Mathematical Society, Providence, RI, 1999.

[12] Dickey L.A., Soliton equations and Hamiltonian systems, 2nd ed., *Adv. Ser. Math. Phys.*, Vol. 26, World Scientific, River Edge, NJ, 2003.

[13] Dijkgraaf R., Witten E., Mean field theory, topological field theory, and multi-matrix models, *Nuclear Phys. B* **342** (1990), 486–522.

[14] Dubrovin B., Integrable systems and classification of 2-dimensional topological field theories, in Integrable Systems (Luminy, 1991), *Progr. Math.*, Vol. 115, Birkhäuser, Boston, MA, 1993, 313–359, arXiv:hep-th/9209040.

[15] Dubrovin B., Geometry of 2D topological field theories, in Integrable Systems and Quantum Groups (Montecatini Terme, 1993), *Lecture Notes in Math.*, Vol. 1620, Springer, Berlin, 1996, 120–348, arXiv:hep-th/9407018.

[16] Dubrovin B., Hamiltonian perturbations of hyperbolic PDEs: from classification results to the properties of solutions, in New Trends in Mathematical Physics, Springer, Dordrecht, 2009, 231–276.

[17] Dubrovin B., Symplectic field theory of a disk, quantum integrable systems, and Schur polynomials, *Ann. Henri Poincaré* **17** (2016), 1595–1613, arXiv:1407.5824.

[18] Dubrovin B., Yang D., Generating series for GUE correlators, *Lett. Math. Phys.* **107** (2017), 1971–2012, arXiv:1604.07628.

[19] Dubrovin B., Yang D., On Gromov–Witten invariants of $\mathbb{P}^1$, *Math. Res. Lett.* **26** (2019), 729–748, arXiv:1702.01669.

[20] Dubrovin B., Yang D., Zagier D., Gromov–Witten invariants of the Riemann sphere, *Pure Appl. Math. Q.* **16** (2020), 153–190, arXiv:1802.00711.

[21] Dubrovin B., Zhang Y., Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov–Witten invariants, arXiv:math.DG/0108160.

[22] Dubrovin B., Zhang Y., Virasoro symmetries of the extended Toda hierarchy, *Comm. Math. Phys.* **250** (2004), 161–193, arXiv:math.DG/0308152.

[23] Eguchi T., Hori K., Yang S.-K., Topological σ models and large-N matrix integral, *Internat. J. Modern Phys. A* **10** (1995), 4203–4224, arXiv:hep-th/9503017.

[24] Eguchi T., Yang S.-K., The topological $\mathbf{C}\mathrm{P}^1$ model and the large-N matrix integral, *Modern Phys. Lett. A* **9** (1994), 2893–2902, arXiv:hep-th/9407134.

[25] Gerasimov A., Marshakov A., Mironov A., Morozov A., Orlov A., Matrix models of two-dimensional gravity and Toda theory, *Nuclear Phys. B* **357** (1991), 565–618.

[26] Getzler E., The Toda conjecture, in Symplectic Geometry and Mirror Symmetry (Seoul, 2000), World Scientific, River Edge, NJ, 2001, 51–79, arXiv:math.AG/0108108.

[27] Han G.-N., Some conjectures and open problems on partition hook lengths, *Experiment. Math.* **18** (2009), 97–106.

[28] Harer J., Zagier D., The Euler characteristic of the moduli space of curves, *Invent. Math.* **85** (1986), 457–485.

[29] Kazakov V.A., Kostov I.K., Nekrasov N., D-particles, matrix integrals and KP hierarchy, *Nuclear Phys. B* **557** (1999), 413–442, arXiv:hep-th/9810035.

[30] Kazarian M., Zograf P., Virasoro constraints and topological recursion for Grothendieck's dessin counting, *Lett. Math. Phys.* **105** (2015), 1057–1084, arXiv:1406.5976.

[31] Kharchev S., Marshakov A., Mironov A., Morozov A., Generalized Kontsevich model versus Toda hierarchy and discrete matrix models, *Nuclear Phys. B* **397** (1993), 339–378, arXiv:hep-th/9203043.

[32] Macdonald I.G., Symmetric functions and Hall polynomials, 2nd ed., *Oxford Math. Monogr.*, The Clarendon Press, New York, 1995.

[33] Marchal O., WKB solutions of difference equations and reconstruction by the topological recursion, *Nonlinearity* **31** (2018), 226–262, arXiv:1703.06152.

[34] Mehta M.L., Random matrices, 2nd ed., Academic Press, Inc., Boston, MA, 1991.

[35] Milanov T.E., Hirota quadratic equations for the extended Toda hierarchy, *Duke Math. J.* **138** (2007), 161–178, arXiv:math.AG/0501336.

[36] Morozov A., Integrability and matrix models, *Phys. Usp.* **37** (1994), 1–55, arXiv:hep-th/9303139.

[37] Okounkov A., Pandharipande R., The equivariant Gromov–Witten theory of $\mathbf{P}^1$, *Ann. of Math.* **163** (2006), 561–605, arXiv:math.AG/0207233.

[38] Okounkov A., Pandharipande R., Gromov–Witten theory, Hurwitz theory, and completed cycles, *Ann. of Math.* **163** (2006), 517–560, arXiv:math.AG/0204305.

[39] Okounkov A., Pandharipande R., Virasoro constraints for target curves, *Invent. Math.* **163** (2006), 47–108, arXiv:math.AG/0308097.

[40] Panova G., Polynomiality of some hook-length statistics, *Ramanujan J.* **27** (2012), 349–356, arXiv:0811.3463.

[41] Shaw J.C., Tu M.H., Yen H.C., Matrix models at finite N and the KP hierarchy, *Chinese J. Phys.* **30** (1992), 497–507.

[42] Stanley R.P., Some combinatorial properties of hook lengths, contents, and parts of partitions, *Ramanujan J.* **23** (2010), 91–105, arXiv:0807.0383.

[43] Ueno K., Takasaki K., Toda lattice hierarchy, in Group Representations and Systems of Differential Equations (Tokyo, 1982), *Adv. Stud. Pure Math.*, Vol. 4, North-Holland, Amsterdam, 1984, 1–95.

[44] Yang D., On tau-functions for the Toda lattice hierarchy, *Lett. Math. Phys.* **110** (2020), 555–583, arXiv:1905.08140.

[45] Yang D., GUE via Frobenius manifolds. I. From matrix gravity to topological gravity and back, *Acta Math. Sin. (Engl. Ser.)* **40** (2024), 383–405, arXiv:2205.01618.

[46] Yang D., Zhou J., Grothendieck's dessins d'enfants in a web of dualities. III, *J. Phys. A* **56** (2023), 055201, 34 pages, arXiv:2204.11074.

[47] Zagier D., Partitions, quasimodular forms, and the Bloch–Okounkov theorem, *Ramanujan J.* **41** (2016), 345–368.

[48] Zhang Y., On the $\mathbf{C}\mathrm{P}^1$ topological sigma model and the Toda lattice hierarchy, *J. Geom. Phys.* **40** (2002), 215–232.

[49] Zhou J., Explicit formula for Witte–Kontsevich tau-function, arXiv:1306.5429.

[50] Zhou J., Emergent geometry and mirror symmetry of a point, arXiv:1507.01679.

[51] Zhou J., Hermitian one-matrix model and KP hierarchy, arXiv:1809.07951.

[52] Zhou J., Grothendieck's dessins d'enfants in a web of dualities, arXiv:1905.10773.

[53] Zhou J., Grothendieck's dessins d'enfants in a web of dualities. II, arXiv:1907.00357.

[54] Zograf P., Enumeration of Grothendieck's dessins and KP hierarchy, *Int. Math. Res. Not.* **2015** (2015), 13533–13544, arXiv:1312.2538.