Trans-Series Asymptotics of Solutions to the Degenerate Painlevé III Equation: A Case Study

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Abstract. A one-parameter family of trans-series asymptotics as $\tau \to \pm\infty$ and $\tau \to \pm i\infty$ for solutions of the degenerate Painlevé III equation (DP3E),

$$u''(\tau) = \frac{(u'(\tau))^2}{u(\tau)} - \frac{u'(\tau)}{\tau} + \frac{1}{\tau}\big(-8\varepsilon(u(\tau))^2 + 2ab\big) + \frac{b^2}{u(\tau)},$$

where $\varepsilon \in \{\pm 1\}$, $a \in \mathbb{C}$, and $b \in \mathbb{R} \setminus \{0\}$, are parametrised in terms of the monodromy data of an associated first-order 2×2 matrix linear ODE via the isomonodromy deformation approach: trans-series asymptotics for the associated Hamiltonian and principal auxiliary functions and the solution of one of the σ-forms of the DP3E are also obtained. The actions of various Lie-point symmetries for the DP3E are derived.

Key words: isomonodromy deformations; Stokes phenomena; symmetries

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1 Introduction

In this section, which is partitioned into five inter-dependent subsections, the reader is given a concise overview of the information subsumed in the text: (i) in Section 1.1, the degenerate Painlevé III equation (DP3E) is introduced, and the qualitative behaviours of the asymptotic results the reader can expect to excise from this work are delineated; (ii) in Section 1.2, the DP3E's associated Hamiltonian and principal auxiliary functions, as well as one of its σ-forms, are introduced; (iii) in Section 1.3, pre- and post-gauge-transformed Lax pairs giving rise to isomonodromy deformations are reviewed; (iv) in Section 1.4, canonical asymptotics of the post-gauge-transformed Lax-pair solution matrix is presented in conjunction with the corresponding monodromy data; and (v) in Section 1.5, the monodromy manifold and the direct and inverse problems of monodromy theory are introduced, and a synopsis of the organisation of this work is given.

1.1 The degenerate Painlevé III equation (DP3E)

This paper continues the studies in [56, 57, 58, 59, 60, 61] of the DP3E,

$$u''(\tau) = \frac{(u'(\tau))^2}{u(\tau)} - \frac{u'(\tau)}{\tau} + \frac{1}{\tau}\left(-8\varepsilon(u(\tau))^2 + 2ab\right) + \frac{b^2}{u(\tau)}, \qquad \varepsilon \in \{\pm 1\}, \tag{1.1}$$

where the prime denotes differentiation with respect to τ, $\mathbb{C} \ni a$ is the parameter of formal monodromy, and $\mathbb{R}\setminus\{0\} \ni b$ is a parameter (see also [33, Chapter 7, Section 33]); in fact, making

This paper is a contribution to the Special Issue on Evolution Equations, Exactly Solvable Models and Random Matrices in honor of Alexander Its' 70th birthday. The full collection is available at https://www.emis.de/journals/SIGMA/Its.html the formal change of variables $\tau \to t^{1/2}$, $u(\tau) \to \tilde{\eta}_0^2 t^{-1/2}\tilde{\lambda}(t)$, $a \to \mp \mathrm{i}\tilde{c}_0\tilde{\eta}_0$, and $b \to \pm \mathrm{i}2\tilde{\eta}_0^3$, where $\tilde{c}_0 \in \mathbb{C}$ and $\mathrm{i}\tilde{\eta}_0 \in \mathbb{R} \setminus \{0\}$, and setting $\varepsilon = +1$, one shows that the DP3E (1.1) transforms into, in the classification scheme of [66], the degenerate third Painlevé equation of type D_7,

$$(P_{\mathrm{III}'})_{D_7}\colon\ \frac{\mathrm{d}^2\tilde{\lambda}}{\mathrm{d}t^2} = \frac{1}{\tilde{\lambda}}\left(\frac{\mathrm{d}\tilde{\lambda}}{\mathrm{d}t}\right)^2 - \frac{1}{t}\frac{\mathrm{d}\tilde{\lambda}}{\mathrm{d}t} + \tilde{\eta}_0^2\left(-2\frac{\tilde{\lambda}^2}{t^2} + \frac{\tilde{c}_0}{t} - \frac{1}{\tilde{\lambda}}\right). \tag{1.2}$$

It is known that, in the complex plane of the independent variable, Painlevé equations admit, in open sectors near the point at infinity containing one special ray, pole-free solutions that are characterised by divergent asymptotic expansions: such solutions, called *tronquée* solutions by Boutroux, usually contain free parameters manifesting in exponentially small terms for large values of the independent variable. There also exist pole-free solutions that are void of parameters in larger open sectors near the point at infinity containing three special rays: such solutions are called *tritronquée* solutions (see, for example, [21, Chapter 3]). In contrast to the asymptotic results of [57, 61], this work entails an analysis of one-parameter families of *trans-series* (see [21, Chapter 5]) asymptotic (as $\tau \to \pm\infty$ and $\tau \to \pm \mathrm{i}\infty$) solutions related to the underlying quasi-linear Stokes phenomenon associated with the DP3E (1.1): such solutions are also referred to as instanton-type solutions in the physics literature [30] (see also [44, 49, 50, 51], and [29, Chapter 11]); in particular, tronquée solutions that are free of poles not only on the real and the imaginary axes of τ, but also in open sectors about the point at infinity, are considered.[1] The existence of one-parameter tronquée solutions for a scaled version of the DP3E (1.1) was proved in [62] via direct asymptotic analysis. A review of recent manifestations of the DP3E (1.1) and $(P_{\mathrm{III}'})_{D_7}$ (1.2) in variegated mathematical and physical settings such as, for example, nonlinear optics, number theory, asymptotics, nonlinear waves, random matrix theory, and differential geometry, is presented in Appendix F.

An effectual approach for studying the asymptotic behaviour of solutions (in particular, the connection formulae for their asymptotics) of the Painlevé equations PI, PII, . . . , PVI is the isomonodromic deformation method (IDM) [29, 41, 42, 43, 45]: specific features of the IDM as applied, in particular, to the DP3E (1.1) can be located in [61, Sections 1 and 2]. It is imperative, within the IDM framework, to mention the seminal rôle played by the recent monograph [29], as it summarizes and reflects not only the key technical and theoretical developments and advances of the IDM since the appearance of [45], but also of an equivalent, technically distinct approach based on the Deift–Zhou nonlinear steepest descent analysis of the associated RHP [20]. The methodological paradigm adopted in this paper is the IDM. Even though the DP3E (1.1) resembles one of the canonical variants of the Painlevé equations PI, PII, . . . , PVI, the associated asymptotic analysis of its solutions via the IDM subsumes additional technical complications due to the necessity of having to extract the explicit functional dependencies of the contributing error terms rather than merely estimating them, which requires a considerably more detailed study of the error functions. By studying the isomonodromic deformations of a first-order 3×3 matrix linear ODE (see also [24, Section 8]) with two irregular singular points, asymptotics as $\tau \to \infty$ and $\tau \to 0$ of solutions to the DP3E (1.1) for the case $a = 0$, as well as the corresponding connection formulae, were obtained in [53] via the IDM. As observed in [52], though, there is an alternative first-order 2×2 matrix linear ODE whose isomonodromy deformations are described, for arbitrary $a \in \mathbb{C}$, by the DP3E (1.1): it is this latter 2×2 ODE system that is adopted in this work.

In order to eschew a flood of superfluous notation and to motivate, in as succinct a manner as possible, the qualitative behaviour of the solution of the DP3E (1.1) that the reader will encounter in this work, consider, for example, asymptotics as $\tau \to +\infty$ with $\varepsilon b > 0$ of $u(\tau)$.

[1] The terms trans-series [3, 26] and tronquée are used interchangeably in this work.

As is well known [2, 5, 21, 29, 63, 67, 68, 69, 70, 78, 81], the Painlevé equations admit a one-parameter family of trans-series solutions of the form "(power series) + (exponentially small terms)". As noted in Remark 1.1 below, $u(\tau)$ admits the 'complete' asymptotic trans-series representation $u(\tau) \underset{\tau\to+\infty}{=} c_{0,k}\big(\tau^{1/3}+v_{0,k}(\tau)\big)$, $k\in\{\pm1\}$, where

$$
\begin{aligned}
&c_{0,k} := \frac{1}{2}\varepsilon(\varepsilon b)^{2/3}\mathrm{e}^{-\mathrm{i}2\pi k/3},\\
&v_{0,k}(\tau) := \tau^{-1/3}\mathrm{u}_{\mathrm{R},k}(\tau)+\mathrm{u}_{\mathrm{E},k}(\tau) \qquad \text{with} \quad \mathbb{C}[\![\tau^{-1/3}]\!] \ni \mathrm{u}_{\mathrm{R},k}(\tau)=\sum_{n=0}^{\infty}\upsilon_{n,k}(\mathcal{M})\big(\tau^{-1/3}\big)^{n},\\
&\mathrm{u}_{\mathrm{E},k}(\tau)=\sum_{m=1}^{\infty}\sum_{j=0}^{\infty}\mathfrak{v}_{m,j,k}(\mathcal{M})\big(\tau^{-1/3}\big)^{j}\big(\mathrm{e}^{-\frac{3\sqrt{3}}{2}(\sqrt{3}+\mathrm{i}k)(\varepsilon b)^{1/3}\tau^{2/3}}\big)^{m},
\end{aligned}
$$

where the monodromy-data-dependent expansion coefficients $\upsilon_{n,k}(\mathcal{M})$ and $\mathfrak{v}_{m,j,k}(\mathcal{M})$ can be determined recursively provided that certain leading coefficients are known *a priori*. The purpose of this work, though, is not to address the complete asymptotic trans-series representation stated above, but, rather, to determine the coefficient of the leading-order exponentially small correction term to the asymptotics of solutions of the DP3E (1.1), which is, to the best of the author's knowledge as at the time of the presents, the decidedly non-trivial task within the IDM paradigm, in which case, the asymptotic trans-series representation for $u(\tau)$ reads[2]

$$
u(\tau) \underset{\tau\to+\infty}{=} c_{0,k}\left(\tau^{1/3}+\sum_{m=0}^{\infty}\frac{\mathfrak{u}_m(k)}{\big(\tau^{1/3}\big)^{m+1}}+\mathrm{A}_k\mathrm{e}^{-\frac{3\sqrt{3}}{2}(\sqrt{3}+\mathrm{i}k)(\varepsilon b)^{1/3}\tau^{2/3}}\big(1+\mathcal{O}\big(\tau^{-1/3}\big)\big)\right),
$$
$$
k\in\{\pm1\}. \tag{1.3}
$$

While the expansion coefficients $\{\mathfrak{u}_m(k)\}_{m=0}^{\infty}$, $k\in\{\pm1\}$, can be determined (not always uniquely) by substituting the trans-series representation (1.3) into the DP3E (1.1) and solving a system of recurrence relations for the $\mathfrak{u}_m(k)$'s, the monodromy-data-dependent expansion coefficients A_k, $k\in\{\pm1\}$, can not, and must, therefore, be determined independently; in fact, the principal technical accomplishment of this work is the determination, via the IDM, of the explicit dependence of the coefficients A_k, $k\in\{\pm1\}$, on the Stokes multiplier s_0^0 (see, in particular, Section 4, equations (4.71) and (4.92)). Even though the motivational discussion above for the introduction of the monodromy-data-dependent expansion coefficients A_k, $k\in\{\pm1\}$, relies on the asymptotics of $u(\tau)$ as $\tau\to+\infty$ for $\varepsilon b>0$, it must be emphasized that, in this work, the coefficients A_k, $k\in\{\pm1\}$, and their analogues, corresponding to trans-series asymptotics of $u(\tau)$, the associated Hamiltonian and principal auxiliary functions, and one of the σ-forms of the DP3E (1.1) as $\tau\to+\infty\mathrm{e}^{\mathrm{i}\pi\varepsilon_1}$ for $\varepsilon b=|\varepsilon b|\mathrm{e}^{\mathrm{i}\pi\varepsilon_2}$, $\varepsilon_1,\varepsilon_2\in\{0,\pm1\}$, and $\tau\to+\infty\mathrm{e}^{\mathrm{i}\pi\hat{\varepsilon}_1/2}$ for $\varepsilon b=|\varepsilon b|\mathrm{e}^{\mathrm{i}\pi\hat{\varepsilon}_2}$, $\hat{\varepsilon}_1\in\{\pm1\}$ and $\hat{\varepsilon}_2\in\{0,\pm1\}$, are obtained (see, in particular, Section 2, Theorems 2.4 and 2.8, respectively).

Remark 1.1. In the seminal work [62], the authors consider, in particular, the existence and uniqueness of tronquée solutions of the PIII equation with parameters $(1,\beta,0,-1)$, denoted by $\mathrm{P}_{\mathrm{III}}^{(\mathrm{ii})}$ in [62, equation (1.5)]:

$$
v''(x)=\frac{(v'(x))^2}{v(x)}-\frac{v'(x)}{x}+\frac{1}{x}\big((v(x))^2+\beta\big)-\frac{1}{v(x)},
$$

where $\mathbb{C}\ni\beta$ is arbitrary; $\mathrm{P}_{\mathrm{III}}^{(\mathrm{ii})}$ can be derived from the DP3E (1.1) via the mapping

$$
\mathcal{S}_{\varepsilon}\colon\ (\tau,u(\tau),a,b)\to\left(\alpha x,\gamma v(x),\frac{\beta}{2}\mathrm{e}^{-\mathrm{i}(2m+1)\pi/2},b\right),\qquad \varepsilon=\pm1,\quad m=0,1,
$$

[2]The notation $\lambda_1(t) \underset{t\to+\infty}{=} \mathcal{O}(\lambda_2(t))$ means that there exists $\widehat{C}>0$ and sufficiently small $\hat{\epsilon}>0$ such that $|\lambda_1(t)/\lambda_2(t)|\leqslant\widehat{C}$ for all $t>1/\hat{\epsilon}$.

where $\alpha := 2^{-3/2}b^{-1/2}\mathrm{e}^{\mathrm{i}(2+\varepsilon)\pi/4}\mathrm{e}^{\mathrm{i}(2m'+m)\pi/2}$, $\gamma := -2^{-3/2}\varepsilon b^{1/2}\mathrm{e}^{-\mathrm{i}(2+\varepsilon)\pi/4}\mathrm{e}^{-\mathrm{i}(2m'+m)\pi/2}$, $m' = 0, 1$. In [62, Theorem 2], the authors prove that, in any open sector of angle less than $3\pi/2$, there exist one-parameter solutions of $\mathrm{P}_{\mathrm{III}}^{(\mathrm{ii})}$ with asymptotic expansion

$$v(x) \sim v_f^{(m_1)}(x) := x^{1/3}\sum_{n=0}^{\infty} a_n^{(m_1)}\big(x^{-2/3}\big)^n \qquad \text{for} \quad S_k^{(m_1)} \ni x \to \infty, \quad m_1 = 0, 1, 2,$$

where the sectors $S_k^{(m_1)}$, $k = 0, 1, 2, 3$, are defined in [62, equation (1.10)], $a_0^{(m_1)} := \exp(\mathrm{i}2\pi m_1/3)$, and the ($x$-independent) coefficients $a_n^{(m_1)}$, $n \in \mathbb{N}$, solve the recursion relations [62, equation (1.12)]; moreover, the authors prove that, for any branch of $x^{1/3}$, there exists a unique solution of $\mathrm{P}_{\mathrm{III}}^{(\mathrm{ii})}$ in $\mathbb{C} \setminus \lambda$ with asymptotic expansion $v_f^{(m_1)}(x)$, where λ is an arbitrary branch cut connecting the singular points 0 and ∞ (they also address the existence of the exponentially small correction term(s) of the tronquée solution of $\mathrm{P}_{\mathrm{III}}^{(\mathrm{ii})}$). This crucial result of [62], in conjunction with the invertibility of the mapping $\mathcal{S}_\varepsilon$, implies the existence and the uniqueness of the asymptotic (as $\tau \to +\infty$ with $\varepsilon b > 0$) trans-series representation (1.3).

1.2 Hamiltonian structure, auxiliary functions, and the σ-form

Herewith follows a brief synopsis of select results from [61] that are relevant for the present work; for complete details, see, in particular, [61, Sections 1, 2 and 6], and [58].

An important formal property of the DP3E (1.1) is its associated Hamiltonian structure; in fact, as shown in [61, Proposition 1.3], upon setting

$$\begin{aligned}\mathcal{H}_{\epsilon_1}(\hat{p}(\tau), \hat{q}(\tau); \tau) := {}& (\hat{p}(\tau)\hat{q}(\tau))^2\tau^{-1} - 2\epsilon_1\hat{p}(\tau)\hat{q}(\tau)(\mathrm{i}a + 1/2)\tau^{-1} + 4\varepsilon\hat{q}(\tau) + \mathrm{i}b\hat{p}(\tau)\\ &+ \frac{1}{2}(\mathrm{i}a + 1/2)^2\tau^{-1},\end{aligned}$$

where the functions $\hat{p}(\tau)$ and $\hat{q}(\tau)$ are the generalised impulse and co-ordinate, respectively, $\epsilon_1 \in \{\pm 1\}$, and $\epsilon_1^2 = \varepsilon^2 = 1$, Hamilton's equations, that is,

$$\hat{p}'(\tau) = -\frac{\partial\mathcal{H}_{\epsilon_1}(\hat{p}(\tau), \hat{q}(\tau); \tau)}{\partial\hat{q}} \qquad \text{and} \qquad \hat{q}'(\tau) = \frac{\partial\mathcal{H}_{\epsilon_1}(\hat{p}(\tau), \hat{q}(\tau); \tau)}{\partial\hat{p}}, \tag{1.4}$$

are equivalent to either one of the degenerate PIII equations

$$\begin{aligned}\hat{p}''(\tau) &= \frac{(\hat{p}'(\tau))^2}{\hat{p}(\tau)} - \frac{\hat{p}'(\tau)}{\tau} + \frac{1}{\tau}\big(-2\mathrm{i}b(\hat{p}(\tau))^2 + 8\varepsilon(\mathrm{i}a\epsilon_1 + (\epsilon_1 - 1)/2)\big) - \frac{16}{\hat{p}(\tau)},\\ \hat{q}''(\tau) &= \frac{(\hat{q}'(\tau))^2}{\hat{q}(\tau)} - \frac{\hat{q}'(\tau)}{\tau} + \frac{1}{\tau}\big(-8\varepsilon(\hat{q}(\tau))^2 - b(2a\epsilon_1 - \mathrm{i}(1 + \epsilon_1))\big) + \frac{b^2}{\hat{q}(\tau)};\end{aligned}$$

it was also noted during the proof of the above-mentioned result that the Hamiltonian system (1.4) can be rewritten as

$$\hat{p}(\tau) = \frac{\tau(\hat{q}'(\tau) - \mathrm{i}b)}{2(\hat{q}(\tau))^2} + \frac{\mathrm{i}\epsilon_1(a - \mathrm{i}/2)}{\hat{q}(\tau)}, \qquad \hat{q}(\tau) = -\frac{\tau(\hat{p}'(\tau) + 4\varepsilon)}{2(\hat{p}(\tau))^2} + \frac{\mathrm{i}\epsilon_1(a - \mathrm{i}/2)}{\hat{p}(\tau)}. \tag{1.5}$$

As shown in [61, Section 2], the *Hamiltonian function*, $\mathcal{H}(\tau)$, is defined as follows

$$\mathcal{H}(\tau) := \mathcal{H}_{\epsilon_1}(\hat{p}(\tau), \hat{q}(\tau); \tau)|_{\epsilon_1 = -1}, \tag{1.6}$$

where $\hat{p}(\tau)$ is calculated from the first (left-most) relation of equations (1.5) with $\hat{q}(\tau) = u(\tau)$; moreover, as shown in [61, Section 2], the definition (1.6) implies the following explicit expression for $\mathcal{H}(\tau)$ in terms of $u(\tau)$:

$$\mathcal{H}(\tau) = (a - \mathrm{i}/2)\frac{b}{u(\tau)} + \frac{1}{2\tau}(a - \mathrm{i}/2)^2 + \frac{\tau}{4(u(\tau))^2}\big((u'(\tau))^2 + b^2\big) + 4\varepsilon u(\tau). \tag{1.7}$$

It was shown in [61, Section 1] that the function $\sigma(\tau)$ defined by

$$\begin{aligned}\sigma(\tau) &:= \tau\mathcal{H}_{\epsilon_1}(\hat{p}(\tau),\hat{q}(\tau);\tau)+\hat{p}(\tau)\hat{q}(\tau)+\frac{1}{2}(\mathrm{i}a+1/2)^2-\epsilon_1(\mathrm{i}a+1/2)+\frac{1}{4}\\ &= (\hat{p}(\tau)\hat{q}(\tau)-\epsilon_1(\mathrm{i}a+(1-\epsilon_1)/2))^2+\tau(4\varepsilon\hat{q}(\tau)+\mathrm{i}b\hat{p}(\tau))\end{aligned} \tag{1.8}$$

satisfies the second-order nonlinear ODE (related to the DP3E (1.1))

$$\begin{aligned}(\tau\sigma''(\tau)-\sigma'(\tau))^2 &= 2(2\sigma(\tau)-\tau\sigma'(\tau))(\sigma'(\tau))^2\\ &\quad -32\mathrm{i}\varepsilon b\tau(((1-\epsilon_1)/2-\mathrm{i}a\epsilon_1)\sigma'(\tau)+2\mathrm{i}\varepsilon b\tau).\end{aligned} \tag{1.9}$$

Equation (1.9) is referred to as the σ-form of the DP3E (1.1). Motivated by the definition (1.6) for the Hamiltonian function, setting $\epsilon_1=-1$, letting the generalised co-ordinate $\hat{q}(\tau)=u(\tau)$, and using the first (left-most) relation of equations (1.5) to calculate the generalised impulse, it suffices, for the purposes of the present work, to define the function (cf. definition (1.8)) $\sigma(\tau)$ and the second-order nonlinear ODE it satisfies as follows

$$\sigma(\tau) := \tau\mathcal{H}(\tau)+\frac{\tau(u'(\tau)-\mathrm{i}b)}{2u(\tau)}+\frac{1}{2}(\mathrm{i}a+1/2)^2+\frac{1}{4}, \tag{1.10}$$

and

$$(\tau\sigma''(\tau)-\sigma'(\tau))^2=2(2\sigma(\tau)-\tau\sigma'(\tau))(\sigma'(\tau))^2-32\mathrm{i}\varepsilon b\tau((1+\mathrm{i}a)\sigma'(\tau)+2\mathrm{i}\varepsilon b\tau). \tag{1.11}$$

Via the Bäcklund transformations given in [61, Section 6.1], let

$$\begin{aligned}u_-(\tau) &:= \frac{\mathrm{i}\varepsilon b}{8(u(\tau))^2}(\tau(u'(\tau)-\mathrm{i}b)+(1-2\mathrm{i}a_-)u(\tau)),\\ u_+(\tau) &:= -\frac{\mathrm{i}\varepsilon b}{8(u(\tau))^2}(\tau(u'(\tau)+\mathrm{i}b)+(1+2\mathrm{i}a_+)u(\tau)),\end{aligned} \tag{1.12}$$

where $u(\tau)$ denotes any solution of the DP3E (1.1), and $a_\pm := a\pm\mathrm{i}$; in fact, as shown in [61, Section 6.1], $u_-(\tau)$ (resp., $u_+(\tau)$) solves the DP3E (1.1) for $a=a_-$ (resp., $a=a_+$). From the results of [58], define the two *principal auxiliary functions*

$$f_-(\tau) := -\frac{2\mathrm{i}}{\varepsilon b}u(\tau)u_-(\tau), \qquad f_+(\tau) := u(\tau)u_+(\tau), \tag{1.13}$$

where $f_-(\tau)$ solves the second-order nonlinear ODE[3]

$$\tau^2(f_-''(\tau)+4\mathrm{i}\varepsilon b)^2-(4f_-(\tau)+2\mathrm{i}a+1)^2\big((f_-'(\tau))^2+8\mathrm{i}\varepsilon bf_-(\tau)\big)=0, \tag{1.14}$$

and $f_+(\tau)$ solves the second-order nonlinear ODE[4]

$$(\varepsilon b\tau)^2\left(f_+''(\tau)-2(\varepsilon b)^2\right)^2+(8f_+(\tau)+\mathrm{i}\varepsilon b(2\mathrm{i}a-1))^2\left((f_+'(\tau))^2-4(\varepsilon b)^2f_+(\tau)\right)=0. \tag{1.15}$$

It follows from the definitions (1.12)–(1.13) that the functions $f_\pm(\tau)$ possess the alternative representations

$$2f_-(\tau)=-\mathrm{i}(a-\mathrm{i}/2)+\frac{\tau(u'(\tau)-\mathrm{i}b)}{2u(\tau)}, \tag{1.16}$$

[3]This is a consequence of the ODE for the function $f(\tau)$ presented on [61, p. 1168] upon making the notational change $f(\tau)\to f_-(\tau)$ and setting $\epsilon_1=-1$.

[4]See [58, equation (2)].

$$\frac{4\mathrm{i}}{\varepsilon b} f_+(\tau) = \mathrm{i}(a + \mathrm{i}/2) + \frac{\tau(u'(\tau) + \mathrm{i}b)}{2u(\tau)}; \tag{1.17}$$

incidentally, equations (1.16) and (1.17) imply the corollary

$$\frac{4\mathrm{i}}{\varepsilon b} f_+(\tau) = 2f_-(\tau) + \mathrm{i}\tau \left(\frac{2a}{\tau} + \frac{b}{u(\tau)} \right).$$

For the monodromy data considered in [56], preliminary asymptotics as $\tau \to +\infty$ with $\varepsilon b > 0$ for $\int_0^\tau \xi^{-1} f_+(\xi) \mathrm{d}\xi$ have been presented in [58].

1.3 Lax pairs and isomonodromic deformations

In this subsection, the reader is reminded about some basic facts regarding the isomonodromy deformation theory for the DP3E (1.1).

Remark 1.2. Pre-gauge-transformed Lax-pair-associated functions are denoted with 'hats', whilst post-gauge-transformed Lax-pair-associated functions are not; in some cases, these functions are equal, and in others, they are not (see the discussion below).

The study of the DP3E (1.1) is based on the following pre-gauge-transformed Lax pair (see [61, Proposition 2.1], with notational amendments):

$$\partial_\mu \widehat{\Psi}(\mu,\tau) = \widehat{\mathcal{U}}(\mu,\tau)\widehat{\Psi}(\mu,\tau), \qquad \partial_\tau \widehat{\Psi}(\mu,\tau) = \widehat{\mathcal{V}}(\mu,\tau)\widehat{\Psi}(\mu,\tau), \tag{1.18}$$

where

$$\begin{aligned} \widehat{\mathcal{U}}(\mu,\tau) = & -2\mathrm{i}\tau\mu\sigma_3 + 2\tau \begin{pmatrix} 0 & \frac{2\mathrm{i}\hat{A}(\tau)}{\sqrt{-\hat{A}(\tau)\hat{B}(\tau)}} \\ -\hat{D}(\tau) & 0 \end{pmatrix} - \frac{1}{\mu}\left(\mathrm{i}a + \frac{1}{2} + \frac{2\tau\hat{A}(\tau)\hat{D}(\tau)}{\sqrt{-\hat{A}(\tau)\hat{B}(\tau)}} \right)\sigma_3 \\ & + \frac{1}{\mu^2} \begin{pmatrix} 0 & \hat{\alpha}(\tau) \\ \mathrm{i}\tau\hat{B}(\tau) & 0 \end{pmatrix}, \end{aligned} \tag{1.19}$$

$$\begin{aligned} \widehat{\mathcal{V}}(\mu,\tau) = & -\mathrm{i}\mu^2\sigma_3 + \mu \begin{pmatrix} 0 & \frac{2\mathrm{i}\hat{A}(\tau)}{\sqrt{-\hat{A}(\tau)\hat{B}(\tau)}} \\ -\hat{D}(\tau) & 0 \end{pmatrix} + \left(\frac{\mathrm{i}a}{2\tau} - \frac{\hat{A}(\tau)\hat{D}(\tau)}{\sqrt{-\hat{A}(\tau)\hat{B}(\tau)}} \right)\sigma_3 \\ & - \frac{1}{\mu}\frac{1}{2\tau} \begin{pmatrix} 0 & \hat{\alpha}(\tau) \\ \mathrm{i}\tau\hat{B}(\tau) & 0 \end{pmatrix}, \end{aligned} \tag{1.20}$$

with $\sigma_3 = \operatorname{diag}(1,-1)$,

$$\hat{\alpha}(\tau) := -2\big(\hat{B}(\tau)\big)^{-1}\big(\mathrm{i}a\sqrt{-\hat{A}(\tau)\hat{B}(\tau)} + \tau(\hat{A}(\tau)\hat{D}(\tau) + \hat{B}(\tau)\hat{C}(\tau))\big), \tag{1.21}$$

and where the differentiable, scalar-valued functions $\hat{A}(\tau)$, $\hat{B}(\tau)$, $\hat{C}(\tau)$, and $\hat{D}(\tau)$ satisfy the system of isomonodromy deformations

$$\begin{aligned} &\hat{A}'(\tau) = 4\hat{C}(\tau)\sqrt{-\hat{A}(\tau)\hat{B}(\tau)}, \qquad \hat{B}'(\tau) = -4\hat{D}(\tau)\sqrt{-\hat{A}(\tau)\hat{B}(\tau)}, \\ &\big(\tau\hat{C}(\tau)\big)' = 2\mathrm{i}a\hat{C}(\tau) - 2\tau\hat{A}(\tau), \qquad \big(\tau\hat{D}(\tau)\big)' = -2\mathrm{i}a\hat{D}(\tau) + 2\tau\hat{B}(\tau), \\ &\big(\sqrt{-\hat{A}(\tau)\hat{B}(\tau)}\big)' = 2(\hat{A}(\tau)\hat{D}(\tau) - \hat{B}(\tau)\hat{C}(\tau)). \end{aligned} \tag{1.22}$$

(Note: the isomonodromy deformations (1.22) are, for arbitrary values of $\mu \in \mathbb{C}$, the Frobenius compatibility condition for the system (1.18).)

Remark 1.3. In fact, $-\mathrm{i}\hat{\alpha}(\tau)\hat{B}(\tau)=\varepsilon b$, $\varepsilon=\pm1$, so that the definition (1.21) is the *first integral* of the system (1.22) (see [61, Lemma 2.1], with notational amendments).

Remark 1.4. With conspicuous changes in notation (cf. [61, system (4)]), whilst transforming from the original Lax pair

$$\begin{aligned}\partial_\lambda\Phi(\lambda,\tau)&=\tau\left(-\mathrm{i}\sigma_3-\frac{1}{\lambda}\frac{\mathrm{i}a}{2\tau}\sigma_3-\frac{1}{\lambda}\begin{pmatrix}0&\hat{C}(\tau)\\ \hat{D}(\tau)&0\end{pmatrix}\right.\\&\quad\left.+\frac{1}{\lambda^2}\frac{\mathrm{i}}{2}\begin{pmatrix}\sqrt{-\hat{A}(\tau)\hat{B}(\tau)}&\hat{A}(\tau)\\ \hat{B}(\tau)&-\sqrt{-\hat{A}(\tau)\hat{B}(\tau)}\end{pmatrix}\right)\Phi(\lambda,\tau),\\ \partial_\tau\Phi(\lambda,\tau)&=\left(-\mathrm{i}\lambda\sigma_3+\frac{\mathrm{i}a}{2\tau}\sigma_3-\begin{pmatrix}0&\hat{C}(\tau)\\ \hat{D}(\tau)&0\end{pmatrix}\right.\\&\quad\left.-\frac{1}{\lambda}\frac{\mathrm{i}}{2}\begin{pmatrix}\sqrt{-\hat{A}(\tau)\hat{B}(\tau)}&\hat{A}(\tau)\\ \hat{B}(\tau)&-\sqrt{-\hat{A}(\tau)\hat{B}(\tau)}\end{pmatrix}\right)\Phi(\lambda,\tau),\end{aligned}$$

to the Fuchs–Garnier pair (1.18), the Fabry-type transformation (cf. [61, Proposition 2.1])

$$\lambda=\mu^2\qquad\text{and}\qquad\Phi(\lambda,\tau):=\sqrt{\mu}\left(\begin{pmatrix}1&0\\0&0\end{pmatrix}+\frac{1}{\mu}\begin{pmatrix}0&-\frac{\hat{A}(\tau)}{\sqrt{-\hat{A}(\tau)\hat{B}(\tau)}}\\0&1\end{pmatrix}\right)\widehat{\Psi}(\mu,\tau)$$

was used; if, instead, one applies the slightly more general transformation

$$\Phi(\lambda,\tau):=\sqrt{\mu}\left(\begin{pmatrix}1&0\\0&0\end{pmatrix}+\frac{1}{\mu}\begin{pmatrix}-\frac{\hat{A}(\tau)\mathbb{P}^*}{\sqrt{-\hat{A}(\tau)\hat{B}(\tau)}}&-\frac{\hat{A}(\tau)}{\sqrt{-\hat{A}(\tau)\hat{B}(\tau)}}\\ \mathbb{P}^*&1\end{pmatrix}\right)\widehat{\Psi}(\mu,\tau)$$

for some constant or τ-dependent $\mathbb{P}^*$, then, in lieu of, say, the μ-part of the Fuchs–Garnier pair (1.18), that is, $\partial_\mu\widehat{\Psi}(\mu,\tau)=\widehat{\mathfrak{U}}(\mu,\tau)\widehat{\Psi}(\mu,\tau)$, one arrives at $\partial_\mu\widehat{\Psi}(\mu,\tau)=\big(\widehat{\mathcal{L}}_{-1}\mu+\widehat{\mathcal{L}}_0+\widehat{\mathcal{L}}_1\mu^{-1}+\widehat{\mathcal{L}}_2\mu^{-2}\big)\widehat{\Psi}(\mu,\tau)$, where

$$\begin{aligned}&\widehat{\mathcal{L}}_{-1}=-2\mathrm{i}\tau\begin{pmatrix}1&0\\-2\mathbb{P}^*&-1\end{pmatrix},\qquad\widehat{\mathcal{L}}_0=-2\tau\begin{pmatrix}0&0\\ \hat{D}(\tau)&0\end{pmatrix}-\frac{4\mathrm{i}\tau\hat{A}(\tau)}{\sqrt{-\hat{A}(\tau)\hat{B}(\tau)}}\begin{pmatrix}-\mathbb{P}^*&-1\\(\mathbb{P}^*)^2&\mathbb{P}^*\end{pmatrix},\\&\widehat{\mathcal{L}}_1=\left(\mathrm{i}a+\frac{1}{2}+\frac{2\tau\hat{A}(\tau)\hat{D}(\tau)}{\sqrt{-\hat{A}(\tau)\hat{B}(\tau)}}\right)\begin{pmatrix}-1&0\\2\mathbb{P}^*&1\end{pmatrix},\\&\widehat{\mathcal{L}}_2=\mathrm{i}\tau\begin{pmatrix}0&0\\ \hat{B}(\tau)&0\end{pmatrix}+\hat{\alpha}(\tau)\begin{pmatrix}\mathbb{P}^*&1\\-(\mathbb{P}^*)^2&-\mathbb{P}^*\end{pmatrix},\end{aligned}$$

with $\hat{\alpha}(\tau)$ defined by equation (1.21). Setting $\mathbb{P}^*=0$, one arrives at the Fuchs–Garnier pair stated in [61, Proposition 2.1], [57, system (1.4)], and system (1.18) of the present work.

A relation between the Fuchs–Garnier pair (1.18) and the DP3E (1.1) is given by (see, in particular, [61, Proposition 1.2], with notational amendments) the following statement.

Proposition 1.5 ([57, 61])**.** *Let $\hat{u}=\hat{u}(\tau)$ and $\hat{\varphi}=\hat{\varphi}(\tau)$ solve the system*

$$\hat{u}''(\tau)=\frac{(\hat{u}'(\tau))^2}{\hat{u}(\tau)}-\frac{\hat{u}'(\tau)}{\tau}+\frac{1}{\tau}\big(-8\varepsilon(\hat{u}(\tau))^2+2ab\big)+\frac{b^2}{\hat{u}(\tau)},\qquad\hat{\varphi}'(\tau)=\frac{2a}{\tau}+\frac{b}{\hat{u}(\tau)},\tag{1.23}$$

where $\varepsilon=\pm1$, and $a,b\in\mathbb{C}$ are independent of τ; then,

$$\hat{A}(\tau):=\frac{\hat{u}(\tau)}{\tau}\mathrm{e}^{\mathrm{i}\hat{\varphi}(\tau)},\qquad\hat{B}(\tau):=-\frac{\hat{u}(\tau)}{\tau}\mathrm{e}^{-\mathrm{i}\hat{\varphi}(\tau)},$$

$$\hat{C}(\tau) := \frac{\varepsilon\tau\hat{A}'(\tau)}{4\hat{u}(\tau)} = \frac{\varepsilon \mathrm{e}^{\mathrm{i}\hat{\varphi}(\tau)}}{2\tau}\left(\mathrm{i}(a+\mathrm{i}/2) + \frac{\tau(\hat{u}'(\tau)+\mathrm{i}b)}{2\hat{u}(\tau)}\right),$$
$$\hat{D}(\tau) := -\frac{\varepsilon\tau\hat{B}'(\tau)}{4\hat{u}(\tau)} = -\frac{\varepsilon \mathrm{e}^{-\mathrm{i}\hat{\varphi}(\tau)}}{2\tau}\left(\mathrm{i}(a-\mathrm{i}/2) - \frac{\tau(\hat{u}'(\tau)-\mathrm{i}b)}{2\hat{u}(\tau)}\right) \tag{1.24}$$

solve the system (1.22). *Conversely, let* $\hat{A}(\tau) \not\equiv 0$, $\hat{B}(\tau) \not\equiv 0$, $\hat{C}(\tau)$, *and* $\hat{D}(\tau)$ *solve the system* (1.22), *and define*

$$\hat{u}(\tau) := \varepsilon\tau\sqrt{-\hat{A}(\tau)\hat{B}(\tau)}, \qquad \hat{\varphi}(\tau) := -\frac{\mathrm{i}}{2}\ln\big(-\hat{A}(\tau)/\hat{B}(\tau)\big),$$
$$b := \hat{u}(\tau)\big(\hat{\varphi}'(\tau) - 2a\tau^{-1}\big); \tag{1.25}$$

then, b is independent of τ, and $\hat{u}(\tau)$ and $\hat{\varphi}(\tau)$ solve the system (1.23).

Proposition 1.6. *Let* (*cf. equation* (1.16))

$$2\hat{f}_-(\tau) := -\mathrm{i}(a-\mathrm{i}/2) + \frac{\tau}{2}\left(\frac{\hat{u}'(\tau)-\mathrm{i}b}{\hat{u}(\tau)}\right), \tag{1.26}$$

and (*cf. equation* (1.17))

$$\frac{4\mathrm{i}}{\varepsilon b}\hat{f}_+(\tau) := \mathrm{i}(a+\mathrm{i}/2) + \frac{\tau}{2}\left(\frac{\hat{u}'(\tau)+\mathrm{i}b}{\hat{u}(\tau)}\right).$$

Then, for $\varepsilon \in \{\pm 1\}$,

$$2\hat{f}_-(\tau) = \frac{2\varepsilon\tau^2\hat{A}(\tau)\hat{D}(\tau)}{\hat{u}(\tau)} = \frac{\tau}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\ln\left(\frac{\hat{u}(\tau)}{\tau}\right) - \mathrm{i}\hat{\varphi}(\tau)\right), \tag{1.27}$$

and

$$\frac{4\mathrm{i}}{\varepsilon b}\hat{f}_+(\tau) = -\frac{2\varepsilon\tau^2\hat{B}(\tau)\hat{C}(\tau)}{\hat{u}(\tau)} = \frac{\tau}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\ln\left(\frac{\hat{u}(\tau)}{\tau}\right) + \mathrm{i}\hat{\varphi}(\tau)\right); \tag{1.28}$$

furthermore,

$$\frac{4\mathrm{i}}{\varepsilon b}\hat{f}_+(\tau) = 2\hat{f}_-(\tau) + \mathrm{i}\tau\hat{\varphi}'(\tau) = 2\hat{f}_-(\tau) + \mathrm{i}\tau\left(\frac{2a}{\tau} + \frac{b}{\hat{u}(\tau)}\right). \tag{1.29}$$

Proof. Without loss of generality, consider, say, the proof for the function $\hat{f}_-(\tau)$: the proof for the function $\hat{f}_+(\tau)$ is analogous. One commences by establishing the following relation:

$$\frac{\hat{u}'(\tau)-\mathrm{i}b}{\hat{u}(\tau)} = \frac{2}{\tau}\left(\frac{2\tau\hat{A}(\tau)\hat{D}(\tau)}{\sqrt{-\hat{A}(\tau)\hat{B}(\tau)}} + (\mathrm{i}a+1/2)\right). \tag{1.30}$$

From definition (1.21), the system of isomonodromy deformations (1.22), Remark 1.3, and the definition of the function $\hat{u}(\tau)$ given by the first (left-most) member of equations (1.25), it follows via differentiation that

$$\frac{\hat{u}'(\tau)-\mathrm{i}b}{\hat{u}(\tau)} = \frac{2\tau(\hat{A}(\tau)\hat{D}(\tau)-\hat{B}(\tau)\hat{C}(\tau)) + \sqrt{-\hat{A}(\tau)\hat{B}(\tau)}}{\tau\sqrt{-\hat{A}(\tau)\hat{B}(\tau)}} - \frac{\mathrm{i}(\varepsilon b)}{\varepsilon\hat{u}(\tau)}$$
$$= \frac{2\tau(\hat{A}(\tau)\hat{D}(\tau)-\hat{B}(\tau)\hat{C}(\tau)) + \sqrt{-\hat{A}(\tau)\hat{B}(\tau)}}{\tau\sqrt{-\hat{A}(\tau)\hat{B}(\tau)}} - \frac{\hat{\alpha}(\tau)\hat{B}(\tau)}{\varepsilon\hat{u}(\tau)}$$

$$= \frac{2\tau(\hat{A}(\tau)\hat{D}(\tau) - \hat{B}(\tau)\hat{C}(\tau)) + \sqrt{-\hat{A}(\tau)\hat{B}(\tau)}}{\tau\sqrt{-\hat{A}(\tau)\hat{B}(\tau)}} + \frac{2\tau(\hat{A}(\tau)\hat{D}(\tau) + \hat{B}(\tau)\hat{C}(\tau)) + 2\mathrm{i}a\sqrt{-\hat{A}(\tau)\hat{B}(\tau)}}{\tau\sqrt{-\hat{A}(\tau)\hat{B}(\tau)}}$$
$$= \frac{2}{\tau}\left(\frac{2\tau\hat{A}(\tau)\hat{D}(\tau)}{\sqrt{-\hat{A}(\tau)\hat{B}(\tau)}} + (\mathrm{i}a + 1/2)\right);$$

conversely, from the system of isomonodromy deformations (1.22), the system (1.23), and the definitions (1.24) and (1.25), it follows that

$$\frac{4\hat{A}(\tau)\hat{D}(\tau)}{\sqrt{-\hat{A}(\tau)\hat{B}(\tau)}} = \frac{4\varepsilon\tau\hat{A}(\tau)\hat{D}(\tau)}{\hat{u}(\tau)} = \frac{4\varepsilon\tau}{\hat{u}(\tau)}\left(-\frac{\varepsilon}{4}\hat{B}'(\tau)\mathrm{e}^{\mathrm{i}\hat{\varphi}(\tau)}\right) = \frac{\tau\mathrm{e}^{\mathrm{i}\hat{\varphi}(\tau)}}{\hat{u}(\tau)}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\hat{u}(\tau)}{\tau}\mathrm{e}^{-\mathrm{i}\hat{\varphi}(\tau)}\right)$$
$$= \frac{\tau}{\hat{u}(\tau)}\left(-\mathrm{i}\hat{\varphi}'(\tau)\frac{\hat{u}(\tau)}{\tau} - \frac{\hat{u}(\tau)}{\tau^2} + \frac{\hat{u}'(\tau)}{\tau}\right) = \frac{\tau}{\hat{u}(\tau)}\left(-\frac{\hat{u}(\tau)}{\tau}\left(\frac{2\mathrm{i}a}{\tau} + \frac{\mathrm{i}b}{\hat{u}(\tau)}\right) - \frac{\hat{u}(\tau)}{\tau^2} + \frac{\hat{u}'(\tau)}{\tau}\right) = \frac{\hat{u}'(\tau) - \mathrm{i}b}{\hat{u}(\tau)} - \frac{2}{\tau}(\mathrm{i}a + 1/2),$$

whence

$$\frac{2}{\tau}\left(\frac{2\tau\hat{A}(\tau)\hat{D}(\tau)}{\sqrt{-\hat{A}(\tau)\hat{B}(\tau)}} + (\mathrm{i}a + 1/2)\right) = \frac{\hat{u}'(\tau) - \mathrm{i}b}{\hat{u}(\tau)},$$

which establishes equation (1.30). Via definition (1.26) and equation (1.30), one shows that

$$\hat{f}_-(\tau) = \frac{\tau\hat{A}(\tau)\hat{D}(\tau)}{\sqrt{-\hat{A}(\tau)\hat{B}(\tau)}},$$

hence, via the definition for $\hat{u}(\tau)$ given by the first (left-most) member of equations (1.25), one arrives at the first (left-most) relation of equation (1.27); moreover, it follows from the ODE for the function $\hat{\varphi}(\tau)$ given in the system (1.23) and definition (1.26) that

$$\tau^{-1}\hat{f}_-(\tau) = \frac{1}{4}\left(\frac{\hat{u}'(\tau)}{\hat{u}(\tau)} + \frac{2\mathrm{i}a}{\tau} - \mathrm{i}\hat{\varphi}'(\tau)\right) - \frac{1}{2\tau}(\mathrm{i}a + 1/2) = \frac{1}{4}\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\ln\left(\frac{\hat{u}(\tau)}{\tau}\right) - \mathrm{i}\hat{\varphi}'(\tau)\right),$$

which implies the second (right-most) relation of equation (1.27). Equations (1.27) and (1.28) imply the corollary (1.29), which is consistent with, and can also be derived from, the definition (1.21) and the first integral of system (1.22) (cf. Remark 1.3). ■

Herewith follows the post-gauge-transformed Fuchs–Garnier pair.

Proposition 1.7. *Let $\widehat{\Psi}(\mu,\tau)$ be a fundamental solution of the system (1.18). Set*

$$A(\tau) := \hat{A}(\tau)\tau^{-\mathrm{i}a}, \qquad B(\tau) := \hat{B}(\tau)\tau^{\mathrm{i}a}, \qquad C(\tau) := \hat{C}(\tau)\tau^{-\mathrm{i}a}, \qquad D(\tau) := \hat{D}(\tau)\tau^{\mathrm{i}a},$$
$$\alpha(\tau) := \hat{\alpha}(\tau)\tau^{-\mathrm{i}a}, \qquad \widehat{\Psi}(\mu,\tau) := \tau^{\frac{\mathrm{i}a}{2}\sigma_3}\Psi(\mu,\tau). \tag{1.31}$$

Then

(i) $\Psi(\mu,\tau)$ *is a fundamental solution of*

$$\partial_\mu\Psi(\mu,\tau) = \widetilde{\mathcal{U}}(\mu,\tau)\Psi(\mu,\tau), \qquad \partial_\tau\Psi(\mu,\tau) = \widetilde{\mathcal{V}}(\mu,\tau)\Psi(\mu,\tau), \tag{1.32}$$

where

$$\widetilde{\mathfrak{U}}(\mu,\tau) = -2\mathrm{i}\tau\mu\sigma_3 + 2\tau\begin{pmatrix} 0 & \frac{2\mathrm{i}A(\tau)}{\sqrt{-A(\tau)B(\tau)}} \\ -D(\tau) & 0 \end{pmatrix} - \frac{1}{\mu}\left(\mathrm{i}a + \frac{1}{2} + \frac{2\tau A(\tau)D(\tau)}{\sqrt{-A(\tau)B(\tau)}}\right)\sigma_3 + \frac{1}{\mu^2}\begin{pmatrix} 0 & \alpha(\tau) \\ \mathrm{i}\tau B(\tau) & 0 \end{pmatrix}, \tag{1.33}$$

$$\widetilde{\mathcal{V}}(\mu,\tau) = -\mathrm{i}\mu^2\sigma_3 + \mu\begin{pmatrix} 0 & \frac{2\mathrm{i}A(\tau)}{\sqrt{-A(\tau)B(\tau)}} \\ -D(\tau) & 0 \end{pmatrix} - \frac{A(\tau)D(\tau)}{\sqrt{-A(\tau)B(\tau)}}\sigma_3 - \frac{1}{\mu}\frac{1}{2\tau}\begin{pmatrix} 0 & \alpha(\tau) \\ \mathrm{i}\tau B(\tau) & 0 \end{pmatrix}, \tag{1.34}$$

with

$$\alpha(\tau) := -2(B(\tau))^{-1}\big(\mathrm{i}a\sqrt{-A(\tau)B(\tau)} + \tau(A(\tau)D(\tau) + B(\tau)C(\tau))\big); \tag{1.35}$$

(*ii*) *if the coefficient functions* $\hat{A}(\tau)$, $\hat{B}(\tau)$, $\hat{C}(\tau)$, *and* $\hat{D}(\tau)$ *satisfy the system of isomonodromy deformations* (1.22) *and the functions* $A(\tau)$, $B(\tau)$, $C(\tau)$, *and* $D(\tau)$ *are defined by equations* (1.31), *then the Frobenius compatibility condition of the system* (1.32), *for arbitrary values of* $\mu \in \mathbb{C}$, *is that the differentiable, scalar-valued functions* $A(\tau)$, $B(\tau)$, $C(\tau)$, *and* $D(\tau)$ *satisfy the corresponding system of isomonodromy deformations*

$$\begin{aligned} &A'(\tau) = -\frac{\mathrm{i}a}{\tau}A(\tau) + 4C(\tau)\sqrt{-A(\tau)B(\tau)}, \qquad B'(\tau) = \frac{\mathrm{i}a}{\tau}B(\tau) - 4D(\tau)\sqrt{-A(\tau)B(\tau)}, \\ &(\tau C(\tau))' = \mathrm{i}aC(\tau) - 2\tau A(\tau), \qquad (\tau D(\tau))' = -\mathrm{i}aD(\tau) + 2\tau B(\tau), \\ &\big(\sqrt{-A(\tau)B(\tau)}\big)' = 2(A(\tau)D(\tau) - B(\tau)C(\tau)). \end{aligned} \tag{1.36}$$

Proof. If $\widehat{\Psi}(\mu,\tau)$ is a fundamental solution of the system (1.18), then it follows from the isomonodromy deformations (1.22) and the definitions (1.31) that $\Psi(\mu,\tau)$ solves the system (1.32), and that the functions $A(\tau)$, $B(\tau)$, $C(\tau)$, and $D(\tau)$ satisfy the corresponding isomonodromy deformations (1.36). One verifies the Frobenius compatibility condition for the system (1.32) by showing that, $\forall \mu \in \mathbb{C}$, $\partial_\tau \widetilde{\mathfrak{U}}(\mu,\tau) - \partial_\mu \widetilde{\mathcal{V}}(\mu,\tau) + \big[\widetilde{\mathfrak{U}}(\mu,\tau), \widetilde{\mathcal{V}}(\mu,\tau)\big] = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, where, for $\mathfrak{X}, \mathfrak{Y} \in \mathrm{M}_2(\mathbb{C})$, $[\mathfrak{X}, \mathfrak{Y}] := \mathfrak{X}\mathfrak{Y} - \mathfrak{Y}\mathfrak{X}$ is the matrix commutator. ∎

Remark 1.8. Definitions (1.21), (1.31), and (1.35), and Remark 1.3 imply that $-\mathrm{i}\alpha(\tau)B(\tau) = \varepsilon b$, $\varepsilon = \pm 1$.

Proposition 1.9. *Let* $u(\tau)$ *and* $\varphi(\tau)$ *solve the system*

$$u''(\tau) = \frac{(u'(\tau))^2}{u(\tau)} - \frac{u'(\tau)}{\tau} + \frac{1}{\tau}\big(-8\varepsilon(u(\tau))^2 + 2ab\big) + \frac{b^2}{u(\tau)}, \qquad \varphi'(\tau) = \frac{a}{\tau} + \frac{b}{u(\tau)}, \tag{1.37}$$

where $\varepsilon = \pm 1$, *and* $a, b \in \mathbb{C}$ *are independent of* τ; *then,*

$$\begin{aligned} &A(\tau) := \frac{u(\tau)}{\tau}\mathrm{e}^{\mathrm{i}\varphi(\tau)}, \qquad B(\tau) := -\frac{u(\tau)}{\tau}\mathrm{e}^{-\mathrm{i}\varphi(\tau)}, \\ &C(\tau) := \frac{\varepsilon\tau}{4u(\tau)}\left(A'(\tau) + \frac{\mathrm{i}a}{\tau}A(\tau)\right) = \frac{\varepsilon\mathrm{e}^{\mathrm{i}\varphi(\tau)}}{2\tau}\left(\mathrm{i}(a + \mathrm{i}/2) + \frac{\tau(u'(\tau) + \mathrm{i}b)}{2u(\tau)}\right), \\ &D(\tau) := -\frac{\varepsilon\tau}{4u(\tau)}\left(B'(\tau) - \frac{\mathrm{i}a}{\tau}B(\tau)\right) = -\frac{\varepsilon\mathrm{e}^{-\mathrm{i}\varphi(\tau)}}{2\tau}\left(\mathrm{i}(a - \mathrm{i}/2) - \frac{\tau(u'(\tau) - \mathrm{i}b)}{2u(\tau)}\right) \end{aligned} \tag{1.38}$$

solve the system (1.36)*. Conversely, let* $A(\tau) \not\equiv 0$, $B(\tau) \not\equiv 0$, $C(\tau)$*, and* $D(\tau)$ *solve the system* (1.36)*, and define*

$$u(\tau) := \varepsilon\tau\sqrt{-A(\tau)B(\tau)}, \qquad \varphi(\tau) := -\frac{\mathrm{i}}{2}\ln(-A(\tau)/B(\tau)),$$
$$b := u(\tau)\big(\varphi'(\tau) - a\tau^{-1}\big); \tag{1.39}$$

then, b *is independent of* τ*, and* $u(\tau)$ *and* $\varphi(\tau)$ *solve the system* (1.37)*.*

Proof. Via the definition of $\hat{u}(\tau)$ given by the first (left-most) member of equations (1.25) and the definitions (1.31), one arrives at the definition for $u(\tau)$ given by the first (left-most) member of equations (1.39); in particular, it follows that $u(\tau) = \hat{u}(\tau)$, and, from the first equation of system (1.23), $u(\tau)$ solves the DP3E (1.1) (see the first equation of the system (1.37)). Let $\varphi(\tau)$ be defined as in equations (1.39), that is, $\varphi(\tau) = -\mathrm{i}\ln\big(\sqrt{-A(\tau)B(\tau)}/B(\tau)\big)$; then, via differentiation, the definition (1.35), and the corresponding system of isomonodromy deformations (1.36), it follows that

$$\begin{aligned}
\varphi'(\tau) &= \mathrm{i}\left(\frac{1}{\sqrt{-A(\tau)B(\tau)}}\big(\sqrt{-A(\tau)B(\tau)}\big)' - \frac{B'(\tau)}{B(\tau)}\right) \\
&= -\mathrm{i}\left(\frac{2(A(\tau)D(\tau) - B(\tau)C(\tau))}{\sqrt{-A(\tau)B(\tau)}} - \frac{1}{B(\tau)}\left(\frac{\mathrm{i}a}{\tau}B(\tau) - 4D(\tau)\sqrt{-A(\tau)B(\tau)}\right)\right) \\
&= -\frac{a}{\tau} + \frac{2\mathrm{i}}{\sqrt{-A(\tau)B(\tau)}}(A(\tau)D(\tau) + B(\tau)C(\tau)) \\
&= -\frac{a}{\tau} + \frac{2\mathrm{i}}{\sqrt{-A(\tau)B(\tau)}}\left(-\frac{\mathrm{i}\varepsilon b}{2\tau} - \frac{\mathrm{i}a}{\tau}\sqrt{-A(\tau)B(\tau)}\right) = \frac{a}{\tau} + \frac{b}{u(\tau)},
\end{aligned}$$

that is, $\varphi(\tau)$ solves the ODE given by the second (right-most) member of the system (1.37); moreover, it also follows from the definitions (1.25), (1.31), and (1.39) that

$$\varphi(\tau) = \hat{\varphi}(\tau) - a\ln\tau. \tag{1.40}$$

The definitions (1.38) for the functions $A(\tau)$, $B(\tau)$, $C(\tau)$, and $D(\tau)$ are a consequence of the definitions (1.24) and (1.31), the fact that $u(\tau) = \hat{u}(\tau)$, and equation (1.40). A series of lengthy, but otherwise straightforward, differentiation arguments completes the proof. ■

Remark 1.10. It also follows from the ODE satisfied by $\hat{\varphi}(\tau)$ given in the system (1.23) and equation (1.40) that $\varphi(\tau)$ solves the corresponding ODE given in the system (1.37).

Proposition 1.11. *Let*

$$2f_-(\tau) := -\mathrm{i}(a - \mathrm{i}/2) + \frac{\tau}{2}\left(\frac{u'(\tau) - \mathrm{i}b}{u(\tau)}\right), \tag{1.41}$$

and

$$\frac{4\mathrm{i}}{\varepsilon b}f_+(\tau) := \mathrm{i}(a + \mathrm{i}/2) + \frac{\tau}{2}\left(\frac{u'(\tau) + \mathrm{i}b}{u(\tau)}\right). \tag{1.42}$$

Then, for $\varepsilon \in \{\pm 1\}$*,*

$$2f_-(\tau) = \frac{2\varepsilon\tau^2 A(\tau)D(\tau)}{u(\tau)} = \frac{\tau}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\ln\left(\frac{u(\tau)}{\tau}\right) - \mathrm{i}(\varphi(\tau) + a\ln\tau)\right),$$

and

$$\frac{4\mathrm{i}}{\varepsilon b}f_+(\tau) = -\frac{2\varepsilon\tau^2 B(\tau)C(\tau)}{u(\tau)} = \frac{\tau}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\ln\left(\frac{u(\tau)}{\tau}\right) + \mathrm{i}(\varphi(\tau) + a\ln\tau)\right);$$

furthermore,

$$\frac{4\mathrm{i}}{\varepsilon b}f_{+}(\tau)=2f_{-}(\tau)+\mathrm{i}\tau\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\varphi(\tau)+a\ln\tau\right)=2f_{-}(\tau)+\mathrm{i}\tau\left(\frac{2a}{\tau}+\frac{b}{u(\tau)}\right). \tag{1.43}$$

Proof. Via definition (1.35), the system (1.37), the corresponding system of isomonodromy deformations (1.36), Remark 1.8, and the definitions (1.38) and (1.39), one establishes the veracity of the relation

$$\frac{u'(\tau)-\mathrm{i}b}{u(\tau)}=\frac{2}{\tau}\left(\frac{2\tau A(\tau)D(\tau)}{\sqrt{-A(\tau)B(\tau)}}+(\mathrm{i}a+1/2)\right), \tag{1.44}$$

and then proceeds, *mutatis mutandis*, as in the proof of Proposition 1.6. The corollary (1.43) follows from, and is consistent with, the definition (1.35) and the first integral of system (1.36) (cf. Remark 1.8). ■

Remark 1.12. One deduces from the definitions (1.31), equation (1.40), and Propositions 1.6 and 1.11 that $f_{\pm}(\tau)=\hat{f}_{\pm}(\tau)$.

Remark 1.13. A lengthy algebraic exercise reveals that, in terms of the coefficient functions $A(\tau)$, $B(\tau)$, $C(\tau)$, and $D(\tau)$ satisfying the corresponding isomonodromy deformations (1.36), the Hamiltonian function (cf. equation (1.7)) reads

$$\mathcal{H}(\tau)=\frac{1}{2\tau}\left(\mathrm{i}a+\frac{1}{2}+\frac{2\tau A(\tau)D(\tau)}{\sqrt{-A(\tau)B(\tau)}}\right)^{2}+4\tau\sqrt{-A(\tau)B(\tau)}-\frac{\mathrm{i}(\varepsilon b)D(\tau)}{B(\tau)}+2\tau C(\tau)D(\tau)$$
$$+\frac{A(\tau)D(\tau)}{\sqrt{-A(\tau)B(\tau)}}.$$

Remark 1.14. Hereafter, explicit τ-dependencies are suppressed, except where imperative.

1.4 Canonical solutions and the monodromy data

A succinct discussion of the monodromy data associated with the system (1.32) is presented in this subsection (see, in particular, [57, 61]).

For $\mu\in\mathbb{C}$, the system (1.32) has two irregular singular points, one being the point at infinity ($\mu=\infty$) and the other being the origin ($\mu=0$). For $\delta_{\infty},\delta_0>0$ and $m\in\mathbb{Z}$, define the (sectorial) neighbourhoods Ω_m^{∞} and Ω_m^{0}, respectively, of these singular points:

$$\Omega_m^{\infty}:=\left\{\mu\in\mathbb{C};\,|\mu|>\delta_{\infty}^{-1},\,\frac{\pi}{2}(m-1)<\arg(\mu)+\frac{1}{2}\arg(\tau)<\frac{\pi}{2}(m+1)\right\},$$
$$\Omega_m^{0}:=\left\{\mu\in\mathbb{C};\,|\mu|<\delta_0,\,\pi(m-1)<\arg(\mu)-\frac{1}{2}\arg(\tau)-\frac{1}{2}\arg(\varepsilon b)<\pi(m+1)\right\}.$$

Proposition 1.15 ([57, 61])**.** *There exist solutions* $\mathbb{Y}_m^{\infty}(\mu)=\mathbb{Y}_m^{\infty}(\mu,\tau)$ *and* $\mathbb{X}_m^{0}(\mu)=\mathbb{X}_m^{0}(\mu,\tau)$, $m\in\mathbb{Z}$, *of the system* (1.32) *that are uniquely defined by the following asymptotic expansions:*

$$\mathbb{Y}_m^{\infty}(\mu)\underset{\Omega_m^{\infty}\ni\mu\to\infty}{:=}\left(\mathrm{I}+\Psi^{(1)}\mu^{-1}+\Psi^{(2)}\mu^{-2}+\cdots\right)\exp\left(-\mathrm{i}\left(\tau\mu^{2}+(a-\mathrm{i}/2)\ln\mu\right)\sigma_3\right),$$
$$\mathbb{X}_m^{0}(\mu)\underset{\Omega_m^{0}\ni\mu\to0}{:=}\Psi_0\left(\mathrm{I}+\hat{\mathcal{Z}}_1\mu+\cdots\right)\exp\left(-\mathrm{i}\sqrt{\tau\varepsilon b}\mu^{-1}\sigma_3\right),$$

where $\mathrm{I}=\operatorname{diag}(1,1)$, $\ln\mu:=\ln|\mu|+\mathrm{i}\arg\mu$,

$$\Psi^{(1)}=\begin{pmatrix}0&\frac{A(\tau)}{\sqrt{-A(\tau)B(\tau)}}\\-\mathrm{i}D(\tau)/2&0\end{pmatrix},\qquad\Psi^{(2)}=\begin{pmatrix}\psi_{11}^{(2)}&0\\0&\psi_{22}^{(2)}\end{pmatrix},$$

$$\psi_{11}^{(2)} := -\frac{\mathrm{i}}{2}\left(\tau\sqrt{-A(\tau)B(\tau)} + \tau C(\tau)D(\tau) + \frac{A(\tau)D(\tau)}{\sqrt{-A(\tau)B(\tau)}}\right),$$
$$\psi_{22}^{(2)} := \frac{\mathrm{i}\tau}{2}\big(\sqrt{-A(\tau)B(\tau)} + C(\tau)D(\tau)\big),$$
$$\Psi_0 = \frac{\mathrm{i}}{\sqrt{2}}\left(\frac{(\varepsilon b)^{1/4}}{\tau^{1/4}\sqrt{B(\tau)}}\right)^{\sigma_3}(\sigma_1+\sigma_3), \qquad \hat{\mathcal{Z}}_1 = \begin{pmatrix} z_1^{(11)} & z_1^{(12)} \\ -z_1^{(12)} & -z_1^{(11)} \end{pmatrix},$$
$$z_1^{(11)} := -\frac{\mathrm{i}\big(\mathrm{i}a + \frac{1}{2} + \frac{2\tau A(\tau)D(\tau)}{\sqrt{-A(\tau)B(\tau)}}\big)^2}{2\sqrt{\tau\varepsilon b}} - \frac{2\mathrm{i}\tau^{3/2}\sqrt{-A(\tau)B(\tau)}}{\sqrt{\varepsilon b}} - \frac{D(\tau)\sqrt{\tau\varepsilon b}}{B(\tau)},$$
$$z_1^{(12)} := -\frac{\mathrm{i}\big(\mathrm{i}a + \frac{1}{2} + \frac{2\tau A(\tau)D(\tau)}{\sqrt{-A(\tau)B(\tau)}}\big)}{2\sqrt{\tau\varepsilon b}},$$

and $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Remark 1.16. The canonical solutions $\mathbb{X}_m^0(\mu)$, $m \in \mathbb{Z}$, are defined uniquely provided that the branch of $(B(\tau))^{1/2}$ is fixed: hereafter, the branch of $(B(\tau))^{1/2}$ is not fixed; therefore, the set of canonical solutions $\{\mathbb{X}_m^0(\mu)\}_{m\in\mathbb{Z}}$ is defined up to a sign. This ambiguity does not affect the definition of the Stokes multipliers (see equations (1.45) below); rather, it results in a sign discrepancy in the definition of the connection matrix, G (see equation (1.48) below).

The *canonical solutions*, $\mathbb{Y}_m^\infty(\mu)$ and $\mathbb{X}_m^0(\mu)$, $m \in \mathbb{Z}$, enable one to define the *Stokes matrices*, S_m^∞ and S_m^0, respectively,

$$\mathbb{Y}_{m+1}^\infty(\mu) = \mathbb{Y}_m^\infty(\mu)S_m^\infty, \qquad \mathbb{X}_{m+1}^0(\mu) = \mathbb{X}_m^0(\mu)S_m^0. \tag{1.45}$$

The Stokes matrices are independent of μ and τ, and have the following structures:

$$S_{2m}^\infty = \begin{pmatrix} 1 & 0 \\ s_{2m}^\infty & 1 \end{pmatrix}, \qquad S_{2m+1}^\infty = \begin{pmatrix} 1 & s_{2m+1}^\infty \\ 0 & 1 \end{pmatrix}, \qquad S_{2m}^0 = \begin{pmatrix} 1 & s_{2m}^0 \\ 0 & 1 \end{pmatrix},$$
$$S_{2m+1}^0 = \begin{pmatrix} 1 & 0 \\ s_{2m+1}^0 & 1 \end{pmatrix}.$$

The parameters s_m^∞ and s_m^0 are called the *Stokes multipliers*: it can be shown that

$$S_{m+4}^\infty = \mathrm{e}^{-2\pi(a-\mathrm{i}/2)\sigma_3}S_m^\infty\mathrm{e}^{2\pi(a-\mathrm{i}/2)\sigma_3}, \qquad S_{m+2}^0 = S_m^0. \tag{1.46}$$

Equations (1.46) imply that the number of independent Stokes multipliers does not exceed six; for example, s_0^0, s_1^0, s_0^∞, s_1^∞, s_2^∞, and s_3^∞. Furthermore, due to the special structure of the system (1.32), that is, the coefficient matrices of odd (resp., even) powers of μ in $\widetilde{\mathcal{U}}(\mu,\tau)$ are diagonal (resp., off-diagonal) and *vice-versa* for $\widetilde{\mathcal{V}}(\mu,\tau)$, one can deduce the following relations for the Stokes matrices:

$$S_{m+2}^\infty = \sigma_3\mathrm{e}^{-\pi(a-\mathrm{i}/2)\sigma_3}S_m^\infty\mathrm{e}^{\pi(a-\mathrm{i}/2)\sigma_3}\sigma_3, \qquad S_{m+1}^0 = \sigma_1 S_m^0\sigma_1. \tag{1.47}$$

Equations (1.47) reduce the number of independent Stokes multipliers by two, that is, all Stokes multipliers can be expressed in terms of s_0^0, s_0^∞, s_1^∞, and a. There is one more relation between the Stokes multipliers that follows from the so-called cyclic relation (see equation (1.49) below). Define the monodromy matrix at the point at infinity, M^∞, and the monodromy matrix at the origin, M^0, via the following relations:

$$\mathbb{Y}_0^\infty\big(\mu\mathrm{e}^{-2\pi\mathrm{i}}\big) := \mathbb{Y}_0^\infty(\mu)M^\infty, \qquad \mathbb{X}_0^0\big(\mu\mathrm{e}^{-2\pi\mathrm{i}}\big) := \mathbb{X}_0^0(\mu)M^0.$$

Since $\mathbb{Y}_0^\infty(\mu)$ and $\mathbb{X}_0^0(\mu)$ are solutions of the system (1.32), they differ by a right-hand (matrix) factor G:

$$\mathbb{Y}_0^\infty(\mu) := \mathbb{X}_0^0(\mu)G, \tag{1.48}$$

where G is called the *connection matrix*. As matrices relating fundamental solutions of the system (1.32), the monodromy, connection, and Stokes matrices are independent of μ and τ; moreover, since $\operatorname{tr}\big(\widetilde{\mathcal{U}}(\mu,\tau)\big) = \operatorname{tr}\big(\widetilde{\mathcal{V}}(\mu,\tau)\big) = 0$, it follows that $\det(M^\infty) = \det\big(M^0\big) = \det(G) = 1$. From the definition of the monodromy and connection matrices, one deduces the following *cyclic relation*:

$$GM^\infty = M^0 G. \tag{1.49}$$

The monodromy matrices can be expressed in terms of the Stokes matrices

$$M^\infty = S_0^\infty S_1^\infty S_2^\infty S_3^\infty \mathrm{e}^{-2\pi(a-\mathrm{i}/2)\sigma_3}, \qquad M^0 = S_0^0 S_1^0.$$

The Stokes multipliers, s_0^0, s_0^∞, and s_1^∞, the elements of the connection matrix, $(G)_{ij} =: g_{ij}$, $i,j \in \{1,2\}$, and the parameter of formal monodromy, a, are called the *monodromy data*.

1.5 The monodromy manifold, the direct and inverse problems of monodromy theory, and organisation of the paper

In this subsection, the monodromy manifold is introduced, the direct and inverse problems of monodromy theory are discussed (see, for example, [10, 29, 45, 54], and [55, Section 2]), and the contents of this work are delineated.

Consider $\mathbb{C}^8$ with co-ordinates $\big(a, s_0^0, s_0^\infty, s_1^\infty, g_{11}, g_{12}, g_{21}, g_{22}\big)$. The (algebraic) variety defined by $\det(G) = 1$ and the *semi-cyclic relation*

$$G^{-1} S_0^0 \sigma_1 G = S_0^\infty S_1^\infty \sigma_3 \mathrm{e}^{-\pi(a-\mathrm{i}/2)\sigma_3} \tag{1.50}$$

are called the *manifold of the monodromy data*, $\mathcal{M}$. Since only three of the four equations in the semi-cyclic relation (1.50) are independent, it follows that $\dim_{\mathbb{C}}(\mathcal{M}) = 4$; more specifically, the system of algebraic equations defining $\mathcal{M}$ reads[5]

$$\begin{aligned} &s_0^\infty s_1^\infty = -1 - \mathrm{e}^{-2\pi a} - \mathrm{i}s_0^0 \mathrm{e}^{-\pi a}, \qquad g_{21}g_{22} - g_{11}g_{12} + s_0^0 g_{11}g_{22} = \mathrm{i}\mathrm{e}^{-\pi a},\\ &g_{11}^2 - g_{21}^2 - s_0^0 g_{11}g_{21} = \mathrm{i}s_0^\infty \mathrm{e}^{-\pi a}, \qquad g_{22}^2 - g_{12}^2 + s_0^0 g_{12}g_{22} = \mathrm{i}s_1^\infty \mathrm{e}^{\pi a},\\ &g_{11}g_{22} - g_{12}g_{21} = 1. \end{aligned} \tag{1.51}$$

Remark 1.17. To achieve a one-to-one correspondence between the coefficients of the system (1.32) and the points on $\mathcal{M}$, one has to factorize $\mathcal{M}$ by the involution $G \to -G$ (cf. Remark 1.16), that is, $G \in \mathrm{PSL}(2,\mathbb{C})$.

As shown in [61, Section 2], equations (1.51) defining $\mathcal{M}$ are equivalent to one of the following three systems: (i)[6] $g_{11}g_{22} \neq 0 \Rightarrow$

$$s_0^\infty = -\frac{(g_{21} + \mathrm{i}\mathrm{e}^{\pi a} g_{11})}{g_{22}}, \qquad s_1^\infty = -\frac{\mathrm{i}(g_{22} + \mathrm{i}g_{12}\mathrm{e}^{-\pi a})\mathrm{e}^{-\pi a}}{g_{11}}, \qquad s_0^0 = \frac{\mathrm{i}\mathrm{e}^{-\pi a} + g_{11}g_{12} - g_{21}g_{22}}{g_{11}g_{22}};$$

[5] In these equations, $\mathrm{e}^{\pi a}$ is considered to be a parameter.

[6] This case does not exclude the possibility that $g_{12} = 0$ or $g_{21} = 0$. There is a misprint in [61, Section 2, p. 1172]: in item (1), below equations (33), the formula for the Stokes multiplier s_1^∞ must be changed to $s_1^\infty = -\frac{\mathrm{i}(g_{22}+\mathrm{i}g_{12}\mathrm{e}^{-\pi a})\mathrm{e}^{-\pi a}}{g_{11}}$.

(ii) $g_{11} \neq 0$ and $g_{22}=0$, in which case the parameters are s_0^0 and g_{11}, and

$$g_{12}=-\frac{\mathrm{i}\mathrm{e}^{-\pi a}}{g_{11}}, \qquad g_{21}=-\mathrm{i}\mathrm{e}^{\pi a}g_{11}, \qquad s_0^{\infty}=-\mathrm{i}g_{11}^2\big(1+\mathrm{e}^{2\pi a}+\mathrm{i}s_0^0\mathrm{e}^{\pi a}\big)\mathrm{e}^{\pi a},$$
$$s_1^{\infty}=-\frac{\mathrm{i}\mathrm{e}^{-3\pi a}}{g_{11}^2}; \tag{1.52}$$

and (iii) $g_{11}=0$ and $g_{22}\neq 0$, in which case the parameters are s_0^0 and g_{22}, and

$$g_{12}=\mathrm{i}\mathrm{e}^{\pi a}g_{22}, \qquad g_{21}=\frac{\mathrm{i}\mathrm{e}^{-\pi a}}{g_{22}}, \qquad s_0^{\infty}=-\frac{\mathrm{i}\mathrm{e}^{-\pi a}}{g_{22}^2},$$
$$s_1^{\infty}=-\mathrm{i}g_{22}^2\big(1+\mathrm{e}^{2\pi a}+\mathrm{i}s_0^0\mathrm{e}^{\pi a}\big)\mathrm{e}^{-\pi a}. \tag{1.53}$$

Asymptotics as $\tau \to \pm 0$ and $\tau \to \pm\mathrm{i}0$ (resp., as $\tau \to \pm\infty$ and $\tau \to \pm\mathrm{i}\infty$) of the general (resp., general regular) solution of the DP3E (1.1), and its associated Hamiltonian function, $\mathcal{H}(\tau)$, parametrised in terms of the proper open subset of $\mathcal{M}$ corresponding to case (i) were presented in [61],[7] and asymptotics as $\tau \to \pm\infty$ and $\tau \to \pm\mathrm{i}\infty$ of general regular and singular solutions of the DP3E (1.1), and its associated Hamiltonian and auxiliary functions, $\mathcal{H}(\tau)$ and $f_-(\tau)$,[8] respectively, parametrised in terms of the proper open subset of $\mathcal{M}$ corresponding to case (i) were obtained in [57]; furthermore, three-real-parameter families of solutions of the DP3E (1.1) that possess infinite sequences of poles and zeros asymptotically located along the imaginary and real axes were identified, and the asymptotic distribution of these poles and zeros were also derived. The purpose of the present work, therefore, is to close the aforementioned gaps, and to continue to cover $\mathcal{M}$ by deriving asymptotics (as $\tau \to \pm\infty$ and $\tau \to \pm\mathrm{i}\infty$) of $u(\tau)$, and the related functions $f_{\pm}(\tau)$, $\mathcal{H}(\tau)$, and $\sigma(\tau)$, that are parametrised in terms of the complementary proper open subsets of $\mathcal{M}$ corresponding to cases (ii) and (iii).[9] For notational consistency with the main body of the text, cases (ii) and (iii) for $\mathcal{M}$ will, henceforth, be referred to via the integer index $k \in \{\pm 1\}$; more specifically, case (ii), that is, $g_{11}\neq 0$, $g_{22}=0$, and $g_{12}g_{21}=-1$, will be designated by $k=+1$, and case (iii), that is, $g_{11}=0$, $g_{22}\neq 0$, and $g_{12}g_{21}=-1$, will be designated by $k=-1$.

Without loss of generality, and with a slight, temporary amendment of the notation, reconsider, for given $a \in \mathbb{C}$, $b \in \mathbb{R}\setminus\{0\}$, and $\varepsilon \in \{\pm 1\}$, the first-order linear matrix ODE that constitutes the μ-part of the post-gauge-transformed Fuchs–Garnier pair given in the system (1.32),[10]

$$\partial_{\mu}\Psi(\mu,\tau)=\widetilde{\mathcal{U}}(\mu,\tau;\vec{\mathbf{y}})\Psi(\mu,\tau), \tag{1.54}$$

where $\mu,\tau \in \mathbb{C}$,

$$\mathbb{C}^5 \ni \vec{\mathbf{y}} := \big(A(\tau),B(\tau),C(\tau),D(\tau),\sqrt{-A(\tau)B(\tau)}\big)$$

is a vector-valued function constructed from the matrix elements of the coefficient matrices in the decomposition of (cf. equation (1.33)) $\mathrm{M}_2(\mathbb{C}) \ni \widetilde{\mathcal{U}}(\mu,\tau;\vec{\mathbf{y}})$ into partial fractions, $\widetilde{\mathcal{U}}(\mu,\tau;\vec{\mathbf{y}})$ is a rational function with respect to the spectral parameter μ with poles that are independent of τ, and $\operatorname{tr}\big(\widetilde{\mathcal{U}}(\mu,\tau;\vec{\mathbf{y}})\big)=0$. The *direct problem of monodromy theory* (DMP) can be

[7] Asymptotics as $\tau \to \pm 0$ and $\tau \to \pm\mathrm{i}0$ for the corresponding $\boldsymbol{\tau}$-function, but without the 'constant term', were also conjectured in [61].

[8] Denoted as $f(\tau)$ in [57].

[9] Asymptotics as $\tau \to \pm 0$ and $\tau \to \pm\mathrm{i}0$ for $u(\tau)$, $\mathcal{H}(\tau)$, $f_{\pm}(\tau)$, and $\sigma(\tau)$ corresponding to cases (ii) and (iii) will be presented elsewhere.

[10] One merely makes the purely notational change $\widetilde{\mathcal{U}}(\mu,\tau) \to \widetilde{\mathcal{U}}(\mu,\tau;\vec{\mathbf{y}})$ in equation (1.33). Analogous statements can be made regarding the μ-part of the pre-gauge-transformed Fuchs–Garnier pair presented in the system (1.18).

stated as follows: using the tuple of coefficients $\big(\tau, A(\tau), B(\tau), C(\tau), D(\tau), \sqrt{-A(\tau)B(\tau)}\big)$, find the monodromy data $\mathfrak{M} := \big(a, s_0^0, s_0^\infty, s_1^\infty, g_{11}, g_{12}, g_{21}, g_{22}\big) \in \mathcal{M}$ (recall that the monodromy data are not independent and are related via the algebraic equations (1.51), which define the complex manifold $\mathcal{M} \in \mathbb{C}^8$ called the manifold of the monodromy data), or, in other words, it is a correspondence $\big(\tau, A(\tau), B(\tau), C(\tau), D(\tau), \sqrt{-A(\tau)B(\tau)}\big) \to$ system (1.54) $\to \mathfrak{M} \in \mathcal{M}$. The *inverse problem of monodromy theory* (IMP) can be stated as follows: using the data set $\{\tau, \mathfrak{M}\}$, find $\vec{\mathbf{y}} \in \mathbb{C}^5$ such that the system (1.54) constructed with the help of the co-ordinate (or coefficient) functions of $\vec{\mathbf{y}}$ has the monodromy data $\mathfrak{M} \in \mathcal{M}$, or, in other words, it is the inverse map[11]

$$\{\tau, \mathfrak{M}\} \to \big(\tau, A(\tau), B(\tau), C(\tau), D(\tau), \sqrt{-A(\tau)B(\tau)}\big).$$

Thus, if one fixes the collection of the monodromy data $\mathfrak{M} \in \mathcal{M}$ and denotes by $\mathcal{T} \subset \mathbb{C}$ the set of all τ for which the IMP is solvable, then the functions $A(\tau)$, $B(\tau)$, $C(\tau)$, $D(\tau)$, $\sqrt{-A(\tau)B(\tau)}$: $\mathcal{T} \to \mathbb{C}$ are determined, and thus, via Proposition 1.9, the 2-tuple $(u(\tau), \varphi(\tau))$ solves the system (1.37).[12] The complete set of the monodromy data corresponding to the system (1.54) (equivalently, the system (1.32)) depends, in general, on both τ and $\vec{\mathbf{y}}$, and will be denoted by $\mathfrak{M}(\tau; \vec{\mathbf{y}})$. As a consequence of the requirement that the monodromy data be independent of τ and $\vec{\mathbf{y}}$, that is, $\mathfrak{M}(\tau; \vec{\mathbf{y}}) = \mathrm{const}$, it is necessary that $\vec{\mathbf{y}} = \vec{\mathbf{y}}(\tau)$ satisfy the system of isomonodromy deformations (nonlinear ODEs) (1.36), which can be presented in the form

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\vec{\mathbf{y}}(\tau) = \bigg(-\frac{\mathrm{i}a}{\tau}A(\tau) + 4C(\tau)\sqrt{-A(\tau)B(\tau)}, \frac{\mathrm{i}a}{\tau}B(\tau) - 4D(\tau)\sqrt{-A(\tau)B(\tau)}, \\ \frac{(\mathrm{i}a-1)}{\tau}C(\tau) - 2A(\tau), -\frac{(\mathrm{i}a+1)}{\tau}D(\tau) + 2B(\tau), 2(A(\tau)D(\tau) - B(\tau)C(\tau))\bigg).$$

Clearly, $\mathfrak{M}(\tau; \vec{\mathbf{y}}) \in \mathcal{M}$. Denote by $\mathbb{M}_3$ the collection of monodromy data for which the IMP is explicitly solvable: for other $\mathfrak{M}(\tau; \vec{\mathbf{y}}) \in \mathcal{M}$, it is possible to solve the IMP asymptotically (as $\tau \to +\infty$, say); this leads to, for example, asymptotic formulae for solutions of the DP3E (1.1). Let $\mathcal{D} \subset \mathcal{M} \setminus \mathbb{M}_3$ be a domain. The IMP is said to be *asymptotically solvable* (as $\tau \to +\infty$, say) if, for any $\mathfrak{M} \in \mathcal{D}$ representing the monodromy data, there exists an asymptotically locally uniform[13] vector-valued function

$$\vec{\mathbf{y}}^{\clubsuit} = \vec{\mathbf{y}}^{\clubsuit}(\tau; \mathfrak{M}) := \big(A(\tau; \mathfrak{M}), B(\tau; \mathfrak{M}), C(\tau; \mathfrak{M}), D(\tau; \mathfrak{M}), \sqrt{-A(\tau; \mathfrak{M})B(\tau; \mathfrak{M})}\big) \in \mathbb{C}^5$$

constructed from the matrix elements of the $\mathrm{M}_2(\mathbb{C})$-coefficients of the system (1.54) that is analytic in $(T, +\infty) \times \mathcal{D}$ and invertible with respect to $\mathfrak{M}$, and the monodromy data $\mathfrak{M}^{\clubsuit}(\tau; \mathfrak{M})$ corresponding to $\vec{\mathbf{y}}^{\clubsuit}(\tau; \mathfrak{M})$ can be represented as

$$\mathfrak{M}^{\clubsuit}(\tau; \mathfrak{M}) = \mathfrak{M} + \mathfrak{G}(\tau; \mathfrak{M}),$$

where $\mathfrak{G}(\tau; \mathfrak{M})$ is a locally uniformly decreasing vector-valued function, that is, $||\mathfrak{M}^{\clubsuit}(\tau; \mathfrak{M}) - \mathfrak{M}|| = ||\mathfrak{G}(\tau; \mathfrak{M})|| < \mathrm{C}|\tau|^{-\delta_*}$ as $\tau \to +\infty$,[14] where $\mathrm{C} > 0$ and $\delta_* > 0$ are the same for all

[11]If there exists a solution of the IMP, then it is unique [10, 29, 45, 54, 55].

[12]As long as the monodromy data is given, the function $\varphi(\tau)$ is fixed modulo $2\pi l$, $l \in \mathbb{Z}$, or, alternatively, the constant of integration in the system (1.37) is defined via the monodromy data modulo $2\pi l$. The function $\varphi(\tau)$ belongs to the class of functions defined by the equivalence relation $\varphi \equiv \varphi + 2\pi l$, $l \in \mathbb{Z}$.

[13]A function $f(\tau, \lambda)$ is said to be *asymptotically locally uniform* (as $\tau \to +\infty$, say) if, for any point λ in the domain of definition of $f(\tau, \lambda)$, there exist functions $h_1(\tau, \lambda)$ and $h_2(\tau, \lambda)$ such that, for any $\tilde{\epsilon}_* > 0$, there exist numbers T and $\tilde{\delta}_* = \tilde{\delta}_*(\lambda, \tilde{\epsilon}_*) > 0$ such that, for any $(T, +\infty) \ni \tau$ and for all $\tilde{\lambda} \in \mathbb{B}_{\tilde{\delta}_*}(\lambda) := \{\tilde{\lambda} \mid |\tilde{\lambda} - \lambda| < \tilde{\delta}_*\}$ (the open ball of radius $\tilde{\delta}_*$ centred at λ), the inequality $h_1(\tau, \lambda)(1 - \tilde{\epsilon}_*) < |f(\tau, \tilde{\lambda})| < h_2(\tau, \lambda)(1 + \tilde{\epsilon}_*)$ is satisfied; furthermore, if $h_1(\tau, \lambda), h_2(\tau, \lambda) \to 0$ (as $\tau \to +\infty$, say) in the latter inequality, then $f(\tau, \lambda)$ is said to be a *locally uniformly decreasing* function [54].

[14]$||\cdot||$ is any norm in $\mathbb{C}^8$.

$\mathfrak{M}^{\clubsuit}(\tau;\mathfrak{M})$ [54, 55].[15] In fact, according to the THEOREM in [54], if the IMP is solvable for the domain $\mathcal{D}$, then, for any $\mathfrak{M}_0 \in \mathcal{D}$ representing the monodromy data for the system (1.54), there exists a *unique* vector-valued function

$$\vec{\mathbf{y}} = \vec{\mathbf{y}}(\tau;\mathfrak{M}_0) := \big(A(\tau;\mathfrak{M}_0), B(\tau;\mathfrak{M}_0), C(\tau;\mathfrak{M}_0), D(\tau;\mathfrak{M}_0), \sqrt{-A(\tau;\mathfrak{M}_0)B(\tau;\mathfrak{M}_0)}\big) \in \mathbb{C}^5$$

formed by the matrix elements of the $\mathrm{M}_2(\mathbb{C})$-coefficients of the system (1.54) that is analytic in $(T,+\infty)\times\mathcal{D}$ such that the monodromy data $\mathfrak{M}(\tau;\mathfrak{M}_0)$ corresponding to $\vec{\mathbf{y}}(\tau;\mathfrak{M}_0)$ coincides with $\mathfrak{M}_0$ for all $\tau \in (T,+\infty)$, namely, $||\mathfrak{M}(\tau;\mathfrak{M}_0)-\mathfrak{M}_0|| = o\big(\tau^{-\delta_*}\big)$ uniformly as $\tau \to +\infty$, $\delta_* > 0$.

Remark 1.18. The explication above of the DMP and IMP for the μ-part of the system (1.32) was formulated within the framework of the $\mathbb{C}$-valued functions $A(\tau)$, $B(\tau)$, $C(\tau)$, $D(\tau)$, and $\sqrt{-A(\tau)B(\tau)}$ (solving the system of isomonodromy deformations (1.36)) which appear as matrix elements of the $\mathrm{M}_2(\mathbb{C})$-coefficients of (cf. equation (1.33)) $\widetilde{\mathfrak{U}}(\mu,\tau)$ in its partial fraction decomposition with respect to the spectral parameter μ. Equivalently, via the definition (1.35), Remark 1.8, and Proposition 1.9, one may eschew the $\mathbb{C}$-valued functions $A(\tau)$, $B(\tau)$, $C(\tau)$, $D(\tau)$, and $\sqrt{-A(\tau)B(\tau)}$ altogether and re-express $\widetilde{\mathfrak{U}}(\mu,\tau) \in \mathrm{M}_2(\mathbb{C})$ solely in terms of the 3-tuple of $\mathbb{C}$-valued functions $(u(\tau),\varphi(\tau),u'(\tau))$, where, in particular, the 2-tuple $(u(\tau),\varphi(\tau))$ solves the system (1.37), that is,

$$\begin{aligned}\widetilde{\mathfrak{U}}(\mu,\tau) = & -2\mathrm{i}\tau\mu\sigma_3 + 2\tau\begin{pmatrix} 0 & 2\mathrm{i}\varepsilon\mathrm{e}^{\mathrm{i}\varphi(\tau)} \\ \frac{\varepsilon\mathrm{e}^{-\mathrm{i}\varphi(\tau)}}{2\tau}\big(\mathrm{i}(a-\frac{\mathrm{i}}{2})-\frac{\tau(u'(\tau)-\mathrm{i}b)}{2u(\tau)}\big) & 0\end{pmatrix} \\ & -\frac{1}{\mu}\frac{\tau(u'(\tau)-\mathrm{i}b)}{2u(\tau)}\sigma_3 + \frac{1}{\mu^2}\begin{pmatrix} 0 & -\frac{\mathrm{i}\varepsilon b\tau}{u(\tau)}\mathrm{e}^{\mathrm{i}\varphi(\tau)} \\ -\mathrm{i}u(\tau)\mathrm{e}^{-\mathrm{i}\varphi(\tau)} & 0\end{pmatrix},\end{aligned}$$

and regurgitate *verbatim* the above discussion of the DMP and IMP in terms of the $\mathbb{C}$-valued functions $u(\tau)$, $\varphi(\tau)$, and $u'(\tau)$; but, since the former, and not the latter, approach has been adopted in the present work, this matter will not be addressed further.

The contents of this paper, the main body of which is devoted to the asymptotic analysis (as $\tau \to +\infty$ for $\varepsilon b > 0$) of $u(\tau)$ and the related, auxiliary functions $f_{\pm}(\tau)$, $\mathcal{H}(\tau)$, and $\sigma(\tau)$, are now described. In Section 2, the main asymptotic results as $\tau \to \pm\infty$ and $\tau \to \pm\mathrm{i}\infty$ for $u(\tau)$, $f_{\pm}(\tau)$, $\mathcal{H}(\tau)$, and $\sigma(\tau)$ parametrised in terms of the monodromy data corresponding to the cases designated by the index $k \in \{\pm 1\}$ (see the discussion above) are stated. In Section 3, the asymptotic (as $\tau \to +\infty$ for $\varepsilon b > 0$) solution of the DMP for the μ-part of the system (1.32), under certain tempered restrictions on its coefficient functions (in some class(es) of functions) that are consistent with the monodromy data corresponding to $k \in \{\pm 1\}$, is presented; in particular, with the coefficient functions satisfying the asymptotic conditions (3.17), the asymptotic representation for the connection matrix, G, corresponding to $k \in \{\pm 1\}$ stated in Theorem 3.23 is obtained, and, in conjunction with the parametrisations (1.52) and (1.53),

[15] There are also asymptotics obtained via the IDM for which the vector-valued function(s) $\vec{\mathbf{y}}^{\clubsuit} = \vec{\mathbf{y}}^{\clubsuit}(\tau;\mathfrak{M})$ have poles for certain $\mathfrak{M} \in \mathcal{D}$ with ∞ (the point at infinity) being an accumulation point of the poles (see, for example, [57]). In such cases, $(T,+\infty)$ must be replaced by $\bigcup_{m=0}^{\infty}(T_{2m},T_{2m+1})$, with $T_m \nearrow \infty$, where the poles lie in the intervals (lacunae) (T_{2m+1},T_{2m+2}), and where the ratio of the lengths of the intervals containing the poles to the lengths of the intervals devoid of poles must tend to zero, that is, $\frac{|T_{2m+2}-T_{2m+1}|}{|T_{2m+1}-T_{2m}|} \to 0$ as $\mathbb{N} \ni m \to \infty$ (see [54] for technical details). In such cases, $\bigcup_{m=0}^{\infty}(T_{2m},T_{2m+1})\times\mathcal{D}$ should be regarded as the domain of definition for $\vec{\mathbf{y}}^{\clubsuit}(\tau;\mathfrak{M})$, and the IDM enables one to prove the existence of an analytic solution for $\tau \in \mathbb{C}$ whose asymptotic behaviour on $\bigcup_{m=0}^{\infty}(T_{2m},T_{2m+1})$ is determined by $\vec{\mathbf{y}}^{\clubsuit}(\tau;\mathfrak{M})$ and with poles in the intervals (T_{2m+1},T_{2m+2}) [54]. For complexified τ with $|\tau| \to +\infty$, $(T,+\infty)$ must be replaced by a Swiss-cheese-like, multiply-connected strip domain (see, for example, [57]).

the complete asymptotic representation for the monodromy data is derived. The latter analysis is predicated on focusing the principal emphasis on the study of the global asymptotic properties of the fundamental solution of the system (1.32) via the possibility of 'matching' different local asymptotic expansions of $\Psi(\mu,\tau)$ at singular and turning points, namely, matching WKB-asymptotics of the fundamental solution of the system (1.32) with its parametrix represented in terms of parabolic-cylinder functions in open neighbourhoods of double-turning points. In Section 4, the asymptotic results derived in Section 3 are inverted in order to solve the IMP for the μ-part of the system (1.32), that is, explicit asymptotics for the coefficient functions of the μ-part of the system (1.32) are parametrised in terms of the monodromy data corresponding to $k \in \{\pm 1\}$; in particular, via the inversion of the asymptotic representation for the connection matrix corresponding to $k \in \{\pm 1\}$, explicit asymptotic expressions for the coefficient functions parametrised in terms of points on $\mathcal{M}$ are obtained. Under the permanency of the isomonodromy condition on the corresponding connection matrices, namely, the monodromy data are constant and satisfy certain conditions, one deduces that the asymptotics obtained via inversion represent an asymptotic solution of the IMP and satisfy all the restrictions imposed in Section 3; however, since it is not immediately apparent that an asymptotic solution of the IMP represents an asymptotic expansion of the functions in the systems (1.36) and (1.37), because the asymptotic solution of the corresponding monodromy problem was obtained via the IDM, one can use the justification scheme presented in [54] (see also [10, 29, 42]) to prove solvability of the corresponding monodromy problem, from which it follows, therefore, that there exist (exact) solutions of the system of isomonodromy deformations (1.36) whose asymptotics coincide with those obtained in this section. In order to extend the results derived in Sections 3 and 4 for asymptotics of $u(\tau)$, $f_{\pm}(\tau)$, $\mathcal{H}(\tau)$, and $\sigma(\tau)$ on the positive semi-axis ($\tau \to +\infty$) for $\varepsilon b > 0$ to asymptotics on the negative semi-axis ($\tau \to -\infty$) and on the imaginary axis ($\tau \to \pm \mathrm{i}\infty$) for both positive and negative values of εb, one applies the (group) action of the Lie-point symmetries changing $\tau \to -\tau$, $\tau \to \tau$, $a \to -a$, and $\tau \to \pm \mathrm{i}\tau$ derived in Appendix D on the proper open subsets of $\mathcal{M}$ corresponding to $k \in \{\pm 1\}$. Finally, in Appendix E, asymptotics as $\tau \to \pm\infty$ and $\tau \to \pm \mathrm{i}\infty$ with $\pm(\varepsilon b) > 0$ for the multi-valued function $\hat{\varphi}(\tau)$ are presented.

2 Summary of results

In this work, the detailed analysis of asymptotics as $\tau \to +\infty$ for $\varepsilon b > 0$ of $u(\tau)$ and the associated functions $f_{\pm}(\tau)$, $\mathcal{H}(\tau)$, $\sigma(\tau)$, and $\hat{\varphi}(\tau)$ is presented. In order to arrive at the corresponding asymptotics of $u(\tau)$, $f_{\pm}(\tau)$, $\mathcal{H}(\tau)$, $\sigma(\tau)$, and $\hat{\varphi}(\tau)$ for positive, negative, and pure-imaginary values of τ for both positive and negative values of εb, one applies the actions of the Lie-point symmetries changing $\tau \to -\tau$, $\tau \to \tau$, $a \to -a$, and $\tau \to \pm \mathrm{i}\tau$ on $\mathcal{M}$ (see Appendices D.1–D.4, respectively). The 'composed' symmetries of these actions on $\mathcal{M}$ are presented in Appendix D.5 in terms of two auxiliary mappings, both of which are isomorphisms on $\mathcal{M}$, denoted by $\mathcal{F}^{\{\ell\}}_{\varepsilon_1,\varepsilon_2,m(\varepsilon_2)}$, which is relevant for real τ, and $\hat{\mathcal{F}}^{\{\hat{\ell}\}}_{\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)}$, which is relevant for pure-imaginary τ; more precisely, from Appendix D.5,[16]

$$
\begin{aligned}
&\mathcal{F}^{\{\ell\}}_{\varepsilon_1,\varepsilon_2,m(\varepsilon_2)} \colon\ \mathcal{M} \to \mathcal{M},\\
&\big(a, s_0^0, s_0^\infty, s_1^\infty, g_{11}, g_{12}, g_{21}, g_{22}\big) \mapsto \big((-1)^{\varepsilon_2} a, s_0^0(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell),\\
&\qquad s_0^\infty(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell), s_1^\infty(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell), g_{11}(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell),\\
&\qquad g_{12}(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell), g_{21}(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell), g_{22}(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)\big),
\end{aligned}
$$

[16] Due to the involution $G \to -G$ (cf. Remarks 1.16 and 1.17), it suffices to take $\tilde{l} = l' = +1$ in equations (D.47)–(D.93).

where

$$\varepsilon_1,\varepsilon_2\in\{0,\pm1\},\qquad m(\varepsilon_2)=\begin{cases}0, & \varepsilon_2=0,\\ \pm\varepsilon_2, & \varepsilon_2\in\{\pm1\},\end{cases}\qquad \ell\in\{0,1\},$$

and the explicit expressions for $s_0^0(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)$, $s_0^\infty(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)$, $s_1^\infty(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)$, and $g_{ij}(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)$, $i,j\in\{1,2\}$, are given in equations (D.47)–(D.61) and (D.71)–(D.85), and

$$\begin{aligned}&\hat{\mathcal{F}}^{\{\hat\ell\}}_{\hat\varepsilon_1,\hat\varepsilon_2,\hat m(\hat\varepsilon_2)}\colon\ \mathcal{M}\to\mathcal{M},\\ &\big(a,s_0^0,s_0^\infty,s_1^\infty,g_{11},g_{12},g_{21},g_{22}\big)\mapsto\big((-1)^{1+\hat\varepsilon_2}a,\hat s_0^0\big(\hat\varepsilon_1,\hat\varepsilon_2,\hat m(\hat\varepsilon_2)|\hat\ell\big),\hat s_0^\infty\big(\hat\varepsilon_1,\hat\varepsilon_2,\hat m(\hat\varepsilon_2)|\hat\ell\big),\\ &\qquad\hat s_1^\infty\big(\hat\varepsilon_1,\hat\varepsilon_2,\hat m(\hat\varepsilon_2)|\hat\ell\big),\hat g_{11}\big(\hat\varepsilon_1,\hat\varepsilon_2,\hat m(\hat\varepsilon_2)|\hat\ell\big),\hat g_{12}\big(\hat\varepsilon_1,\hat\varepsilon_2,\hat m(\hat\varepsilon_2)|\hat\ell\big),\hat g_{21}\big(\hat\varepsilon_1,\hat\varepsilon_2,\hat m(\hat\varepsilon_2)|\hat\ell\big),\\ &\qquad\hat g_{22}\big(\hat\varepsilon_1,\hat\varepsilon_2,\hat m(\hat\varepsilon_2)|\hat\ell\big)\big),\end{aligned}$$

where

$$\hat\varepsilon_1\in\{\pm1\},\qquad\hat\varepsilon_2\in\{0,\pm1\},\qquad\hat m(\hat\varepsilon_2)=\begin{cases}0, & \hat\varepsilon_2\in\{\pm1\},\\ \pm\hat\varepsilon_1, & \hat\varepsilon_2=0,\end{cases}\qquad\hat\ell\in\{0,1\},$$

and the expressions for $\hat s_0^0\big(\hat\varepsilon_1,\hat\varepsilon_2,\hat m(\hat\varepsilon_2)|\hat\ell\big)$, $\hat s_0^\infty\big(\hat\varepsilon_1,\hat\varepsilon_2,\hat m(\hat\varepsilon_2)|\hat\ell\big)$, $\hat s_1^\infty\big(\hat\varepsilon_1,\hat\varepsilon_2,\hat m(\hat\varepsilon_2)|\hat\ell\big)$, and $\hat g_{ij}\big(\hat\varepsilon_1,\hat\varepsilon_2,\hat m(\hat\varepsilon_2)|\hat\ell\big)$, $i,j\in\{1,2\}$, are given in equations (D.62)–(D.70) and (D.86)–(D.93).

Remark 2.1. It is worth noting that $s_0^0(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)=s_0^0=\hat s_0^0\big(\hat\varepsilon_1,\hat\varepsilon_2,\hat m(\hat\varepsilon_2)|\hat\ell\big)$; furthermore, it follows that $\operatorname{card}\{(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)\}=30$ and $\operatorname{card}\big\{\big(\hat\varepsilon_1,\hat\varepsilon_2,\hat m(\hat\varepsilon_2)|\hat\ell\big)\big\}=16$, that is, for $\ell,\hat\ell\in\{0,1\}$,

$$(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)=\begin{cases}(0,0,0|\ell),\\(-1,0,0|\ell),\\(1,0,0|\ell),\\(0,-1,-1|\ell),\\(0,-1,1|\ell),\\(0,1,-1|\ell),\\(0,1,1|\ell),\\(-1,-1,-1|\ell),\\(1,-1,-1|\ell),\\(-1,-1,1|\ell),\\(1,-1,1|\ell),\\(-1,1,-1|\ell),\\(1,1,-1|\ell),\\(-1,1,1|\ell),\\(1,1,1|\ell),\end{cases}\qquad\text{and}\qquad\big(\hat\varepsilon_1,\hat\varepsilon_2,\hat m(\hat\varepsilon_2)|\hat\ell\big)=\begin{cases}\big(1,1,0|\hat\ell\big),\\\big(1,-1,0|\hat\ell\big),\\\big(-1,1,0|\hat\ell\big),\\\big(-1,-1,0|\hat\ell\big),\\\big(1,0,-1|\hat\ell\big),\\\big(-1,0,-1|\hat\ell\big),\\\big(1,0,1|\hat\ell\big),\\\big(-1,0,1|\hat\ell\big).\end{cases}$$

Via the above-defined notation(s) and Remark 2.1, asymptotics as $\tau\to\pm\infty$ (resp., $\tau\to\pm\mathrm{i}\infty$) for $\pm(\varepsilon b)>0$ of $u(\tau)$, $f_\pm(\tau)$, $\mathcal{H}(\tau)$, and $\sigma(\tau)$ are presented in Theorem 2.4 (resp., Theorem 2.8) below, whilst asymptotics as $\tau\to\pm\infty$ (resp., $\tau\to\pm\mathrm{i}\infty$) for $\pm(\varepsilon b)>0$ of $\hat\varphi(\tau)$ are presented in Appendix E, Theorem E.3 (resp., Theorem E.6).

Remark 2.2. The roots and fractional powers of positive quantities are assumed positive, whilst the branches of the roots of complex quantities can be taken arbitrarily, unless stated otherwise; moreover, it is assumed that, for negative real z, the following branches are always taken: $z^{1/3}:=-|z|^{1/3}$ and $z^{2/3}:=\big(z^{1/3}\big)^2$.

Remark 2.3. If one is only interested in the asymptotics as $\tau \to +\infty$ for $\varepsilon b > 0$ of the functions $u(\tau)$, $f_{\pm}(\tau)$, $\mathcal{H}(\tau)$, and $\sigma(\tau)$, then, in Theorem 2.4 below, one sets $(\varepsilon_1, \varepsilon_2, m(\varepsilon_2)|\ell) = (0,0,0|0)$ and uses the fact that (see Appendix D.5, the identity map (D.47)) $s_0^0(0,0,0|0) = s_0^0$, $s_0^{\infty}(0,0,0|0) = s_0^{\infty}$, $s_1^{\infty}(0,0,0|0) = s_1^{\infty}$, and $g_{ij}(0,0,0|0) = g_{ij}$, $i,j \in \{1,2\}$.

Theorem 2.4. *Let $u(\tau)$ be a solution of the* DP3E (1.1) *and $\hat{\varphi}(\tau)$ be the general solution of the* ODE *$\hat{\varphi}'(\tau) = 2a\tau^{-1} + b(u(\tau))^{-1}$ for $\varepsilon b > 0$ corresponding to the monodromy data $\big(a, s_0^0, s_0^{\infty}, s_1^{\infty}, g_{11}, g_{12}, g_{21}, g_{22}\big)$. Let*

$$\varepsilon_1, \varepsilon_2 \in \{0, \pm 1\}, \qquad m(\varepsilon_2) = \begin{cases} 0, & \varepsilon_2 = 0, \\ \pm\varepsilon_2, & \varepsilon_2 \in \{\pm 1\}, \end{cases} \qquad \ell \in \{0,1\},$$

and $\varepsilon b = |\varepsilon b| \mathrm{e}^{\mathrm{i}\pi\varepsilon_2}$.[17] *For $k = +1$, let*

$$g_{11}(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell) g_{12}(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell) g_{21}(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell) \neq 0, \qquad g_{22}(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell) = 0,$$

and, for $k = -1$, let

$$g_{11}(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell) = 0, \qquad g_{12}(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell) g_{21}(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell) g_{22}(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell) \neq 0.$$

Then, for $s_0^0(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell) \neq \mathrm{i}\mathrm{e}^{(-1)^{1+\varepsilon_2}\pi a}$,[18]

$$u(\tau) \underset{\tau \to +\infty \mathrm{e}^{\mathrm{i}\pi\varepsilon_1}}{=} u_{0,k}^*(\tau) - \frac{(-1)^{\varepsilon_1} \mathrm{i}\varepsilon \big(\varepsilon b \mathrm{e}^{-\mathrm{i}\pi\varepsilon_2}\big)^{1/2} \mathrm{e}^{\mathrm{i}\pi k/4} \big(s_0^0(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell) - \mathrm{i}\mathrm{e}^{(-1)^{1+\varepsilon_2}\pi a}\big)}{\sqrt{\pi} 2^{3/2} 3^{1/4} \big(2+\sqrt{3}\big)^{\mathrm{i}k(-1)^{1+\varepsilon_2} a}}$$
$$\times \mathrm{e}^{-(\beta(\tau) + \mathrm{i}k\vartheta(\tau))} \big(1 + \mathcal{O}\big(\tau^{-1/3}\big)\big), \qquad k \in \{\pm 1\}, \tag{2.1}$$

where

$$u_{0,k}^*(\tau) = c_{0,k} \tau^{1/3} \left(1 + \tau^{-2/3} \sum_{m=0}^{\infty} \frac{\mathfrak{u}_m(k)}{\big((-1)^{\varepsilon_1} \tau^{1/3}\big)^m}\right),$$

with

$$c_{0,k} := \frac{\varepsilon (\varepsilon b)^{2/3}}{2} \mathrm{e}^{-\mathrm{i}2\pi k/3}, \tag{2.2}$$

$$\mathfrak{u}_0(k) = \frac{a \mathrm{e}^{-\mathrm{i}2\pi k/3}}{3(\varepsilon b)^{1/3}} = \frac{a}{6\alpha_k^2}, \qquad \mathfrak{u}_1(k) = \mathfrak{u}_2(k) = \mathfrak{u}_3(k) = \mathfrak{u}_5(k) = \mathfrak{u}_7(k) = \mathfrak{u}_9(k) = 0, \tag{2.3}$$

$$\mathfrak{u}_4(k) = -\frac{a\big(a^2+1\big)}{3^4(\varepsilon b)}, \qquad \mathfrak{u}_6(k) = \frac{a^2\big(a^2+1\big)\mathrm{e}^{-\mathrm{i}2\pi k/3}}{3^5(\varepsilon b)^{4/3}}, \qquad \mathfrak{u}_8(k) = \frac{a\big(a^2+1\big)\mathrm{e}^{\mathrm{i}2\pi k/3}}{3^5(\varepsilon b)^{5/3}}, \tag{2.4}$$

where

$$\alpha_k := 2^{-1/2} (\varepsilon b)^{1/6} \mathrm{e}^{\mathrm{i}\pi k/3}, \tag{2.5}$$

and, for $m \in \mathbb{Z}_+ := \{0\} \cup \mathbb{N}$,

$$\mathfrak{u}_{2(m+5)}(k) = \frac{1}{27} \left(\frac{c_{0,k}}{b}\right)^2 \bigg(\mathfrak{w}_{2(m+3)}(k) - 2\mathfrak{u}_0(k)\mathfrak{w}_{2(m+2)}(k) + \eta_{2(m+2)}(k) - \mathfrak{u}_0(k)\eta_{2(m+1)}(k)$$

[17] See Remark 2.5 below.

[18] For $s_0^0(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell) = \mathrm{i}\mathrm{e}^{(-1)^{1+\varepsilon_2}\pi a}$, the exponentially small correction terms in the asymptotics (2.1), (2.11), (2.12), (2.15), and (2.16) are absent.

$$+\sum_{p=0}^{2m}\eta_p(k)\mathfrak{w}_{2(m+1)-p}(k)\Bigg)-\frac{1}{3}\sum_{p=0}^{2(m+4)}(\mathfrak{u}_p(k)+\mathfrak{w}_p(k))\mathfrak{u}_{2(m+4)-p}(k)$$
$$-\frac{1}{3}\left(\frac{c_{0,k}}{b}\right)^2\left(\frac{2m+7}{3}\right)^2\mathfrak{u}_{2(m+3)}(k), \tag{2.6}$$

$$\mathfrak{u}_{2(m+5)+1}(k)=0, \tag{2.7}$$

where

$$\mathfrak{w}_0(k)=-\mathfrak{u}_0(k), \qquad \mathfrak{w}_1(k)=0,$$
$$\mathfrak{w}_{n+2}(k)=-\mathfrak{u}_{n+2}(k)-\sum_{p=0}^{n}\mathfrak{w}_p(k)\mathfrak{u}_{n-p}(k), \qquad n\in\mathbb{Z}_+, \tag{2.8}$$

with

$$\eta_j(k):=-2(j+3)\mathfrak{u}_{j+2}(k)+\sum_{p=0}^{j}(p+1)(j-p+1)\mathfrak{u}_p(k)\mathfrak{u}_{j-p}(k), \qquad j\in\mathbb{Z}_+, \tag{2.9}$$

$$\vartheta(\tau):=\frac{3\sqrt{3}}{2}(-1)^{\varepsilon_2}(\varepsilon b)^{1/3}\tau^{2/3}, \qquad \beta(\tau):=\frac{9}{2}(-1)^{\varepsilon_2}(\varepsilon b)^{1/3}\tau^{2/3}. \tag{2.10}$$

Let the auxiliary function $f_-(\tau)$ *(corresponding to* $u(\tau)$ *above) defined by equation* (1.41) *solve the* ODE (1.14)*, and let the auxiliary function* $f_+(\tau)$ *(corresponding to* $u(\tau)$ *above) defined by equation* (1.42) *solve the* ODE (1.15)*. Then, for* $s_0^0(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)\neq \mathrm{i}\mathrm{e}^{(-1)^{1+\varepsilon_2}\pi a}$,

$$2f_-(\tau)\underset{\tau\to+\infty\mathrm{e}^{\mathrm{i}\pi\varepsilon_1}}{=}f_{0,k}^*(\tau)$$
$$-\frac{(-1)^{\varepsilon_1}k\big(\varepsilon b\mathrm{e}^{-\mathrm{i}\pi\varepsilon_2}\big)^{1/6}\mathrm{e}^{\mathrm{i}\pi k/4}\mathrm{e}^{\mathrm{i}\pi k/3}\big(s_0^0(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)-\mathrm{i}\mathrm{e}^{(-1)^{1+\varepsilon_2}\pi a}\big)}{\sqrt{\pi}2^{k/2}3^{1/4}\big(\sqrt{3}+1\big)^{-k}\big(2+\sqrt{3}\big)^{\mathrm{i}k(-1)^{1+\varepsilon_2}a}}$$
$$\times\tau^{1/3}\mathrm{e}^{-(\beta(\tau)+\mathrm{i}k\vartheta(\tau))}\big(1+\mathcal{O}\big(\tau^{-1/3}\big)\big), \qquad k\in\{\pm1\}, \tag{2.11}$$

where

$$f_{0,k}^*(\tau)=-\mathrm{i}\left((-1)^{\varepsilon_2}a-\mathrm{i}/2\right)$$
$$+\frac{\mathrm{i}(-1)^{\varepsilon_2}(\varepsilon b)^{1/3}\mathrm{e}^{\mathrm{i}2\pi k/3}}{2}\tau^{2/3}\left(-2+\tau^{-2/3}\sum_{m=0}^{\infty}\frac{\mathfrak{r}_m(k)}{\big((-1)^{\varepsilon_1}\tau^{1/3}\big)^m}\right),$$

and

$$\frac{4\mathrm{i}(-1)^{\varepsilon_2}}{\varepsilon b}f_+(\tau)\underset{\tau\to+\infty\mathrm{e}^{\mathrm{i}\pi\varepsilon_1}}{=}\mathfrak{f}_{0,k}^*(\tau)+\frac{(-1)^{\varepsilon_1}\big(\varepsilon b\mathrm{e}^{-\mathrm{i}\pi\varepsilon_2}\big)^{1/6}\mathrm{e}^{\mathrm{i}\pi k/4}\mathrm{e}^{\mathrm{i}\pi k/3}\big(2^{(k+1)/2}-k\big(\sqrt{3}+1\big)^k\big)}{\sqrt{\pi}2^{k/2}3^{1/4}\big(2+\sqrt{3}\big)^{\mathrm{i}k(-1)^{1+\varepsilon_2}a}}$$
$$\times\big(s_0^0(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)-\mathrm{i}\mathrm{e}^{(-1)^{1+\varepsilon_2}\pi a}\big)\tau^{1/3}\mathrm{e}^{-(\beta(\tau)+\mathrm{i}k\vartheta(\tau))}$$
$$\times\big(1+\mathcal{O}\big(\tau^{-1/3}\big)\big), \qquad k\in\{\pm1\}, \tag{2.12}$$

where

$$\mathfrak{f}_{0,k}^*(\tau)=\mathrm{i}\left((-1)^{\varepsilon_2}a+\mathrm{i}/2\right)$$
$$+\mathrm{i}(-1)^{\varepsilon_2}(\varepsilon b)^{1/3}\mathrm{e}^{\mathrm{i}2\pi k/3}\tau^{2/3}\left(1+\tau^{-2/3}\sum_{m=0}^{\infty}\frac{\big(\frac{1}{2}\mathfrak{r}_m(k)+2\mathfrak{w}_m(k)\big)}{\big((-1)^{\varepsilon_1}\tau^{1/3}\big)^m}\right),$$

with

$$\mathfrak{r}_0(k)=\frac{a-\mathrm{i}(-1)^{\varepsilon_2}/2}{3\alpha_k^2},\qquad \mathfrak{r}_1(k)=0,\qquad \mathfrak{r}_2(k)=\frac{\mathrm{i}(-1)^{\varepsilon_2}a(1+\mathrm{i}(-1)^{\varepsilon_2}a)}{18\alpha_k^4},$$

$$\mathfrak{r}_3(k)=0, \tag{2.13}$$

$$\begin{aligned}\mathrm{i}2\alpha_k^2\mathfrak{r}_{m+4}(k)=&\sum_{p=0}^{m}\bigg(\mathrm{i}4\alpha_k^2(\mathfrak{u}_{m+2-p}(k)-\mathfrak{u}_0(k)\mathfrak{u}_{m-p}(k))\\&-\frac{(-1)^{\varepsilon_2}}{3}(m-p+2)\mathfrak{u}_{m-p}(k)\bigg)\mathfrak{w}_p(k)+\mathrm{i}4\alpha_k^2(\mathfrak{u}_{m+4}(k)\\&-\mathfrak{u}_0(k)\mathfrak{u}_{m+2}(k))-\frac{(-1)^{\varepsilon_2}}{3}(m+4)\mathfrak{u}_{m+2}(k),\qquad m\in\mathbb{Z}_+.\end{aligned} \tag{2.14}$$

Let the Hamiltonian function $\mathcal{H}(\tau)$ *(corresponding to* $u(\tau)$ *above) be defined by equation (1.7). Then, for* $s_0^0(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)\neq\mathrm{i}\mathrm{e}^{(-1)^{1+\varepsilon_2}\pi a}$,

$$\begin{aligned}\mathcal{H}(\tau)\underset{\tau\to+\infty\mathrm{e}^{\mathrm{i}\pi\varepsilon_1}}{=}&\mathcal{H}_{0,k}^{*}(\tau)\\&-\frac{(-1)^{\varepsilon_1}\big(\varepsilon b\mathrm{e}^{-\mathrm{i}\pi\varepsilon_2}\big)^{1/6}\mathrm{e}^{\mathrm{i}\pi k/4}\mathrm{e}^{\mathrm{i}\pi k/3}\big(s_0^0(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)-\mathrm{i}\mathrm{e}^{(-1)^{1+\varepsilon_2}\pi a}\big)}{\sqrt{\pi}2^{k/2}3^{3/4}\big(\sqrt{3}+1\big)^{-k}\big(2+\sqrt{3}\big)^{\mathrm{i}k(-1)^{1+\varepsilon_2}a}}\\&\times\tau^{-2/3}\mathrm{e}^{-(\beta(\tau)+\mathrm{i}k\vartheta(\tau))}\big(1+\mathcal{O}\big(\tau^{-1/3}\big)\big),\qquad k\in\{\pm1\},\end{aligned} \tag{2.15}$$

where

$$\begin{aligned}\mathcal{H}_{0,k}^{*}(\tau)=&\,3(\varepsilon b)^{2/3}\mathrm{e}^{-\mathrm{i}2\pi k/3}\tau^{1/3}+2(\varepsilon b)^{1/3}\mathrm{e}^{\mathrm{i}2\pi k/3}(a-\mathrm{i}(-1)^{\varepsilon_2}/2)\tau^{-1/3}+\frac{1}{6}\big((a-\mathrm{i}(-1)^{\varepsilon_2}/2)^2\\&-1/3\big)\tau^{-1}+\alpha_k^2\big(\tau^{-1/3}\big)^5\sum_{m=0}^{\infty}\bigg(-4(a-\mathrm{i}(-1)^{\varepsilon_2}/2)\mathfrak{u}_{m+2}(k)+\alpha_k^2\mathfrak{d}_m(k)\\&+\sum_{p=0}^{m}\big(\tilde{\mathfrak{h}}_p(k)-4(a-\mathrm{i}(-1)^{\varepsilon_2}/2)\mathfrak{u}_p(k)\big)\mathfrak{w}_{m-p}(k)\bigg)\big((-1)^{\varepsilon_1}\tau^{-1/3}\big)^m,\end{aligned}$$

with

$$\begin{aligned}\mathfrak{d}_m(k):=&\sum_{p=0}^{m+2}(8\mathfrak{u}_p(k)\mathfrak{u}_{m+2-p}(k)+(4\mathfrak{u}_p(k)-\mathfrak{r}_p(k))\mathfrak{r}_{m+2-p}(k))\\&-\sum_{p_1=0}^{m}\sum_{m_1=0}^{p_1}\mathfrak{r}_{m_1}(k)\mathfrak{r}_{p_1-m_1}(k)\mathfrak{u}_{m-p_1}(k),\qquad m\in\mathbb{Z}_+,\end{aligned}$$

and

$$\tilde{\mathfrak{h}}_0(k)=-\frac{\big(12a^2+1\big)\mathrm{e}^{\mathrm{i}\pi k/3}}{18(\varepsilon b)^{1/3}},\qquad \tilde{\mathfrak{h}}_1(k)=0,\qquad \tilde{\mathfrak{h}}_{m+2}(k)=\alpha_k^2\mathfrak{d}_m(k).$$

Let the auxiliary function $\sigma(\tau)$ *(corresponding to* $u(\tau)$ *above) defined by equation (1.10) solve the* ODE *(1.11). Then, for* $s_0^0(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)\neq\mathrm{i}\mathrm{e}^{(-1)^{1+\varepsilon_2}\pi a}$,

$$\begin{aligned}\sigma(\tau)\underset{\tau\to+\infty\mathrm{e}^{\mathrm{i}\pi\varepsilon_1}}{=}&\sigma_{0,k}^{*}(\tau)-\frac{(-1)^{\varepsilon_1}\big(\varepsilon b\mathrm{e}^{-\mathrm{i}\pi\varepsilon_2}\big)^{1/6}\mathrm{e}^{\mathrm{i}\pi k/4}\mathrm{e}^{\mathrm{i}\pi k/3}\big(s_0^0(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)-\mathrm{i}\mathrm{e}^{(-1)^{1+\varepsilon_2}\pi a}\big)}{\sqrt{\pi}2^{k/2}3^{3/4}\big(\sqrt{3}+1\big)^{-k}\big(1+k\sqrt{3}\big)^{-1}\big(2+\sqrt{3}\big)^{\mathrm{i}k(-1)^{1+\varepsilon_2}a}}\\&\times\tau^{1/3}\mathrm{e}^{-(\beta(\tau)+\mathrm{i}k\vartheta(\tau))}\big(1+\mathcal{O}\big(\tau^{-1/3}\big)\big),\qquad k\in\{\pm1\},\end{aligned} \tag{2.16}$$

where

$$\sigma^{*}_{0,k}(\tau)=3(\varepsilon b)^{2/3}\mathrm{e}^{-\mathrm{i}2\pi k/3}\tau^{4/3}-\mathrm{i}(-1)^{\varepsilon_2}2(\varepsilon b)^{1/3}\mathrm{e}^{\mathrm{i}2\pi k/3}(1+\mathrm{i}(-1)^{\varepsilon_2}a)\tau^{2/3} + \frac{1}{3}\big((1+\mathrm{i}(-1)^{\varepsilon_2}a)^2+1/3\big)+\alpha_k^2\tau^{-2/3}\sum_{m=0}^{\infty}\bigg(-4(a-\mathrm{i}(-1)^{\varepsilon_2}/2)\mathfrak{u}_{m+2}(k) + \alpha_k^2\mathfrak{d}_m(k)+\sum_{p=0}^{m}(\tilde{\mathfrak{h}}_p(k)-4(a-\mathrm{i}(-1)^{\varepsilon_2}/2)\mathfrak{u}_p(k))\mathfrak{w}_{m-p}(k)+\mathrm{i}(-1)^{\varepsilon_2}\mathfrak{r}_{m+2}(k)\bigg) \times\big((-1)^{\varepsilon_1}\tau^{-1/3}\big)^m.$$

Remark 2.5. To be unequivocally clear, the first two sentences of the formulation of Theorem 2.4 do not imply that $\varepsilon_2=0$ (similar comments apply, *mutatis mutandis*, to Theorems 2.8, E.3, and E.6). The first sentence of Theorem 2.4 states that $u(\tau)$ is a solution of the DP3E (1.1) and $\hat{\varphi}(\tau)$ is the general solution of the ODE $\hat{\varphi}'(\tau)=2a\tau^{-1}+b(u(\tau))^{-1}$ for $\varepsilon b>0$ corresponding to the monodromy data $\big(a,s_0^0,s_0^{\infty},s_1^{\infty},g_{11},g_{12},g_{21},g_{22}\big)$. Taking into account Remarks 2.1 and 2.3, these monodromy co-ordinates are ascribed a clearer notational sense, namely, $s_0^0:=s_0^0(0,0,0|0)$, $s_0^{\infty}:=s_0^{\infty}(0,0,0|0)$, $s_1^{\infty}:=s_1^{\infty}(0,0,0|0)$, and $g_{ij}:=g_{ij}(0,0,0|0)$, $i,j\in\{1,2\}$. This means that one first solves the DP3E (1.1) for $u(\tau)$ as $\tau\to+\infty$ ($\varepsilon_1=0$) for $\varepsilon b>0$ ($\varepsilon_2=0$) corresponding to the monodromy data satisfying the restrictions (take, say, the case $k=+1$) $g_{11}(0,0,0|0)g_{12}(0,0,0|0)g_{21}(0,0,0|0)=g_{11}g_{12}g_{21}\neq 0$ and $g_{22}(0,0,0|0)=g_{22}=0$, that is,

$$u(\tau)\underset{\tau\to+\infty}{=}u^{*}_{0,1}(\tau)+\frac{\varepsilon(\varepsilon b)^{1/2}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}\big(2+\sqrt{3}\big)^{\mathrm{i}a}\big(s_0^0-\mathrm{i}\mathrm{e}^{-\pi a}\big)}{\sqrt{\pi}2^{3/2}3^{1/4}}\mathrm{e}^{-(\beta(\tau)+\mathrm{i}\vartheta(\tau))} \times\big(1+\mathcal{O}\big(\tau^{-1/3}\big)\big), \tag{2.17}$$

and then use this $(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)=(0,0,0|0)$ asymptotics (2.17) as a "base", "seed", or "germ" solution to which Lie-point symmetries can be applied (akin to Darboux transformations in the theory of solitons); for example, if one wants the solution $u(\tau)$ of the DP3E (1.1) as $\tau\to-\infty$ for $\varepsilon b<0$, which corresponds to any one of the parameter values $(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)=(-1,1,\pm1|\ell)$, $\ell=0,1$, provided that the corresponding monodromy data satisfy the restrictions

$$g_{22}(-1,1,\pm1|\ell)=0, g_{11}(-1,1,\pm1|\ell)g_{12}(-1,1,\pm1|\ell)g_{21}(-1,1,\pm1|\ell)\neq 0, \qquad \ell=0,1,$$

where explicit formulae for $s_0^0(-1,1,\pm1|\ell)$, $s_0^{\infty}(-1,1,\pm1|\ell)$, $s_1^{\infty}(-1,1,\pm1|\ell)$, and $g_{ij}(-1,1,\pm1|\ell)$, $i,j\in\{1,2\}$, in terms of s_0^0, s_0^{∞}, s_1^{∞}, and g_{ij}, are given in Appendix D.5, equations (D.58), (D.60), (D.82), and (D.84), one makes the changes $\tau\to\tau\mathrm{e}^{\mathrm{i}\pi}$ ($\varepsilon_1=-1$) and $\varepsilon b\to|\varepsilon b|\mathrm{e}^{\mathrm{i}\pi}$ ($\varepsilon_2=+1$) in equation (2.17), and, taking into account Remark 2.2, arrives at the asymptotics of $u(\tau)$ as $\tau\to-\infty$ for $\varepsilon b<0$.

Remark 2.6. For $\mathrm{i}a\in\mathbb{Z}$, a separate analysis based on Bäcklund transformations is required in order to generate the analogue of the sequence of $\mathbb{C}$-valued expansion coefficients $\{\mathfrak{u}_m(k)\}$, $m\in\mathbb{Z}_+$, $k=\pm1$, and the corresponding function $u^{*}_{0,k}(\tau)$; this comment applies, *mutatis mutandis*, to the $\mathbb{C}$-valued expansion coefficients $\{\hat{\mathfrak{u}}_m(k)\}$ and the corresponding function $\hat{u}^{*}_{0,k}(\tau)$ given in Theorem 2.8 below (see also Theorems E.3 and E.6). In fact, as discussed in [61, Section 1], for fixed values of $\mathrm{i}a=n\in\mathbb{Z}$, ε, and b, there is only one algebraic solution (rational function of $\tau^{1/3}$) of the DP3E (1.1) which is a multi-valued function with three branches (see also [65]): this solution can be derived via the $|n|$-fold iteration of the Bäcklund transformations given in [61, Section 6.1] to the simplest solution of the DP3E (1.1) (for $a=0$), namely, $u(\tau)=\frac{1}{2}\varepsilon(\varepsilon b)^{2/3}\tau^{1/3}$. The case $\mathrm{i}a\in\mathbb{Z}$ will be considered elsewhere. In this context, it must be mentioned that an expansive analysis, based on the RHP approach, of algebraic solutions of the PIII equation of D7 type has recently appeared in [13]; in particular, the authors present a study of algebraic solutions of the DP3E (1.1) for the parameter values $\varepsilon = -1$, $b = \mathrm{i}$, and $a = -\mathrm{i}n$, $n \in \mathbb{Z}$.

Remark 2.7. Define the simply-connected strip domain

$$\mathfrak{D}_u^{\blacktriangledown} := \left\{\tau \in \mathbb{C} \mid \operatorname{Re}\left(\theta^{\ddagger}(\tau)\right) > d_{1,*}^{\diamond},\ \left|\operatorname{Im}\left(\theta^{\ddagger}(\tau)\right)\right| < d_{2,*}^{\diamond}\right\}, \tag{2.18}$$

where $\theta^{\ddagger}(\tau) = 3^{3/2}(-1)^{\varepsilon_2}(\varepsilon b)^{1/3}\tau^{2/3}$, and $d_{1,*}^{\diamond}, d_{2,*}^{\diamond} > 0$ are some (τ-independent) constants. The asymptotics of $u(\tau)$, $f_{\pm}(\tau)$, $\mathcal{H}(\tau)$, and $\sigma(\tau)$ stated in Theorem 2.4 are actually valid in $\mathfrak{D}_u^{\blacktriangledown}$.

Theorem 2.8. *Let $u(\tau)$ be a solution of the* DP3E (1.1) *and $\hat{\varphi}(\tau)$ be the general solution of the* ODE $\hat{\varphi}'(\tau) = 2a\tau^{-1} + b(u(\tau))^{-1}$ *for $\varepsilon b > 0$ corresponding to the monodromy data $\left(a, s_0^0, s_0^{\infty}, s_1^{\infty}, g_{11}, g_{12}, g_{21}, g_{22}\right)$. Let*

$$\hat{\varepsilon}_1 \in \{\pm 1\}, \qquad \hat{\varepsilon}_2 \in \{0, \pm 1\}, \qquad \hat{m}(\hat{\varepsilon}_2) = \begin{cases} 0, & \hat{\varepsilon}_2 \in \{\pm 1\}, \\ \pm\hat{\varepsilon}_1, & \hat{\varepsilon}_2 = 0, \end{cases} \qquad \hat{\ell} \in \{0, 1\},$$

and $\varepsilon b = |\varepsilon b|\mathrm{e}^{\mathrm{i}\pi\hat{\varepsilon}_2}$. For $k = +1$, let

$$\hat{g}_{11}\left(\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\right)\hat{g}_{12}\left(\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\right)\hat{g}_{21}\left(\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\right) \neq 0 \qquad \hat{g}_{22}\left(\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\right) = 0,$$

and, for $k = -1$, let

$$\hat{g}_{11}\left(\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\right) = 0, \qquad \hat{g}_{12}\left(\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\right)\hat{g}_{21}\left(\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\right)\hat{g}_{22}\left(\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\right) \neq 0.$$

Then, for $\hat{s}_0^0\left(\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\right) \neq \mathrm{i}\mathrm{e}^{(-1)^{\hat{\varepsilon}_2}\pi a}$,[19]

$$\begin{aligned} u(\tau) \underset{\tau \to +\infty \mathrm{e}^{\mathrm{i}\pi\hat{\varepsilon}_1/2}}{=} \hat{u}_{0,k}^{*}(\tau) &- \frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}\pi\hat{\varepsilon}_1/2}\varepsilon\left(\varepsilon b\mathrm{e}^{-\mathrm{i}\pi\hat{\varepsilon}_2}\right)^{1/2}\mathrm{e}^{\mathrm{i}\pi k/4}\left(\hat{s}_0^0\left(\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\right) - \mathrm{i}\mathrm{e}^{(-1)^{\hat{\varepsilon}_2}\pi a}\right)}{\sqrt{\pi}2^{3/2}3^{1/4}\left(2+\sqrt{3}\right)^{\mathrm{i}k(-1)^{\hat{\varepsilon}_2}a}} \\ &\times \mathrm{e}^{-(\hat{\beta}(\tau_*) + \mathrm{i}k\hat{\vartheta}(\tau_*))}\left(1 + \mathcal{O}\left(\tau^{-1/3}\right)\right), \qquad k \in \{\pm 1\}, \end{aligned} \tag{2.19}$$

where

$$\hat{u}_{0,k}^{*}(\tau) = \mathrm{e}^{-\mathrm{i}\pi\hat{\varepsilon}_1/2}c_{0,k}\tau_*^{1/3}\left(1 + \tau_*^{-2/3}\sum_{m=0}^{\infty}\frac{\hat{\mathfrak{u}}_m(k)}{\left(\tau_*^{1/3}\right)^m}\right),$$

with $c_{0,k}$ defined by equation (2.2),

$$\tau_* := \tau\mathrm{e}^{-\mathrm{i}\pi\hat{\varepsilon}_1/2}, \tag{2.20}$$

$$\hat{\mathfrak{u}}_0(k) = -\frac{a\mathrm{e}^{-\mathrm{i}2\pi k/3}}{3(\varepsilon b)^{1/3}} = -\frac{a}{6\alpha_k^2}, \tag{2.21}$$

$$\hat{\mathfrak{u}}_1(k) = \hat{\mathfrak{u}}_2(k) = \hat{\mathfrak{u}}_3(k) = \hat{\mathfrak{u}}_5(k) = \hat{\mathfrak{u}}_7(k) = \hat{\mathfrak{u}}_9(k) = 0, \tag{2.22}$$

$$\hat{\mathfrak{u}}_4(k) = \frac{a\left(a^2+1\right)}{3^4(\varepsilon b)}, \qquad \hat{\mathfrak{u}}_6(k) = \frac{a^2\left(a^2+1\right)\mathrm{e}^{-\mathrm{i}2\pi k/3}}{3^5(\varepsilon b)^{4/3}}, \qquad \hat{\mathfrak{u}}_8(k) = -\frac{a\left(a^2+1\right)\mathrm{e}^{\mathrm{i}2\pi k/3}}{3^5(\varepsilon b)^{5/3}}, \tag{2.23}$$

where α_k is defined by equation (2.5), *and, for $m \in \mathbb{Z}_+$,*

$$\hat{\mathfrak{u}}_{2(m+5)}(k) = \frac{1}{27}\left(\frac{c_{0,k}}{b}\right)^2\Bigg(\hat{\mathfrak{w}}_{2(m+3)}(k) - 2\hat{\mathfrak{u}}_0(k)\hat{\mathfrak{w}}_{2(m+2)}(k) + \hat{\eta}_{2(m+2)}(k) - \hat{\mathfrak{u}}_0(k)\hat{\eta}_{2(m+1)}(k)$$

[19] For $\hat{s}_0^0\left(\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\right) = \mathrm{i}\mathrm{e}^{(-1)^{\hat{\varepsilon}_2}\pi a}$, the exponentially small correction terms in the asymptotics (2.19), (2.28), (2.29), (2.32), and (2.33) are absent.

$$+\sum_{p=0}^{2m}\hat{\eta}_p(k)\hat{\mathfrak{w}}_{2(m+1)-p}(k)\Bigg)-\frac{1}{3}\sum_{p=0}^{2(m+4)}(\hat{\mathfrak{u}}_p(k)+\hat{\mathfrak{w}}_p(k))\hat{\mathfrak{u}}_{2(m+4)-p}(k)$$

$$-\frac{1}{3}\left(\frac{c_{0,k}}{b}\right)^2\left(\frac{2m+7}{3}\right)^2\hat{\mathfrak{u}}_{2(m+3)}(k), \tag{2.24}$$

$$\hat{\mathfrak{u}}_{2(m+5)+1}(k)=0, \tag{2.25}$$

where

$$\hat{\mathfrak{w}}_0(k)=-\hat{\mathfrak{u}}_0(k), \qquad \hat{\mathfrak{w}}_1(k)=0, \qquad \hat{\mathfrak{w}}_{n+2}(k)=-\hat{\mathfrak{u}}_{n+2}(k)-\sum_{p=0}^{n}\hat{\mathfrak{w}}_p(k)\hat{\mathfrak{u}}_{n-p}(k), \tag{2.26}$$

$$n\in\mathbb{Z}_+,$$

with

$$\hat{\eta}_j(k):=-2(j+3)\hat{\mathfrak{u}}_{j+2}(k)+\sum_{p=0}^{j}(p+1)(j-p+1)\hat{\mathfrak{u}}_p(k)\hat{\mathfrak{u}}_{j-p}(k), \qquad j\in\mathbb{Z}_+, \tag{2.27}$$

and $\hat{\vartheta}(\tau):=\frac{3\sqrt{3}}{2}(-1)^{\hat{\varepsilon}_2}(\varepsilon b)^{1/3}\tau^{2/3}$, $\hat{\beta}(\tau):=\frac{9}{2}(-1)^{\hat{\varepsilon}_2}(\varepsilon b)^{1/3}\tau^{2/3}$.

Let the auxiliary function $f_-(\tau)$ *(corresponding to* $u(\tau)$ *above) defined by equation* (1.41) *solve the* ODE (1.14)*, and let the auxiliary function* $f_+(\tau)$ *(corresponding to* $u(\tau)$ *above) defined by equation* (1.42) *solve the* ODE (1.15)*. Then, for* $\hat{s}_0^0(\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)|\hat{\ell})\neq \mathrm{i}\mathrm{e}^{(-1)^{\hat{\varepsilon}_2}\pi a}$,

$$2f_-(\tau)\underset{\tau\to+\infty\mathrm{e}^{\mathrm{i}\pi\hat{\varepsilon}_1/2}}{=}\hat{f}^*_{0,k}(\tau)-\frac{k\big(\varepsilon b\mathrm{e}^{-\mathrm{i}\pi\hat{\varepsilon}_2}\big)^{1/6}\mathrm{e}^{\mathrm{i}\pi k/4}\mathrm{e}^{\mathrm{i}\pi k/3}\big(\hat{s}_0^0(\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)|\hat{\ell})-\mathrm{i}\mathrm{e}^{(-1)^{\hat{\varepsilon}_2}\pi a}\big)}{\sqrt{\pi}2^{k/2}3^{1/4}\big(\sqrt{3}+1\big)^{-k}\big(2+\sqrt{3}\big)^{\mathrm{i}k(-1)^{\hat{\varepsilon}_2}a}}$$
$$\times\,\tau_*^{1/3}\mathrm{e}^{-(\hat{\beta}(\tau_*)+\mathrm{i}k\hat{\vartheta}(\tau_*))}\big(1+\mathcal{O}\big(\tau^{-1/3}\big)\big), \qquad k\in\{\pm1\}, \tag{2.28}$$

where

$$\hat{f}^*_{0,k}(\tau)=-\mathrm{i}\big((-1)^{1+\hat{\varepsilon}_2}a-\mathrm{i}/2\big)+\frac{\mathrm{i}(-1)^{\hat{\varepsilon}_2}(\varepsilon b)^{1/3}\mathrm{e}^{\mathrm{i}2\pi k/3}}{2}\tau_*^{2/3}\left(-2+\tau_*^{-2/3}\sum_{m=0}^{\infty}\frac{\hat{\mathfrak{r}}_m(k)}{\big(\tau_*^{1/3}\big)^m}\right),$$

and

$$\frac{4\mathrm{i}(-1)^{\hat{\varepsilon}_2}}{\varepsilon b}f_+(\tau)\underset{\tau\to+\infty\mathrm{e}^{\mathrm{i}\pi\hat{\varepsilon}_1/2}}{=}\hat{\mathfrak{f}}^*_{0,k}(\tau)+\frac{\big(\varepsilon b\mathrm{e}^{-\mathrm{i}\pi\hat{\varepsilon}_2}\big)^{1/6}\mathrm{e}^{\mathrm{i}\pi k/4}\mathrm{e}^{\mathrm{i}\pi k/3}\big(2^{(k+1)/2}-k\big(\sqrt{3}+1\big)^k\big)}{\sqrt{\pi}2^{k/2}3^{1/4}\big(2+\sqrt{3}\big)^{\mathrm{i}k(-1)^{\hat{\varepsilon}_2}a}}$$
$$\times\,\big(\hat{s}_0^0(\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)|\hat{\ell})-\mathrm{i}\mathrm{e}^{(-1)^{\hat{\varepsilon}_2}\pi a}\big)\tau_*^{1/3}\mathrm{e}^{-(\hat{\beta}(\tau_*)+\mathrm{i}k\hat{\vartheta}(\tau_*))}$$
$$\times\,\big(1+\mathcal{O}\big(\tau^{-1/3}\big)\big), \qquad k\in\{\pm1\}, \tag{2.29}$$

where

$$\hat{\mathfrak{f}}^*_{0,k}(\tau)=\mathrm{i}\big((-1)^{1+\hat{\varepsilon}_2}a+\mathrm{i}/2\big)$$
$$+\,\mathrm{i}(-1)^{\hat{\varepsilon}_2}(\varepsilon b)^{1/3}\mathrm{e}^{\mathrm{i}2\pi k/3}\tau_*^{2/3}\left(1+\tau_*^{-2/3}\sum_{m=0}^{\infty}\frac{\big(\frac{1}{2}\hat{\mathfrak{r}}_m(k)+2\hat{\mathfrak{w}}_m(k)\big)}{\big(\tau_*^{1/3}\big)^m}\right),$$

with

$$\hat{\mathfrak{r}}_0(k)=-\frac{\big(a+\mathrm{i}(-1)^{\hat{\varepsilon}_2}/2\big)}{3\alpha_k^2}, \qquad \hat{\mathfrak{r}}_1(k)=0,$$

$$\hat{\mathfrak{r}}_2(k)=\frac{\mathrm{i}a\big((-1)^{1+\hat{\varepsilon}_2}+\mathrm{i}a\big)}{18\alpha_k^4},\qquad \hat{\mathfrak{r}}_3(k)=0, \tag{2.30}$$

$$\begin{aligned}\mathrm{i}2\alpha_k^2\hat{\mathfrak{r}}_{m+4}(k)&=\sum_{p=0}^{m}\bigg(\mathrm{i}4\alpha_k^2(\hat{\mathfrak{u}}_{m+2-p}(k)-\hat{\mathfrak{u}}_0(k)\hat{\mathfrak{u}}_{m-p}(k))-\frac{(-1)^{\hat{\varepsilon}_2}}{3}(m-p+2)\hat{\mathfrak{u}}_{m-p}(k)\bigg)\\&\quad\times\hat{\mathfrak{w}}_p(k)+\mathrm{i}4\alpha_k^2(\hat{\mathfrak{u}}_{m+4}(k)-\hat{\mathfrak{u}}_0(k)\hat{\mathfrak{u}}_{m+2}(k))-\frac{(-1)^{\hat{\varepsilon}_2}}{3}(m+4)\hat{\mathfrak{u}}_{m+2}(k),\\ m&\in\mathbb{Z}_+.\end{aligned} \tag{2.31}$$

Let the Hamiltonian function $\mathcal{H}(\tau)$ (corresponding to $u(\tau)$ above) be defined by equation (1.7). Then, for $\hat{s}_0^0\big(\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\big)\neq\mathrm{i}\mathrm{e}^{(-1)^{\hat{\varepsilon}_2}\pi a}$,

$$\begin{aligned}\mathcal{H}(\tau)\underset{\tau\to+\infty\mathrm{e}^{\mathrm{i}\pi\hat{\varepsilon}_1/2}}{=}&\hat{\mathcal{H}}_{0,k}^{*}(\tau)\\&-\frac{\mathrm{e}^{-\mathrm{i}\pi\hat{\varepsilon}_1/2}\big(\varepsilon b\mathrm{e}^{-\mathrm{i}\pi\hat{\varepsilon}_2}\big)^{1/6}\mathrm{e}^{\mathrm{i}\pi k/4}\mathrm{e}^{\mathrm{i}\pi k/3}\big(\hat{s}_0^0\big(\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\big)-\mathrm{i}\mathrm{e}^{(-1)^{\hat{\varepsilon}_2}\pi a}\big)}{\sqrt{\pi}2^{k/2}3^{3/4}\big(\sqrt{3}+1\big)^{-k}\big(2+\sqrt{3}\big)^{\mathrm{i}k(-1)^{\hat{\varepsilon}_2}a}}\\&\times\tau_*^{-2/3}\mathrm{e}^{-(\hat{\beta}(\tau_*)+\mathrm{i}k\hat{\vartheta}(\tau_*))}\big(1+\mathcal{O}\big(\tau^{-1/3}\big)\big),\qquad k\in\{\pm1\},\end{aligned} \tag{2.32}$$

where

$$\begin{aligned}\hat{\mathcal{H}}_{0,k}^{*}(\tau)=\mathrm{e}^{-\mathrm{i}\pi\hat{\varepsilon}_1/2}\bigg(&3(\varepsilon b)^{2/3}\mathrm{e}^{-\mathrm{i}2\pi k/3}\tau_*^{1/3}+(-1)^{\hat{\varepsilon}_2}2(\varepsilon b)^{1/3}\mathrm{e}^{\mathrm{i}2\pi k/3}\big((-1)^{1+\hat{\varepsilon}_2}a-\mathrm{i}/2\big)\tau_*^{-1/3}\\&+\frac{1}{6}\big(\big((-1)^{1+\hat{\varepsilon}_2}a-\mathrm{i}/2\big)^2-1/3\big)\tau_*^{-1}\\&+(-1)^{\hat{\varepsilon}_2}\alpha_k^2\big(\tau_*^{-1/3}\big)^5\sum_{m=0}^{\infty}\bigg(-4\big((-1)^{1+\hat{\varepsilon}_2}a-\mathrm{i}/2\big)\hat{\mathfrak{u}}_{m+2}(k)+(-1)^{\hat{\varepsilon}_2}\alpha_k^2\hat{\mathfrak{d}}_m(k)\\&+\sum_{p=0}^{m}\big(\hat{\mathfrak{h}}_p^{*}(k)-4\big((-1)^{1+\hat{\varepsilon}_2}a-\mathrm{i}/2\big)\hat{\mathfrak{u}}_p(k)\big)\hat{\mathfrak{w}}_{m-p}(k)\bigg)\big(\tau_*^{-1/3}\big)^m\bigg),\end{aligned}$$

with

$$\begin{aligned}\hat{\mathfrak{d}}_m(k):=&\sum_{p=0}^{m+2}(8\hat{\mathfrak{u}}_p(k)\hat{\mathfrak{u}}_{m+2-p}(k)+(4\hat{\mathfrak{u}}_p(k)-\hat{\mathfrak{r}}_p(k))\hat{\mathfrak{r}}_{m+2-p}(k))\\&-\sum_{p_1=0}^{m}\sum_{m_1=0}^{p_1}\hat{\mathfrak{r}}_{m_1}(k)\hat{\mathfrak{r}}_{p_1-m_1}(k)\hat{\mathfrak{u}}_{m-p_1}(k),\qquad m\in\mathbb{Z}_+,\end{aligned}$$

and

$$\hat{\mathfrak{h}}_0^{*}(k)=\frac{(-1)^{1+\hat{\varepsilon}_2}\big(12a^2+1\big)\mathrm{e}^{\mathrm{i}\pi k/3}}{18(\varepsilon b)^{1/3}},\qquad\hat{\mathfrak{h}}_1^{*}(k)=0,\qquad\hat{\mathfrak{h}}_{m+2}^{*}(k)=(-1)^{\hat{\varepsilon}_2}\alpha_k^2\hat{\mathfrak{d}}_m(k).$$

Let the auxiliary function $\sigma(\tau)$ (corresponding to $u(\tau)$ above) defined by equation (1.10) solve the ODE (1.11)*. Then, for $\hat{s}_0^0\big(\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\big)\neq\mathrm{i}\mathrm{e}^{(-1)^{\hat{\varepsilon}_2}\pi a}$,*

$$\begin{aligned}\sigma(\tau)\underset{\tau\to+\infty\mathrm{e}^{\mathrm{i}\pi\hat{\varepsilon}_1/2}}{=}&\hat{\sigma}_{0,k}^{*}(\tau)-\frac{\big(\varepsilon b\mathrm{e}^{-\mathrm{i}\pi\hat{\varepsilon}_2}\big)^{1/6}\mathrm{e}^{\mathrm{i}\pi k/4}\mathrm{e}^{\mathrm{i}\pi k/3}\big(\hat{s}_0^0\big(\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\big)-\mathrm{i}\mathrm{e}^{(-1)^{\hat{\varepsilon}_2}\pi a}\big)}{\sqrt{\pi}2^{k/2}3^{3/4}\big(\sqrt{3}+1\big)^{-k}\big(1+k\sqrt{3}\big)^{-1}\big(2+\sqrt{3}\big)^{\mathrm{i}k(-1)^{\hat{\varepsilon}_2}a}}\\&\times\tau_*^{1/3}\mathrm{e}^{-(\hat{\beta}(\tau_*)+\mathrm{i}k\hat{\vartheta}(\tau_*))}\big(1+\mathcal{O}\big(\tau^{-1/3}\big)\big),\qquad k\in\{\pm1\},\end{aligned} \tag{2.33}$$

where

$$\hat{\sigma}^{*}_{0,k}(\tau)=3(\varepsilon b)^{2/3}\mathrm{e}^{-\mathrm{i}2\pi k/3}\tau_{*}^{4/3}-\mathrm{i}(-1)^{\hat{\varepsilon}_2}2(\varepsilon b)^{1/3}\mathrm{e}^{\mathrm{i}2\pi k/3}\big(1+\mathrm{i}(-1)^{1+\hat{\varepsilon}_2}a\big)\tau_{*}^{2/3}$$
$$+\frac{1}{3}\Big(\big(1+\mathrm{i}(-1)^{1+\hat{\varepsilon}_2}a\big)^2+1/3\Big)$$
$$+(-1)^{\hat{\varepsilon}_2}\alpha_k^2\tau_{*}^{-2/3}\sum_{m=0}^{\infty}\Bigg(-4\big((-1)^{1+\hat{\varepsilon}_2}a-\mathrm{i}/2\big)\hat{\mathfrak{u}}_{m+2}(k)+(-1)^{\hat{\varepsilon}_2}\alpha_k^2\hat{\mathfrak{d}}_m(k)$$
$$+\sum_{p=0}^{m}\big(\hat{\mathfrak{h}}^{*}_{p}(k)-4\big((-1)^{1+\hat{\varepsilon}_2}a-\mathrm{i}/2\big)\hat{\mathfrak{u}}_p(k)\big)\hat{\mathfrak{w}}_{m-p}(k)+\mathrm{i}\hat{\mathfrak{r}}_{m+2}(k)\Bigg)\big(\tau_{*}^{-1/3}\big)^m.$$

Remark 2.9. Define the simply-connected strip domain

$$\widehat{\mathfrak{D}}_u^{\blacktriangle}:=\big\{\tau\in\mathbb{C}\mid \operatorname{Re}\big(\hat{\theta}^{\ddagger}\big(\tau\mathrm{e}^{-\mathrm{i}\pi\hat{\varepsilon}_1/2}\big)\big)>\hat{d}^{\diamond}_{1,*},\ \big|\operatorname{Im}\big(\hat{\theta}^{\ddagger}\big(\tau\mathrm{e}^{-\mathrm{i}\pi\hat{\varepsilon}_1/2}\big)\big)\big|<\hat{d}^{\diamond}_{2,*}\big\},$$

where $\hat{\theta}^{\ddagger}(\tau)=3^{3/2}(-1)^{\hat{\varepsilon}_2}(\varepsilon b)^{1/3}\tau^{2/3}$, and $\hat{d}^{\diamond}_{1,*},\hat{d}^{\diamond}_{2,*}>0$ are some (τ-independent) constants. The asymptotics of $u(\tau)$, $f_{\pm}(\tau)$, $\mathcal{H}(\tau)$, and $\sigma(\tau)$ stated in Theorem 2.8 are actually valid in $\widehat{\mathfrak{D}}_u^{\blacktriangle}$.

3 Asymptotic solution of the direct problem of monodromy theory

In this section, the monodromy data introduced in Section 1.4 is calculated as $\tau\to+\infty$ for $\varepsilon b>0$ (corresponding to $(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)=(0,0,0|0)$; cf. Section 2): this constitutes the first step towards the proof of the results stated in Theorems 2.4, 2.8, E.3, and E.6.

The aforementioned calculation consists of three components: (i) the matrix WKB analysis for the μ-part of the system (1.32), that is,

$$\partial_{\mu}\Psi(\mu)=\widetilde{\mathfrak{U}}(\mu,\tau)\Psi(\mu),\tag{3.1}$$

where $\Psi(\mu)=\Psi(\mu,\tau)$ (see Section 3.1); (ii) the approximation of $\Psi(\mu)$ in the neighbourhoods of the turning points (see Section 3.2); and (iii) the matching of these asymptotics (see Section 3.3).

Before commencing the asymptotic analysis, the notation used throughout this work is introduced:

(1) $\mathrm{I}=\operatorname{diag}(1,1)$ is the 2×2 identity matrix,

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad \sigma_2=\begin{pmatrix}0&-\mathrm{i}\\\mathrm{i}&0\end{pmatrix},\qquad\text{and}\qquad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

are the Pauli matrices, $\sigma_{\pm}:=\frac{1}{2}(\sigma_1\pm\mathrm{i}\sigma_2)$, $\mathbb{R}_{\pm}:=\{x\in\mathbb{R}\mid\pm x>0\}$, and $\mathbb{C}_{\pm}:=\{z\in\mathbb{C}\mid\pm\operatorname{Im}(z)>0\}$;

(2) for $(\varsigma_1,\varsigma_2)\in\mathbb{R}\times\mathbb{R}$, the function $(z-\varsigma_1)^{\mathrm{i}\varsigma_2}\colon\mathbb{C}\setminus(-\infty,\varsigma_1]\to\mathbb{C}$, $z\mapsto\exp(\mathrm{i}\varsigma_2\ln(z-\varsigma_1))$, with the branch cut taken along $(-\infty,\varsigma_1]$ and the principal branch of the logarithm chosen (that is, $\arg(z-\varsigma_1)\in(-\pi,\pi]$);

(3) for $\omega_o\in\mathbb{C}$ and $\widehat{\Upsilon}\in\mathrm{M}_2(\mathbb{C})$, $\omega_o^{\operatorname{ad}(\sigma_3)}\widehat{\Upsilon}:=\omega_o^{\sigma_3}\widehat{\Upsilon}\omega_o^{-\sigma_3}$;

(4) for $\mathrm{M}_2(\mathbb{C})\ni\mathfrak{I}(z)$, $(\mathfrak{I}(z))_{ij}$ or $\mathfrak{I}_{ij}(z)$, $i,j\in\{1,2\}$, denotes the (ij)-element of $\mathfrak{I}(z)$;

(5) $\hat{w}(t)\underset{t\to+\infty}{=}o(1)$ means there exists $C_1>0$ and $\epsilon_1>0$ such that $|\hat{w}(t)|\leqslant C_1|t|^{-\epsilon_1}$;

(6) for $\mathrm{M}_2(\mathbb{C})\ni\hat{\mathfrak{Y}}(z)$, $\hat{\mathfrak{Y}}(z)\underset{z\to z_0}{=}\mathcal{O}(\ast)$ (resp., $o(\ast)$) means $\hat{\mathfrak{Y}}_{ij}(z)\underset{z\to z_0}{=}\mathcal{O}(\ast_{ij})$ (resp., $o(\ast_{ij})$), $i,j\in\{1,2\}$;

(7) for $\mathrm{M}_2(\mathbb{C}) \ni \hat{\mathfrak{B}}(z)$,

$$||\hat{\mathfrak{B}}(\cdot)|| := \left(\sum_{i,j=1}^{2} \hat{\mathfrak{B}}_{ij}(\cdot)\overline{\hat{\mathfrak{B}}_{ij}(\cdot)}\right)^{1/2}$$

denotes the Hilbert–Schmidt norm, where $\overline{\star}$ denotes complex conjugation of $\star$; and

(8) for some $\delta_* > 0$, $\mathcal{O}_{\delta_*}(z_0)$ denotes the (open) δ_*-neighbourhood of the point z_0, that is, for $z_0 \in \mathbb{C}$, $\mathcal{O}_{\delta_*}(z_0) := \{z \in \mathbb{C} \mid |z - z_0| < \delta_*\}$, and, for z_0 the point at infinity, $\mathcal{O}_{\delta_*}(\infty) := \left\{z \in \mathbb{C} \mid |z| > \delta_*^{-1}\right\}$.

3.1 Matrix WKB analysis

This subsection is devoted to the WKB analysis of equation (3.1) as $\tau \to +\infty$ for $\varepsilon b > 0$.

In order to transform equation (3.1) into a form amenable to WKB analysis, one uses the result of [61, Proposition 4.1.1] (see also [57, Proposition 3.2.1]), which is summarised here for the reader's convenience.

Proposition 3.1 ([57, 61])**.** *In the system* (1.32)*, let*

$$A(\tau) = a(\tau)\tau^{-2/3}, \qquad B(\tau) = b(\tau)\tau^{-2/3}, \qquad C(\tau) = c(\tau)\tau^{-1/3}, \qquad D(\tau) = d(\tau)\tau^{-1/3},$$
$$\widetilde{\mu} = \mu\tau^{1/6}, \qquad \widetilde{\Psi}(\widetilde{\mu}) := \tau^{-\frac{1}{12}\sigma_3}\Psi\big(\widetilde{\mu}\tau^{-1/6}\big), \tag{3.2}$$

where $\widetilde{\Psi}(\widetilde{\mu}) = \widetilde{\Psi}(\widetilde{\mu}, \tau)$*. Then, the* μ*-part of the system* (1.32) *transforms as follows:*

$$\partial_{\widetilde{\mu}}\widetilde{\Psi}(\widetilde{\mu}) = \tau^{2/3}\mathcal{A}(\widetilde{\mu}, \tau)\widetilde{\Psi}(\widetilde{\mu}), \tag{3.3}$$

where

$$\begin{aligned}\mathcal{A}(\widetilde{\mu}, \tau) := &-2\mathrm{i}\widetilde{\mu}\sigma_3 + \begin{pmatrix} 0 & -\frac{4\mathrm{i}\sqrt{-a(\tau)b(\tau)}}{b(\tau)} \\ -2d(\tau) & 0 \end{pmatrix} - \frac{1}{\widetilde{\mu}}\frac{\mathrm{i}r(\tau)(\varepsilon b)^{1/3}}{2}\sigma_3 \\ &+ \frac{1}{\widetilde{\mu}^2}\begin{pmatrix} 0 & \frac{\mathrm{i}(\varepsilon b)}{b(\tau)} \\ \mathrm{i}b(\tau) & 0 \end{pmatrix},\end{aligned} \tag{3.4}$$

with

$$\frac{\mathrm{i}r(\tau)(\varepsilon b)^{1/3}}{2} = \mathrm{i}(a - \mathrm{i}/2)\tau^{-2/3} + \frac{2a(\tau)d(\tau)}{\sqrt{-a(\tau)b(\tau)}}. \tag{3.5}$$

As in [57, Section 3.2], define the functions $h_0(\tau)$, $\hat{r}_0(\tau)$, and $\hat{u}_0(\tau)$ via the relations

$$\begin{aligned}&\sqrt{-a(\tau)b(\tau)} + c(\tau)d(\tau) + \frac{a(\tau)d(\tau)\tau^{-2/3}}{2\sqrt{-a(\tau)b(\tau)}} - \frac{1}{4}(a - \mathrm{i}/2)^2\tau^{-4/3} \\ &\quad = \frac{3}{4}(\varepsilon b)^{2/3} - h_0(\tau)\tau^{-2/3},\end{aligned} \tag{3.6}$$

$$r(\tau) = -2 + \hat{r}_0(\tau), \tag{3.7}$$

$$\sqrt{-a(\tau)b(\tau)} = \frac{(\varepsilon b)^{2/3}}{2}(1 + \hat{u}_0(\tau)). \tag{3.8}$$

As follows from the first integral (1.35) (cf. Remark 1.8), the functions $a(\tau)$, $b(\tau)$, $c(\tau)$, and $d(\tau)$ are related via the formula

$$a(\tau)d(\tau) + b(\tau)c(\tau) + \mathrm{i}a\sqrt{-a(\tau)b(\tau)}\tau^{-2/3} = -\mathrm{i}\varepsilon b/2, \qquad \varepsilon \in \{\pm 1\}. \tag{3.9}$$

It is worth noting that equations (3.6)–(3.9) are self-consistent; in fact, a calculation reveals that they are equivalent to

$$a(\tau)d(\tau) = \frac{(\varepsilon b)^{2/3}}{2}(1+\hat{u}_0(\tau))\left(-\frac{\mathrm{i}(\varepsilon b)^{1/3}}{2} + \frac{\mathrm{i}(\varepsilon b)^{1/3}\hat{r}_0(\tau)}{4} - \frac{\mathrm{i}}{2}(a-\mathrm{i}/2)\tau^{-2/3}\right), \tag{3.10}$$

$$b(\tau)c(\tau) = \frac{(\varepsilon b)^{2/3}}{2}(1+\hat{u}_0(\tau))\left(-\frac{\mathrm{i}(\varepsilon b)^{1/3}}{2} + \mathrm{i}(\varepsilon b)^{1/3}\left(\frac{\hat{u}_0(\tau)}{1+\hat{u}_0(\tau)} - \frac{\hat{r}_0(\tau)}{4}\right)\right.$$
$$\left. - \frac{\mathrm{i}}{2}(a+\mathrm{i}/2)\tau^{-2/3}\right), \tag{3.11}$$

$$-h_0(\tau)\tau^{-2/3} = \frac{(\varepsilon b)^{2/3}}{2}\left(\frac{(\hat{u}_0(\tau))^2 + \frac{1}{2}\hat{u}_0(\tau)\hat{r}_0(\tau)}{1+\hat{u}_0(\tau)} - \frac{(\hat{r}_0(\tau))^2}{8}\right)$$
$$+ \frac{(\varepsilon b)^{1/3}(a-\mathrm{i}/2)\tau^{-2/3}}{2(1+\hat{u}_0(\tau))}; \tag{3.12}$$

moreover, via equations (3.8), (3.10), and (3.11), one shows that

$$-c(\tau)d(\tau) = \left(\frac{\mathrm{i}(\varepsilon b)^{1/3}}{2} - \mathrm{i}(\varepsilon b)^{1/3}\left(\frac{\hat{u}_0(\tau)}{1+\hat{u}_0(\tau)} - \frac{\hat{r}_0(\tau)}{4}\right) + \frac{\mathrm{i}}{2}(a+\mathrm{i}/2)\tau^{-2/3}\right)$$
$$\times\left(\frac{\mathrm{i}(\varepsilon b)^{1/3}}{2} - \frac{\mathrm{i}(\varepsilon b)^{1/3}\hat{r}_0(\tau)}{4} + \frac{\mathrm{i}}{2}(a-\mathrm{i}/2)\tau^{-2/3}\right). \tag{3.13}$$

In this work, in lieu of the functions $h_0(\tau)$, $\hat{r}_0(\tau)$, and $\hat{u}_0(\tau)$, it is more convenient to work with the functions $\hat{h}_0(\tau)$, $\tilde{r}_0(\tau)$, and $v_0(\tau)$, respectively, which are defined as follows: for $k=\pm 1$,

$$h_0(\tau) := \left(\frac{3(\varepsilon b)^{2/3}}{4}\left(1-\mathrm{e}^{-\mathrm{i}2\pi k/3}\right) + \hat{h}_0(\tau)\right)\tau^{2/3}, \tag{3.14}$$

$$-2+\hat{r}_0(\tau) := \mathrm{e}^{\mathrm{i}2\pi k/3}\left(-2+\tilde{r}_0(\tau)\tau^{-1/3}\right), \tag{3.15}$$

$$1+\hat{u}_0(\tau) := \mathrm{e}^{-\mathrm{i}2\pi k/3}\left(1+v_0(\tau)\tau^{-1/3}\right). \tag{3.16}$$

The WKB analysis of equation (3.3) is predicated on the assumption that the functions $\hat{h}_0(\tau)$, $\tilde{r}_0(\tau)$, and $v_0(\tau)$ satisfy the (asymptotic) conditions

$$\left|\hat{h}_0(\tau)\right| \underset{\tau\to+\infty}{=} \mathcal{O}\left(\tau^{-2/3}\right), \qquad \left|\tilde{r}_0(\tau)\right| \underset{\tau\to+\infty}{=} \mathcal{O}\left(\tau^{-1/3}\right), \qquad \left|v_0(\tau)\right| \underset{\tau\to+\infty}{=} \mathcal{O}\left(\tau^{-1/3}\right). \tag{3.17}$$

Remark 3.2. Some solutions $u(\tau)$ of the DP3E (1.1) may, and in fact do, have poles and zeros located on the positive real line. In order to be able to study such solutions, one must consider a slightly more general, complex domain $\widetilde{\mathfrak{D}}_u$; however, since, *a priori*, one does not know the solutions $u(\tau)$ which possess such poles and zeros, nor their exact locations, it is necessary to introduce a formal definition for $\widetilde{\mathfrak{D}}_u$. Denote by $\mathcal{P}_u$ and $\mathcal{Z}_u$, respectively, the countable sets of poles and zeros of the function $u(\tau)$. As a consequence of the Painlevé property, these sets may have accumulation points at the origin and at the point at infinity. Define neighbourhoods of $\mathcal{P}_u$ and $\mathcal{Z}_u$, respectively, as follows:[20] for some $\epsilon_* > 0$, let

$$\mathcal{P}_u(\epsilon_*) := \left\{\tau\in\mathbb{C} \mid \left|\theta^{\ddagger}(\tau) - \theta^{\ddagger}(\tau_{\mathfrak{p}})\right| < C_*|\tau_{\mathfrak{p}}|^{-\epsilon_*},\ \tau_{\mathfrak{p}}\in\mathcal{P}_u\right\},$$
$$\mathcal{Z}_u(\epsilon_*) := \left\{\tau\in\mathbb{C} \mid \left|\theta^{\ddagger}(\tau) - \theta^{\ddagger}(\tau_{\mathfrak{z}})\right| < C_*|\tau_{\mathfrak{z}}|^{-\epsilon_*},\ \tau_{\mathfrak{z}}\in\mathcal{Z}_u\right\},$$

[20] There is a misprint in [57, Section 3.1]: in the definitions (3.2) and (3.3), the inequality $>$ must be changed to $<$.

where $\theta^{\ddagger}(\tau)$ is given in Remark 2.7, and $C_{*} > 0$ is some (τ-independent) constant. Now, define the Swiss-cheese-like, multiply-connected domain $\widetilde{\mathfrak{D}}_u := \mathfrak{D}_u^{\blacktriangledown} \setminus (\mathcal{P}_u(\epsilon_*) \cup \mathcal{Z}_u(\epsilon_*))$, where the simply-connected strip domain $\mathfrak{D}_u^{\blacktriangledown}$ is defined by equation (2.18). Theoretically speaking, therefore, it is to be understood that the asymptotic analysis is undertaken in the sense that $\widetilde{\mathfrak{D}}_u \ni \tau$ and $\mathrm{Re}(\tau) \to +\infty$ (with $\varepsilon b > 0$); however, due to the (asymptotic) conditions (3.17), which reflect the sought-after class(es) of functions analysed herein, it turns out that $\mathcal{P}_u(\epsilon_*) = \mathcal{Z}_u(\epsilon_*) = \varnothing$ (see [57, Section 4]), in which case ϵ_* is vacuous and may be set equal to zero, and $\widetilde{\mathfrak{D}}_u = \mathfrak{D}_u^{\blacktriangledown}$. Henceforth, in the asymptotics of all expressions, formulae, etc., depending on $u(\tau)$, the 'notation' $\tau \to +\infty$ means $\mathfrak{D}_u^{\blacktriangledown} \ni \tau$ and $\mathrm{Re}(\tau) \to +\infty$.

Remark 3.3. The function $\hat{h}_0(\tau)$ defined by equation (3.14) plays a prominent rôle in the asymptotic estimates of this work; for further reference, therefore, a compact expression for it, which simplifies several of the ensuing estimates, is presented here: via equation (3.12) and the definition (3.14), one shows that

$$\hat{h}_0(\tau) = \alpha_k^2 \tau^{-2/3} \left(\frac{\varkappa_0^2(\tau)}{4} - \frac{(a - \mathrm{i}/2)}{1 + v_0(\tau)\tau^{-1/3}} \right), \qquad k = \pm 1, \tag{3.18}$$

where α_k is defined by equation (2.5), and the function $\varkappa_0^2(\tau)$ has the following equivalent representations:

$$\begin{aligned}
\left(\frac{\varkappa_0(\tau)}{\tau^{1/3}} \right)^2 &= \left(\frac{1}{\alpha_k^2} + \frac{r(\tau)}{(\varepsilon b)^{1/3}(1 + \hat{u}_0(\tau))} \right) \left(-2(\varepsilon b)^{2/3}(1 + \hat{u}_0(\tau)) + \frac{(\varepsilon b)}{\alpha_k^2} \right) \\
&\quad + \left(2\alpha_k + \frac{(\varepsilon b)^{1/3} r(\tau)}{2\alpha_k} \right)^2 \\
&= -\frac{\varepsilon b}{8\alpha_k^4} \left(\frac{\big(8v_0^2(\tau) + 4\tilde{r}_0(\tau)v_0(\tau) - (\tilde{r}_0(\tau))^2\big)\tau^{-2/3} - (\tilde{r}_0(\tau))^2 v_0(\tau)\tau^{-1}}{1 + v_0(\tau)\tau^{-1/3}} \right) \\
&= -\left(2\alpha_k - \frac{(\varepsilon b)^{1/3} r(\tau)}{2\alpha_k} \right) \left(2\alpha_k + \frac{(\varepsilon b)^{1/3} r(\tau)}{2\alpha_k} \right) \\
&\quad + \frac{1}{\alpha_k^2} \left(\frac{2\varepsilon b}{\alpha_k^2} + (\varepsilon b)^{2/3} \left(-2(1 + \hat{u}_0(\tau)) + \frac{r(\tau)}{1 + \hat{u}_0(\tau)} \right) \right).
\end{aligned} \tag{3.19}$$

It follows from the conditions (3.17) that $\big|\varkappa_0^2(\tau)\big| \underset{\tau \to +\infty}{=} \mathcal{O}\big(\tau^{-2/3}\big)$.

From Proposition 1.5, the definitions (1.31), equations (3.2), equation (3.8), and the definition (3.16), one deduces that, in terms of the function $v_0(\tau)$, the solution of the DP3E (1.1) is given by

$$u(\tau) = c_{0,k} \tau^{1/3} \big(1 + \tau^{-1/3} v_0(\tau)\big), \qquad k = \pm 1, \tag{3.20}$$

where $c_{0,k}$ is defined by equation (2.2). As per the argument at the end of Section 1.1 regarding the particular form of the asymptotics for $u(\tau)$ as $\tau \to +\infty$ with $\varepsilon b > 0$ (cf. equation (1.3) and Remark 1.1), it follows that, in conjunction with the representation (3.20), the function $v_0(\tau)$ can be presented in the form

$$\begin{aligned}
&v_0(\tau) := v_{0,k}(\tau) \underset{\tau \to +\infty}{=} \sum_{m=0}^{\infty} \frac{\mathfrak{u}_m(k)}{\big(\tau^{1/3}\big)^{m+1}} + \mathrm{A}_k \mathrm{e}^{-(\beta(\tau) + \mathrm{i}k\vartheta(\tau))} \big(1 + \mathcal{O}\big(\tau^{-1/3}\big)\big), \\
&k = \pm 1,
\end{aligned} \tag{3.21}$$

where the sequence of $\mathbb{C}$-valued expansion coefficients $\{\mathfrak{u}_m(k)\}_{m=0}^{\infty}$ are determined in Proposition 3.4 below, $\vartheta(\tau)$ and $\beta(\tau)$ are defined in equations (2.10), and, in the course of the ensuing analysis, it will be established that A_k depends on the Stokes multiplier s_0^0 (see Section 4, equations (4.71) and (4.92)). From equation (3.20) and the expansion (3.21), it follows that the associated solution of the DP3E (1.1) has asymptotics

$$u(\tau) \underset{\tau\to+\infty}{=} c_{0,k}\tau^{1/3}\left(1+\sum_{m=0}^{\infty}\frac{\mathfrak{u}_m(k)}{(\tau^{1/3})^{m+2}}+\mathrm{A}_k\tau^{-1/3}\mathrm{e}^{-(\beta(\tau)+\mathrm{i}k\vartheta(\tau))}\left(1+\mathcal{O}(\tau^{-1/3})\right)\right),$$
$$k=\pm1. \tag{3.22}$$

Proposition 3.4. *For $u(\tau)$ the corresponding solution of the* DP3E (1.1)*, let the function $v_0(\tau):=v_{0,k}(\tau)$, $k=\pm1$, have the asymptotic expansion stated in equation* (3.21)*; then, the expansion coefficients $\mathfrak{u}_m(k)$, $m\in\mathbb{Z}_+$, are given in equations* (2.2)*–*(2.9)*.*

Proof. See Appendix A. ■

It follows from equations (1.44), (3.2), (3.5), and (3.7) that

$$\frac{u'(\tau)-\mathrm{i}b}{u(\tau)}=\frac{2}{\tau^{1/3}}\left(\frac{2a(\tau)d(\tau)}{\sqrt{-a(\tau)b(\tau)}}+\tau^{-2/3}(\mathrm{i}a+1/2)\right)=\mathrm{i}(\varepsilon b)^{1/3}\tau^{-1/3}(-2+\hat{r}_0(\tau));$$

thus, via the definition (3.15), it follows that

$$\tilde{r}_0(\tau)=2\tau^{1/3}-\frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}2\pi k/3}\tau^{2/3}}{(\varepsilon b)^{1/3}}\left(\frac{u'(\tau)-\mathrm{i}b}{u(\tau)}\right),\qquad k=\pm1. \tag{3.23}$$

Proposition 3.5. *For $u(\tau)$ the corresponding solution of the* DP3E (1.1) *having the differentiable asymptotics* (3.22)*, with $\mathfrak{u}_m(k)$, $m\in\mathbb{Z}_+$, $k=\pm1$, given in Proposition* 3.4*, let the function $\tilde{r}_0(\tau)$ be given by equation* (3.23)*; then, $\tilde{r}_0(\tau)$ has the following asymptotic expansion:*

$$\tilde{r}_0(\tau):=\tilde{r}_{0,k}(\tau)\underset{\tau\to+\infty}{=}\sum_{m=0}^{\infty}\frac{\mathfrak{r}_m(k)}{(\tau^{1/3})^{m+1}}+2\left(1+k\sqrt{3}\right)\mathrm{A}_k\mathrm{e}^{-(\beta(\tau)+\mathrm{i}k\vartheta(\tau))}\left(1+\mathcal{O}(\tau^{-1/3})\right),$$
$$k=\pm1, \tag{3.24}$$

where the expansion coefficients $\mathfrak{r}_m(k)$, $m\in\mathbb{Z}_+$, are given in equations (2.13) *and* (2.14)*.*

Proof. Substituting the differentiable asymptotics (3.22) for $u(\tau)$ into equation (3.23) and using the expressions for the coefficients $c_{0,k}$, $\mathfrak{u}_m(k)$, and $\mathfrak{w}_m(k)$, $k=\pm1$, $m\in\mathbb{Z}_+$, given in the proof of Proposition 3.4 (cf. Appendix A), one arrives at, after a lengthy, but otherwise straightforward, algebraic calculation, the asymptotics for $\tilde{r}_0(\tau):=\tilde{r}_{0,k}(\tau)$ stated in the proposition. ■

Remark 3.6. Hereafter, explicit k dependencies for the subscripts of the functions $v_0(\tau)$ and $\tilde{r}_0(\tau)$ (cf. equations (3.21) and (3.24), respectively) will be suppressed, except where absolutely necessary and/or where confusion may arise.

In certain domains of the complex $\widetilde{\mu}$-plane (see the discussion below), the leading term of asymptotics (as $\tau\to+\infty$ for $\varepsilon b>0$) of a fundamental solution of equation (3.3) is given by the following matrix WKB formula (see, for example, [28, Chapter 5]),[21]

$$T(\widetilde{\mu})\exp\left(-\sigma_3\mathrm{i}\tau^{2/3}\int^{\widetilde{\mu}}l(\xi)\mathrm{d}\xi-\int^{\widetilde{\mu}}\mathrm{diag}\left((T(\xi))^{-1}\partial_\xi T(\xi)\right)\mathrm{d}\xi\right):=\widetilde{\Psi}_{\mathrm{WKB}}(\widetilde{\mu}), \tag{3.25}$$

[21]Hereafter, for simplicity of notation, explicit τ dependencies will be suppressed, except where absolutely necessary.

where

$$l(\widetilde{\mu}) := \sqrt{\det(\mathcal{A}(\widetilde{\mu}))}, \tag{3.26}$$

and the matrix $T(\widetilde{\mu})$, which diagonalizes $\mathcal{A}(\widetilde{\mu})$, that is, $(T(\widetilde{\mu}))^{-1}\mathcal{A}(\widetilde{\mu})T(\widetilde{\mu}) = -\mathrm{i}l(\widetilde{\mu})\sigma_3$, is given by

$$T(\widetilde{\mu}) = \frac{\mathrm{i}}{\sqrt{2\mathrm{i}l(\widetilde{\mu})(\mathcal{A}_{11}(\widetilde{\mu}) - \mathrm{i}l(\widetilde{\mu}))}} \left(\mathcal{A}(\widetilde{\mu}) - \mathrm{i}l(\widetilde{\mu})\sigma_3\right)\sigma_3. \tag{3.27}$$

Proposition 3.7 ([57]). *Let $T(\widetilde{\mu})$ be given in equation (3.27), with $\mathcal{A}(\widetilde{\mu})$ and $l(\widetilde{\mu})$ defined by equations (3.4) and (3.26), respectively. Then,* $\det(T(\widetilde{\mu})) = 1$ *and* $\operatorname{tr}\big((T(\widetilde{\mu}))^{-1}\partial_{\widetilde{\mu}}T(\widetilde{\mu})\big) = 0$*; moreover,*

$$\operatorname{diag}\big((T(\widetilde{\mu}))^{-1}\partial_{\widetilde{\mu}}T(\widetilde{\mu})\big) = -\frac{1}{2}\left(\frac{\mathcal{A}_{12}(\widetilde{\mu})\partial_{\widetilde{\mu}}\mathcal{A}_{21}(\widetilde{\mu}) - \mathcal{A}_{21}(\widetilde{\mu})\partial_{\widetilde{\mu}}\mathcal{A}_{12}(\widetilde{\mu})}{2l(\widetilde{\mu})(\mathrm{i}\mathcal{A}_{11}(\widetilde{\mu}) + l(\widetilde{\mu}))}\right)\sigma_3. \tag{3.28}$$

Corollary 3.8. *Let $\widetilde{\Psi}_{\mathrm{WKB}}(\widetilde{\mu})$ be defined by equation (3.25), with $l(\widetilde{\mu})$ defined by equation (3.26) and $T(\widetilde{\mu})$ given in equation (3.27); then,* $\det\big(\widetilde{\Psi}_{\mathrm{WKB}}(\widetilde{\mu})\big) = 1$.

The domains in the complex $\widetilde{\mu}$-plane where equation (3.25) gives the (leading) asymptotic approximation of solutions to equation (3.3) are defined in terms of the *Stokes graph* (see, for example, [28, 63, 80]). The vertices of the Stokes graph are the singular points of equation (3.3), that is, $\widetilde{\mu} = 0$ and $\widetilde{\mu} = \infty$, and the *turning points*, which are the roots of the equation $l^2(\widetilde{\mu}) = 0$. The edges of the Stokes graph are the *Stokes curves*, defined as $\operatorname{Im}\big(\int_{\widetilde{\mu}_{\mathrm{TP}}}^{\widetilde{\mu}} l(\xi)\mathrm{d}\xi\big) = 0$, where $\widetilde{\mu}_{\mathrm{TP}}$ denotes a turning point. *Canonical domains* are those domains in the complex $\widetilde{\mu}$-plane containing one, and only one, Stokes curve and bounded by two adjacent Stokes curves. (Note that the restriction of any branch of $l(\widetilde{\mu})$ to a canonical domain is a single-valued function.) In each canonical domain, for any choice of the branch of $l(\widetilde{\mu})$, there exists a fundamental solution of equation (3.3) which has asymptotics whose leading term is given by equation (3.25). From the definition of $l(\widetilde{\mu})$ given by equation (3.26), one arrives at

$$l^2(\widetilde{\mu}) := l_k^2(\widetilde{\mu}) = \frac{4}{\widetilde{\mu}^4}\big((\widetilde{\mu}^2 - \alpha_k^2)^2(\widetilde{\mu}^2 + 2\alpha_k^2) + \widetilde{\mu}^2\hat{h}_0(\tau) + \widetilde{\mu}^4(a - \mathrm{i}/2)\tau^{-2/3}\big),$$
$$k = \pm 1, \tag{3.29}$$

where α_k is defined by equation (2.5). It follows from equation (3.29) that there are six turning points. For $k = \pm 1$, the conditions (3.17) imply that one pair of turning points coalesce at α_k with asymptotics $\mathcal{O}\big(\tau^{-1/3}\big)$, another pair has asymptotics $-\alpha_k + \mathcal{O}\big(\tau^{-1/3}\big)$, and the two remaining turning points have the asymptotic behaviour $\pm\mathrm{i}\sqrt{2}\alpha_k + \mathcal{O}\big(\tau^{-2/3}\big)$. For simplicity of notation, denote by $\widetilde{\mu}_1(k)$ any one of the turning points coalescing at α_k, and denote by $\widetilde{\mu}_2(k)$ the turning point approaching $\mathrm{i}k\sqrt{2}\alpha_k$. Let $\mathcal{G}_{\mathbb{S}}(k)$, $k = \pm 1$, be the part of the Stokes graph that consists of the vertices 0, ∞, $\widetilde{\mu}_1(k)$ and $\widetilde{\mu}_2(k)$, and the union of the (oriented) edges $\operatorname{arc}(\mathrm{i}k\infty, \widetilde{\mu}_2(k))$, $\operatorname{arc}(\widetilde{\mu}_2(k), 0)$ and $\operatorname{arc}(\widetilde{\mu}_2(k), -\infty)$, and $\operatorname{arc}(\mathrm{i}k\infty, \widetilde{\mu}_1(k))$, $\operatorname{arc}(\widetilde{\mu}_1(k), 0)$, $\operatorname{arc}(0, \widetilde{\mu}_1(k))$ and $\operatorname{arc}(\widetilde{\mu}_1(k), +\infty)$; the complete Stokes graph is given by $\mathcal{G}_{\mathbb{S}}(k) \cup \mathrm{e}^{\mathrm{i}\pi}\mathcal{G}_{\mathbb{S}}(k)$ (see Figure 1 (resp., Figure 2) for the case $k = +1$ (resp., $k = -1$)).

Proposition 3.9. *Let $l_k^2(\widetilde{\mu})$, $k = \pm 1$, be given in equation (3.29); then,*

$$\int_{\widetilde{\mu}_{0,k}}^{\widetilde{\mu}} l_k(\xi)\mathrm{d}\xi \underset{\tau\to+\infty}{=} \Upsilon_k(\widetilde{\mu}) - \Upsilon_k(\widetilde{\mu}_{0,k}) + \mathcal{O}(\mathcal{E}_k(\widetilde{\mu})) + \mathcal{O}(\mathcal{E}_k(\widetilde{\mu}_{0,k})), \tag{3.30}$$

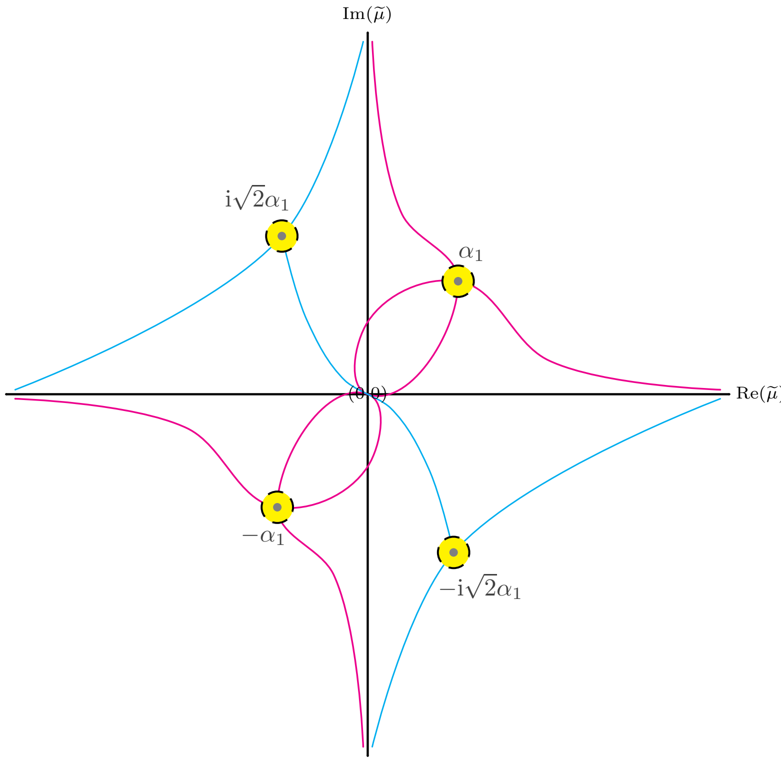

Figure 1. The Stokes graph for $k=+1$.

where, for $\delta>0$, $\widetilde{\mu},\widetilde{\mu}_{0,k}\in\mathbb{C}\setminus\big(\mathcal{O}_{\tau^{-1/3+\delta}}(\pm\alpha_k)\cup\mathcal{O}_{\tau^{-2/3+2\delta}}\big(\pm\mathrm{i}\sqrt{2}\alpha_k\big)\cup\{0,\infty\}\big)$ *and the path of integration lies in the corresponding canonical domain,*

$$\begin{aligned}\Upsilon_k(\xi):=&\,\big(\xi+2\alpha_k^2\xi^{-1}\big)\big(\xi^2+2\alpha_k^2\big)^{1/2}+\tau^{-2/3}(a-\mathrm{i}/2)\ln\big(\xi+\big(\xi^2+2\alpha_k^2\big)^{1/2}\big)\\&+\frac{\tau^{-2/3}}{2\sqrt{3}}\left((a-\mathrm{i}/2)+\frac{\tau^{2/3}}{\alpha_k^2}\hat{h}_0(\tau)\right)\\&\times\ln\left(\left(\frac{3^{1/2}\big(\xi^2+2\alpha_k^2\big)^{1/2}-\xi+2\alpha_k}{3^{1/2}\big(\xi^2+2\alpha_k^2\big)^{1/2}+\xi+2\alpha_k}\right)\left(\frac{\xi-\alpha_k}{\xi+\alpha_k}\right)\right),\end{aligned}\tag{3.31}$$

and

$$\tau^{4/3}\mathcal{E}_k(\xi):=\begin{cases}\dfrac{\Big((a-\mathrm{i}/2)+\frac{\tau^{2/3}}{\alpha_k^2}\hat{h}_0(\tau)\Big)^2}{192\sqrt{3}(\xi\mp\alpha_k)^2}+\mathcal{O}\left(\dfrac{c_{1,k}+c_{2,k}\tau^{2/3}\hat{h}_0(\tau)+c_{3,k}\big(\tau^{2/3}\hat{h}_0(\tau)\big)^2}{\xi\mp\alpha_k}\right), & \xi\in\mathbb{U}_k^1,\\ \dfrac{\Big((a-\mathrm{i}/2)-\frac{\tau^{2/3}}{2\alpha_k^2}\hat{h}_0(\tau)\Big)^2}{d_{0,k}(\xi\mp\mathrm{i}\sqrt{2}\alpha_k)^{1/2}}+\mathcal{O}\big(\big(\xi\mp\mathrm{i}\sqrt{2}\alpha_k\big)^{1/2} & \\ \quad\times\big(c_{4,k}+c_{5,k}\tau^{2/3}\hat{h}_0(\tau)+c_{6,k}\big(\tau^{2/3}\hat{h}_0(\tau)\big)^2\big)\big), & \xi\in\mathbb{U}_k^2,\\ \mathfrak{f}_{1,k}\big(\xi^{-1}\big)+\tau^{2/3}\hat{h}_0(\tau)\mathfrak{f}_{2,k}\big(\xi^{-1}\big)+\big(\tau^{2/3}\hat{h}_0(\tau)\big)^2\mathfrak{f}_{3,k}\big(\xi^{-1}\big), & \xi\to\infty,\\ \mathfrak{f}_{4,k}(\xi)+\tau^{2/3}\hat{h}_0(\tau)\mathfrak{f}_{5,k}(\xi)+\big(\tau^{2/3}\hat{h}_0(\tau)\big)^2\mathfrak{f}_{6,k}(\xi), & \xi\to 0,\end{cases}\tag{3.32}$$

where $\mathbb{U}_k^1:=\mathcal{O}_{\tau^{-1/3+\delta_k}}(\pm\alpha_k)$, $\mathbb{U}_k^2:=\mathcal{O}_{\tau^{-2/3+2\delta_k}}\big(\pm\mathrm{i}\sqrt{2}\alpha_k\big)$, *the parameter* δ_k *satisfies* (*see Corollary* 3.10) $0<\delta<\delta_k<1/9$, $d_{0,k}^{-1}:=2^{-1/4}\mathrm{e}^{\mp\mathrm{i}3\pi/4}\alpha_k^{-3/2}/27$, $\mathfrak{f}_{j,k}(z)$, $j=1,2,\dots,6$, *are analytic functions of* z *in a neighbourhood of* $z=0$ *given in equations* (3.38)–(3.43), *and* $c_{m,k}$, $m=1,2,\dots,6$, *are* $\mathcal{O}(1)$.

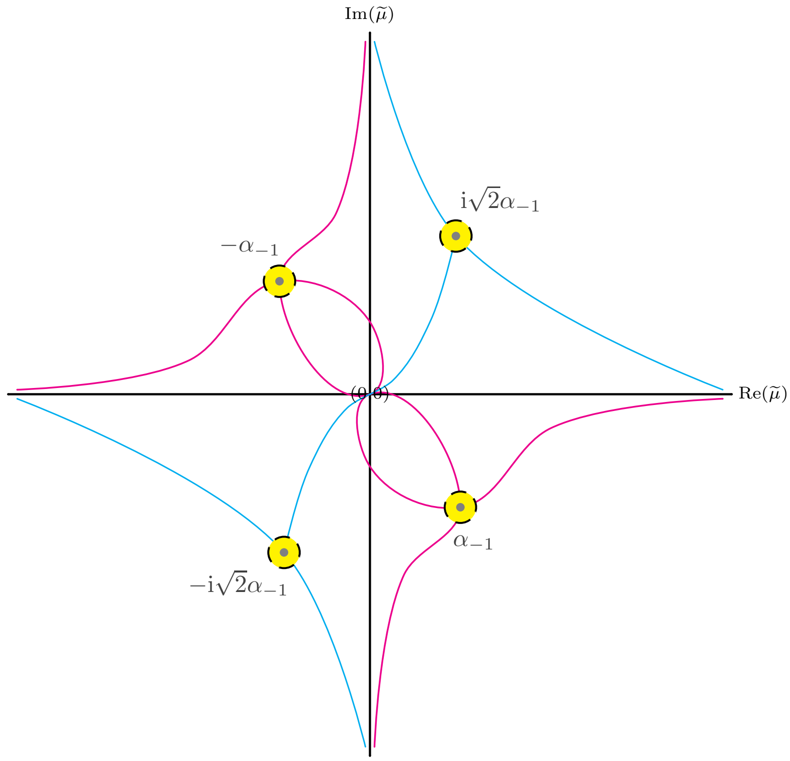

Figure 2. The Stokes graph for $k=-1$.

Proof. Let $l_k^2(\widetilde{\mu})$, $k=\pm1$, be given in equation (3.29), with α_k defined by equation (2.5). Recalling from the conditions (3.17) that $\big|\hat{h}_0(\tau)\big| \underset{\tau\to+\infty}{=} \mathcal{O}\big(\tau^{-2/3}\big)$, set

$$l_{k,\infty}^2(\widetilde{\mu}) = 4\widetilde{\mu}^{-4}\big(\widetilde{\mu}^2-\alpha_k^2\big)^2\big(\widetilde{\mu}^2+2\alpha_k^2\big). \tag{3.33}$$

Define

$$\Delta_{k,\tau}(\widetilde{\mu}) := \frac{l_k^2(\widetilde{\mu})-l_{k,\infty}^2(\widetilde{\mu})}{l_{k,\infty}^2(\widetilde{\mu})} = \frac{\widetilde{\mu}^2\hat{h}_0(\tau)+\widetilde{\mu}^4(a-\mathrm{i}/2)\tau^{-2/3}}{\big(\widetilde{\mu}^2-\alpha_k^2\big)^2\big(\widetilde{\mu}^2+2\alpha_k^2\big)}; \tag{3.34}$$

hence, presenting $l_k(\widetilde{\mu})$ as $l_k(\widetilde{\mu}) = l_{k,\infty}(\widetilde{\mu})(1+\Delta_{k,\tau}(\widetilde{\mu}))^{1/2}$, a straightforward calculation, via the conditions (3.17), shows that, for $k=\pm1$,

$$\begin{aligned} l_k(\widetilde{\mu}) &\underset{\tau\to+\infty}{=} l_{k,\infty}(\widetilde{\mu})\big(1+\Delta_{k,\tau}(\widetilde{\mu})/2+\mathcal{O}(-(\Delta_{k,\tau}(\widetilde{\mu}))^2/8)\big) \\ &\underset{\tau\to+\infty}{=} 2\big(1-\alpha_k^2/\widetilde{\mu}^2\big)\big(\widetilde{\mu}^2+2\alpha_k^2\big)^{1/2} + \frac{\hat{h}_0(\tau)+\widetilde{\mu}^2(a-\mathrm{i}/2)\tau^{-2/3}}{\big(\widetilde{\mu}^2-\alpha_k^2\big)\big(\widetilde{\mu}^2+2\alpha_k^2\big)^{1/2}} \\ &\quad + \mathcal{O}\left(-\frac{\widetilde{\mu}^2\big(\hat{h}_0(\tau)+\widetilde{\mu}^2(a-\mathrm{i}/2)\tau^{-2/3}\big)^2}{4\big(\widetilde{\mu}^2-\alpha_k^2\big)^3\big(\widetilde{\mu}^2+2\alpha_k^2\big)^{3/2}}\right). \end{aligned} \tag{3.35}$$

Integration of the two terms in the second line of equation (3.35) gives rise to the leading term of asymptotics in equation (3.30), and integration of the error term in the third line of equation (3.35) leads to an explicit expression for the error function, $\mathcal{E}_k(\cdot)$, whose asymptotics at the turning and the singular points read: (i) for $\xi \in \mathcal{O}_{\tau^{-1/3+\delta_k}}(\pm\alpha_k)$, $0<\delta<\delta_k<1/9$,

$$\tau^{4/3}\mathcal{E}_k(\xi) \underset{\tau\to+\infty}{=} \frac{\big((a-\mathrm{i}/2)+\alpha_k^{-2}\tau^{2/3}\hat{h}_0(\tau)\big)^2}{192\sqrt{3}(\xi\mp\alpha_k)^2} + \frac{\hat{d}_{-1,k}(\tau)}{\xi\mp\alpha_k} + \hat{d}_{0,k}(\tau)\ln(\xi\mp\alpha_k)$$

$$+\sum_{m\in\mathbb{Z}_+}\hat{d}_{m+1,k}(\tau)(\xi\mp\alpha_k)^{m+1}, \tag{3.36}$$

where

$$\hat{d}_{m,k}(\tau):=\hat{\mathfrak{c}}^{\flat}_{m,k}+\hat{\mathfrak{c}}^{\natural}_{m,k}\tau^{2/3}\hat{h}_0(\tau)+\hat{\mathfrak{c}}^{\sharp}_{m,k}\big(\tau^{2/3}\hat{h}_0(\tau)\big)^2, \qquad m\in\{-1\}\cup\mathbb{Z}_+,$$

and $\hat{\mathfrak{c}}^{r}_{m,k}$, $r\in\{\flat,\natural,\sharp\}$, are $\mathcal{O}(1)$, and thus, retaining only the first two terms of the expansion (3.36), one arrives at the representation for $\mathcal{E}_k(\xi)$ stated in the first line of equation (3.32); (ii) for $\xi\in\mathbb{O}_{\tau^{-2/3+2\delta_k}}\big(\pm\mathrm{i}\sqrt{2}\alpha_k\big)$,

$$\begin{aligned}\tau^{4/3}\mathcal{E}_k(\xi) \underset{\tau\to+\infty}{=} & \frac{2^{-1/4}\big((a-\mathrm{i}/2)-\tau^{2/3}\hat{h}_0(\tau)/2\alpha_k^2\big)^2}{27\alpha_k^{3/2}\mathrm{e}^{\pm\mathrm{i}3\pi/4}\big(\xi\mp\mathrm{i}\sqrt{2}\alpha_k\big)^{1/2}}\\ &+\big(\xi\mp\mathrm{i}\sqrt{2}\alpha_k\big)^{1/2}\sum_{m\in\mathbb{Z}_+}\tilde{d}_{m,k}(\tau)\big(\xi\mp\mathrm{i}\sqrt{2}\alpha_k\big)^m,\end{aligned} \tag{3.37}$$

where

$$\tilde{d}_{m,k}(\tau):=\tilde{\mathfrak{c}}^{\flat}_{m,k}+\tilde{\mathfrak{c}}^{\natural}_{m,k}\tau^{2/3}\hat{h}_0(\tau)+\tilde{\mathfrak{c}}^{\sharp}_{m,k}\big(\tau^{2/3}\hat{h}_0(\tau)\big)^2, \qquad m\in\mathbb{Z}_+,$$

and $\tilde{\mathfrak{c}}^{r}_{m,k}$, $r\in\{\flat,\natural,\sharp\}$, are $\mathcal{O}(1)$, and thus, keeping only the first two terms of the expansion (3.37), one arrives at the representation for $\mathcal{E}_k(\xi)$ stated in the second line of equation (3.32); (iii) as $\xi\to\infty$, one arrives at the representation for $\mathcal{E}_k(\xi)$ stated in the third line of equation (3.32), where

$$\mathfrak{f}_{1,k}(z)=\frac{(a-\mathrm{i}/2)^2}{8}z^2+(a-\mathrm{i}/2)^2z^6\sum_{m\in\mathbb{Z}_+}\hat{\mathfrak{c}}^{\circ,1}_{m,k}z^{2m}, \tag{3.38}$$

$$\mathfrak{f}_{2,k}(z)=\frac{(a-\mathrm{i}/2)}{8}z^4+(a-\mathrm{i}/2)z^8\sum_{m\in\mathbb{Z}_+}\hat{\mathfrak{c}}^{\circ,2}_{m,k}z^{2m}, \tag{3.39}$$

$$\mathfrak{f}_{3,k}(z)=\frac{1}{24}z^6+z^{10}\sum_{m\in\mathbb{Z}_+}\hat{\mathfrak{c}}^{\circ,3}_{m,k}z^{2m}, \tag{3.40}$$

and $\hat{\mathfrak{c}}^{\circ,r}_{m,k}$, $r=1,2,3$, $m\in\mathbb{Z}_+$, are $\mathcal{O}(1)$; and (iv) as $\xi\to0$, one arrives at the representation for $\mathcal{E}_k(\xi)$ stated in the fourth line of equation (3.32), where

$$\mathfrak{f}_{4,k}(z)=\frac{(a-\mathrm{i}/2)^2}{56\sqrt{2}\alpha_k^9}z^7+(a-\mathrm{i}/2)^2z^9\sum_{m\in\mathbb{Z}_+}\tilde{d}^{\circ,4}_{m,k}z^{2m}, \tag{3.41}$$

$$\mathfrak{f}_{5,k}(z)=\frac{(a-\mathrm{i}/2)}{20\sqrt{2}\alpha_k^9}z^5+(a-\mathrm{i}/2)z^7\sum_{m\in\mathbb{Z}_+}\tilde{d}^{\circ,5}_{m,k}z^{2m}, \tag{3.42}$$

$$\mathfrak{f}_{6,k}(z)=\frac{1}{24\sqrt{2}\alpha_k^9}z^3+z^5\sum_{m\in\mathbb{Z}_+}\tilde{d}^{\circ,6}_{m,k}z^{2m}, \tag{3.43}$$

and $\tilde{d}^{\circ,r}_{m,k}$, $r=4,5,6$, $m\in\mathbb{Z}_+$, are $\mathcal{O}(1)$. ■

Corollary 3.10. *Set* $\widetilde{\mu}_{0,k}=\alpha_k+\tau^{-1/3}\widetilde{\Lambda}$, $k=\pm1$, *where* $\widetilde{\Lambda}\underset{\tau\to+\infty}{=}\mathcal{O}\big(\tau^{\delta_k}\big)$, $0<\delta<\delta_k<1/9$; *then,*

$$\int_{\widetilde{\mu}_{0,k}}^{\widetilde{\mu}}l_k(\xi)\mathrm{d}\xi \underset{\tau\to+\infty}{=} \Upsilon_k(\widetilde{\mu})+\Upsilon_k^{\sharp}+\mathcal{O}(\mathcal{E}_k(\widetilde{\mu}))+\mathcal{O}\big(\tau^{-1}\widetilde{\Lambda}^3\big)+\mathcal{O}\big(\tau^{-1}\widetilde{\Lambda}\big)$$

$$+\,\mathcal{O}\left(\frac{\tau^{-1}}{\widetilde{\Lambda}}\big(\mathfrak{c}_{1,k}+\mathfrak{c}_{2,k}\tau^{2/3}\hat{h}_0(\tau)+\mathfrak{c}_{3,k}\big(\tau^{2/3}\hat{h}_0(\tau)\big)^2\big)\right), \tag{3.44}$$

where $\Upsilon_k(\widetilde{\mu})$ *and* $\mathcal{E}_k(\widetilde{\mu})$ *are defined by equations* (3.31) *and* (3.32)*, respectively,*

$$\begin{aligned}\Upsilon_k^{\sharp}:=\;&\mp 3\sqrt{3}\alpha_k^2\mp 2\sqrt{3}\tau^{-2/3}\widetilde{\Lambda}^2-\tau^{-2/3}(a-\mathrm{i}/2)\ln\big(\big(\sqrt{3}\pm 1\big)\alpha_k\mathrm{e}^{\mathrm{i}\pi(1\mp 1)/2}\big)\\&\mp\frac{\tau^{-2/3}}{2\sqrt{3}}\big((a-\mathrm{i}/2)+\alpha_k^{-2}\tau^{2/3}\hat{h}_0(\tau)\big)\left(\ln\widetilde{\Lambda}-\frac{1}{3}\ln\tau-\ln(3\alpha_k)\right),\end{aligned} \tag{3.45}$$

with the upper (resp., lower) signs taken according to the branch of the square-root function $\lim_{\xi^2\to+\infty}\big(\xi^2+2\alpha_k^2\big)^{1/2}=+\infty$ *(resp.,* $\lim_{\xi^2\to+\infty}\big(\xi^2+2\alpha_k^2\big)^{1/2}=-\infty$*), and* $\mathfrak{c}_{m,k}$*,* $m=1,2,3$*, are* $\mathcal{O}(1)$*.*

Proof. Substituting $\widetilde{\mu}_{0,k}$, as given in the corollary, for the argument of the functions $\Upsilon_k(\xi)$ and $\mathcal{E}_k(\xi)$ (cf. equation (3.31) and the first line of equation (3.32), respectively) and expanding with respect to the "small parameter" $\tau^{-1/3}\widetilde{\Lambda}$, one arrives at the following estimates:

$$\begin{aligned}-\Upsilon_k(\widetilde{\mu}_{0,k})\underset{\tau\to+\infty}{=}\;&\Upsilon_k^{\sharp}+\mathcal{O}\big(\tau^{-1}\widetilde{\Lambda}^3\big)+\mathcal{O}\big(\tau^{-1}\widetilde{\Lambda}\big)\\&+\mathcal{O}\big(\tau^{-1}\widetilde{\Lambda}((a-\mathrm{i}/2)+\alpha_k^{-2}\tau^{2/3}\hat{h}_0(\tau))\big),\end{aligned} \tag{3.46}$$

where $\Upsilon_k^{\sharp}$ is defined by equation (3.45),

$$\begin{aligned}\mathcal{O}(\mathcal{E}_k(\widetilde{\mu}_{0,k}))\underset{\tau\to+\infty}{=}\;&\mathcal{O}\left(\frac{\tau^{-2/3}}{\widetilde{\Lambda}^2}\big((a-\mathrm{i}/2)+\alpha_k^{-2}\tau^{2/3}\hat{h}_0(\tau)\big)^2\right)\\&+\mathcal{O}\left(\frac{\tau^{-1}}{\widetilde{\Lambda}}\big(c_{1,k}+c_{2,k}\tau^{2/3}\hat{h}_0(\tau)+c_{3,k}\big(\tau^{2/3}\hat{h}_0(\tau)\big)^2\big)\right),\end{aligned} \tag{3.47}$$

and $c_{m,k}$, $m=1,2,3$, are $\mathcal{O}(1)$. From equations (3.12), (3.14), (3.15), and (3.16), one shows that

$$\begin{aligned}-\tau^{2/3}\hat{h}_0(\tau)=\;&\frac{\alpha_k^2(a-\mathrm{i}/2)}{1+v_0(\tau)\tau^{-1/3}}\\&+\frac{\alpha_k^4\big(8v_0^2(\tau)+4\tilde{r}_0(\tau)v_0(\tau)-(\tilde{r}_0(\tau))^2-v_0(\tau)(\tilde{r}_0(\tau))^2\tau^{-1/3}\big)}{4\big(1+v_0(\tau)\tau^{-1/3}\big)},\end{aligned}$$

whence, via the conditions (3.17),

$$\begin{aligned}&(a-\mathrm{i}/2)+\frac{\tau^{2/3}}{\alpha_k^2}\hat{h}_0(\tau)\\&\underset{\tau\to+\infty}{=}-\frac{\alpha_k^2}{4}\big(8v_0^2(\tau)+4v_0(\tau)\tilde{r}_0(\tau)-(\tilde{r}_0(\tau))^2\big)+(a-\mathrm{i}/2)v_0(\tau)\tau^{-1/3}\\&\qquad+\mathcal{O}\big(\big(2v_0^2(\tau)+v_0(\tau)\tilde{r}_0(\tau)\big)v_0(\tau)\tau^{-1/3}\big)+\mathcal{O}\big(v_0^2(\tau)\tau^{-2/3}\big).\end{aligned} \tag{3.48}$$

Note from the conditions (3.17) and the expansion (3.48) that

$$\begin{aligned}&(a-\mathrm{i}/2)+\frac{\tau^{2/3}}{\alpha_k^2}\hat{h}_0(\tau)\underset{\tau\to+\infty}{=}\mathcal{O}\big(\tau^{-2/3}\big),\\&c_{1,k}+c_{2,k}\tau^{2/3}\hat{h}_0(\tau)+c_{3,k}\big(\tau^{2/3}\hat{h}_0(\tau)\big)^2\underset{\tau\to+\infty}{=}\mathcal{O}(1):\end{aligned}$$

from the expansions (3.46) and (3.47) and the latter two estimates, it follows that

$$-\varUpsilon_k(\widetilde{\mu}_{0,k}) \underset{\tau\to+\infty}{=} \varUpsilon_k^{\sharp} + \mathcal{O}\big(\tau^{-1}\widetilde{\Lambda}^3\big) + \mathcal{O}\big(\tau^{-1}\widetilde{\Lambda}\big) + \mathcal{O}\big(\tau^{-5/3}\widetilde{\Lambda}\big), \tag{3.49}$$

$$\mathcal{O}(\mathcal{E}_k(\widetilde{\mu}_{0,k})) \underset{\tau\to+\infty}{=} \mathcal{O}\big(\tau^{-1}\widetilde{\Lambda}^{-1}\big) + \mathcal{O}\big(\tau^{-2}\widetilde{\Lambda}^{-2}\big), \tag{3.50}$$

whence, introducing the inequality $0 < \delta < \delta_k < 1/9$ in order to guarantee that the error estimates in the expansions (3.49) and (3.50) are $o(1)$ after multiplication by the "large parameter" $\tau^{2/3}$ (cf. equation (3.25)), retaining only leading-order contributions, one arrives at

$$-\varUpsilon_k(\widetilde{\mu}_{0,k}) + \mathcal{O}(\mathcal{E}_k(\widetilde{\mu}_{0,k})) \underset{\tau\to+\infty}{=} \varUpsilon_k^{\sharp} + \mathcal{O}\left(\frac{\tau^{-1}}{\widetilde{\Lambda}}\big(c_{1,k} + c_{2,k}\tau^{2/3}\hat{h}_0(\tau) + c_{3,k}\big(\tau^{2/3}\hat{h}_0(\tau)\big)^2\big)\right) + \mathcal{O}\big(\tau^{-1}\widetilde{\Lambda}^3\big) + \mathcal{O}\big(\tau^{-1}\widetilde{\Lambda}\big),$$

which, via equation (3.30), implies the result stated in the corollary. ∎

Corollary 3.11. *Let the conditions stated in Corollary 3.10 be valid; then, for the branch of $l_k(\xi)$, $k = \pm 1$, that is positive for large and small positive ξ,*

$$\begin{aligned}
-\mathrm{i}\tau^{2/3}\int_{\widetilde{\mu}_{0,k}}^{\widetilde{\mu}} l_k(\xi)\mathrm{d}\xi
\underset{\substack{\tau\to+\infty\\ \widetilde{\mu}\to\infty}}{=} & -\mathrm{i}\big(\tau^{2/3}\widetilde{\mu}^2 + (a-\mathrm{i}/2)\ln\widetilde{\mu}\big) + \mathrm{i}3\big(\sqrt{3}-1\big)\alpha_k^2\tau^{2/3} + \mathrm{i}2\sqrt{3}\widetilde{\Lambda}^2 + C_{\infty,k}^{\mathrm{WKB}} \\
& - \frac{\mathrm{i}}{2\sqrt{3}}\big((a-\mathrm{i}/2) + \alpha_k^{-2}\tau^{2/3}\hat{h}_0(\tau)\big)\left(\frac{1}{3}\ln\tau - \ln\widetilde{\Lambda} + \ln\left(\frac{6\alpha_k}{\big(\sqrt{3}+1\big)^2}\right)\right) \\
& + \mathcal{O}\left(\frac{\tau^{-1/3}}{\widetilde{\Lambda}}\big(\mathfrak{c}_{1,k} + \mathfrak{c}_{2,k}\tau^{2/3}\hat{h}_0(\tau) + \mathfrak{c}_{3,k}\big(\tau^{2/3}\hat{h}_0(\tau)\big)^2\big)\right) \\
& + \mathcal{O}\big(\tau^{-1/3}\widetilde{\Lambda}^3\big) + \mathcal{O}\big(\tau^{-1/3}\widetilde{\Lambda}\big) + \mathcal{O}\big(\tau^{-2/3}\widetilde{\mu}^{-3}\big),
\end{aligned} \tag{3.51}$$

where

$$C_{\infty,k}^{\mathrm{WKB}} := \mathrm{i}(a-\mathrm{i}/2)\ln\big(\big(\sqrt{3}+1\big)\alpha_k/2\big), \tag{3.52}$$

and

$$\begin{aligned}
-\mathrm{i}\tau^{2/3}\int_{\widetilde{\mu}_{0,k}}^{\widetilde{\mu}} l_k(\xi)\mathrm{d}\xi \underset{\substack{\tau\to+\infty\\ \widetilde{\mu}\to0}}{=} & \frac{1}{\widetilde{\mu}}\mathrm{i}2\sqrt{2}\alpha_k^3\tau^{2/3} - \mathrm{i}3\sqrt{3}\alpha_k^2\tau^{2/3} - \mathrm{i}2\sqrt{3}\widetilde{\Lambda}^2 + \frac{\mathrm{i}}{2\sqrt{3}}\big((a-\mathrm{i}/2) \\
& + \alpha_k^{-2}\tau^{2/3}\hat{h}_0(\tau)\big)\left(\frac{1}{3}\ln\tau - \ln\widetilde{\Lambda} + \ln\big(3\alpha_k\mathrm{e}^{-\mathrm{i}\pi k}\big)\right) + C_{0,k}^{\mathrm{WKB}} \\
& + \mathcal{O}\left(\frac{\tau^{-1/3}}{\widetilde{\Lambda}}\big(\mathfrak{c}_{4,k} + \mathfrak{c}_{5,k}\tau^{2/3}\hat{h}_0(\tau) + \mathfrak{c}_{6,k}\big(\tau^{2/3}\hat{h}_0(\tau)\big)^2\big)\right) \\
& + \mathcal{O}\big(\tau^{-1/3}\widetilde{\Lambda}^3\big) + \mathcal{O}\big(\tau^{-1/3}\widetilde{\Lambda}\big) + \mathcal{O}\big(\tau^{2/3}\big(\hat{h}_0(\tau)\big)^2\widetilde{\mu}^3\big),
\end{aligned} \tag{3.53}$$

where

$$C_{0,k}^{\mathrm{WKB}} := -\mathrm{i}(a-\mathrm{i}/2)\ln\big(\big(\sqrt{3}+1\big)/\sqrt{2}\big), \tag{3.54}$$

and $\mathfrak{c}_{m,k}$, $m = 1, 2, \ldots, 6$, are $\mathcal{O}(1)$.

Proof. Consequence of Corollary 3.10, equation (3.44), upon choosing consistently the corresponding branches in equations (3.31) and (3.45) and taking the limits $\widetilde{\mu} \to \infty$ and $\widetilde{\mu} \to 0$: the error estimate $\mathcal{O}(\mathcal{E}_k(\xi))$ in equation (3.44) is given in equation (3.32); in particular, from the last two lines of equation (3.32),

$$\mathcal{O}\big(\tau^{2/3}\mathcal{E}_k(\widetilde{\mu})\big) \underset{\substack{\tau\to+\infty \\ \widetilde{\mu}\to\infty}}{=} \mathcal{O}\big(\tau^{-2/3}\widetilde{\mu}^{-3}\big) \qquad \text{and} \qquad \mathcal{O}\big(\tau^{2/3}\mathcal{E}_k(\widetilde{\mu})\big) \underset{\substack{\tau\to+\infty \\ \widetilde{\mu}\to 0}}{=} \mathcal{O}\big(\tau^{2/3}\big(\hat{h}_0(\tau)\big)^2\widetilde{\mu}^3\big),$$

which implies the results stated in the corollary. ■

Proposition 3.12. *Let $T(\widetilde{\mu})$ be given in equation (3.27), with $\mathcal{A}(\widetilde{\mu})$ defined by equation (3.4) and $l_k^2(\widetilde{\mu})$, $k = \pm 1$, given in equation (3.29); then,*

$$\int_{\widetilde{\mu}_{0,k}}^{\widetilde{\mu}} \operatorname{diag}\big((T(\xi))^{-1}\partial_\xi T(\xi)\big)\,\mathrm{d}\xi \underset{\tau\to+\infty}{=} \left(\mathcal{I}_{\tau,k}(\widetilde{\mu}) + \mathcal{O}(\mathcal{E}_{T,k}(\widetilde{\mu})) + \mathcal{O}(\mathcal{E}_{T,k}(\widetilde{\mu}_{0,k}))\right)\sigma_3, \tag{3.55}$$

where, for $\delta > 0$, $\widetilde{\mu}, \widetilde{\mu}_{0,k} \in \mathbb{C} \setminus \big(\mathbb{O}_{\tau^{-1/3+\delta}}(\pm\alpha_k) \cup \mathbb{O}_{\tau^{-2/3+2\delta}}\big(\pm\mathrm{i}\sqrt{2}\alpha_k\big) \cup \{0,\infty\}\big)$ and the path of integration lies in the corresponding canonical domain,

$$\mathcal{I}_{\tau,k}(\widetilde{\mu}) = \mathfrak{p}_k(\tau)(F_{\tau,k}(\widetilde{\mu}) - F_{\tau,k}(\widetilde{\mu}_{0,k})), \tag{3.56}$$

with

$$\mathfrak{p}_k(\tau) := \frac{\alpha_k^2\left(-2 + \tilde{r}_0(\tau)\tau^{-1/3} + 2(1+v_0(\tau)\tau^{-1/3})^2\right) - (a - \mathrm{i}/2)\tau^{-2/3}}{8\big(-2+\tilde{r}_0(\tau)\tau^{-1/3}\big)\big(1+v_0(\tau)\tau^{-1/3}\big)}, \tag{3.57}$$

$$\begin{aligned} F_{\tau,k}(\xi) :={}& \frac{2}{\xi^2-\alpha_k^2} + \frac{2}{3\sqrt{3}\alpha_k^2}\ln\left(\left(\frac{3^{1/2}\big(\xi^2+2\alpha_k^2\big)^{1/2} - \xi + 2\alpha_k}{3^{1/2}\big(\xi^2+2\alpha_k^2\big)^{1/2} + \xi + 2\alpha_k}\right)\left(\frac{\xi-\alpha_k}{\xi+\alpha_k}\right)\right) \\ &- \frac{2}{3\alpha_k^2}\frac{\xi\big(\xi^2+2\alpha_k^2\big)^{1/2}}{\xi^2-\alpha_k^2}, \end{aligned} \tag{3.58}$$

and

$$\mathcal{E}_{T,k}(\xi) := \begin{cases} \mathfrak{p}_k(\tau)\left(\frac{\mathfrak{c}^{\blacklozenge}_{1,k}\tilde{r}_0(\tau)\tau^{-1/3}+\mathfrak{c}^{\blacklozenge}_{2,k}\hat{\mathfrak{f}}_{1,k}(\tau)}{(\xi\mp\alpha_k)^2} + \frac{\mathfrak{c}^{\blacklozenge}_{3,k}\tilde{r}_0(\tau)\tau^{-1/3}}{\xi\mp\alpha_k}\right), & \xi \in \mathbb{U}_k^1, \\ \mathfrak{p}_k(\tau)\hat{\mathfrak{f}}_{3,k}(\tau)\left(\frac{\mathfrak{c}^{\blacklozenge}_{4,k}}{(\xi\mp\mathrm{i}\sqrt{2}\alpha_k)^{1/2}} + \mathfrak{c}^{\blacklozenge}_{5,k}\ln\big(\xi\mp\mathrm{i}\sqrt{2}\alpha_k\big)\right), & \xi \in \mathbb{U}_k^2, \\ \mathfrak{p}_k(\tau)\xi^{-4}\big(\mathfrak{c}^{\blacklozenge}_{6,k}\tilde{r}_0(\tau)\tau^{-1/3} + \mathcal{O}\big((\mathfrak{c}^{\blacklozenge}_{7,k}\tilde{r}_0(\tau)\tau^{-1/3} & \\ \quad +\mathfrak{c}^{\blacklozenge}_{8,k}\tau^{-2/3})\xi^{-2}\big)\big), & \xi\to\infty, \\ \mathfrak{p}_k(\tau)\tilde{r}_0(\tau)\tau^{-1/3}\xi^2(\mathfrak{c}^{\blacklozenge}_{9,k} + \mathcal{O}(\xi)), & \xi \to 0, \end{cases} \tag{3.59}$$

where $\mathbb{U}_k^1 := \mathbb{O}_{\tau^{-1/3+\delta_k}}(\pm\alpha_k)$, $\mathbb{U}_k^2 := \mathbb{O}_{\tau^{-2/3+2\delta_k}}\big(\pm\mathrm{i}\sqrt{2}\alpha_k\big)$, the parameter δ_k satisfies (cf. Corollary 3.10) $0 < \delta < \delta_k < 1/9$, the functions $\hat{\mathfrak{f}}_{1,k}(\tau)$ and $\hat{\mathfrak{f}}_{3,k}(\tau)$ are given in equation (3.74), and $\mathfrak{c}^{\blacklozenge}_{m,k}$, $m = 1,2,\dots,9$, are $\mathcal{O}(1)$.

Proof. From equations (3.4), (3.15), and (3.33)–(3.35), one shows that

$$\begin{aligned} 2l_k(\xi)(\mathrm{i}\mathcal{A}_{11}(\xi) + l_k(\xi)) \underset{\tau\to+\infty}{=}{}& \mathcal{P}_{\infty,k}(\xi) + \mathcal{P}_{1,k}(\xi)\Delta_{k,\tau}(\xi) + \mathcal{O}\big(l_{k,\infty}^2(\xi)\Delta_{k,\tau}^2(\xi)\big) \\ &+ \mathcal{O}\left(l_{k,\infty}(\xi)\Delta_{k,\tau}^2(\xi)\left(2\xi + \frac{(\varepsilon b)^{1/3}}{2\xi}(-2+\hat{r}_0(\tau))\right)\right), \end{aligned} \tag{3.60}$$

where

$$\mathcal{P}_{\infty,k}(\xi):=2l_{k,\infty}^{2}(\xi)+2l_{k,\infty}(\xi)\left(2\xi+\frac{(\varepsilon b)^{1/3}}{2\xi}(-2+\hat{r}_0(\tau))\right),$$

$$\mathcal{P}_{1,k}(\xi):=2l_{k,\infty}^{2}(\xi)+l_{k,\infty}(\xi)\left(2\xi+\frac{(\varepsilon b)^{1/3}}{2\xi}(-2+\hat{r}_0(\tau))\right), \tag{3.61}$$

and, via equations (3.4), (3.10), (3.15), and (3.16),

$$\mathcal{A}_{12}(\xi)\partial_{\xi}\mathcal{A}_{21}(\xi)-\mathcal{A}_{21}(\xi)\partial_{\xi}\mathcal{A}_{12}(\xi)=-\frac{4(\varepsilon b)^{2/3}}{\xi^3}\left(\frac{2(1+\hat{u}_0(\tau))^2+(-2+\hat{r}_0(\tau))}{2(1+\hat{u}_0(\tau))}\right)+\frac{4(\varepsilon b)^{1/3}(a-\mathrm{i}/2)\tau^{-2/3}}{\xi^3(1+\hat{u}_0(\tau))}. \tag{3.62}$$

Substituting equations (3.60) and (3.62) into equation (3.28) and expanding $(2l_k(\xi)(\mathrm{i}\mathcal{A}_{11}(\xi)+l_k(\xi)))^{-1}$ into a series of powers of $\Delta_{k,\tau}(\xi)$, one arrives at (cf. equation (3.25))

$$\int_{\widetilde{\mu}_{0,k}}^{\widetilde{\mu}}\operatorname{diag}\left((T(\xi))^{-1}\partial_{\xi}T(\xi)\right)\mathrm{d}\xi\underset{\tau\to+\infty}{=}\left(\varkappa_k(\tau)\int_{\widetilde{\mu}_{0,k}}^{\widetilde{\mu}}\frac{1}{\xi^3\mathcal{P}_{\infty,k}(\xi)}\mathrm{d}\xi+\mathcal{O}\left(\varkappa_k(\tau)\int_{\widetilde{\mu}_{0,k}}^{\widetilde{\mu}}\frac{\xi^3\mathcal{P}_{1,k}(\xi)\Delta_{k,\tau}(\xi)}{(\xi^3\mathcal{P}_{\infty,k}(\xi))^2}\mathrm{d}\xi\right)\right)\sigma_3, \tag{3.63}$$

where

$$\varkappa_k(\tau):=(\varepsilon b)^{2/3}\left(\frac{2(1+\hat{u}_0(\tau))^2+(-2+\hat{r}_0(\tau))}{1+\hat{u}_0(\tau)}\right)-\frac{2(\varepsilon b)^{1/3}(a-\mathrm{i}/2)\tau^{-2/3}}{1+\hat{u}_0(\tau)}.$$

Via equations (3.33) and (3.61), a calculation reveals that

$$\frac{\varkappa_k(\tau)}{\xi^3\mathcal{P}_{\infty,k}(\xi)}=\mathfrak{p}_k(\tau)\left(\frac{\xi\big(\xi\big(4\xi^2+(\varepsilon b)^{1/3}(-2+\hat{r}_0(\tau))\big)-4\big(\xi^2-\alpha_k^2\big)\big(\xi^2+2\alpha_k^2\big)^{1/2}\big)}{\big(\xi^2-\alpha_k^2\big)\big(\xi^2+2\alpha_k^2\big)^{1/2}\big(\xi^2+\hat{\mathfrak{z}}_k^{+}(\tau)\big)\big(\xi^2+\hat{\mathfrak{z}}_k^{-}(\tau)\big)}\right), \tag{3.64}$$

where $\mathfrak{p}_k(\tau)$ is defined by equation (3.57), and

$$\hat{\mathfrak{z}}_k^{\pm}(\tau):=\frac{(\varepsilon b)^{1/3}}{4(-2+\hat{r}_0(\tau))}\left(\left(\frac{-2+\hat{r}_0(\tau)}{2}\right)^2-3\mathrm{e}^{\mathrm{i}\pi k/3}\mp\sqrt{\left(\left(\frac{-2+\hat{r}_0(\tau)}{2}\right)^2-3\mathrm{e}^{\mathrm{i}\pi k/3}\right)^2+8(-2+\hat{r}_0(\tau))}\right). \tag{3.65}$$

One shows from equations (3.15) and (3.16), the conditions (3.17), and the definition (3.65) that

$$\hat{\mathfrak{z}}_k^{\pm}(\tau)\underset{\tau\to+\infty}{=}\frac{(\varepsilon b)^{1/3}\mathrm{e}^{-\mathrm{i}\pi k/3}}{2}\left(1+\left(\frac{1\pm\sqrt{3}}{4}\right)\tilde{r}_0(\tau)\tau^{-1/3}+\left(\frac{3\sqrt{3}\pm 5}{16\sqrt{3}}\right)\left(\tilde{r}_0(\tau)\tau^{-1/3}\right)^2+\mathcal{O}\big(\big(\tilde{r}_0(\tau)\tau^{-1/3}\big)^3\big)\right),$$

whence, via equation (3.64), the first term on the right-hand side of equation (3.63) can be presented as follows:

$$\varkappa_k(\tau)\int_{\widetilde{\mu}_{0,k}}^{\widetilde{\mu}}\frac{1}{\xi^3\mathcal{P}_{\infty,k}(\xi)}\mathrm{d}\xi\underset{\tau\to+\infty}{=}\mathcal{I}_{\tau,k}(\widetilde{\mu})+\mathcal{I}_{A,k}(\widetilde{\mu})+\mathcal{O}(\mathcal{I}_{B,k}(\widetilde{\mu})),$$

where

$$\mathcal{I}_{\tau,k}(\widetilde{\mu}) := \mathfrak{p}_k(\tau) \int_{\widetilde{\mu}_{0,k}}^{\widetilde{\mu}} \left(\frac{4\xi^2\big(\xi^2+2\alpha_k^2\big)^{1/2}}{\big(\xi^2+2\alpha_k^2\big)\big(\xi^2-\alpha_k^2\big)^2} - \frac{4\xi}{\big(\xi^2-\alpha_k^2\big)^2} \right) \mathrm{d}\xi, \tag{3.66}$$

$$\mathcal{I}_{A,k}(\widetilde{\mu}) := \mathfrak{p}_k(\tau)\tilde{r}_0(\tau)\tau^{-1/3} \int_{\widetilde{\mu}_{0,k}}^{\widetilde{\mu}} \left(\frac{4\alpha_k^2\xi^2\big(\xi^2+2\alpha_k^2\big)^{1/2}}{\big(\xi^2+2\alpha_k^2\big)\big(\xi^2-\alpha_k^2\big)^3} - \frac{2\alpha_k^2\xi}{\big(\xi^2-\alpha_k^2\big)^3} \right) \mathrm{d}\xi, \tag{3.67}$$

$$\mathcal{I}_{B,k}(\widetilde{\mu}) := \mathfrak{p}_k(\tau)\big(\tilde{r}_0(\tau)\tau^{-1/3}\big)^2 \int_{\widetilde{\mu}_{0,k}}^{\widetilde{\mu}} \left(\frac{\alpha_k^4\xi^2\big(\xi^2+2\alpha_k^2\big)^{1/2}}{\big(\xi^2+2\alpha_k^2\big)\big(\xi^2-\alpha_k^2\big)^4} - \frac{4\xi^3}{\big(\xi^2-\alpha_k^2\big)^4} \right.$$
$$\left. + \frac{4\xi^4\big(\xi^2+2\alpha_k^2\big)^{1/2}}{\big(\xi^2+2\alpha_k^2\big)\big(\xi^2-\alpha_k^2\big)^4} \right) \mathrm{d}\xi. \tag{3.68}$$

A partial fraction decomposition shows that

$$\frac{\xi^2}{\big(\xi^2+2\alpha_k^2\big)\big(\xi^2-\alpha_k^2\big)^2} = \frac{\alpha_k^{-3}}{36}\frac{1}{\xi-\alpha_k} + \frac{\alpha_k^{-2}}{12}\frac{1}{(\xi-\alpha_k)^2} - \frac{\alpha_k^{-3}}{36}\frac{1}{\xi+\alpha_k}$$
$$+ \frac{\alpha_k^{-2}}{12}\frac{1}{(\xi+\alpha_k)^2} - \frac{2\alpha_k^{-2}}{9}\frac{1}{\xi^2+2\alpha_k^2}; \tag{3.69}$$

substituting equation (3.69) into equation (3.66) and integrating, one arrives at equations (3.56)–(3.58).

Equations (3.67) and (3.68) contribute to the error function, $\mathcal{E}_{T,k}(\cdot)$, in equation (3.55); therefore, only its asymptotics at the turning and the singular points are requisite. Evaluating the integrals in equations (3.67) and (3.68), one shows that

$$\mathcal{I}_{A,k}(\widetilde{\mu}) \underset{\tau\to+\infty}{=} \begin{cases} \mathfrak{p}_k(\tau)\tilde{r}_0(\tau)\tau^{-1/3}\big(\hat{\mathfrak{h}}_{1,k}(\widetilde{\mu})-\hat{\mathfrak{h}}_{1,k}(\widetilde{\mu}_{0,k})\big), & \widetilde{\mu}\in\mathcal{O}_{\tau^{-1/3+\delta_k}}(\pm\alpha_k), \\ \mathfrak{p}_k(\tau)\tilde{r}_0(\tau)\tau^{-1/3}\big(\hat{\mathfrak{h}}_{2,k}(\widetilde{\mu})-\hat{\mathfrak{h}}_{2,k}(\widetilde{\mu}_{0,k})\big), & \widetilde{\mu}\in\mathcal{O}_{\tau^{-2/3+2\delta_k}}\big(\pm\mathrm{i}\sqrt{2}\alpha_k\big), \\ \mathfrak{p}_k(\tau)\tilde{r}_0(\tau)\tau^{-1/3}\big(\hat{\mathfrak{h}}_{3,k}(\widetilde{\mu})-\hat{\mathfrak{h}}_{3,k}(\widetilde{\mu}_{0,k})\big), & \widetilde{\mu}\to\infty, \\ \mathfrak{p}_k(\tau)\tilde{r}_0(\tau)\tau^{-1/3}\big(\hat{\mathfrak{h}}_{4,k}(\widetilde{\mu})-\hat{\mathfrak{h}}_{4,k}(\widetilde{\mu}_{0,k})\big), & \widetilde{\mu}\to 0, \end{cases} \tag{3.70}$$

where

$$\hat{\mathfrak{h}}_{1,k}(\xi) := c_{1,k}^{\flat}(\xi\mp\alpha_k)^{-2} + c_{2,k}^{\flat}(\xi\mp\alpha_k)^{-1} + c_{3,k}^{\flat}\ln(\xi\mp\alpha_k) + \sum_{m\in\mathbb{Z}_+} d_{m,k}^{\flat}(\xi\mp\alpha_k)^m,$$

$$\hat{\mathfrak{h}}_{2,k}(\xi) := \big(\xi\mp\mathrm{i}\sqrt{2}\alpha_k\big)^{1/2} \sum_{m\in\mathbb{Z}_+} c_{m,k}^{\natural}\big(\xi\mp\mathrm{i}\sqrt{2}\alpha_k\big)^m + \sum_{m\in\mathbb{Z}_+} d_{m,k}^{\natural}\big(\xi\mp\mathrm{i}\sqrt{2}\alpha_k\big)^m,$$

$$\hat{\mathfrak{h}}_{3,k}(\xi) := \xi^{-4}\sum_{m\in\mathbb{Z}_+} c_{m,k}^{\sharp,\infty}\xi^{-2m}, \qquad \hat{\mathfrak{h}}_{4,k}(\xi) := \xi^2\sum_{m\in\mathbb{Z}_+} c_{m,k}^{\sharp,0}\xi^m,$$

and $c_{1,k}^{\flat}$, $c_{2,k}^{\flat}$, $c_{3,k}^{\flat}$, $d_{m,k}^{\flat}$, $c_{m,k}^{\natural}$, $d_{m,k}^{\natural}$, $c_{m,k}^{\sharp,\infty}$, and $c_{m,k}^{\sharp,0}$ are $\mathcal{O}(1)$, and

$$\mathcal{I}_{B,k}(\widetilde{\mu}) \underset{\tau\to+\infty}{=} \begin{cases} \mathfrak{p}_k(\tau)\big(\tilde{r}_0(\tau)\tau^{-1/3}\big)^2 \\ \quad\times\big(\hat{\mathfrak{h}}_{5,k}(\widetilde{\mu})-\hat{\mathfrak{h}}_{5,k}(\widetilde{\mu}_{0,k})\big), & \widetilde{\mu}\in\mathcal{O}_{\tau^{-1/3+\delta_k}}(\pm\alpha_k), \\ \mathfrak{p}_k(\tau)\big(\tilde{r}_0(\tau)\tau^{-1/3}\big)^2 \\ \quad\times\big(\hat{\mathfrak{h}}_{6,k}(\widetilde{\mu})-\hat{\mathfrak{h}}_{6,k}(\widetilde{\mu}_{0,k})\big), & \widetilde{\mu}\in\mathcal{O}_{\tau^{-2/3+2\delta_k}}\big(\pm\mathrm{i}\sqrt{2}\alpha_k\big), \\ \mathfrak{p}_k(\tau)\big(\tilde{r}_0(\tau)\tau^{-1/3}\big)^2\big(\hat{\mathfrak{h}}_{7,k}(\widetilde{\mu})-\hat{\mathfrak{h}}_{7,k}(\widetilde{\mu}_{0,k})\big), & \widetilde{\mu}\to\infty, \\ \mathfrak{p}_k(\tau)\big(\tilde{r}_0(\tau)\tau^{-1/3}\big)^2\big(\hat{\mathfrak{h}}_{8,k}(\widetilde{\mu})-\hat{\mathfrak{h}}_{8,k}(\widetilde{\mu}_{0,k})\big), & \widetilde{\mu}\to 0, \end{cases} \tag{3.71}$$

where

$$\hat{\mathfrak{h}}_{5,k}(\xi) := \hat{c}^{\flat}_{1,k}(\xi \mp \alpha_k)^{-3} + \hat{c}^{\flat}_{2,k}(\xi \mp \alpha_k)^{-2} + \hat{c}^{\flat}_{3,k}(\xi \mp \alpha_k)^{-1} + \hat{c}^{\flat}_{4,k}\ln(\xi \mp \alpha_k) + \sum_{m \in \mathbb{Z}_+} \hat{d}^{\flat}_{m,k}(\xi \mp \alpha_k)^m,$$

$$\hat{\mathfrak{h}}_{6,k}(\xi) := \big(\xi \mp \mathrm{i}\sqrt{2}\alpha_k\big)^{1/2} \sum_{m \in \mathbb{Z}_+} \hat{c}^{\natural}_{m,k}\big(\xi \mp \mathrm{i}\sqrt{2}\alpha_k\big)^m + \sum_{m \in \mathbb{Z}_+} \hat{d}^{\natural}_{m,k}\big(\xi \mp \mathrm{i}\sqrt{2}\alpha_k\big)^m,$$

$$\hat{\mathfrak{h}}_{7,k}(\xi) := \xi^{-6} \sum_{m \in \mathbb{Z}_+} \hat{c}^{\sharp,\infty}_{m,k}\xi^{-2m}, \qquad \hat{\mathfrak{h}}_{8,k}(\xi) := \xi^3 \sum_{m \in \mathbb{Z}_+} \hat{c}^{\sharp,0}_{m,k}\xi^m,$$

and $\hat{c}^{\flat}_{1,k}$, $\hat{c}^{\flat}_{2,k}$, $\hat{c}^{\flat}_{3,k}$, $\hat{c}^{\flat}_{4,k}$, $d^{\flat}_{m,k}$, $\hat{c}^{\natural}_{m,k}$, $\hat{d}^{\natural}_{m,k}$, $\hat{c}^{\sharp,\infty}_{m,k}$, and $\hat{c}^{\sharp,0}_{m,k}$ are $\mathcal{O}(1)$.

One now estimates the second term on the right-hand side of equation (3.63). From equations (3.33)–(3.35), it follows, after simplification, that

$$\begin{aligned}
&\int_{\widetilde{\mu}_{0,k}}^{\widetilde{\mu}} \frac{\xi^3\mathcal{P}_{1,k}(\xi)\Delta_{k,\tau}(\xi)}{\big(\xi^3\mathcal{P}_{\infty,k}(\xi)\big)^2}\mathrm{d}\xi \\
&\quad = \int_{\widetilde{\mu}_{0,k}}^{\widetilde{\mu}} \frac{\xi\big(\xi\big(4\xi^2+(\varepsilon b)^{1/3}(-2+\hat{r}_0(\tau))\big)+8\big(\xi^2-\alpha_k^2\big)\big(\xi^2+2\alpha_k^2\big)^{1/2}\big)}{\big(\xi\big(4\xi^2+(\varepsilon b)^{1/3}(-2+\hat{r}_0(\tau))\big)+4\big(\xi^2-\alpha_k^2\big)\big(\xi^2+2\alpha_k^2\big)^{1/2}\big)^2} \\
&\qquad \times \frac{\big(\xi^2\hat{h}_0(\tau)+\xi^4(a-\mathrm{i}/2)\tau^{-2/3}\big)}{4\big(\xi^2-\alpha_k^2\big)^3\big(\xi^2+2\alpha_k^2\big)^{3/2}}\mathrm{d}\xi.
\end{aligned} \tag{3.72}$$

Evaluating the integral in equation (3.72), a lengthy calculation shows that its asymptotics at the turning and the singular points are given by

$$\varkappa_k(\tau)\int_{\widetilde{\mu}_{0,k}}^{\widetilde{\mu}} \frac{\xi^3\mathcal{P}_{1,k}(\xi)\Delta_{k,\tau}(\xi)}{\big(\xi^3\mathcal{P}_{\infty,k}(\xi)\big)^2}\mathrm{d}\xi \underset{\tau\to+\infty}{=} \begin{cases} \hat{\mathfrak{h}}_{9,k}(\widetilde{\mu}) - \hat{\mathfrak{h}}_{9,k}(\widetilde{\mu}_{0,k}), & \widetilde{\mu} \in \mathcal{O}_{\tau^{-1/3+\delta_k}}(\pm\alpha_k), \\ \hat{\mathfrak{h}}_{10,k}(\widetilde{\mu}) - \hat{\mathfrak{h}}_{10,k}(\widetilde{\mu}_{0,k}), & \widetilde{\mu} \in \mathcal{O}_{\tau^{-2/3+2\delta_k}}\big(\pm\mathrm{i}\sqrt{2}\alpha_k\big), \\ \hat{\mathfrak{h}}_{11,k}(\widetilde{\mu}) - \hat{\mathfrak{h}}_{11,k}(\widetilde{\mu}_{0,k}), & \widetilde{\mu} \to \infty, \\ \hat{\mathfrak{h}}_{12,k}(\widetilde{\mu}) - \hat{\mathfrak{h}}_{12,k}(\widetilde{\mu}_{0,k}), & \widetilde{\mu} \to 0, \end{cases} \tag{3.73}$$

where

$$\begin{aligned}
\hat{\mathfrak{h}}_{9,k}(\xi) &:= \tilde{c}^{\sharp}_{1,k}\mathfrak{p}_k(\tau)\hat{\mathfrak{f}}_{1,k}(\tau)(\xi \mp \alpha_k)^{-2} + \mathfrak{p}_k(\tau)\big(\tilde{c}^{\sharp}_{2,k}\hat{\mathfrak{f}}_{2,k}(\tau) \\
&\quad + \tilde{c}^{\sharp}_{3,k}\tilde{r}_0(\tau)\tau^{-1/3}\hat{\mathfrak{f}}_{1,k}(\tau)\big)(\xi \mp \alpha_k)^{-3}, \\
\hat{\mathfrak{h}}_{10,k}(\xi) &:= \tilde{c}^{\sharp}_{4,k}\mathfrak{p}_k(\tau)\hat{\mathfrak{f}}_{3,k}(\tau)\big(\xi \mp \mathrm{i}\sqrt{2}\alpha_k\big)^{-1/2} + \tilde{c}^{\sharp}_{5,k}\mathfrak{p}_k(\tau)\hat{\mathfrak{f}}_{3,k}(\tau)\ln\big(\xi \mp \mathrm{i}\sqrt{2}\alpha_k\big), \\
\hat{\mathfrak{h}}_{11,k}(\xi) &:= \mathfrak{p}_k(\tau)\tau^{-2/3}\xi^{-6}\big(\tilde{c}^{\sharp}_{6,k} + \xi^{-2}\big(\tilde{c}^{\sharp}_{7,k} + \tilde{c}^{\sharp}_{8,k}\tau^{2/3}\hat{h}_0(\tau)\big) \\
&\quad + \mathcal{O}\big(\tilde{r}_0(\tau)\tau^{-1/3}\big(\tilde{c}^{\sharp}_{9,k} + \xi^{-2}\big(\tilde{c}^{\sharp}_{10,k} + \tilde{c}^{\sharp}_{11,k}\tau^{2/3}\hat{h}_0(\tau)\big)\big)\big)\big), \\
\hat{\mathfrak{h}}_{12,k}(\xi) &:= \mathfrak{p}_k(\tau)\tau^{-2/3}\xi^4\big(\tilde{c}^{\sharp}_{12,k}\tau^{2/3}\hat{h}_0(\tau) + \xi\tilde{c}^{\sharp}_{13,k}\tau^{2/3}\hat{h}_0(\tau) + \xi^2\big(\tilde{c}^{\sharp}_{14,k} + \tilde{c}^{\sharp}_{15,k}\tau^{2/3}\hat{h}_0(\tau)\big) \\
&\quad + \mathcal{O}\big(\tilde{r}_0(\tau)\tau^{-1/3}\big(\tilde{c}^{\sharp}_{16,k}\tau^{2/3}\hat{h}_0(\tau) + \xi\tilde{c}^{\sharp}_{17,k}\tau^{2/3}\hat{h}_0(\tau) + \xi^2\big(\tilde{c}^{\sharp}_{18,k} \\
&\quad + \tilde{c}^{\sharp}_{19,k}\tau^{2/3}\hat{h}_0(\tau)\big)\big)\big)\big),
\end{aligned}$$

and $\tilde{c}^{\sharp}_{m,k}$, $m = 1, 2, \dots, 19$, are $\mathcal{O}(1)$, and

$$\hat{\mathfrak{f}}_{j,k}(\tau) = \left((a-\mathrm{i}/2) + \frac{2\hat{s}(j)\hat{h}_0(\tau)\tau^{2/3}}{(3+(-1)^{j+1})\alpha_k^2}\right)\tau^{-2/3}, \qquad j = 1, 2, 3, \tag{3.74}$$

where $\hat{s}(1)=\hat{s}(2)=+1$ and $\hat{s}(3)=-1$. Thus, assembling the error estimates (3.70), (3.71), and (3.73), and retaining only leading-order terms, one arrives at the error function defined by equation (3.59). ■

Corollary 3.13. *Set* $\widetilde{\mu}_{0,k}=\alpha_k+\tau^{-1/3}\widetilde{\Lambda}$, $k=\pm 1$, *where* $\widetilde{\Lambda}\underset{\tau\to+\infty}{=}\mathcal{O}(\tau^{\delta_k})$, $0<\delta<\delta_k<1/9$; *then,*

$$\begin{aligned}\int_{\widetilde{\mu}_{0,k}}^{\widetilde{\mu}}\operatorname{diag}\big((T(\xi))^{-1}\partial_{\xi}T(\xi)\big)\mathrm{d}\xi \underset{\tau\to+\infty}{=}&\Bigg(\mathfrak{p}_k(\tau)\big(F_{\tau,k}(\widetilde{\mu})+F^{\sharp}_{\tau,k}(\tau)\big)+\mathcal{O}(\mathcal{E}_{T,k}(\widetilde{\mu}))\\&+\mathcal{O}\Bigg(\big(\mathfrak{c}_{3,k}\tau^{-1/3}+\mathfrak{c}_{4,k}(\tilde{r}_0(\tau)+4v_0(\tau))\big)\\&\times\left(\frac{\mathfrak{c}_{1,k}\tau^{-1/3}+\mathfrak{c}_{2,k}\tilde{r}_0(\tau)}{\widetilde{\Lambda}^2}\right)\Bigg)\Bigg)\sigma_3,\end{aligned}\tag{3.75}$$

where $\mathfrak{p}_k(\tau)$, $F_{\tau,k}(\xi)$ *and* $\mathcal{E}_{T,k}(\xi)$ *are defined by equations* (3.57), (3.58), *and* (3.59), *respectively,*

$$\begin{aligned}F^{\sharp}_{\tau,k}(\tau):=&-\frac{\tau^{1/3}}{\alpha_k\widetilde{\Lambda}}\left(\frac{\sqrt{3}\mp 1}{\sqrt{3}}\right)\mp\frac{2}{3\sqrt{3}\alpha_k^2}\left(-\frac{1}{3}\ln\tau+\ln\widetilde{\Lambda}\right)\pm\frac{(5\pm 3\sqrt{3})}{6\sqrt{3}\alpha_k^2}\\&\pm\frac{2}{3\sqrt{3}\alpha_k^2}\ln(3\alpha_k),\end{aligned}\tag{3.76}$$

with the upper (resp., lower) signs taken according to the branch of the square-root function $\lim_{\xi^2\to+\infty}(\xi^2+2\alpha_k^2)^{1/2}=+\infty$ *(resp.,* $\lim_{\xi^2\to+\infty}(\xi^2+2\alpha_k^2)^{1/2}=-\infty$*), and* $\mathfrak{c}_{m,k}$, $m=1,2,3,4$, *are* $\mathcal{O}(1)$.

Proof. Substituting $\widetilde{\mu}_{0,k}$, as given in the corollary, for the argument of the functions $F_{\tau,k}(\xi)$ and $\mathcal{E}_{T,k}(\xi)$ (cf. equation (3.58) and the first line of equation (3.59), respectively) and expanding with respect to the small parameter $\tau^{-1/3}\widetilde{\Lambda}$, one arrives at the following estimates:

$$-F_{\tau,k}(\widetilde{\mu}_{0,k})\underset{\tau\to+\infty}{=}F^{\sharp}_{\tau,k}(\tau)+\mathcal{O}\big(\tau^{-1/3}\widetilde{\Lambda}\big),\tag{3.77}$$

where $F^{\sharp}_{\tau,k}(\tau)$ is defined by equation (3.76), and

$$\mathcal{O}(\mathcal{E}_{T,k}(\widetilde{\mu}_{0,k}))\underset{\tau\to+\infty}{=}\mathcal{O}\left(\frac{\mathfrak{p}_k(\tau)\tilde{r}_0(\tau)}{\tau^{-1/3}\widetilde{\Lambda}^2}\right)+\mathcal{O}\left(\frac{\mathfrak{p}_k(\tau)\hat{\mathfrak{f}}_{1,k}(\tau)}{\tau^{-2/3}\widetilde{\Lambda}^2}\right)+\mathcal{O}\left(\frac{\mathfrak{p}_k(\tau)\tilde{r}_0(\tau)}{\widetilde{\Lambda}}\right).\tag{3.78}$$

From the conditions (3.17) and the definitions (3.57) and (3.74) (for $j=1$), one shows that

$$\begin{aligned}\mathfrak{p}_k(\tau)\underset{\tau\to+\infty}{=}&\mathfrak{p}_k^{\infty}(\tau)+\mathcal{O}\big((\tilde{r}_0(\tau)-2v_0(\tau))\tau^{-1}\big)\\&+\mathcal{O}\big(\big((\tilde{r}_0(\tau)-2v_0(\tau))(\tilde{r}_0(\tau)+4v_0(\tau))+4v_0^2(\tau)\big)\tau^{-2/3}\big),\end{aligned}\tag{3.79}$$

where

$$\mathfrak{p}_k^{\infty}(\tau):=\frac{\tau^{-1/3}}{16}\big(-\alpha_k^2(\tilde{r}_0(\tau)+4v_0(\tau))+(a-\mathrm{i}/2)\tau^{-1/3}\big),\tag{3.80}$$

$$\begin{aligned}\hat{\mathfrak{f}}_{1,k}(\tau)\underset{\tau\to+\infty}{=}&\tau^{-2/3}\bigg(\frac{1}{2}(a-\mathrm{i}/2)+\mathcal{O}\big(v_0(\tau)\tau^{-1/3}\big)\\&+\mathcal{O}\big(8v_0^2(\tau)+4v_0(\tau)\tilde{r}_0(\tau)-(\tilde{r}_0(\tau))^2\big)\bigg);\end{aligned}\tag{3.81}$$

thus, from the conditions (3.17) and the asymptotics (3.78)–(3.81), it follows that, for $c_{m,k}$, $m=1,2,\dots,6$, that are $\mathcal{O}(1)$,

$$\begin{aligned}\mathcal{O}(\mathcal{E}_{T,k}(\widetilde{\mu}_{0,k})) \underset{\tau\to+\infty}{=} &\mathcal{O}\left(\left(\frac{c_{1,k}\tau^{-1/3}+c_{2,k}\tilde{r}_0(\tau)}{\widetilde{\Lambda}^2}\right)\left(c_{3,k}\tau^{-1/3}+c_{4,k}(\tilde{r}_0(\tau)+4v_0(\tau))\right)\right)\\ &+\mathcal{O}\left(\frac{\tau^{-1/3}}{\widetilde{\Lambda}}\left(c_{5,k}\tilde{r}_0(\tau)\tau^{-1/3}+c_{6,k}\tilde{r}_0(\tau)(\tilde{r}_0(\tau)+4v_0(\tau))\right)\right)\\ \underset{\tau\to+\infty}{=}&\mathcal{O}\big(\tau^{-2/3}\widetilde{\Lambda}^{-2}\big)+\mathcal{O}\big(\tau^{-1}\widetilde{\Lambda}^{-1}\big). \end{aligned}\tag{3.82}$$

From the conditions (3.17), equation (3.56), and the asymptotics (3.77) and (3.79), it follows that

$$\mathcal{I}_{\tau,k}(\widetilde{\mu}) \underset{\tau\to+\infty}{=} \mathfrak{p}_k(\tau)\big(F_{\tau,k}(\widetilde{\mu})+F^{\sharp}_{\tau,k}(\tau)\big)+\mathcal{O}\big((\tilde{r}_0(\tau)+4v_0(\tau))\tau^{-2/3}\widetilde{\Lambda}\big)+\mathcal{O}\big(\tau^{-1}\widetilde{\Lambda}\big). \tag{3.83}$$

Therefore, via the asymptotic estimates (3.82) and (3.83), and the fact that $\widetilde{\Lambda}\underset{\tau\to+\infty}{=}\mathcal{O}\big(\tau^{\delta_k}\big)$, $0<\delta<\delta_k<1/9$, the result stated in the corollary (cf. equation (3.75)) is a consequence of Proposition 3.12 (cf. equation (3.55)), upon retaining only leading-order contributions. ■

Corollary 3.14. *Let the conditions stated in Corollary 3.13 be valid; then, for the branch of $l_k(\xi)$, $k=\pm1$, that is positive for large and small positive ξ,*

$$\begin{aligned}&\int_{\widetilde{\mu}_{0,k}}^{\widetilde{\mu}} \operatorname{diag}\left((T(\xi))^{-1}\partial_\xi T(\xi)\right)\mathrm{d}\xi\\ &\underset{\substack{\tau\to+\infty\\ \widetilde{\mu}\to\infty}}{=}\Bigg(\mathfrak{p}_k(\tau)F^{\sharp,\infty}_{\tau,k}(\tau)+\mathcal{O}\bigg(\bigg(\frac{\mathfrak{c}_{1,k}\tau^{-1/3}+\mathfrak{c}_{2,k}\tilde{r}_0(\tau)}{\widetilde{\Lambda}^2}\bigg)\\ &\qquad\times\left(\mathfrak{c}_{3,k}\tau^{-1/3}+\mathfrak{c}_{4,k}(\tilde{r}_0(\tau)+4v_0(\tau))\right)\bigg)\\ &\qquad+\mathcal{O}\big(\widetilde{\mu}^{-2}\tau^{-1/3}\big(\mathfrak{c}_{5,k}\tau^{-1/3}+\mathfrak{c}_{6,k}(\tilde{r}_0(\tau)+4v_0(\tau))\big)\big)\Bigg)\sigma_3,\end{aligned}\tag{3.84}$$

where $\mathfrak{p}_k(\tau)$ is defined by equation (3.57),

$$\begin{aligned}F^{\sharp,\infty}_{\tau,k}(\tau):=&-\frac{(\sqrt{3}-1)\tau^{1/3}}{\sqrt{3}\alpha_k\widetilde{\Lambda}}-\frac{2}{3\sqrt{3}\alpha_k^2}\left(-\frac{1}{3}\ln\tau+\ln\widetilde{\Lambda}\right)+\frac{5-\sqrt{3}}{6\sqrt{3}\alpha_k^2}\\ &+\frac{2}{3\sqrt{3}\alpha_k^2}\ln\big(3(2-\sqrt{3})\alpha_k\big),\end{aligned}\tag{3.85}$$

and

$$\begin{aligned}&\int_{\widetilde{\mu}_{0,k}}^{\widetilde{\mu}} \operatorname{diag}\big((T(\xi))^{-1}\partial_\xi T(\xi)\big)\mathrm{d}\xi\\ &\underset{\substack{\tau\to+\infty\\ \widetilde{\mu}\to 0}}{=}\Bigg(\mathfrak{p}_k(\tau)F^{\sharp,0}_{\tau,k}(\tau)+\mathcal{O}\bigg(\bigg(\frac{\mathfrak{c}_{7,k}\tau^{-1/3}+\mathfrak{c}_{8,k}\tilde{r}_0(\tau)}{\widetilde{\Lambda}^2}\bigg)\\ &\qquad\times\left(\mathfrak{c}_{9k}\tau^{-1/3}+\mathfrak{c}_{10,k}(\tilde{r}_0(\tau)+4v_0(\tau))\right)\bigg)\end{aligned}$$

$$+\,\mathcal{O}\big(\widetilde{\mu}^{2}\tau^{-1/3}\big(\mathfrak{c}_{11,k}\tau^{-1/3}+\mathfrak{c}_{12,k}(\tilde{r}_{0}(\tau)+4v_{0}(\tau))\big)\big)\Big)\sigma_{3}, \tag{3.86}$$

where

$$F^{\sharp,0}_{\tau,k}(\tau):=-\frac{(\sqrt{3}+1)\tau^{1/3}}{\sqrt{3}\alpha_{k}\widetilde{\Lambda}}+\frac{2}{3\sqrt{3}\alpha_{k}^{2}}\left(-\frac{1}{3}\ln\tau+\ln\widetilde{\Lambda}\right)-\frac{(5+9\sqrt{3})}{6\sqrt{3}\alpha_{k}^{2}}+\frac{2}{3\sqrt{3}\alpha_{k}^{2}}\ln\big(\mathrm{e}^{\mathrm{i}k\pi}/3\alpha_{k}\big), \tag{3.87}$$

and $\mathfrak{c}_{m,k}$*,* $m=1,2,\dots,12$*, are* $\mathcal{O}(1)$*.*

Proof. Choosing consistently the corresponding branches in equations (3.58) and (3.76), and via the third and fourth lines of equation (3.59), respectively, one shows, via the conditions (3.17) and the asymptotics (3.79), that (cf. equation (3.75))

$$F_{\tau,k}(\widetilde{\mu})\underset{\substack{\tau\to+\infty\\ \widetilde{\mu}\to\infty}}{=}-\frac{2}{3\alpha_{k}^{2}}+\frac{2}{3\sqrt{3}\alpha_{k}^{2}}\ln\big(2-\sqrt{3}\big)+\mathcal{O}\big(\widetilde{\mu}^{-2}\big), \tag{3.88}$$

$$F_{\tau,k}(\widetilde{\mu})\underset{\substack{\tau\to+\infty\\ \widetilde{\mu}\to0}}{=}-\frac{2}{\alpha_{k}^{2}}+\frac{2}{3\sqrt{3}\alpha_{k}^{2}}\ln\big(\mathrm{e}^{\mathrm{i}k\pi}\big)+\mathcal{O}\big(\widetilde{\mu}^{2}\big), \tag{3.89}$$

$$\mathcal{O}(\mathcal{E}_{\mathcal{T},k}(\widetilde{\mu}))\underset{\substack{\tau\to+\infty\\ \widetilde{\mu}\to\infty}}{=}\mathcal{O}\big(\widetilde{\mu}^{-4}\tilde{r}_{0}(\tau)(\tilde{r}_{0}(\tau)+4v_{0}(\tau))\tau^{-2/3}\big)+\mathcal{O}\big(\widetilde{\mu}^{-4}\tilde{r}_{0}(\tau)\tau^{-1}\big), \tag{3.90}$$

$$\mathcal{O}(\mathcal{E}_{\mathcal{T},k}(\widetilde{\mu}))\underset{\substack{\tau\to+\infty\\ \widetilde{\mu}\to0}}{=}\mathcal{O}\big(\widetilde{\mu}^{2}\tilde{r}_{0}(\tau)(\tilde{r}_{0}(\tau)+4v_{0}(\tau))\tau^{-2/3}\big)+\mathcal{O}\left(\widetilde{\mu}^{2}\tilde{r}_{0}(\tau)\tau^{-1}\right). \tag{3.91}$$

Via the conditions (3.17), equation (3.76), and the asymptotics (3.79) and (3.88)–(3.91), it follows that (cf. equation (3.75))

$$\mathfrak{p}_{k}(\tau)\big(F_{\tau,k}(\widetilde{\mu})+F^{\sharp}_{\tau,k}(\tau)\big)\underset{\substack{\tau\to+\infty\\ \widetilde{\mu}\to\infty}}{=}\mathfrak{p}_{k}(\tau)F^{\sharp,\infty}_{\tau,k}(\tau)+\mathcal{O}\big(\widetilde{\mu}^{-2}(\tilde{r}_{0}(\tau)+4v_{0}(\tau))\tau^{-1/3}\big)+\mathcal{O}\big(\widetilde{\mu}^{-2}\tau^{-2/3}\big), \tag{3.92}$$

$$\mathfrak{p}_{k}(\tau)\big(F_{\tau,k}(\widetilde{\mu})+F^{\sharp}_{\tau,k}(\tau)\big)\underset{\substack{\tau\to+\infty\\ \widetilde{\mu}\to0}}{=}\mathfrak{p}_{k}(\tau)F^{\sharp,0}_{\tau,k}(\tau)+\mathcal{O}\big(\widetilde{\mu}^{2}(\tilde{r}_{0}(\tau)+4v_{0}(\tau))\tau^{-1/3}\big)+\mathcal{O}\big(\widetilde{\mu}^{2}\tau^{-2/3}\big), \tag{3.93}$$

where $F^{\sharp,\infty}_{\tau,k}(\tau)$ and $F^{\sharp,0}_{\tau,k}(\tau)$ are defined by equations (3.85) and (3.87), respectively. The results stated in the corollary are now a consequence of the conditions (3.17), equation (3.75), and the asymptotic expansions (3.90)–(3.93), upon retaining only leading-order terms. ■

Proposition 3.15. *Let* $T(\widetilde{\mu})$ *be given in equation (3.27), with* $\mathcal{A}(\widetilde{\mu})$ *defined by equation (3.4) and* $l_{k}^{2}(\widetilde{\mu})$*,* $k=\pm1$*, given in equation (3.29), with the branches defined as in Corollary 3.11; then,*

$$T(\widetilde{\mu})\underset{\substack{\tau\to+\infty\\ \widetilde{\mu}\to\infty}}{=}(b(\tau))^{-\frac{1}{2}\,\mathrm{ad}(\sigma_{3})}\left(\mathrm{I}+\frac{1}{\widetilde{\mu}}\begin{pmatrix}0 & -\frac{(\varepsilon b)^{2/3}}{2}(1+\hat{u}_{0}(\tau))\\ \frac{2(a-\mathrm{i}/2)\tau^{-2/3}-(\varepsilon b)^{1/3}(-2+\hat{r}_{0}(\tau))}{4(\varepsilon b)^{2/3}(1+\hat{u}_{0}(\tau))} & 0\end{pmatrix}+\mathcal{O}\left(\frac{1}{\widetilde{\mu}^{2}}\begin{pmatrix}\mathfrak{c}_{1}(\tau) & 0\\ 0 & \mathfrak{c}_{1}(\tau)\end{pmatrix}\right)\right), \tag{3.94}$$

and

$$T(\widetilde{\mu}) \underset{\substack{\tau\to+\infty \\ \widetilde{\mu}\to 0}}{=} \frac{1}{\sqrt{2}}\left(\frac{b(\tau)}{\sqrt{\varepsilon b}}\right)^{-\frac{1}{2}\operatorname{ad}(\sigma_3)}\left(\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} + \widetilde{\mu}\frac{(-2+\hat{r}_0(\tau))}{4(\varepsilon b)^{1/6}}\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} + \mathcal{O}\left(\widetilde{\mu}^2\begin{pmatrix} \mathfrak{c}_2(\tau) & \mathfrak{c}_3(\tau) \\ \mathfrak{c}_4(\tau) & \mathfrak{c}_2(\tau) \end{pmatrix}\right)\right), \tag{3.95}$$

where $\mathfrak{c}_1(\tau)$, $\mathfrak{c}_2(\tau)$, $\mathfrak{c}_3(\tau)$, *and* $\mathfrak{c}_4(\tau)$, *respectively, are defined by equations* (3.99)–(3.102).

Proof. The proof is presented for the asymptotics (3.94). Let the conditions stated in the proposition be valid. Then, via equations (3.10), (3.15), and (3.16), and the conditions (3.17), one shows that

$$l_k(\widetilde{\mu}) \underset{\substack{\tau\to+\infty \\ \widetilde{\mu}\to\infty}}{=} 2\widetilde{\mu} + \frac{1}{\widetilde{\mu}}(a-\mathrm{i}/2)\tau^{-2/3} + \mathcal{O}\big(\widetilde{\mu}^{-3}\hat{\lambda}_1(\tau)\big), \tag{3.96}$$

$$\mathrm{i}(\mathcal{A}(\widetilde{\mu}) - \mathrm{i}l_k(\widetilde{\mu})\sigma_3)\sigma_3 \underset{\substack{\tau\to+\infty \\ \widetilde{\mu}\to\infty}}{=} 4\widetilde{\mu}\mathrm{I} + \begin{pmatrix} 0 & -\frac{4\sqrt{-a(\tau)b(\tau)}}{b(\tau)} \\ -\mathrm{i}2d(\tau) & 0 \end{pmatrix} + \frac{1}{\widetilde{\mu}}\mathfrak{d}^{\diamondsuit}_{0,0}(\tau)\mathrm{I} + \frac{1}{\widetilde{\mu}^2}\begin{pmatrix} 0 & \frac{(\varepsilon b)}{b(\tau)} \\ -b(\tau) & 0 \end{pmatrix} + \mathcal{O}\left(\widetilde{\mu}^{-3}\hat{\lambda}_1(\tau)\begin{pmatrix} c_{1,k} & 0 \\ 0 & c_{2,k} \end{pmatrix}\right), \tag{3.97}$$

$$\frac{1}{\sqrt{2\mathrm{i}l_k(\widetilde{\mu})(\mathcal{A}_{11}(\widetilde{\mu}) - \mathrm{i}l_k(\widetilde{\mu}))}} \underset{\substack{\tau\to+\infty \\ \widetilde{\mu}\to\infty}}{=} \frac{1}{4\widetilde{\mu}}\left(1 - \frac{1}{\widetilde{\mu}^2}\frac{\mathfrak{d}^{\diamondsuit}_{1,0}(\tau)}{8} + \mathcal{O}\big(\widetilde{\mu}^{-4}\hat{\lambda}_2(\tau)\big)\right), \tag{3.98}$$

where

$$\mathfrak{d}^{\diamondsuit}_{m,j}(\tau) := \frac{(\varepsilon b)^{1/3}}{2}(-2+\hat{r}_0(\tau)) + (-1)^j(2m+1)(a-\mathrm{i}/2)\tau^{-2/3}, \qquad m,j\in\{0,1\},$$
$$\hat{\lambda}_1(\tau) := -3\alpha_k^4 + \hat{h}_0(\tau) - \frac{1}{4}(a-\mathrm{i}/2)^2\tau^{-4/3},$$
$$\hat{\lambda}_2(\tau) := c_{3,k}\hat{\lambda}_1(\tau) + c_{4,k}\big(\mathfrak{d}^{\diamondsuit}_{1,0}(\tau)\big)^2 + c_{5,k}\tau^{-2/3}\mathfrak{d}^{\diamondsuit}_{0,0}(\tau),$$

and $c_{m,k}$, $m=1,2,\ldots,5$, are $\mathcal{O}(1)$; thus, via the conditions (3.17), equation (3.27), and the expansions (3.96)–(3.98), one arrives at the asymptotics (3.94), where

$$\mathfrak{c}_1(\tau) := \mathfrak{d}^{\diamondsuit}_{0,1}(\tau)/8. \tag{3.99}$$

Proceeding analogously, one arrives at the asymptotics (3.95), where

$$\mathfrak{c}_2(\tau) := -\frac{(-2+\hat{r}_0(\tau))^2}{32(\varepsilon b)^{1/3}}, \tag{3.100}$$

$$\mathfrak{c}_3(\tau) := \frac{-3\alpha_k^4 + \hat{h}_0(\tau)}{4\alpha_k^6} - \frac{3(-2+\hat{r}_0(\tau))^2}{32(\varepsilon b)^{1/3}} + \frac{2(1+\hat{u}_0(\tau))}{(\varepsilon b)^{1/3}}, \tag{3.101}$$

$$\mathfrak{c}_4(\tau) := \frac{3\alpha_k^4 - \hat{h}_0(\tau)}{4\alpha_k^6} + \frac{3(-2+\hat{r}_0(\tau))^2}{32(\varepsilon b)^{1/3}} + \frac{2\mathfrak{d}^{\diamondsuit}_{0,1}(\tau)}{(\varepsilon b)^{2/3}(1+\hat{u}_0(\tau))}, \tag{3.102}$$

with $\mathfrak{d}^{\diamondsuit}_{0,1}(\tau)$ defined above. ■

Proposition 3.16. *Let $T(\widetilde{\mu})$ be given in equation (3.27), with $\mathcal{A}(\widetilde{\mu})$ defined by equation (3.4) and $l_k^2(\widetilde{\mu})$, $k=\pm 1$, given in equation (3.29). Set $\widetilde{\mu}_{0,k}=\alpha_k+\tau^{-1/3}\widetilde{\Lambda}$, where $\widetilde{\Lambda} \underset{\tau\to+\infty}{=} \mathcal{O}(\tau^{\delta_k})$, $0<\delta<\delta_k<1/9$; then,*

$$T(\widetilde{\mu}_{0,k}) \underset{\tau\to+\infty}{=} \frac{(b(\tau))^{-\frac{1}{2}\operatorname{ad}(\sigma_3)}}{\big(2\sqrt{3}(\varpi+\sqrt{3})\big)^{1/2}}\left(\begin{pmatrix} \varpi+\sqrt{3} & (2\varepsilon b)^{1/2}\varpi \\ -\frac{\sqrt{2}\varpi}{(\varepsilon b)^{1/2}} & \varpi+\sqrt{3}\end{pmatrix} + \begin{pmatrix} \frac{\varpi}{3\alpha_k} & -\frac{(2\varepsilon b)^{1/2}(2\varpi+\sqrt{3})\varpi}{3(\varpi+\sqrt{3})\alpha_k} \\ \frac{\sqrt{2}(2\varpi+\sqrt{3})\varpi}{3(\varepsilon b)^{1/2}(\varpi+\sqrt{3})\alpha_k} & \frac{\varpi}{3\alpha_k}\end{pmatrix}\tau^{-1/3}\widetilde{\Lambda} + \begin{pmatrix} \mathbb{T}_{11,k}(\varpi;\tau) & \mathbb{T}_{12,k}(\varpi;\tau) \\ \mathbb{T}_{21,k}(\varpi;\tau) & \mathbb{T}_{22,k}(\varpi;\tau)\end{pmatrix}\frac{1}{\widetilde{\Lambda}} + \mathcal{O}\left(\begin{pmatrix} \mathfrak{c}_{1,k} & \mathfrak{c}_{2,k} \\ \mathfrak{c}_{3,k} & \mathfrak{c}_{1,k}\end{pmatrix}\big(\tau^{-1/3}\widetilde{\Lambda}\big)^2\right)\right), \tag{3.103}$$

where

$$\mathbb{T}_{11,k}(\varpi;\tau)=\mathbb{T}_{22,k}(\varpi;\tau):=\frac{\varpi}{4}\left(\frac{\alpha_k\tilde{r}_0(\tau)}{2}-\frac{\tau^{-1/3}\hat{\mathfrak{g}}_k^*(\tau)}{3\alpha_k}\right),$$

$$\mathbb{T}_{12,k}(\varpi;\tau):=\left(\frac{\varepsilon b}{2}\right)^{1/2}\left(\varpi\alpha_k v_0(\tau)-\frac{\alpha_k\tilde{r}_0(\tau)}{4(\varpi+\sqrt{3})}-\frac{(1+2\sqrt{3}\varpi)\tau^{-1/3}\hat{\mathfrak{g}}_k^*(\tau)}{6(\varpi+\sqrt{3})\alpha_k}\right),$$

$$\mathbb{T}_{21,k}(\varpi;\tau):=\frac{\varpi}{(2\varepsilon b)^{1/2}}\left(\frac{(\varepsilon b)^{1/3}(\tilde{r}_0(\tau)+2v_0(\tau))+2(a-\mathrm{i}/2)\mathrm{e}^{\mathrm{i}\pi k/3}\tau^{-1/3}}{2^{3/2}(\varepsilon b)^{1/6}\mathrm{e}^{-\mathrm{i}\pi k/3}\big(1+v_0(\tau)\tau^{-1/3}\big)} + \frac{\alpha_k\tilde{r}_0(\tau)+\frac{2(1+2\sqrt{3}\varpi)\tau^{-1/3}\hat{\mathfrak{g}}_k^*(\tau)}{3\alpha_k}}{4(\varpi+\sqrt{3})\varpi}\right),$$

with $\hat{\mathfrak{g}}_k^*(\tau):=\tau^{2/3}\hat{\mathfrak{f}}_{1,k}(\tau)$, where $\hat{\mathfrak{f}}_{1,k}(\tau)$ is given in equation (3.74) (for $j=1$), $\big(\widetilde{\Lambda}^2\big)^{1/2}:=\varpi\widetilde{\Lambda}$, $\varpi=\pm 1$, and $\mathfrak{c}_{m,k}$, $m=1,2,3$, are $\mathcal{O}(1)$.

Proof. Set $T(\widetilde{\mu})=(T(\widetilde{\mu}))_{i,j=1,2}$. From the formula for $T(\widetilde{\mu})$ given in equation (3.27), with $\mathcal{A}(\widetilde{\mu})$ defined by equation (3.4) and $l_k^2(\widetilde{\mu})$, $k=\pm 1$, given in equation (3.29), one shows that

$$T_{11}(\widetilde{\mu})=T_{22}(\widetilde{\mu})=\frac{\mathrm{i}(\mathcal{A}_{11}(\widetilde{\mu})-\mathrm{i}l_k(\widetilde{\mu}))}{\sqrt{2\mathrm{i}l_k(\widetilde{\mu})(\mathcal{A}_{11}(\widetilde{\mu})-\mathrm{i}l_k(\widetilde{\mu}))}}, \qquad T_{12}(\widetilde{\mu})=-\frac{\mathrm{i}\mathcal{A}_{12}(\widetilde{\mu})}{\sqrt{2\mathrm{i}l_k(\widetilde{\mu})(\mathcal{A}_{11}(\widetilde{\mu})-\mathrm{i}l_k(\widetilde{\mu}))}},$$
$$T_{21}(\widetilde{\mu})=\frac{\mathrm{i}\mathcal{A}_{21}(\widetilde{\mu})}{\sqrt{2\mathrm{i}l_k(\widetilde{\mu})(\mathcal{A}_{11}(\widetilde{\mu})-\mathrm{i}l_k(\widetilde{\mu}))}}. \tag{3.104}$$

From equations (3.4), (3.10), (3.15), and (3.16), the conditions (3.17), and equation (3.74) for $\hat{\mathfrak{f}}_{1,k}(\tau)$ (with associated asymptotics (3.81)), one shows, upon taking $\widetilde{\mu}_{0,k}$ as stated in the proposition, that

$$\frac{1}{\sqrt{2\mathrm{i}l_k(\widetilde{\mu}_{0,k})(\mathcal{A}_{11}(\widetilde{\mu}_{0,k})-\mathrm{i}l_k(\widetilde{\mu}_{0,k}))}} \underset{\tau\to+\infty}{=} \frac{\big(\varpi\tau^{-1/3}\widetilde{\Lambda}\big)^{-1}}{4\big(2\sqrt{3}(\varpi+\sqrt{3})\big)^{1/2}}\left(1+\frac{(5\varpi+7\sqrt{3})}{6(\varpi+\sqrt{3})\alpha_k}\tau^{-1/3}\widetilde{\Lambda} - \left(\frac{\alpha_k\tilde{r}_0(\tau)+2\big(1+2\sqrt{3}\varpi\big)(3\alpha_k)^{-1}\hat{\mathfrak{g}}_k^*(\tau)\tau^{-1/3}}{8\varpi\big(\varpi+\sqrt{3}\big)}\right)\frac{1}{\widetilde{\Lambda}} + \mathcal{O}\big(\big(\tau^{-1/3}\widetilde{\Lambda}\big)^2\big)\right), \tag{3.105}$$

$$\begin{aligned}
&\mathrm{i}\mathcal{A}_{11}(\widetilde{\mu}_{0,k})+l_k(\widetilde{\mu}_{0,k})\\
&\quad\underset{\tau\to+\infty}{=}4\varpi(\varpi+\sqrt{3})\tau^{-1/3}\widetilde{\Lambda}\left(1-\frac{\sqrt{3}(7+\sqrt{3}\varpi)}{6(\varpi+\sqrt{3})\alpha_k}\tau^{-1/3}\widetilde{\Lambda}\right.\\
&\qquad\left.+\mathcal{O}\big((\tau^{-1/3}\widetilde{\Lambda})^2\big)+\left(\frac{\alpha_k\tilde{r}_0(\tau)+2\varpi\big(\sqrt{3}\alpha_k\big)^{-1}\hat{\mathfrak{g}}_k^{*}(\tau)\tau^{-1/3}}{4\varpi(\varpi+\sqrt{3})}\right)\frac{1}{\widetilde{\Lambda}}\right), &(3.106)\\
&-\mathrm{i}\mathcal{A}_{12}(\widetilde{\mu}_{0,k})\underset{\tau\to+\infty}{=}(b(\tau))^{-1}\big(-2(\varepsilon b)\alpha_k^{-3}\tau^{-1/3}\widetilde{\Lambda}+3(\varepsilon b)\alpha_k^{-4}(\tau^{-1/3}\widetilde{\Lambda})^2+\mathcal{O}\big((\tau^{-1/3}\widetilde{\Lambda})^3\big)\\
&\qquad-2(\varepsilon b)^{2/3}\mathrm{e}^{-\mathrm{i}2\pi k/3}v_0(\tau)\tau^{-1/3}\big), &(3.107)\\
&\mathrm{i}\mathcal{A}_{21}(\widetilde{\mu}_{0,k})\underset{\tau\to+\infty}{=}b(\tau)\left(2\alpha_k^{-3}\tau^{-1/3}\widetilde{\Lambda}-3\alpha_k^{-4}(\tau^{-1/3}\widetilde{\Lambda})^2+\mathcal{O}\big((\tau^{-1/3}\widetilde{\Lambda})^3\big)\right.\\
&\qquad\left.+\frac{\mathrm{e}^{\mathrm{i}\pi k/3}\tau^{-1/3}\left((\varepsilon b)^{1/3}(\tilde{r}_0(\tau)+2v_0(\tau))+2(a-\mathrm{i}/2)\mathrm{e}^{\mathrm{i}\pi k/3}\tau^{-1/3}\right)}{(\varepsilon b)^{2/3}\big(1+v_0(\tau)\tau^{-1/3}\big)}\right), &(3.108)
\end{aligned}$$

where $\hat{\mathfrak{g}}_k^{*}(\tau)$ and ϖ are defined in the proposition. Substituting the expansions (3.105)–(3.108) into equations (3.104) (with $\widetilde{\mu}=\widetilde{\mu}_{0,k}$), one arrives at the asymptotics for $T(\widetilde{\mu}_{0,k})$ stated in the proposition. ■

3.2 Parametrix near the double-turning points

The matrix WKB formula (cf. equation (3.25)) does not provide an approximation for solutions of equation (3.3) in shrinking (as $\tau\to+\infty$ with $\varepsilon b>0$) neighbourhoods of the turning points, where a more refined approximation must be constructed. There are two simple turning points approaching $\pm\mathrm{i}\sqrt{2}\alpha_k$, $k=\pm1$: the approximate solution of equation (3.3) in the neighbourhoods of these turning points is representable in terms of Airy functions (see, for example, [29, 45], *Riemann–Hilbert Problem* 4 in [11, 13], and [71, Sections 3.5 and 3.6]). There are, additionally, two pairs of double-turning points, one pair coalescing at $-\alpha_k$, and another pair coalescing at α_k: in neighbourhoods of $\pm\alpha_k$, the approximate solution of equation (3.3) is expressed in terms of parabolic-cylinder functions (see, for example, [28, 29, 41, 45, 80]). In order to obtain asymptotics for $u(\tau)$ and the associated, auxiliary functions $f_{\pm}(\tau)$, $\mathcal{H}(\tau)$, $\sigma(\tau)$, and $\hat{\varphi}(\tau)$, it is sufficient to study a subset of the complete set of the monodromy data, which can be calculated via the approximation of the general solution of equation (3.3) in a neighbourhood of the double-turning point α_k, because the remaining monodromy data can be calculated via equations (1.51), which define the monodromy manifold.[22] For the asymptotic conditions (3.17) on the functions $\hat{h}_0(\tau)$, $\tilde{r}_0(\tau)$, and $v_0(\tau)$, this parametrix (approximation) is given in Lemma 3.17 below.

Lemma 3.17. *Set*

$$\nu(k)+1:=-\frac{p_k(\tau)q_k(\tau)}{2\mu_k(\tau)},\qquad k=\pm1, \tag{3.109}$$

where $\mu_k(\tau)$, $p_k(\tau)$, and $q_k(\tau)$ are defined by equations (3.157), (3.160), and (3.161), respectively,[23] *and let $\widetilde{\mu}=\widetilde{\mu}_{0,k}=\alpha_k+\tau^{-1/3}\widetilde{\Lambda}$, where $\widetilde{\Lambda}\underset{\tau\to+\infty}{=}\mathcal{O}(\tau^{\delta_k})$, $0<\delta<\delta_k<1/9$. Concomitant with equations (3.6)–(3.9), the definitions (3.14)–(3.16), and the conditions (3.17), impose the following restrictions*

$$0\underset{\tau\to+\infty}{<}\operatorname{Re}(\nu(k)+1)\underset{\tau\to+\infty}{<}1,\qquad \operatorname{Im}(\nu(k)+1)\underset{\tau\to+\infty}{\leqslant}\mathcal{O}(1),$$

[22] More precisely, equations (1.52) (resp., equations (1.53)) for $k=+1$ (resp., $k=-1$).

[23] See, also, the corresponding definitions (3.121), (3.124)–(3.129), (3.130), (3.136)–(3.138), (3.143), (3.148), (3.149), and (3.156).

$$0 \underset{\tau\to+\infty}{<} \delta_k \underset{\tau\to+\infty}{<} \frac{1}{6(3+\mathrm{Re}(\nu(k)+1))}, \qquad k=\pm1. \tag{3.110}$$

Then, there exists a fundamental solution of equation (3.3), $\widetilde{\Psi}(\widetilde{\mu}) = \widetilde{\Psi}_k(\widetilde{\mu},\tau)$, $k=\pm1$, *with asymptotics*

$$\widetilde{\Psi}_k(\widetilde{\mu},\tau) \underset{\tau\to+\infty}{=} (b(\tau))^{-\frac{1}{2}\sigma_3}\mathcal{G}_{0,k}\mathfrak{B}_k^{\frac{1}{2}\sigma_3}\begin{pmatrix} 1 & 0 \\ \big(\frac{4\mathrm{i}\sqrt{3}\mathcal{Z}_k}{\chi_k(\tau)}-1\big)\mathfrak{A}_k & 1 \end{pmatrix}\big(\mathrm{I}+\daleth_{A,k}(\tau)\widetilde{\Lambda}+\daleth_{B,k}(\tau)\widetilde{\Lambda}^2\big)$$
$$\times\left(\mathrm{I}+\mathcal{O}\left(\tilde{\mathfrak{C}}_k(\tau)\frac{|\nu(k)+1|^2}{|p_k(\tau)|^2}\tau^{-(\frac{1}{3}-2(3+\mathrm{Re}(\nu(k)+1))\delta_k)}\right)\right)\Phi_{M,k}\big(\widetilde{\Lambda}\big), \tag{3.111}$$

where

$$\daleth_{A,k}(\tau) := \begin{pmatrix} \frac{4\mathrm{i}\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\ell_{0,k}^+}{\chi_k(\tau)} & \ell_{0,k}^+ \\ \big(\frac{4\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k}{\chi_k(\tau)}\big)^2\ell_{0,k}^+ + \ell_{1,k}^+ + \ell_{2,k}^+ & -\frac{4\mathrm{i}\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\ell_{0,k}^+}{\chi_k(\tau)} \end{pmatrix}, \tag{3.112}$$

$$\daleth_{B,k}(\tau) := \ell_{0,k}^+\big(\ell_{1,k}^+ + \ell_{2,k}^+\big)\begin{pmatrix} 1 & 0 \\ -\frac{4\mathrm{i}\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k}{\chi_k(\tau)} & 0 \end{pmatrix}, \tag{3.113}$$

with $\mathcal{G}_{0,k}$, $\mathcal{Z}_k$, $\mathfrak{A}_k$, $\mathfrak{B}_k$, $\ell_{0,k}^+$, $\ell_{1,k}^+$, $\chi_k(\tau)$, *and* $\ell_{2,k}^+$ *defined by equations* (3.120), (3.121), (3.124), (3.125), (3.143), (3.148), (3.149), *and* (3.156), *respectively,*[24] $\mathrm{M}_2(\mathbb{C}) \ni \tilde{\mathfrak{C}}_k(\tau) \underset{\tau\to+\infty}{=} \mathcal{O}(1)$, *and* $\Phi_{M,k}\big(\widetilde{\Lambda}\big)$ *is a fundamental solution of*

$$\frac{\partial\Phi_{M,k}\big(\widetilde{\Lambda}\big)}{\partial\widetilde{\Lambda}} = \big(\mu_k(\tau)\widetilde{\Lambda}\sigma_3 + p_k(\tau)\sigma_+ + q_k(\tau)\sigma_-\big)\Phi_{M,k}\big(\widetilde{\Lambda}\big), \tag{3.114}$$

$\Phi_{M,k}\big(\widetilde{\Lambda}\big)$ *has the explicit representation*

$$\Phi_{M,k}\big(\widetilde{\Lambda}\big) = \begin{pmatrix} D_{-\nu(k)-1}\big(\mathrm{i}(2\mu_k(\tau))^{1/2}\widetilde{\Lambda}\big) & D_{\nu(k)}\big((2\mu_k(\tau))^{1/2}\widetilde{\Lambda}\big) \\ \mathbb{D}_k^*\big(\tau,\widetilde{\Lambda}\big)D_{-\nu(k)-1}\big(\mathrm{i}(2\mu_k(\tau))^{1/2}\widetilde{\Lambda}\big) & \mathbb{D}_k^*\big(\tau,\widetilde{\Lambda}\big)D_{\nu(k)}\big((2\mu_k(\tau))^{1/2}\widetilde{\Lambda}\big) \end{pmatrix}, \tag{3.115}$$

where $\mathbb{D}_k^*\big(\tau,\widetilde{\Lambda}\big) := \frac{1}{p_k(\tau)}\big(\frac{\partial}{\partial\widetilde{\Lambda}} - \mu_k(\tau)\widetilde{\Lambda}\big)$, *and* $D_*(\cdot)$ *is the parabolic-cylinder function* [32].

Proof. The derivation of the parametrix (3.111) for a fundamental solution of equation (3.3) consists of applying the sequence of invertible linear transformations $\mathfrak{F}_j$, $j=1,2,\dots,11$; for $k=\pm1$,

(i) $\mathfrak{F}_1\colon\ \mathrm{SL}_2(\mathbb{C}) \ni \widetilde{\Psi}(\widetilde{\mu}) \mapsto \widetilde{\Psi}_k\big(\widetilde{\Lambda}\big) := \widetilde{\Psi}\big(\alpha_k + \tau^{-1/3}\widetilde{\Lambda}\big)$,

(ii) $\mathfrak{F}_2\colon\ \mathrm{SL}_2(\mathbb{C}) \ni \widetilde{\Psi}_k\big(\widetilde{\Lambda}\big) \mapsto \tilde{\Phi}_k\big(\widetilde{\Lambda}\big) := (b(\tau))^{\frac{1}{2}\sigma_3}\widetilde{\Psi}_k\big(\widetilde{\Lambda}\big)$,

(iii) $\mathfrak{F}_3\colon\ \mathrm{SL}_2(\mathbb{C}) \ni \tilde{\Phi}_k\big(\widetilde{\Lambda}\big) \mapsto \Phi_k^{\sharp}\big(\widetilde{\Lambda}\big) := \mathcal{G}_{0,k}^{-1}\tilde{\Phi}_k\big(\widetilde{\Lambda}\big)$,

(iv) $\mathfrak{F}_4\colon\ \mathrm{SL}_2(\mathbb{C}) \ni \Phi_k^{\sharp}\big(\widetilde{\Lambda}\big) \mapsto \hat{\Phi}_k\big(\widetilde{\Lambda}\big) := \mathcal{G}_{1,k}^{-1}\Phi_k^{\sharp}\big(\widetilde{\Lambda}\big)$,

(v) $\mathfrak{F}_5\colon\ \mathrm{SL}_2(\mathbb{C}) \ni \hat{\Phi}_k\big(\widetilde{\Lambda}\big) \mapsto \hat{\Phi}_{0,k}\big(\widetilde{\Lambda}\big) := \tau^{-\frac{1}{6}\sigma_3}\hat{\Phi}_k\big(\widetilde{\Lambda}\big)$,

(vi) $\mathfrak{F}_6\colon\ \mathrm{SL}_2(\mathbb{C}) \ni \hat{\Phi}_{0,k}\big(\widetilde{\Lambda}\big) \mapsto \Phi_{0,k}\big(\widetilde{\Lambda}\big) := (\mathrm{I}+\mathrm{i}\omega_{0,k}\sigma_-)\hat{\Phi}_{0,k}\big(\widetilde{\Lambda}\big)$,

(vii) $\mathfrak{F}_7\colon\ \mathrm{SL}_2(\mathbb{C}) \ni \Phi_{0,k}\big(\widetilde{\Lambda}\big) \mapsto \Phi_{0,k}^{\flat}\big(\widetilde{\Lambda}\big) := \big(\mathrm{I}-\ell_{0,k}\widetilde{\Lambda}\sigma_+\big)\Phi_{0,k}\big(\widetilde{\Lambda}\big)$,

(viii) $\mathfrak{F}_8\colon\ \mathrm{SL}_2(\mathbb{C}) \ni \Phi_{0,k}^{\flat}\big(\widetilde{\Lambda}\big) \mapsto \Phi_{0,k}^{\sharp}\big(\widetilde{\Lambda}\big) := \big(\mathrm{I}-\ell_{1,k}\widetilde{\Lambda}\sigma_-\big)\Phi_{0,k}^{\flat}\big(\widetilde{\Lambda}\big)$,

[24] See, also, the corresponding definition (3.117).

(ix) $\mathfrak{F}_9\colon \mathrm{SL}_2(\mathbb{C}) \ni \Phi^{\sharp}_{0,k}\big(\widetilde{\Lambda}\big) \mapsto \Phi^{\natural}_{0,k}\big(\widetilde{\Lambda}\big) := \mathcal{G}^{-1}_{2,k}\Phi^{\sharp}_{0,k}\big(\widetilde{\Lambda}\big)$,

(x) $\mathfrak{F}_{10}\colon \mathrm{SL}_2(\mathbb{C}) \ni \Phi^{\natural}_{0,k}\big(\widetilde{\Lambda}\big) \mapsto \Phi^{*}_{k}\big(\widetilde{\Lambda}\big) := \big(\mathrm{I} - \ell_{2,k}\widetilde{\Lambda}\sigma_-\big)\Phi^{\natural}_{0,k}\big(\widetilde{\Lambda}\big)$,

(xi) $\mathfrak{F}_{11}\colon \mathrm{SL}_2(\mathbb{C}) \ni \Phi^{*}_{k}\big(\widetilde{\Lambda}\big) \mapsto \Phi_{M,k}\big(\widetilde{\Lambda}\big) := \hat{\chi}^{-1}_{k}\big(\widetilde{\Lambda}\big)\Phi^{*}_{k}\big(\widetilde{\Lambda}\big) \in \mathrm{M}_2(\mathbb{C})$,

where the $\mathrm{M}_2(\mathbb{C})$-valued, τ-dependent functions $\mathcal{G}_{0,k}$, $\mathcal{G}_{1,k}$, $\mathrm{I}+\mathrm{i}\omega_{0,k}\sigma_-$, $\mathcal{G}_{2,k}$, and $\hat{\chi}_k\big(\widetilde{\Lambda}\big)$, and the τ-dependent parameters $\ell_{0,k}$, $\ell_{1,k}$, and $\ell_{2,k}$ are described in steps (iii), (iv), (vi), (ix), (xi), (vii), (viii), and (x), respectively, below, and $\mathrm{M}_2(\mathbb{C}) \ni \Phi_{M,k}\big(\widetilde{\Lambda}\big)$ is given in equation (3.115).

(i) The gist of this step is to simplify the system (3.3) in a proper neighbourhood of the (coalescing) double-turning point α_k, $k \in \{\pm1\}$. Let $\widetilde{\Psi}(\widetilde{\mu})$ solve equation (3.3); then, using equations (3.7), (3.8), (3.10), (3.15), and (3.16), the conditions (3.17), and applying the transformation $\mathfrak{F}_1$, one shows that, for $k=\pm1$,

$$\frac{\partial\widetilde{\Psi}_k\big(\widetilde{\Lambda}\big)}{\partial\widetilde{\Lambda}} \underset{\tau\to+\infty}{=} (b(\tau))^{-\frac{1}{2}\operatorname{ad}(\sigma_3)} \times \big(\hat{\mathcal{P}}_{0,k}(\tau) + \hat{\mathcal{P}}_{1,k}(\tau)\widetilde{\Lambda} + \hat{\mathcal{P}}_{2,k}(\tau)\widetilde{\Lambda}^2 + \mathcal{O}\big(\hat{\mathbb{E}}_k(\tau)\widetilde{\Lambda}^3\big)\big)\widetilde{\Psi}_k\big(\widetilde{\Lambda}\big), \tag{3.116}$$

where

$$\hat{\mathcal{P}}_{0,k}(\tau) = \begin{pmatrix} -\mathrm{i}\alpha_k\tilde{r}_0(\tau) & -2\mathrm{i}(\varepsilon b)^{2/3}\mathrm{e}^{-\mathrm{i}2\pi k/3}v_0(\tau) \\ -\frac{(\mathrm{i}(\varepsilon b)^{1/3}\mathrm{e}^{\mathrm{i}\pi k/3}(\tilde{r}_0(\tau)+2v_0(\tau))+2\mathrm{i}(a-\mathrm{i}/2)\mathrm{e}^{\mathrm{i}2\pi k/3}\tau^{-1/3})}{(\varepsilon b)^{2/3}(1+v_0(\tau)\tau^{-1/3})} & \mathrm{i}\alpha_k\tilde{r}_0(\tau) \end{pmatrix} =: \begin{pmatrix} \hat{\mathcal{A}}_0 & \hat{\mathcal{B}}_0 \\ \hat{\mathcal{C}}_0 & -\hat{\mathcal{A}}_0 \end{pmatrix},$$

$$\hat{\mathcal{P}}_{1,k}(\tau) = \begin{pmatrix} \mathrm{i}\big(-4+\tilde{r}_0(\tau)\tau^{-1/3}\big) & 4\mathrm{i}\sqrt{2}(\varepsilon b)^{1/2} \\ 4\mathrm{i}\sqrt{2}(\varepsilon b)^{-1/2} & -\mathrm{i}\big(-4+\tilde{r}_0(\tau)\tau^{-1/3}\big) \end{pmatrix} =: \begin{pmatrix} \hat{\mathcal{A}}_1 & \hat{\mathcal{B}}_1 \\ \hat{\mathcal{C}}_1 & -\hat{\mathcal{A}}_1 \end{pmatrix}, \tag{3.117}$$

$$\hat{\mathcal{P}}_{2,k}(\tau) = \begin{pmatrix} \frac{\mathrm{i}\sqrt{2}\mathrm{e}^{\mathrm{i}2\pi k/3}}{(\varepsilon b)^{1/6}}\big(-2+\tilde{r}_0(\tau)\tau^{-1/3}\big)\tau^{-1/3} & -12\mathrm{i}(\varepsilon b)^{1/3}\mathrm{e}^{-\mathrm{i}\pi k/3}\tau^{-1/3} \\ -12\mathrm{i}(\varepsilon b)^{-2/3}\mathrm{e}^{-\mathrm{i}\pi k/3}\tau^{-1/3} & -\frac{\mathrm{i}\sqrt{2}\mathrm{e}^{\mathrm{i}2\pi k/3}}{(\varepsilon b)^{1/6}}\big(-2+\tilde{r}_0(\tau)\tau^{-1/3}\big)\tau^{-1/3} \end{pmatrix} =: \begin{pmatrix} \hat{\mathcal{A}}_2 & \hat{\mathcal{B}}_2 \\ \hat{\mathcal{C}}_2 & -\hat{\mathcal{A}}_2 \end{pmatrix},$$

and

$$\hat{\mathbb{E}}_k(\tau) = \begin{pmatrix} \mathrm{i}\alpha_k^{-2}\big(-2+\tilde{r}_0(\tau)\tau^{-1/3}\big)\tau^{-2/3} & -32\mathrm{i}\alpha_k\tau^{-2/3} \\ -4\mathrm{i}\alpha_k^{-5}\tau^{-2/3} & -\mathrm{i}\alpha_k^{-2}\big(-2+\tilde{r}_0(\tau)\tau^{-1/3}\big)\tau^{-2/3} \end{pmatrix}. \tag{3.118}$$

Observe that $\operatorname{tr}\big(\hat{\mathcal{P}}_{0,k}(\tau)\big) = \operatorname{tr}\big(\hat{\mathcal{P}}_{1,k}(\tau)\big) = \operatorname{tr}\big(\hat{\mathcal{P}}_{2,k}(\tau)\big) = \operatorname{tr}\big(\hat{\mathbb{E}}_k(\tau)\big) = 0$.

(ii) This intermediate step removes the scalar-valued function $b(\tau)$ from equation (3.116). Let $\widetilde{\Psi}_k\big(\widetilde{\Lambda}\big)$ solve equation (3.116); then, applying the transformation $\mathfrak{F}_2$, one shows that, for $k=\pm1$,

$$\frac{\partial\tilde{\Phi}_k\big(\widetilde{\Lambda}\big)}{\partial\widetilde{\Lambda}} \underset{\tau\to+\infty}{=} \big(\hat{\mathcal{P}}_{0,k}(\tau) + \hat{\mathcal{P}}_{1,k}(\tau)\widetilde{\Lambda} + \hat{\mathcal{P}}_{2,k}(\tau)\widetilde{\Lambda}^2 + \mathcal{O}\big(\hat{\mathbb{E}}_k(\tau)\widetilde{\Lambda}^3\big)\big)\tilde{\Phi}_k\big(\widetilde{\Lambda}\big). \tag{3.119}$$

(iii) The essence of this step is to transform the coefficient matrix $\hat{\mathcal{P}}_{1,k}(\tau)$ (cf. definition (3.117)) into diagonal form. Let $\tilde{\Phi}_k\big(\widetilde{\Lambda}\big)$ be a solution of equation (3.119); then, applying the transformation $\mathfrak{F}_3$, where

$$\mathcal{G}_{0,k} := \left(\frac{\hat{\mathcal{C}}_1}{2\lambda_1^{*}(k)}\right)^{1/2} \begin{pmatrix} \frac{\hat{\mathcal{A}}_1+\lambda_1^{*}(k)}{\hat{\mathcal{C}}_1} & \frac{\hat{\mathcal{A}}_1-\lambda_1^{*}(k)}{\hat{\mathcal{C}}_1} \\ 1 & 1 \end{pmatrix}, \qquad k=\pm1, \tag{3.120}$$

with $\hat{\mathcal{A}}_1$ and $\hat{\mathcal{C}}_1$ given in equation (3.117), and

$$\lambda_1^*(k) := \mathrm{i}4\sqrt{3}\mathcal{Z}_k = \mathrm{i}4\sqrt{3}\left(1 - \frac{1}{6}\tilde{r}_0(\tau)\tau^{-1/3} + \frac{1}{48}\big(\tilde{r}_0(\tau)\tau^{-1/3}\big)^2\right)^{1/2}, \tag{3.121}$$

one shows that

$$\frac{\partial \Phi_k^{\sharp}(\widetilde{\Lambda})}{\partial \widetilde{\Lambda}} \underset{\tau\to+\infty}{=} \big(\mathcal{P}_{0,k}^{\triangle}(\tau) + \mathcal{P}_{1,k}^{\triangle}(\tau)\widetilde{\Lambda} + \mathcal{P}_{2,k}^{\triangle}(\tau)\widetilde{\Lambda}^2 + \mathcal{O}\big(\mathcal{G}_{0,k}^{-1}\hat{\mathbb{E}}_k(\tau)\mathcal{G}_{0,k}\widetilde{\Lambda}^3\big)\big)\Phi_k^{\sharp}(\widetilde{\Lambda}), \tag{3.122}$$

where

$$\begin{aligned} &\mathcal{P}_{0,k}^{\triangle}(\tau) := \mathcal{G}_{0,k}^{-1}\hat{\mathcal{P}}_{0,k}(\tau)\mathcal{G}_{0,k} = \mathfrak{A}_k\sigma_3 + \mathfrak{B}_k\sigma_+ + \mathfrak{C}_k\sigma_-, \\ &\mathcal{P}_{1,k}^{\triangle}(\tau) := \mathcal{G}_{0,k}^{-1}\hat{\mathcal{P}}_{1,k}(\tau)\mathcal{G}_{0,k} = \mathrm{i}4\sqrt{3}\mathcal{Z}_k\sigma_3, \\ &\mathcal{P}_{2,k}^{\triangle}(\tau) := \mathcal{G}_{0,k}^{-1}\hat{\mathcal{P}}_{2,k}(\tau)\mathcal{G}_{0,k} = \mathfrak{A}_{0,k}^{\sharp}\sigma_3 + \mathfrak{B}_{0,k}^{\sharp}\sigma_+ + \mathfrak{C}_{0,k}^{\sharp}\sigma_-, \end{aligned} \tag{3.123}$$

with

$$\begin{aligned} \mathfrak{A}_k = \frac{1}{(6\varepsilon b)^{1/2}\mathcal{Z}_k}\Bigg(&-\frac{\mathrm{i}\alpha_k(\varepsilon b)^{1/2}}{2\sqrt{2}}\tilde{r}_0(\tau)\big(-4 + \tilde{r}_0(\tau)\tau^{-1/3}\big) - 2\mathrm{i}(\varepsilon b)^{2/3}\mathrm{e}^{-\mathrm{i}2\pi k/3}v_0(\tau) \\ &- \mathrm{i}(\varepsilon b)^{1/3}\left(\frac{(\varepsilon b)^{1/3}\mathrm{e}^{\mathrm{i}\pi k/3}(\tilde{r}_0(\tau) + 2v_0(\tau)) + 2(a - \mathrm{i}/2)\mathrm{e}^{\mathrm{i}2\pi k/3}\tau^{-1/3}}{1 + v_0(\tau)\tau^{-1/3}}\right)\Bigg), \end{aligned} \tag{3.124}$$

$$\begin{aligned} \mathfrak{B}_k = \frac{1}{(6\varepsilon b)^{1/2}\mathcal{Z}_k}\Bigg(&-\frac{\mathrm{i}\alpha_k(\varepsilon b)^{1/2}}{2\sqrt{2}}\tilde{r}_0(\tau)\big(-4 + \tilde{r}_0(\tau)\tau^{-1/3} - 4\sqrt{3}\mathcal{Z}_k\big) - 2\mathrm{i}(\varepsilon b)^{2/3}\mathrm{e}^{-\mathrm{i}2\pi k/3}v_0(\tau) \\ &+ \mathrm{i}(\varepsilon b)^{1/3}\left(\frac{(\varepsilon b)^{1/3}\mathrm{e}^{\mathrm{i}\pi k/3}(\tilde{r}_0(\tau) + 2v_0(\tau)) + 2(a - \mathrm{i}/2)\mathrm{e}^{\mathrm{i}2\pi k/3}\tau^{-1/3}}{1 + v_0(\tau)\tau^{-1/3}}\right) \\ &\times\left(1 + \frac{1}{16}\big(-4 + \tilde{r}_0(\tau)\tau^{-1/3}\big)\big(-4 + \tilde{r}_0(\tau)\tau^{-1/3} - 4\sqrt{3}\mathcal{Z}_k\big)\right)\Bigg), \end{aligned} \tag{3.125}$$

$$\begin{aligned} \mathfrak{C}_k = \frac{1}{(6\varepsilon b)^{1/2}\mathcal{Z}_k}\Bigg(&\frac{\mathrm{i}\alpha_k(\varepsilon b)^{1/2}}{2\sqrt{2}}\tilde{r}_0(\tau)\big(-4 + \tilde{r}_0(\tau)\tau^{-1/3} + 4\sqrt{3}\mathcal{Z}_k\big) + 2\mathrm{i}(\varepsilon b)^{2/3}\mathrm{e}^{-\mathrm{i}2\pi k/3}v_0(\tau) \\ &- \mathrm{i}(\varepsilon b)^{1/3}\left(\frac{(\varepsilon b)^{1/3}\mathrm{e}^{\mathrm{i}\pi k/3}(\tilde{r}_0(\tau) + 2v_0(\tau)) + 2(a - \mathrm{i}/2)\mathrm{e}^{\mathrm{i}2\pi k/3}\tau^{-1/3}}{1 + v_0(\tau)\tau^{-1/3}}\right) \\ &\times\left(1 + \frac{1}{16}\big(-4 + \tilde{r}_0(\tau)\tau^{-1/3}\big)\big(-4 + \tilde{r}_0(\tau)\tau^{-1/3} + 4\sqrt{3}\mathcal{Z}_k\big)\right)\Bigg), \end{aligned} \tag{3.126}$$

$$\mathfrak{A}_{0,k}^{\sharp} = -\frac{\mathrm{i}(\varepsilon b)^{1/3}\mathrm{e}^{-\mathrm{i}\pi k/3}\tau^{-1/3}}{2(6\varepsilon b)^{1/2}\mathcal{Z}_k}\big(48 + \big(-2 + \tilde{r}_0(\tau)\tau^{-1/3}\big)\big(-4 + \tilde{r}_0(\tau)\tau^{-1/3}\big)\big), \tag{3.127}$$

$$\mathfrak{B}_{0,k}^{\sharp} = \frac{\mathrm{i}(\varepsilon b)^{1/3}\mathrm{e}^{-\mathrm{i}\pi k/3}\tau^{-1/3}}{2(6\varepsilon b)^{1/2}\mathcal{Z}_k}\big(-4 + \tilde{r}_0(\tau)\tau^{-1/3} - 4\sqrt{3}\mathcal{Z}_k\big)\left(-4 + \frac{1}{2}\tilde{r}_0(\tau)\tau^{-1/3}\right), \tag{3.128}$$

$$\mathfrak{C}_{0,k}^{\sharp} = -\frac{\mathrm{i}(\varepsilon b)^{1/3}\mathrm{e}^{-\mathrm{i}\pi k/3}\tau^{-1/3}}{2(6\varepsilon b)^{1/2}\mathcal{Z}_k}\big(-4 + \tilde{r}_0(\tau)\tau^{-1/3} + 4\sqrt{3}\mathcal{Z}_k\big)\left(-4 + \frac{1}{2}\tilde{r}_0(\tau)\tau^{-1/3}\right). \tag{3.129}$$

Observe that $\mathrm{tr}(\mathcal{P}_{0,k}^{\triangle}(\tau)) = \mathrm{tr}\big(\mathcal{P}_{1,k}^{\triangle}(\tau)\big) = \mathrm{tr}\big(\mathcal{P}_{2,k}^{\triangle}(\tau)\big) = \mathrm{tr}\big(\mathcal{G}_{0,k}^{-1}\hat{\mathbb{E}}_k(\tau)\mathcal{G}_{0,k}\big) = 0$.

(iv) The idea behind the transformation for equation (3.122) that is subsumed in this step is to put the coefficient matrix $\mathcal{P}_{0,k}^{\triangle}(\tau)$ (cf. definition (3.123)) into Jordan canonical form, namely, to find a unimodular, τ-dependent function $\mathcal{G}_{1,k}$ such that

$$\mathcal{G}_{1,k}^{-1}\mathcal{P}_{0,k}^{\triangle}(\tau)\mathcal{G}_{1,k} = \mathrm{i}\omega_{0,k}\sigma_3 + \tau^{1/3}\sigma_+, \qquad k = \pm 1,$$

where (cf. equations (3.18), (3.19), and (3.124)–(3.126))

$$\begin{aligned}\omega_{0,k}^2 &:= \det\big(\mathcal{P}_{0,k}^{\triangle}(\tau)\big) = \varkappa_0^2(\tau) + \frac{4(a-\mathrm{i}/2)v_0(\tau)\tau^{-1/3}}{1+v_0(\tau)\tau^{-1/3}} = 4\big((a-\mathrm{i}/2)+\alpha_k^{-2}\tau^{2/3}\hat{h}_0(\tau)\big)\\ &= -\,\alpha_k^2\left(\frac{8v_0^2(\tau)+4v_0(\tau)\tilde{r}_0(\tau)-(\tilde{r}_0(\tau))^2-v_0(\tau)(\tilde{r}_0(\tau))^2\tau^{-1/3}}{1+v_0(\tau)\tau^{-1/3}}\right)\\ &\quad+\frac{4(a-\mathrm{i}/2)v_0(\tau)\tau^{-1/3}}{1+v_0(\tau)\tau^{-1/3}};\end{aligned} \tag{3.130}$$

the following lower-triangular solution for $\mathcal{G}_{1,k}$ is chosen:

$$\mathcal{G}_{1,k} = \mathfrak{B}_k^{\frac{1}{2}\sigma_3}\tau^{-\frac{1}{6}\sigma_3}\big(\mathrm{I}+(\mathrm{i}\omega_{0,k}-\mathfrak{A}_k)\tau^{-1/3}\sigma_-\big), \qquad k=\pm1. \tag{3.131}$$

Let $\Phi_k^{\sharp}(\widetilde{\Lambda})$ solve equation (3.122); then, applying the transformation $\mathfrak{F}_4$, one shows that

$$\begin{aligned}\frac{\partial\hat{\Phi}_k(\widetilde{\Lambda})}{\partial\widetilde{\Lambda}} \underset{\tau\to+\infty}{=} &\big(\mathcal{P}_{0,k}^{\triangledown}(\tau)+\mathcal{P}_{1,k}^{\triangledown}(\tau)\widetilde{\Lambda}+\mathcal{P}_{2,k}^{\triangledown}(\tau)\widetilde{\Lambda}^2+\mathcal{O}\big(\mathcal{G}_{1,k}^{-1}\mathcal{G}_{0,k}^{-1}\mathbb{E}_k(\tau)\mathcal{G}_{0,k}\mathcal{G}_{1,k}\widetilde{\Lambda}^3\big)\big)\\ &\times\hat{\Phi}_k(\widetilde{\Lambda}),\end{aligned} \tag{3.132}$$

where

$$\begin{aligned}\mathcal{P}_{0,k}^{\triangledown}(\tau) &:= \mathcal{G}_{1,k}^{-1}\mathcal{P}_{0,k}^{\triangle}(\tau)\mathcal{G}_{1,k} = \mathrm{i}\omega_{0,k}\sigma_3+\tau^{1/3}\sigma_+,\\ \mathcal{P}_{1,k}^{\triangledown}(\tau) &:= \mathcal{G}_{1,k}^{-1}\mathcal{P}_{1,k}^{\triangle}(\tau)\mathcal{G}_{1,k} = \mathrm{i}4\sqrt{3}\mathcal{Z}_k\sigma_3-\mathrm{i}8\sqrt{3}(\mathrm{i}\omega_{0,k}-\mathfrak{A}_k)\mathcal{Z}_k\tau^{-1/3}\sigma_-,\\ \mathcal{P}_{2,k}^{\triangledown}(\tau) &:= \mathcal{G}_{1,k}^{-1}\mathcal{P}_{2,k}^{\triangle}(\tau)\mathcal{G}_{1,k}\\ &= \begin{pmatrix}\mathfrak{A}_{0,k}^{\sharp}+\frac{(\mathrm{i}\omega_{0,k}-\mathfrak{A}_k)\mathfrak{B}_{0,k}^{\sharp}}{\mathfrak{B}_k} & \frac{\mathfrak{B}_{0,k}^{\sharp}\tau^{1/3}}{\mathfrak{B}_k}\\ \frac{2(\mathrm{i}\omega_{0,k}-\mathfrak{A}_k)(\mathfrak{A}_k\mathfrak{B}_{0,k}^{\sharp}-\mathfrak{B}_k\mathfrak{A}_{0,k}^{\sharp})+(\mathfrak{B}_k\mathfrak{C}_{0,k}^{\sharp}-\mathfrak{C}_k\mathfrak{B}_{0,k}^{\sharp})\mathfrak{B}_k}{\mathfrak{B}_k\tau^{1/3}} & -\big(\mathfrak{A}_{0,k}^{\sharp}+\frac{(\mathrm{i}\omega_{0,k}-\mathfrak{A}_k)\mathfrak{B}_{0,k}^{\sharp}}{\mathfrak{B}_k}\big)\end{pmatrix}.\end{aligned}$$

Note that, at this stage, the matrix $\mathcal{P}_{1,k}^{\triangledown}(\tau)$ is not diagonal; instead, it now contains an additional, lower off-diagonal contribution.

(v) This step entails a straightforward τ-dependent scaling. Let $\hat{\Phi}_k(\widetilde{\Lambda})$ solve equation (3.132); then, applying the transformation $\mathfrak{F}_5$, one shows that, for $k=\pm1$,

$$\begin{aligned}\frac{\partial\hat{\Phi}_{0,k}(\widetilde{\Lambda})}{\partial\widetilde{\Lambda}} \underset{\tau\to+\infty}{=} &\big(\tilde{\mathcal{P}}_{0,k}^{\blacktriangle}(\tau)+\tilde{\mathcal{P}}_{1,k}^{\blacktriangle}(\tau)\widetilde{\Lambda}+\tilde{\mathcal{P}}_{2,k}^{\blacktriangle}(\tau)\widetilde{\Lambda}^2\\ &+\mathcal{O}\big(\tau^{-\frac{1}{6}\sigma_3}\mathcal{G}_{1,k}^{-1}\mathcal{G}_{0,k}^{-1}\mathbb{E}_k(\tau)\mathcal{G}_{0,k}\mathcal{G}_{1,k}\tau^{\frac{1}{6}\sigma_3}\widetilde{\Lambda}^3\big)\big)\hat{\Phi}_{0,k}(\widetilde{\Lambda}),\end{aligned} \tag{3.133}$$

where

$$\begin{aligned}\tilde{\mathcal{P}}_{0,k}^{\blacktriangle}(\tau) &:= \tau^{-\frac{1}{6}\sigma_3}\mathcal{P}_{0,k}^{\triangledown}(\tau)\tau^{\frac{1}{6}\sigma_3} = \mathrm{i}\omega_{0,k}\sigma_3+\sigma_+,\\ \tilde{\mathcal{P}}_{1,k}^{\blacktriangle}(\tau) &:= \tau^{-\frac{1}{6}\sigma_3}\mathcal{P}_{1,k}^{\triangledown}(\tau)\tau^{\frac{1}{6}\sigma_3} = \mathrm{i}4\sqrt{3}\mathcal{Z}_k\sigma_3-\mathrm{i}8\sqrt{3}(\mathrm{i}\omega_{0,k}-\mathfrak{A}_k)\mathcal{Z}_k\sigma_-,\\ \tilde{\mathcal{P}}_{2,k}^{\blacktriangle}(\tau) &:= \tau^{-\frac{1}{6}\sigma_3}\mathcal{P}_{2,k}^{\triangledown}(\tau)\tau^{\frac{1}{6}\sigma_3}\\ &= \begin{pmatrix}\mathfrak{A}_{0,k}^{\sharp}+\frac{(\mathrm{i}\omega_{0,k}-\mathfrak{A}_k)\mathfrak{B}_{0,k}^{\sharp}}{\mathfrak{B}_k} & \frac{\mathfrak{B}_{0,k}^{\sharp}}{\mathfrak{B}_k}\\ \frac{2(\mathrm{i}\omega_{0,k}-\mathfrak{A}_k)(\mathfrak{A}_k\mathfrak{B}_{0,k}^{\sharp}-\mathfrak{B}_k\mathfrak{A}_{0,k}^{\sharp})+(\mathfrak{B}_k\mathfrak{C}_{0,k}^{\sharp}-\mathfrak{C}_k\mathfrak{B}_{0,k}^{\sharp})\mathfrak{B}_k}{\mathfrak{B}_k} & -\big(\mathfrak{A}_{0,k}^{\sharp}+\frac{(\mathrm{i}\omega_{0,k}-\mathfrak{A}_k)\mathfrak{B}_{0,k}^{\sharp}}{\mathfrak{B}_k}\big)\end{pmatrix}.\end{aligned} \tag{3.134}$$

(vi) The purpose of this step is to transform the coefficient matrix $\tilde{\mathcal{P}}_{0,k}^{\blacktriangle}(\tau)$ (cf. (3.134)) into off-diagonal form. Let $\hat{\Phi}_{0,k}(\widetilde{\Lambda})$ solve equation (3.133); then, applying the transformation $\mathfrak{F}_6$, one shows that, for $k=\pm1$,

$$\frac{\partial\Phi_{0,k}(\widetilde{\Lambda})}{\partial\widetilde{\Lambda}} \underset{\tau\to+\infty}{=} \left(\begin{pmatrix}0 & 1\\ -\omega_{0,k}^2 & 0\end{pmatrix}+\begin{pmatrix}\mathrm{i}4\sqrt{3}\mathcal{Z}_k & 0\\ \mathrm{i}8\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k & -\mathrm{i}4\sqrt{3}\mathcal{Z}_k\end{pmatrix}\widetilde{\Lambda}+\begin{pmatrix}\mathfrak{P}_{0,k}^{*} & \mathfrak{Q}_{0,k}^{*}\\ \mathfrak{R}_{0,k}^{*} & -\mathfrak{P}_{0,k}^{*}\end{pmatrix}\widetilde{\Lambda}^2\right.$$

$$+ \mathcal{O}\big(\mathbb{E}_k^*(\tau)\widetilde{\Lambda}^3\big)\Big)\,\Phi_{0,k}\big(\widetilde{\Lambda}\big), \tag{3.135}$$

where

$$\mathfrak{P}_{0,k}^* := \mathfrak{A}_{0,k}^\sharp - \mathfrak{B}_{0,k}^\sharp \mathfrak{A}_k \mathfrak{B}_k^{-1}, \tag{3.136}$$

$$\mathfrak{Q}_{0,k}^* := \mathfrak{B}_{0,k}^\sharp \mathfrak{B}_k^{-1}, \tag{3.137}$$

$$\mathfrak{R}_{0,k}^* := -\mathfrak{B}_{0,k}^\sharp \mathfrak{A}_k^2 \mathfrak{B}_k^{-1} + 2\mathfrak{A}_k \mathfrak{A}_{0,k}^\sharp + \mathfrak{B}_k \mathfrak{C}_{0,k}^\sharp, \tag{3.138}$$

and

$$\mathbb{E}_k^*(\tau) := (\mathrm{I} + \mathrm{i}\omega_{0,k}\sigma_-)\,\tau^{-\frac{1}{6}\sigma_3}\mathcal{G}_{1,k}^{-1}\mathcal{G}_{0,k}^{-1}\hat{\mathbb{E}}_k(\tau)\mathcal{G}_{0,k}\mathcal{G}_{1,k}\tau^{\frac{1}{6}\sigma_3}\,(\mathrm{I} - \mathrm{i}\omega_{0,k}\sigma_-)\,. \tag{3.139}$$

(vii) This step, in conjunction with steps (viii) and (x) below, is precipitated by the fact that, in order to derive a (canonical) model problem solvable in terms of parabolic-cylinder functions (see step (xi) below), one must eliminate the coefficient matrix of the $\widetilde{\Lambda}^2$ term from equation (3.135); in particular, this step focuses on the excision of the (12)-element. Let $\Phi_{0,k}\big(\widetilde{\Lambda}\big)$ solve equation (3.135); then, applying the transformation $\mathfrak{F}_7$, with τ-dependent parameter $\ell_{0,k}$, one shows, via the conditions (3.17), that, for $k = \pm 1$,

$$\begin{aligned}\frac{\partial \Phi_{0,k}^\flat\big(\widetilde{\Lambda}\big)}{\partial \widetilde{\Lambda}} \underset{\tau \to +\infty}{=} &\left(\begin{pmatrix} 0 & -\ell_{0,k}+1 \\ -\omega_{0,k}^2 & 0 \end{pmatrix} + \begin{pmatrix} \mathrm{i}4\sqrt{3}\mathcal{Z}_k + \omega_{0,k}^2\ell_{0,k} & 0 \\ \mathrm{i}8\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k & -\mathrm{i}4\sqrt{3}\mathcal{Z}_k - \omega_{0,k}^2\ell_{0,k} \end{pmatrix}\widetilde{\Lambda}\right. \\ &+ \begin{pmatrix} -\mathrm{i}8\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\ell_{0,k} + \mathfrak{P}_{0,k}^* & \omega_{0,k}^2\ell_{0,k}^2 + \mathrm{i}8\sqrt{3}\mathcal{Z}_k\ell_{0,k} + \mathfrak{Q}_{0,k}^* \\ \mathfrak{R}_{0,k}^* & \mathrm{i}8\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\ell_{0,k} - \mathfrak{P}_{0,k}^* \end{pmatrix}\widetilde{\Lambda}^2 \\ &+ \left.\mathcal{O}\big(\mathbb{E}_k^\triangledown(\ell_{0,k};\tau)\widetilde{\Lambda}^3\big)\right)\Phi_{0,k}^\flat\big(\widetilde{\Lambda}\big),\end{aligned} \tag{3.140}$$

where

$$\mathbb{E}_k^\triangledown(\ell_{0,k};\tau) := \mathbb{E}_k^*(\tau) + \begin{pmatrix} -\mathfrak{R}_{0,k}^*\ell_{0,k} & -\mathrm{i}8\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\ell_{0,k}^2 + 2\mathfrak{P}_{0,k}^*\ell_{0,k} \\ 0 & \mathfrak{R}_{0,k}^*\ell_{0,k} \end{pmatrix}, \tag{3.141}$$

with $\mathbb{E}_k^*(\tau)$ defined by equation (3.139). One now chooses $\ell_{0,k}$ so that the (12)-element of the coefficient matrix of the $\widetilde{\Lambda}^2$ term in equation (3.140) is equal to zero, that is, $\omega_{0,k}^2\ell_{0,k}^2 + \mathrm{i}8\sqrt{3}\mathcal{Z}_k\ell_{0,k} + \mathfrak{Q}_{0,k}^* = 0$; the roots are given by

$$\ell_{0,k}^\pm = \frac{-\mathrm{i}8\sqrt{3}\mathcal{Z}_k \pm \sqrt{\big(\mathrm{i}8\sqrt{3}\mathcal{Z}_k\big)^2 - 4\omega_{0,k}^2\mathfrak{Q}_{0,k}^*}}{2\omega_{0,k}^2}, \qquad k = \pm 1. \tag{3.142}$$

Noting from the conditions (3.17), the asymptotics (3.21) and (3.24), equations (3.125) and (3.128), and the definitions (3.121), (3.130), and (3.137) that $\mathcal{Z}_k \underset{\tau \to +\infty}{=} 1 + \mathcal{O}\big(\tau^{-2/3}\big)$, $\omega_{0,k}^2 \underset{\tau \to +\infty}{=} \mathcal{O}\big(\tau^{-2/3}\big)$, and $\mathfrak{Q}_{0,k}^* \underset{\tau \to +\infty}{=} \mathcal{O}(1)$, it follows that, for the class of functions consistent with the conditions (3.17), the '+-root' in equation (3.142) is chosen

$$\ell_{0,k} := \ell_{0,k}^+ = \frac{-\mathrm{i}8\sqrt{3}\mathcal{Z}_k + \sqrt{\big(\mathrm{i}8\sqrt{3}\mathcal{Z}_k\big)^2 - 4\omega_{0,k}^2\mathfrak{Q}_{0,k}^*}}{2\omega_{0,k}^2}. \tag{3.143}$$

Via the formula for the τ-dependent parameter $\ell_{0,k} := \ell_{0,k}^+$ given in equation (3.143), one rewrites equation (3.140) as follows: for $k = \pm 1$,

$$\frac{\partial \Phi_{0,k}^\flat\big(\widetilde{\Lambda}\big)}{\partial \widetilde{\Lambda}} \underset{\tau \to +\infty}{=} \left(\begin{pmatrix} 0 & -\ell_{0,k}^+ + 1 \\ -\omega_{0,k}^2 & 0 \end{pmatrix} + \begin{pmatrix} \mathrm{i}4\sqrt{3}\mathcal{Z}_k + \omega_{0,k}^2\ell_{0,k}^+ & 0 \\ \mathrm{i}8\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k & -\mathrm{i}4\sqrt{3}\mathcal{Z}_k - \omega_{0,k}^2\ell_{0,k}^+ \end{pmatrix}\widetilde{\Lambda}\right.$$

$$+\begin{pmatrix} -\mathrm{i}8\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\ell_{0,k}^{+}+\mathfrak{P}_{0,k}^{*} & 0 \\ \mathfrak{R}_{0,k}^{*} & \mathrm{i}8\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\ell_{0,k}^{+}-\mathfrak{P}_{0,k}^{*} \end{pmatrix}\widetilde{\Lambda}^2$$
$$\left.+\mathcal{O}\big(\mathbb{E}_k^{\triangledown}(\ell_{0,k}^{+};\tau)\widetilde{\Lambda}^3\big)\right)\Phi_{0,k}^{\flat}(\widetilde{\Lambda}). \tag{3.144}$$

(viii) This step focuses on the excision of the (21)-element from the coefficient matrix of the $\widetilde{\Lambda}^2$ term in equation (3.144). Let $\Phi_{0,k}^{\flat}(\widetilde{\Lambda})$ solve equation (3.144); then, under the action of the transformation $\mathfrak{F}_8$, with τ-dependent parameter $\ell_{1,k}$, one shows that, for $k=\pm1$,

$$\frac{\partial\Phi_{0,k}^{\sharp}(\widetilde{\Lambda})}{\partial\widetilde{\Lambda}} \underset{\tau\to+\infty}{=} \left(\begin{pmatrix} 0 & -\ell_{0,k}^{+}+1 \\ -\omega_{0,k}^2-\ell_{1,k} & 0 \end{pmatrix}+\big(\big(\mathrm{i}4\sqrt{3}\mathcal{Z}_k+\omega_{0,k}^2\ell_{0,k}^{+}+\ell_{1,k}(-\ell_{0,k}^{+}+1)\big)\sigma_3\right.$$
$$+\mathrm{i}8\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\sigma_-\big)\widetilde{\Lambda}+\big(\big(\mathfrak{R}_{0,k}^{*}-2\big(\mathrm{i}4\sqrt{3}\mathcal{Z}_k+\omega_{0,k}^2\ell_{0,k}^{+}\big)\ell_{1,k}$$
$$-\ell_{1,k}^2(-\ell_{0,k}^{+}+1)\big)\sigma_--\big(\mathrm{i}8\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\ell_{0,k}^{+}-\mathfrak{P}_{0,k}^{*}\big)\sigma_3\big)\widetilde{\Lambda}^2$$
$$\left.+\mathcal{O}\big(\overset{*}{\mathbb{E}}{}_k^{\blacktriangle}(\ell_{0,k}^{+},\ell_{1,k};\tau)\widetilde{\Lambda}^3\big)\right)\Phi_{0,k}^{\sharp}(\widetilde{\Lambda}), \tag{3.145}$$

where

$$\overset{*}{\mathbb{E}}{}_k^{\blacktriangle}(\ell_{0,k}^{+},\ell_{1,k};\tau):=\mathbb{E}_k^{\triangledown}(\ell_{0,k}^{+};\tau)+2\ell_{1,k}\big(-\mathfrak{P}_{0,k}^{*}+\mathrm{i}8\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\ell_{0,k}^{+}\big)\sigma_-. \tag{3.146}$$

One now chooses $\ell_{1,k}$ so that the (21)-element of the coefficient matrix of the $\widetilde{\Lambda}^2$ term in equation (3.145) vanishes, that is, $(-\ell_{0,k}^{+}+1)\ell_{1,k}^2+2\big(\mathrm{i}4\sqrt{3}\mathcal{Z}_k+\omega_{0,k}^2\ell_{0,k}^{+}\big)\ell_{1,k}-\mathfrak{R}_{0,k}^{*}=0$; the roots are given by

$$\ell_{1,k}^{\pm}=\frac{-\big(\mathrm{i}4\sqrt{3}\mathcal{Z}_k+\omega_{0,k}^2\ell_{0,k}^{+}\big)\pm\sqrt{\big(\mathrm{i}4\sqrt{3}\mathcal{Z}_k+\omega_{0,k}^2\ell_{0,k}^{+}\big)^2+\mathfrak{R}_{0,k}^{*}(-\ell_{0,k}^{+}+1)}}{-\ell_{0,k}^{+}+1},$$
$$k=\pm1. \tag{3.147}$$

Noting from the conditions (3.17), the asymptotics (3.21) and (3.24), equations (3.124)–(3.129), and the definition (3.138) that $\mathfrak{R}_{0,k}^{*}\underset{\tau\to+\infty}{=}\mathcal{O}(\tau^{-2/3})$, and, recalling (from step (vii) above) the asymptotics $\mathcal{Z}_k\underset{\tau\to+\infty}{=}1+\mathcal{O}(\tau^{-2/3})$, $\omega_{0,k}^2\underset{\tau\to+\infty}{=}\mathcal{O}(\tau^{-2/3})$, and $\mathfrak{Q}_{0,k}^{*}\underset{\tau\to+\infty}{=}\mathcal{O}(1)$, it follows from the definition (3.143) for $\ell_{0,k}^{+}$ that, for the class of functions consistent with the conditions (3.17), the '+-root' in equation (3.147) is taken

$$\ell_{1,k}:=\ell_{1,k}^{+}=\frac{-\big(\mathrm{i}4\sqrt{3}\mathcal{Z}_k+\omega_{0,k}^2\ell_{0,k}^{+}\big)+\chi_k(\tau)}{-\ell_{0,k}^{+}+1}, \tag{3.148}$$

where

$$\chi_k(\tau):=\sqrt{\big(\mathrm{i}4\sqrt{3}\mathcal{Z}_k+\omega_{0,k}^2\ell_{0,k}^{+}\big)^2+\mathfrak{R}_{0,k}^{*}(-\ell_{0,k}^{+}+1)}. \tag{3.149}$$

Via the formula for the τ-dependent parameter $\ell_{1,k}:=\ell_{1,k}^{+}$ defined by equations (3.148) and (3.149), one rewrites equation (3.145) as follows: for $k=\pm1$,

$$\frac{\partial\Phi_{0,k}^{\sharp}(\widetilde{\Lambda})}{\partial\widetilde{\Lambda}} \underset{\tau\to+\infty}{=} \left(\begin{pmatrix} 0 & -\ell_{0,k}^{+}+1 \\ -\omega_{0,k}^2-\ell_{1,k}^{+} & 0 \end{pmatrix}+\big(\chi_k(\tau)\sigma_3+\mathrm{i}8\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\sigma_-\big)\widetilde{\Lambda}\right.$$
$$\left.+\big(\mathfrak{P}_{0,k}^{*}-\mathrm{i}8\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\ell_{0,k}^{+}\big)\widetilde{\Lambda}^2\sigma_3+\mathcal{O}\big(\overset{*}{\mathbb{E}}{}_k^{\blacktriangle}(\ell_{0,k}^{+},\ell_{1,k}^{+};\tau)\widetilde{\Lambda}^3\big)\right)\Phi_{0,k}^{\sharp}(\widetilde{\Lambda}). \tag{3.150}$$

(ix) This step is necessitated by the fact that the coefficient matrix of the $\widetilde{\Lambda}$ term in equation (3.150) remains to be re-diagonalised. Let $\Phi^{\sharp}_{0,k}(\widetilde{\Lambda})$ solve equation (3.150); then, under the action of the transformation $\mathfrak{F}_9$, where

$$\mathcal{G}_{2,k} := \begin{pmatrix} 1 & 0 \\ \frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k}{\chi_k(\tau)} & 1 \end{pmatrix}, \qquad k=\pm 1, \tag{3.151}$$

with $\mathcal{Z}_k$, $\mathfrak{A}_k$, and $\chi_k(\tau)$ defined by equations (3.121), (3.124), and (3.149), respectively, one shows that

$$\begin{aligned}\frac{\partial \Phi^{\natural}_{0,k}(\widetilde{\Lambda})}{\partial \widetilde{\Lambda}} \underset{\tau\to+\infty}{=} \Bigg(&\begin{pmatrix} \frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k}{\chi_k(\tau)}(-\ell^{+}_{0,k}+1) & -\ell^{+}_{0,k}+1 \\ -\big(\frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k}{\chi_k(\tau)}\big)^2(-\ell^{+}_{0,k}+1)-\ell^{+}_{1,k}-\omega^2_{0,k} & -\frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k}{\chi_k(\tau)}(-\ell^{+}_{0,k}+1) \end{pmatrix} \\ &+\chi_k(\tau)\widetilde{\Lambda}\sigma_3 + \big(\mathfrak{P}^{*}_{0,k}-\mathrm{i}8\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\ell^{+}_{0,k}\big)\begin{pmatrix} 1 & 0 \\ -\frac{\mathrm{i}8\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k}{\chi_k(\tau)} & -1\end{pmatrix}\widetilde{\Lambda}^2 \\ &+\mathcal{O}\big(\mathcal{G}^{-1}_{2,k}\overset{*}{\mathbb{E}}{}^{\blacktriangle}_k(\ell^{+}_{0,k},\ell^{+}_{1,k};\tau)\mathcal{G}_{2,k}\widetilde{\Lambda}^3\big)\Bigg)\Phi^{\natural}_{0,k}(\widetilde{\Lambda}).\end{aligned} \tag{3.152}$$

(x) This penultimate step focuses on the annihilation of the nilpotent coefficient sub-matrix of the $\widetilde{\Lambda}^2$ term in equation (3.152). Let $\Phi^{\natural}_{0,k}(\widetilde{\Lambda})$ solve equation (3.152); then, under the action of the transformation $\mathfrak{F}_{10}$, with τ-dependent parameter $\ell_{2,k}$, one shows that, for $k=\pm 1$,

$$\begin{aligned}&\frac{\partial \Phi^{*}_{k}(\widetilde{\Lambda})}{\partial \widetilde{\Lambda}} \\ &\underset{\tau\to+\infty}{=} \Bigg(\begin{pmatrix} \frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k}{\chi_k(\tau)}(-\ell^{+}_{0,k}+1) & -\ell^{+}_{0,k}+1 \\ -\big(\frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k}{\chi_k(\tau)}\big)^2(-\ell^{+}_{0,k}+1)-\ell^{+}_{1,k}-\ell_{2,k}-\omega^2_{0,k} & -\frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k}{\chi_k(\tau)}(-\ell^{+}_{0,k}+1) \end{pmatrix} \\ &+\begin{pmatrix} \chi_k(\tau)+\ell_{2,k}(-\ell^{+}_{0,k}+1) & 0 \\ -\frac{\mathrm{i}8\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k}{\chi_k(\tau)}\ell_{2,k}(-\ell^{+}_{0,k}+1) & -\big(\chi_k(\tau)+\ell_{2,k}(-\ell^{+}_{0,k}+1)\big)\end{pmatrix}\widetilde{\Lambda} \\ &+\Bigg(\bigg(-\ell^2_{2,k}(-\ell^{+}_{0,k}+1)-2\ell_{2,k}\chi_k(\tau)-\frac{\mathrm{i}8\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k}{\chi_k(\tau)}\big(\mathfrak{P}^{*}_{0,k}-\mathrm{i}8\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\ell^{+}_{0,k}\big)\bigg) \\ &\quad\times\sigma_{-}+\big(\mathfrak{P}^{*}_{0,k}-\mathrm{i}8\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\ell^{+}_{0,k}\big)\sigma_3\Bigg)\widetilde{\Lambda}^2 \\ &+\mathcal{O}\big(\overset{*}{\mathbb{E}}{}^{*}_k(\ell^{+}_{0,k},\ell^{+}_{1,k},\ell_{2,k};\tau)\widetilde{\Lambda}^3\big)\Bigg)\Phi^{*}_{k}(\widetilde{\Lambda}),\end{aligned} \tag{3.153}$$

where

$$\overset{*}{\mathbb{E}}{}^{*}_k(\ell^{+}_{0,k},\ell^{+}_{1,k},\ell_{2,k};\tau) := \mathcal{G}^{-1}_{2,k}\overset{*}{\mathbb{E}}{}^{\blacktriangle}_k(\ell^{+}_{0,k},\ell^{+}_{1,k};\tau)\mathcal{G}_{2,k}-2\ell_{2,k}\big(\mathfrak{P}^{*}_{0,k}-\mathrm{i}8\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\ell^{+}_{0,k}\big)\sigma_{-}. \tag{3.154}$$

One now chooses $\ell_{2,k}$ so that the (21)-element of the nilpotent coefficient matrix of the $\widetilde{\Lambda}^2$ terms in equation (3.153) is equal to zero, that is,

$$\big(-\ell^{+}_{0,k}+1\big)\ell^2_{2,k}+2\chi_k(\tau)\ell_{2,k}+\mathrm{i}8\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\chi^{-1}_k(\tau)\big(\mathfrak{P}^{*}_{0,k}-\mathrm{i}8\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\ell^{+}_{0,k}\big)=0;$$

the roots are given by

$$\ell^{\pm}_{2,k} = \frac{-\chi_k(\tau)\pm\sqrt{\chi^2_k(\tau)-\mathrm{i}8\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\chi^{-1}_k(\tau)\big(-\ell^{+}_{0,k}+1\big)\big(\mathfrak{P}^{*}_{0,k}-\mathrm{i}8\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\ell^{+}_{0,k}\big)}}{-\ell^{+}_{0,k}+1}, \quad k=\pm 1. \tag{3.155}$$

Arguing as in steps (vii) and (viii) above, for the class of functions consistent with the conditions (3.17), the '+-root' in equation (3.155) is taken

$$\ell_{2,k} := \ell^{+}_{2,k} = \frac{-\chi_k(\tau) + \mu_k(\tau)}{-\ell^{+}_{0,k} + 1}, \tag{3.156}$$

where

$$\mu_k(\tau) := \sqrt{\chi_k^2(\tau) - \mathrm{i}8\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\chi_k^{-1}(\tau)\big(-\ell^{+}_{0,k}+1\big)\big(\mathfrak{P}^{*}_{0,k} - \mathrm{i}8\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\ell^{+}_{0,k}\big)}, \tag{3.157}$$

with $\chi_k(\tau)$ defined by equation (3.149). Via the formula for the τ-dependent parameter $\ell_{2,k} := \ell^{+}_{2,k}$ defined by equations (3.156) and (3.157), one simplifies equation (3.153) to read

$$\frac{\partial \Phi^{*}_k(\widetilde{\Lambda})}{\partial \widetilde{\Lambda}} \underset{\tau\to+\infty}{=} \big(\daleth_k(\tau,\widetilde{\Lambda}) + \mathcal{O}\big(\beth_k(\tau,\widetilde{\Lambda})\big)\big)\Phi^{*}_k(\widetilde{\Lambda}), \qquad k=\pm1, \tag{3.158}$$

where

$$\daleth_k(\tau,\widetilde{\Lambda}) := \mu_k(\tau)\widetilde{\Lambda}\sigma_3 + p_k(\tau)\sigma_+ + q_k(\tau)\sigma_-, \tag{3.159}$$

with

$$p_k(\tau) := -\ell^{+}_{0,k} + \hat{\mathbb{L}}_k(\tau) + 1, \tag{3.160}$$

$$q_k(\tau) := \big(4\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\chi_k^{-1}(\tau)\big)^2\big(-\ell^{+}_{0,k}+1\big) - \ell^{+}_{1,k} - \ell^{+}_{2,k} - \omega^2_{0,k}, \tag{3.161}$$

and

$$\begin{aligned}\beth_k(\tau,\widetilde{\Lambda}) :=\,& \frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k}{\chi_k(\tau)}\big(-\ell^{+}_{0,k}+1\big)\sigma_3 - \hat{\mathbb{L}}_k(\tau)\sigma_+ - \frac{\mathrm{i}8\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k}{\chi_k(\tau)}\ell^{+}_{2,k}\big(-\ell^{+}_{0,k}+1\big)\widetilde{\Lambda}\sigma_- \\ &+ \big(\mathfrak{P}^{*}_{0,k} - \mathrm{i}8\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\ell^{+}_{0,k}\big)\widetilde{\Lambda}^2\sigma_3 + \overset{*}{\mathbb{E}}{}^{*}_k\big(\ell^{+}_{0,k},\ell^{+}_{1,k},\ell^{+}_{2,k};\tau\big)\widetilde{\Lambda}^3,\end{aligned} \tag{3.162}$$

where the yet-to-be-determined scalar function $\hat{\mathbb{L}}_k(\tau)$ is chosen in the proof of Lemma 4.1.[25]

(xi) The rationale for this (final) step is to transform equation (3.158) into a 'model' matrix linear ODE describing the coalescence of turning points. Let $\Phi_{M,k}(\widetilde{\Lambda})$, $k=\pm1$, be a fundamental solution of equation (3.114); then, changing variables according to $\widetilde{\Lambda} = \widetilde{\Lambda}(z) = a^{*}_k(\tau)b^{*}z$, where $a^{*}_k(\tau) := \big(\mathrm{i}4\sqrt{3}/\mu_k(\tau)\big)^{1/2}$ and $b^{*} := 2^{-3/2}3^{-1/4}\mathrm{e}^{-\mathrm{i}\pi/4}$, and defining $\phi_{M,k}(z) := \Phi_{M,k}(\widetilde{\Lambda}(z))$, one shows that $\phi_{M,k}(z)$ solves the canonical matrix ODE

$$\partial_z\phi_{M,k}(z) = \Big(\frac{z}{2}\sigma_3 + P_k(\tau)\sigma_+ + Q_k(\tau)\sigma_-\Big)\phi_{M,k}(z), \qquad k=\pm1,$$

where $P_k(\tau) := a^{*}_k(\tau)b^{*}p_k(\tau)$ and $Q_k(\tau) := a^{*}_k(\tau)b^{*}q_k(\tau)$, with fundamental solution expressed in terms of the parabolic-cylinder function $D_{\star}(\cdot)$,[26]

$$\phi_{M,k}(z) = \begin{pmatrix} D_{-\nu(k)-1}(\mathrm{i}z) & D_{\nu(k)}(z) \\ \frac{1}{P_k(\tau)}\big(\frac{\partial}{\partial z} - \frac{z}{2}\big)D_{-\nu(k)-1}(\mathrm{i}z) & \frac{1}{P_k(\tau)}\big(\frac{\partial}{\partial z} - \frac{z}{2}\big)D_{\nu(k)}(z)\end{pmatrix}, \tag{3.163}$$

where $-(\nu(k)+1) := P_k(\tau)Q_k(\tau)$. Inverting the dependent- and independent-variable linear transformations given above, one arrives at the formula for the parameter $\nu(k)+1$ defined by equation (3.109) and the representation for $\Phi_{M,k}(\widetilde{\Lambda})$ given in equation (3.115).

[25]It will be shown that $\hat{\mathbb{L}}_k(\tau) \underset{\tau\to+\infty}{=} \mathcal{O}(\tau^{-2/3})$, $k\in\{\pm1\}$: this fact will be used throughout the remainder of the proof.

[26]See, for example, [29, 41, 45].

Finally, in order to establish the asymptotic representation (3.111), one has to estimate the unimodular function $\hat{\chi}_k(\widetilde{\Lambda})$ defined in the transformation $\mathfrak{F}_{11}$. Under the action of the transformation $\mathfrak{F}_{11}$, one rewrites equation (3.158) as follows:

$$\frac{\partial \hat{\chi}_k(\widetilde{\Lambda})}{\partial \widetilde{\Lambda}} \underset{\tau\to+\infty}{=} \daleth_k(\tau,\widetilde{\Lambda})\hat{\chi}_k(\widetilde{\Lambda}) + \big[\gimel_k(\tau,\widetilde{\Lambda}), \hat{\chi}_k(\widetilde{\Lambda})\big], \qquad k=\pm 1, \tag{3.164}$$

where $\gimel_k(\tau,\widetilde{\Lambda})$ is defined by (3.159)–(3.161), and $\daleth_k(\tau,\widetilde{\Lambda})$ is defined by equation (3.162). The normalised solution of equation (3.164), that is, the one for which $\hat{\chi}_k(0)=\mathrm{I}$, is given by

$$\hat{\chi}_k(\widetilde{\Lambda}) = \mathrm{I} + \int_0^{\widetilde{\Lambda}} \Phi_{M,k}(\widetilde{\Lambda})\Phi_{M,k}^{-1}(\xi)\daleth_k(\tau,\xi)\hat{\chi}_k(\xi)\Phi_{M,k}(\xi)\Phi_{M,k}^{-1}(\widetilde{\Lambda})\mathrm{d}\xi, \qquad k=\pm 1.$$

In order to prove the required estimate for $\hat{\chi}_k(\widetilde{\Lambda})$, one uses the method of successive approximations, namely,

$$\hat{\chi}_k^{(m)}(\widetilde{\Lambda}) = \mathrm{I} + \int_0^{\widetilde{\Lambda}} \Phi_{M,k}(\widetilde{\Lambda})\Phi_{M,k}^{-1}(\xi)\daleth_k(\tau,\xi)\hat{\chi}_k^{(m-1)}(\xi)\Phi_{M,k}(\xi)\Phi_{M,k}^{-1}(\widetilde{\Lambda})\mathrm{d}\xi,$$
$$k=\pm 1, \qquad m\in\mathbb{N},$$

with $\hat{\chi}_k^{(0)}(\widetilde{\Lambda}) \equiv \mathrm{I}$, to construct a Neumann series solution for $\hat{\chi}_k(\widetilde{\Lambda})$ $\big(\hat{\chi}_k(\widetilde{\Lambda}) := \lim_{m\to\infty}\hat{\chi}_k^{(m)}(\widetilde{\Lambda})\big)$; in this instance, however, it suffices to estimate the matrix norm of the associated resolvent kernel. Via the above iteration argument, a calculation shows that, for $k=\pm 1$,

$$\big|\big|\hat{\chi}_k(\widetilde{\Lambda}) - \mathrm{I}\big|\big| \underset{\tau\to+\infty}{\leqslant} \exp\left(\int_0^{\widetilde{\Lambda}} \big|\big|\Phi_{M,k}(\widetilde{\Lambda})\big|\big| ||\Phi_{M,k}^{-1}(\xi)|| ||\daleth_k(\tau,\xi)|| ||\Phi_{M,k}(\xi)|| \big|\big|\Phi_{M,k}^{-1}(\widetilde{\Lambda})\big|\big| |\mathrm{d}\xi|\right) - 1, \tag{3.165}$$

where $|\mathrm{d}\xi|$ denotes integration with respect to arc length. Noting that (see Remark 3.20) $\det(\Phi_{M,k}(z)) = -\mathrm{e}^{-\mathrm{i}\pi(\nu(k)+1)/2}(2\mu_k(\tau))^{1/2}p_k^{-1}(\tau)$, it follows from the estimate (3.165) that, for $k=\pm 1$,

$$\big|\big|\hat{\chi}_k(\widetilde{\Lambda}) - \mathrm{I}\big|\big| \underset{\tau\to+\infty}{\leqslant} \exp\left(\frac{|p_k(\tau)|^2\big|\big|\Phi_{M,k}(\widetilde{\Lambda})\big|\big|^2}{|2\mu_k(\tau)|\big(\mathrm{e}^{\pi\,\mathrm{Im}(\nu(k)+1)/2}\big)^2}\int_0^{\widetilde{\Lambda}} ||\Phi_{M,k}(\xi)||^2 ||\daleth_k(\tau,\xi)|| |\mathrm{d}\xi|\right) - 1. \tag{3.166}$$

One now proceeds to estimate the respective norms in equation (3.166).

One commences with the estimation of the norm $||\daleth_k(\tau,\xi)||$, $k=\pm 1$, appearing in equation (3.166). Via equations (3.118), (3.122), (3.132), (3.133), (3.135), (3.139), (3.140), (3.141), (3.144), (3.145), (3.146), (3.150), (3.152), (3.153), (3.154), and (3.162), one shows that, for $k=\pm 1$, in terms of the composition of the linear transformations $\mathfrak{F}_j$, $j=1,2,\dots,11$,

$$\begin{aligned}\daleth_k(\tau,\widetilde{\Lambda}) &:= (\mathfrak{F}_{11}\circ\mathfrak{F}_{10}\circ\mathfrak{F}_9\circ\mathfrak{F}_8\circ\mathfrak{F}_7\circ\mathfrak{F}_6\circ\mathfrak{F}_5\circ\mathfrak{F}_4\circ\mathfrak{F}_3\circ\mathfrak{F}_2\circ\mathfrak{F}_1)\big(\widetilde{\Psi}(\widetilde{\mu},\tau) - \widetilde{\Psi}_k(\widetilde{\mu},\tau)\big) \\ &= \frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k}{\chi_k(\tau)}\big(-\ell_{0,k}^+ + 1\big)\sigma_3 - \hat{\mathbb{L}}_k(\tau)\sigma_+ - \frac{\mathrm{i}8\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k}{\chi_k(\tau)}\ell_{2,k}^+\big(-\ell_{0,k}^+ + 1\big)\widetilde{\Lambda}\sigma_- \\ &\quad + \big(\mathfrak{P}_{0,k}^* - \mathrm{i}8\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\ell_{0,k}^+\big)\widetilde{\Lambda}^2\sigma_3 + \Bigg(-2\ell_{2,k}^+\big(\mathfrak{P}_{0,k}^* - \mathrm{i}8\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\ell_{0,k}^+\big)\sigma_- \\ &\quad + \mathcal{G}_{2,k}^{-1}\Bigg(\begin{pmatrix}1 & 0\\ \mathrm{i}\omega_{0,k} & 1\end{pmatrix}\tau^{-\frac{1}{6}\sigma_3}\mathcal{G}_{1,k}^{-1}\mathcal{G}_{0,k}^{-1}\hat{\mathbb{E}}_k(\tau)\mathcal{G}_{0,k}\mathcal{G}_{1,k}\tau^{\frac{1}{6}\sigma_3}\begin{pmatrix}1 & 0\\ -\mathrm{i}\omega_{0,k} & 1\end{pmatrix}\end{aligned}$$

$$+\begin{pmatrix} -\mathfrak{R}^{*}_{0,k}\ell^{+}_{0,k} & \ell^{+}_{0,k}\big(2\mathfrak{P}^{*}_{0,k}-\mathrm{i}8\sqrt{3}\mathcal{Z}_{k}\mathfrak{A}_{k}\ell^{+}_{0,k}\big) \\ -2\ell^{+}_{1,k}\big(\mathfrak{P}^{*}_{0,k}-\mathrm{i}8\sqrt{3}\mathcal{Z}_{k}\mathfrak{A}_{k}\ell^{+}_{0,k}\big) & \mathfrak{R}^{*}_{0,k}\ell^{+}_{0,k}\end{pmatrix}\Bigg)\mathcal{G}_{2,k}\Bigg)\widetilde{\Lambda}^{3},$$

whence, via the definitions (3.120), (3.131), (3.136)–(3.138), (3.143), and (3.151), and a matrix-multiplication argument, one arrives at, for $k=\pm1$,

$$\begin{aligned}\beth_{k}(\tau,\widetilde{\Lambda}) &= \frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_{k}\mathfrak{A}_{k}}{\chi_{k}(\tau)}\big(-\ell^{+}_{0,k}+1\big)\sigma_{3}-\hat{\mathbb{L}}_{k}(\tau)\sigma_{+}-\frac{\mathrm{i}8\sqrt{3}\mathcal{Z}_{k}\mathfrak{A}_{k}}{\chi_{k}(\tau)}\ell^{+}_{2,k}\big(-\ell^{+}_{0,k}+1\big)\widetilde{\Lambda}\sigma_{-}\\ &+\big(\mathfrak{A}^{\sharp}_{0,k}+\mathfrak{A}_{k}\omega^{2}_{0,k}\big(\ell^{+}_{0,k}\big)^{2}\big)\widetilde{\Lambda}^{2}\sigma_{3}\\ &+\begin{pmatrix}\mathcal{N}^{*}_{11}(\tau)+\mathcal{M}^{*}_{11}(\tau) & \mathcal{N}^{*}_{12}(\tau)+\mathcal{M}^{*}_{12}(\tau)\\ \mathcal{N}^{*}_{21}(\tau)+\mathcal{M}^{*}_{21}(\tau) & -(\mathcal{N}^{*}_{11}(\tau)+\mathcal{M}^{*}_{11}(\tau))\end{pmatrix}\widetilde{\Lambda}^{3},\end{aligned} \tag{3.167}$$

where

$$\begin{aligned}\mathcal{N}^{*}_{11}(\tau) &:= \ell^{+}_{0,k}\mathfrak{A}_{k}\left(\frac{\mathfrak{A}_{k}\mathfrak{B}^{\sharp}_{0,k}}{\mathfrak{B}_{k}}-2\mathfrak{A}^{\sharp}_{0,k}\right)\left(1-\frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_{k}}{\chi_{k}(\tau)}\right)\\ &-\ell^{+}_{0,k}\left(\mathfrak{B}_{k}\mathfrak{C}^{\sharp}_{0,k}-\frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_{k}}{\chi_{k}(\tau)}\mathfrak{A}^{2}_{k}\omega^{2}_{0,k}\big(\ell^{+}_{0,k}\big)^{2}\right),\end{aligned} \tag{3.168}$$

$$\mathcal{N}^{*}_{12}(\tau) := \ell^{+}_{0,k}\left(2\mathfrak{A}^{\sharp}_{0,k}+\mathfrak{A}_{k}\omega^{2}_{0,k}\big(\ell^{+}_{0,k}\big)^{2}-\frac{\mathfrak{A}_{k}\mathfrak{B}^{\sharp}_{0,k}}{\mathfrak{B}_{k}}\right), \tag{3.169}$$

$$\begin{aligned}\mathcal{N}^{*}_{21}(\tau) &:= -\frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_{k}\mathfrak{A}_{k}}{\chi_{k}(\tau)}\left(\ell^{+}_{0,k}\mathfrak{A}_{k}\left(\frac{\mathfrak{A}_{k}\mathfrak{B}^{\sharp}_{0,k}}{\mathfrak{B}_{k}}-2\mathfrak{A}^{\sharp}_{0,k}\right)\left(2-\frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_{k}}{\chi_{k}(\tau)}\right)\right.\\ &\left.-\ell^{+}_{0,k}\left(2\mathfrak{B}_{k}\mathfrak{C}^{\sharp}_{0,k}-\frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_{k}}{\chi_{k}(\tau)}\mathfrak{A}^{2}_{k}\omega^{2}_{0,k}\big(\ell^{+}_{0,k}\big)^{2}\right)\right)\\ &-2\big(\mathfrak{A}^{\sharp}_{0,k}+\mathfrak{A}_{k}\omega^{2}_{0,k}\big(\ell^{+}_{0,k}\big)^{2}\big)\big(\ell^{+}_{1,k}+\ell^{+}_{2,k}\big),\end{aligned} \tag{3.170}$$

$$\begin{aligned}\mathcal{M}^{*}_{11}(\tau) &:= \frac{\hat{\mathcal{C}}_{1}}{2\lambda^{*}_{1}(k)\mathfrak{B}_{k}}\left(\big(\hat{\mathbb{E}}_{k}(\tau)\big)_{11}\left(\hat{\mathfrak{g}}_{11}\mathfrak{B}_{k}+\hat{\mathfrak{g}}_{12}\mathfrak{A}_{k}\left(\frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_{k}}{\chi_{k}(\tau)}-1\right)\right.\right.\\ &\left.+\hat{\mathfrak{g}}_{12}\left(\mathfrak{B}_{k}+\mathfrak{A}_{k}\left(\frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_{k}}{\chi_{k}(\tau)}-1\right)\right)\right)+\big(\hat{\mathbb{E}}_{k}(\tau)\big)_{12}\left(\mathfrak{B}_{k}+\mathfrak{A}_{k}\left(\frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_{k}}{\chi_{k}(\tau)}-1\right)\right)\\ &\left.-\big(\hat{\mathbb{E}}_{k}(\tau)\big)_{21}\hat{\mathfrak{g}}_{12}\left(\hat{\mathfrak{g}}_{11}\mathfrak{B}_{k}+\hat{\mathfrak{g}}_{12}\mathfrak{A}_{k}\left(\frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_{k}}{\chi_{k}(\tau)}-1\right)\right)\right),\end{aligned} \tag{3.171}$$

$$\mathcal{M}^{*}_{12}(\tau) := \frac{\hat{\mathcal{C}}_{1}}{2\lambda^{*}_{1}(k)\mathfrak{B}_{k}}\big(2\big(\hat{\mathbb{E}}_{k}(\tau)\big)_{11}\hat{\mathfrak{g}}_{12}+\big(\hat{\mathbb{E}}_{k}(\tau)\big)_{12}-\big(\hat{\mathbb{E}}_{k}(\tau)\big)_{21}(\hat{\mathfrak{g}}_{12})^{2}\big), \tag{3.172}$$

$$\begin{aligned}\mathcal{M}^{*}_{21}(\tau) &:= \frac{\hat{\mathcal{C}}_{1}}{2\lambda^{*}_{1}(k)\mathfrak{B}_{k}}\left(-2\big(\hat{\mathbb{E}}_{k}(\tau)\big)_{11}\left(\mathfrak{B}_{k}+\mathfrak{A}_{k}\left(\frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_{k}}{\chi_{k}(\tau)}-1\right)\right)\right.\\ &\times\left(\hat{\mathfrak{g}}_{11}\mathfrak{B}_{k}+\hat{\mathfrak{g}}_{12}\mathfrak{A}_{k}\left(\frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_{k}}{\chi_{k}(\tau)}-1\right)\right)\\ &-\big(\hat{\mathbb{E}}_{k}(\tau)\big)_{12}\left(\mathfrak{B}_{k}+\mathfrak{A}_{k}\left(\frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_{k}}{\chi_{k}(\tau)}-1\right)\right)^{2}\\ &\left.+\big(\hat{\mathbb{E}}_{k}(\tau)\big)_{21}\left(\hat{\mathfrak{g}}_{11}\mathfrak{B}_{k}+\hat{\mathfrak{g}}_{12}\mathfrak{A}_{k}\left(\frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_{k}}{\chi_{k}(\tau)}-1\right)\right)^{2}\right),\end{aligned} \tag{3.173}$$

with

$$\hat{\mathfrak{g}}_{11} := \frac{\hat{\mathcal{A}}_1 + \lambda_1^*(k)}{\hat{\mathcal{C}}_1} \qquad \text{and} \qquad \hat{\mathfrak{g}}_{12} := \frac{\hat{\mathcal{A}}_1 - \lambda_1^*(k)}{\hat{\mathcal{C}}_1}. \tag{3.174}$$

In order to realise the $\tau \to +\infty$ asymptotics for $\beth_k\big(\tau, \widetilde{\Lambda}\big)$, $k = \pm 1$, via equation (3.167) (cf. definitions (3.168)–(3.174)), and subsequently estimate the norm $||\beth_k(\tau,\xi)||$, the $\tau \to +\infty$ asymptotics of the functions $\mathcal{Z}_k$, $\mathcal{G}_{0,k}$, $\mathfrak{A}_k$, $\mathfrak{B}_k$, $\mathfrak{C}_k$, $\mathfrak{A}_{0,k}^\sharp$, $\mathfrak{B}_{0,k}^\sharp$, $\mathfrak{C}_{0,k}^\sharp$, $\omega_{0,k}^2$, $\chi_k(\tau)$, and $\mu_k(\tau)$, and the τ-dependent parameters $\ell_{0,k}^+$, $\ell_{1,k}^+$, and $\ell_{2,k}^+$ are required: for the reader's convenience, they are presented in Appendix B. Substituting the asymptotics (3.21), (3.24), and (B.1)–(B.19) into the definitions (3.168)–(3.174), recalling the definitions (3.136) and (3.138), and using equation (3.167), one arrives at the estimate

$$||\beth_k(\tau,\xi)|| \underset{\tau\to+\infty}{=} \mathcal{O}\big(\tau^{-1/3}|\xi|^3\big), \qquad k = \pm 1. \tag{3.175}$$

There remains the matter of estimating the norm of the unimodular function $\hat{\chi}_k(\xi)$, $k = \pm 1$. In order to do so, one has to derive a uniform approximation for $\hat{\chi}_k(\xi)$ on $\mathbb{R} \cup \mathrm{i}\mathbb{R}$. Towards this goal, one uses the following integral representation for the parabolic-cylinder function (see, for example, [27]); for $k = \pm 1$,

$$\begin{aligned} &D_{\nu(k)}(z) = \frac{2^{\frac{\nu(k)}{2}} \mathrm{e}^{-\frac{z^2}{4}}}{\Gamma(-\nu(k)/2)} \int_0^{+\infty} \mathrm{e}^{-\frac{\xi z^2}{2}} \xi^{-\frac{\nu(k)}{2}-1} (1+\xi)^{\frac{\nu(k)-1}{2}} \mathrm{d}\xi, \\ &\mathrm{Re}(\nu(k)) < 0, \qquad |\arg(z)| \leqslant \pi/4, \end{aligned} \tag{3.176}$$

where $\Gamma(\cdot)$ is the gamma function. As the integral representation (3.176) will be applied simultaneously to the entries of the $\mathrm{M}_2(\mathbb{C})$-valued function (cf. equation (3.115)) $\Phi_{M,k}(\xi)$ in order to arrive at a uniform approximation for $\hat{\chi}_k(\xi)$ on the Stokes rays $\arg(\xi) = 0, \pm\pi/2, \pm\pi, \ldots$, $0 \leqslant |\xi| < +\infty$, it implies the restrictions (3.110) on $\nu(k) + 1$; in fact, for the purposes of this proof, it suffices to have a uniform approximation for $\hat{\chi}_k(\xi)$ on, say, the Stokes rays $\widehat{\boldsymbol{\mathcal{S}}} := \{\xi \in \mathbb{C} \mid 0 \leqslant |\xi| < +\infty, \arg(\xi) = 0, -\pi/2, -\pi, -3\pi/2\}$. Using the following functional relations and values for the gamma function (see, for example, [32])

$$\begin{aligned} &\Gamma(z+1) = z\Gamma(z), \qquad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \qquad \sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+1/2), \\ &\Gamma(1/2) = \sqrt{\pi}, \qquad \int_0^{+\infty} \frac{t^{x-1}}{(1+t)^{x+y}} \mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \qquad \mathrm{Re}(x), \mathrm{Re}(y) > 0, \end{aligned}$$

the linear relations relating any three of the four parabolic-cylinder functions (cf. (3.163)) $D_{-\nu(k)-1}(\pm\mathrm{i}z)$ and $D_{\nu(k)}(\pm z)$,

$$\begin{aligned} &\sqrt{2\pi} D_{\nu(k)}(z) = \Gamma(\nu(k)+1)\big(\mathrm{e}^{\mathrm{i}\pi\nu(k)/2} D_{-\nu(k)-1}(\mathrm{i}z) + \mathrm{e}^{-\mathrm{i}\pi\nu(k)/2} D_{-\nu(k)-1}(-\mathrm{i}z)\big), \\ &D_{\nu(k)}(z) = \mathrm{e}^{-\mathrm{i}\pi\nu(k)} D_{\nu(k)}(-z) + \frac{\sqrt{2\pi}\mathrm{e}^{-\mathrm{i}\pi(\nu(k)+1)/2}}{\Gamma(-\nu(k))} D_{-\nu(k)-1}(\mathrm{i}z), \\ &D_{\nu(k)}(z) = \mathrm{e}^{\mathrm{i}\pi\nu(k)} D_{\nu(k)}(-z) + \frac{\sqrt{2\pi}\mathrm{e}^{\mathrm{i}\pi(\nu(k)+1)/2}}{\Gamma(-\nu(k))} D_{-\nu(k)-1}(-\mathrm{i}z), \end{aligned}$$

and the fact that (see the asymptotics (4.12) below) $\nu(k) + 1 \to 0$ as $\tau \to +\infty$, one arrives at, via the restrictions (3.110) on $\nu(k)+1$, equation (3.115), and the integral representation (3.176), estimates for the moduli $|(\Phi_{M,k}(\xi))_{ij}|$, $k = \pm 1$, $i,j = 1,2$, on the Stokes rays $\widehat{\boldsymbol{\mathcal{S}}}$: for the convenience of the reader, they are stated in Appendix C. To eschew technical redundancies, consider, say, the case $k=+1$, and, without loss of generality, $\arg(\widetilde{\Lambda})=\pm\pi/2$:[27] the case $k=-1$ is analogous. Using the asymptotic expansions for the parabolic-cylinder functions (see Remark 3.20), one shows that: **(a)** for $\arg(\widetilde{\Lambda}) \underset{\tau\to+\infty}{=} \pi/2+\mathcal{O}(\tau^{-2/3})$,

$$\begin{aligned}
&\big|\big(\Phi_{M,1}(\widetilde{\Lambda})\big)_{11}\big| \underset{\tau\to+\infty}{=} \mathcal{O}\big(\tilde{\rho}_0|\widetilde{\Lambda}|^{-\operatorname{Re}(\nu(1)+1)}\big), \qquad \big|\big(\Phi_{M,1}(\widetilde{\Lambda})\big)_{12}\big| \underset{\tau\to+\infty}{=} \mathcal{O}\big(\tilde{\rho}_1|\widetilde{\Lambda}|^{-\operatorname{Re}(\nu(1)+1)}\big),\\
&\big|\big(\Phi_{M,1}(\widetilde{\Lambda})\big)_{21}\big| \underset{\tau\to+\infty}{=} \mathcal{O}\left(\tilde{\rho}_2\frac{|\widetilde{\Lambda}|^{\operatorname{Re}(\nu(1)+1)}}{|p_1(\tau)|}\right),\\
&\big|\big(\Phi_{M,1}(\widetilde{\Lambda})\big)_{22}\big| \underset{\tau\to+\infty}{=} \mathcal{O}\left(\tilde{\rho}_3\frac{|\widetilde{\Lambda}|^{\operatorname{Re}(\nu(1)+1)}}{|p_1(\tau)|}\right),
\end{aligned}\tag{3.177}$$

where

$$\begin{aligned}
&\tilde{\rho}_0=\eta_+\mathrm{e}^{-3\pi\operatorname{Im}(\nu(1)+1)/2}, \qquad \tilde{\rho}_3=2^{3/2}3^{1/4}/\eta_+,\\
&\tilde{\rho}_1=\frac{2^{3/2}\eta_+}{\sqrt{\pi}}\mathrm{e}^{-\pi\operatorname{Im}(\nu(1)+1)}\big|\cos\big(\tfrac{\pi}{2}(\nu(1)+1)\big)\big|\big|\sin\big(\tfrac{\pi}{2}(\nu(1)+1)\big)\big|\Gamma(\operatorname{Re}(\nu(1)+1)),\\
&\tilde{\rho}_2=\frac{8\cdot 3^{1/4}}{\sqrt{\pi}\eta_+}\mathrm{e}^{\pi\operatorname{Im}(\nu(1)+1)/2}\big|\cos\big(\tfrac{\pi}{2}(\nu(1)+1)\big)\big|\big|\sin\big(\tfrac{\pi}{2}(\nu(1)+1)\big)\big|\Gamma(-\operatorname{Re}(\nu(1))),
\end{aligned}$$

with $\eta_+:=\big(2^{3/2}3^{1/4}\big)^{-\operatorname{Re}(\nu(1)+1)}\mathrm{e}^{3\pi\operatorname{Im}(\nu(1)+1)/4}$; and **(b)** for $\arg(\widetilde{\Lambda}) \underset{\tau\to+\infty}{=} -\pi/2+\mathcal{O}(\tau^{-2/3})$,

$$\begin{aligned}
&\big|\big(\Phi_{M,1}(\widetilde{\Lambda})\big)_{11}\big| \underset{\tau\to+\infty}{=} \mathcal{O}\big(\hat{\rho}_0|\widetilde{\Lambda}|^{-\operatorname{Re}(\nu(1)+1)}\big),\\
&\big|\big(\Phi_{M,1}(\widetilde{\Lambda})\big)_{12}\big| \underset{\tau\to+\infty}{=} \mathcal{O}\left(\hat{\rho}_1\frac{|\widetilde{\Lambda}|^{\operatorname{Re}(\nu(1)+1)}}{|\widetilde{\Lambda}|}\right),\\
&\big|\big(\Phi_{M,1}(\widetilde{\Lambda})\big)_{21}\big| \underset{\tau\to+\infty}{=} \mathcal{O}\left(\hat{\rho}_2\frac{|\widetilde{\Lambda}|^{-\operatorname{Re}(\nu(1)+1)}}{|p_1(\tau)||\widetilde{\Lambda}|}\right),\\
&\big|\big(\Phi_{M,1}(\widetilde{\Lambda})\big)_{22}\big| \underset{\tau\to+\infty}{=} \mathcal{O}\left(\hat{\rho}_3\frac{|\widetilde{\Lambda}|^{\operatorname{Re}(\nu(1)+1)}}{|p_1(\tau)|}\right),
\end{aligned}\tag{3.178}$$

where

$$\begin{aligned}
&\hat{\rho}_0=\eta_-\mathrm{e}^{\pi\operatorname{Im}(\nu(1)+1)/2}, \qquad \hat{\rho}_1=2^{-3/2}3^{-1/4}/\eta_-,\\
&\hat{\rho}_2=\eta_-\mathrm{e}^{\pi\operatorname{Im}(\nu(1)+1)/2}|\nu(1)+1|, \qquad \hat{\rho}_3=2^{3/2}3^{1/4}/\eta_-,
\end{aligned}$$

with $\eta_-:=\big(2^{3/2}3^{1/4}\big)^{-\operatorname{Re}(\nu(1)+1)}\mathrm{e}^{-\pi\operatorname{Im}(\nu(1)+1)/4}$. Hence, via the elementary inequalities $|\operatorname{Re}(\nu(1)+1)|\leqslant|\nu(1)+1|$ and $|\operatorname{Im}(\nu(1)+1)|\leqslant|\nu(1)+1|$, it follows from the estimates (3.177) and (C.5)–(C.8) that, for $\arg(\widetilde{\Lambda}) \underset{\tau\to+\infty}{=} \pi/2+\mathcal{O}(\tau^{-2/3})$,

$$||\Phi_{M,1}(\xi)||^2 \underset{\tau\to+\infty}{=} \mathcal{O}\big(\tilde{\mathfrak{c}}_M^{\sharp}\big)+\mathcal{O}\left(\frac{\tilde{\mathfrak{c}}_M^{\sharp}|\nu(1)+1|^2|\xi|^2}{|p_1(\tau)|^2}\right), \tag{3.179}$$

$$\big|\big|\Phi_{M,1}(\widetilde{\Lambda})\big|\big|^2 \underset{\tau\to+\infty}{=} \mathcal{O}\left(|\widetilde{\Lambda}|^{2\operatorname{Re}(\nu(1)+1)}\left(\frac{\tilde{\mathfrak{c}}_M}{|p_1(\tau)|^2}+\mathcal{O}\left(\frac{\tilde{\mathfrak{c}}_M}{|\widetilde{\Lambda}|^{4\operatorname{Re}(\nu(1)+1)}}\right)\right)\right), \tag{3.180}$$

[27]The pair of values $\arg(\widetilde{\Lambda})=\pm\pi/2$ on the Stokes rays are chosen for illustrative purposes only, in order to present the general scheme of the calculations: for any of the remaining $\binom{4}{2}-1=5$ pairs of values of $\arg(\widetilde{\Lambda})$ on the Stokes rays, one arrives at the same estimate (see equation (3.183)) for $||\hat{\chi}_k(\widetilde{\Lambda})-\mathrm{I}||$, $k=\pm1$, but with different $\mathcal{O}(1)$ constants.

where $\tilde{\mathfrak{c}}_M^{\sharp} := 2\max_{m=0,1,2,3}\left\{(\tilde{\varrho}_m(1))^2\right\}$, and $\tilde{\mathfrak{c}}_M := 2\max_{m=0,1,2,3}\left\{\tilde{\rho}_m^2\right\}$, and, from the estimates (3.178) and (C.1)–(C.4), it follows that, for $\arg\big(\widetilde{\Lambda}\big) \underset{\tau\to+\infty}{=} -\pi/2+\mathcal{O}\big(\tau^{-2/3}\big)$,

$$||\Phi_{M,1}(\xi)||^2 \underset{\tau\to+\infty}{=} \mathcal{O}\big(\hat{\mathfrak{c}}_M^{\sharp}\big)+\mathcal{O}\left(\frac{\hat{\mathfrak{c}}_M^{\sharp}|\nu(1)+1|^2|\xi|^2}{|p_1(\tau)|^2}\right), \tag{3.181}$$

$$\big|\big|\Phi_{M,1}\big(\widetilde{\Lambda}\big)\big|\big|^2 \underset{\tau\to+\infty}{=} \mathcal{O}\left(|\widetilde{\Lambda}|^{2\,\mathrm{Re}(\nu(1)+1)}\left(\frac{\hat{\mathfrak{c}}_M}{|p_1(\tau)|^2}+\mathcal{O}\left(\frac{\hat{\mathfrak{c}}_M}{|\widetilde{\Lambda}|^{2\min\{1,2\,\mathrm{Re}(\nu(1)+1)\}}}\right)\right)\right), \tag{3.182}$$

where $\hat{\mathfrak{c}}_M^{\sharp} := 2\max_{m=0,1,2,3}\left\{(\hat{\varrho}_m(1))^2\right\}$, and $\hat{\mathfrak{c}}_M := \max_{m=0,1,2,3}\left\{\hat{\rho}_m^2\right\}$. Assembling the asymptotics (3.179)–(3.182) and invoking the restriction (3.110) on δ_k (for $k=+1$), one deduces from the asymptotics (3.166) and (3.175) that, for $\arg\big(\widetilde{\Lambda}\big) \underset{\tau\to+\infty}{=} \pm\pi/2+\mathcal{O}\big(\tau^{-2/3}\big)$,

$$||\hat{\chi}_k\big(\widetilde{\Lambda}\big)-\mathrm{I}|| \underset{\tau\to+\infty}{\leqslant} \mathcal{O}\big(\mathfrak{c}_k^{\curlyvee}(\tau)|\nu(k)+1|^2|p_k(\tau)|^{-2}\tau^{-(\frac{1}{3}-2(3+\mathrm{Re}(\nu(k)+1))\delta_k)}\big),$$
$$k=+1, \tag{3.183}$$

where, for $\arg\big(\widetilde{\Lambda}\big) \underset{\tau\to+\infty}{=} \frac{\pi}{2}+\mathcal{O}\big(\tau^{-2/3}\big)$, $\mathfrak{c}_1^{\curlyvee}(\tau) := \tilde{\mathfrak{c}}_M^{\sharp}\tilde{\mathfrak{c}}_M\big(2^{3/2}3^{1/4}\mathrm{e}^{\pi\,\mathrm{Im}(\nu(1)+1)/2}\big)^{-2}$, and, for $\arg\big(\widetilde{\Lambda}\big) \underset{\tau\to+\infty}{=} -\frac{\pi}{2}+\mathcal{O}\big(\tau^{-2/3}\big)$, $\mathfrak{c}_1^{\curlyvee}(\tau) := \hat{\mathfrak{c}}_M^{\sharp}\hat{\mathfrak{c}}_M\big(2^{3/2}3^{1/4}\mathrm{e}^{\pi\,\mathrm{Im}(\nu(1)+1)/2}\big)^{-2}$ (see Remark 3.19). Via an analogous series of calculations, one arrives at a similar estimate (cf. asymptotics (3.183)) for the case $k=-1$.

Forming the composition of the inverses of the linear transformations $\mathfrak{F}_j$, $j=1,2,\dots,11$, that is,

$$\begin{aligned}\widetilde{\Psi}_k(\widetilde{\mu},\tau) &:= \big(\mathfrak{F}_1^{-1}\circ\mathfrak{F}_2^{-1}\circ\mathfrak{F}_3^{-1}\circ\mathfrak{F}_4^{-1}\circ\mathfrak{F}_5^{-1}\circ\mathfrak{F}_6^{-1}\circ\mathfrak{F}_7^{-1}\circ\mathfrak{F}_8^{-1}\circ\mathfrak{F}_9^{-1}\circ\mathfrak{F}_{10}^{-1}\circ\mathfrak{F}_{11}^{-1}\big)\,\Phi_{M,k}\big(\widetilde{\Lambda}\big)\\ &= (b(\tau))^{-\frac{1}{2}\sigma_3}\mathcal{G}_{0,k}\mathcal{G}_{1,k}\tau^{\frac{1}{6}\sigma_3}\begin{pmatrix}1 & 0\\ -\mathrm{i}\omega_{0,k} & 1\end{pmatrix}\begin{pmatrix}1 & \ell_{0,k}^{+}\widetilde{\Lambda}\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \ell_{1,k}^{+}\widetilde{\Lambda} & 1\end{pmatrix}\mathcal{G}_{2,k}\begin{pmatrix}1 & 0\\ \ell_{2,k}^{+}\widetilde{\Lambda} & 1\end{pmatrix}\\ &\quad\times\hat{\chi}_k\big(\widetilde{\Lambda}\big)\Phi_{M,k}\big(\widetilde{\Lambda}\big),\qquad k=\pm1,\end{aligned}$$

one arrives at the asymptotic representation for $\widetilde{\Psi}_k(\widetilde{\mu},\tau)$ given in equation (3.111). ■

Remark 3.18. Heretofore, it was assumed that (cf. Corollaries 3.10, 3.11, 3.13 and 3.14) $0<\delta<\delta_k<1/9$, $k=\pm1$; however, the set of restrictions (3.110) implies the following, more stringent restriction on δ_k[28]

$$0 \underset{\tau\to+\infty}{<} \delta_k \underset{\tau\to+\infty}{<} 1/24, \qquad k=\pm1. \tag{3.184}$$

Since $(0,1/24)\subset(0,1/9)$, the latter restriction (3.184) on δ_k implies, and is consistent with, the earlier one; henceforth, the restriction (3.184) on δ_k will be enforced.

Remark 3.19. Using the fact that (see the asymptotics (4.12) below) $\nu(k)+1\to0$ as $\tau\to+\infty$, $k=\pm1$, one shows, via the expansion for the gamma function [32]

$$\frac{1}{\Gamma(z+1)}=\sum_{j=0}^{\infty}\mathfrak{d}_j^{*}z^j,$$

$|z|<1$, where $\mathfrak{d}_0^{*}=1$ and

$$\mathfrak{d}_{n+1}^{*}=\frac{1}{(n+1)}\sum_{j=0}^{n}(-1)^j s_{j+1}\mathfrak{d}_{n-j}^{*},$$

[28]Note: $18 \underset{\tau\to+\infty}{<} 6(3+\mathrm{Re}(\nu(k)+1)) \underset{\tau\to+\infty}{<} 24$.

$n \in \mathbb{Z}_+$, with $s_1 = -\psi(1)$ the Euler–Mascheroni constant,[29] and $s_m = \zeta(m)$, $\mathbb{N} \ni m \geqslant 2$, where $\zeta(z)$ is the Riemann zeta function, and well-known inequalities for complex-valued trigonometric functions, that the auxiliary parameters introduced in step (xi) of the proof of Lemma 3.17 have (for the case $k = +1$) the following asymptotics: **(1)** for $\arg\big(\widetilde{\Lambda}\big) \underset{\tau\to+\infty}{=} \frac{\pi}{2} + \mathcal{O}\big(\tau^{-2/3}\big)$,

$$\begin{aligned}
&(\tilde{\varrho}_0(1))^2 \underset{\tau\to+\infty}{=} (2+|\sec\theta|)^2(1+\mathcal{O}(|\nu(1)+1|)),\\
&(\tilde{\varrho}_1(1))^2 \underset{\tau\to+\infty}{=} \frac{\pi}{2}\big(1+2\sec^2\theta\big)^2(1+\mathcal{O}(|\nu(1)+1|)),\\
&(\tilde{\varrho}_2(1))^2 \underset{\tau\to+\infty}{=} 192\big(2\sqrt{\pi}+|\sec\theta|\big)^2(1+\mathcal{O}(|\nu(1)+1|)),\\
&(\tilde{\varrho}_3(1))^2 \underset{\tau\to+\infty}{=} 96\pi\big(1+2\sec^2\theta\big)^2(1+\mathcal{O}(|\nu(1)+1|)),\\
&\tilde{\rho}_0^2 \underset{\tau\to+\infty}{=} 1+\mathcal{O}(|\nu(1)+1|), \qquad \tilde{\rho}_1^2 \underset{\tau\to+\infty}{=} 2\pi\sec^2(\theta)(1+\mathcal{O}(|\nu(1)+1|)),\\
&\tilde{\rho}_2^2 \underset{\tau\to+\infty}{=} 16\sqrt{3}\pi|\nu(1)+1|^2(1+\mathcal{O}(|\nu(1)+1|)), \qquad \tilde{\rho}_3^2 \underset{\tau\to+\infty}{=} 8\sqrt{3}(1+\mathcal{O}(|\nu(1)+1|)),
\end{aligned}$$

where $\theta := \arg(\nu(1)+1)$, whence $\tilde{\mathfrak{c}}^{\sharp}_M \underset{\tau\to+\infty}{=} \mathcal{O}(1)$ and $\tilde{\mathfrak{c}}_M \underset{\tau\to+\infty}{=} \mathcal{O}(1) \Rightarrow \mathfrak{c}^{\curlyvee}_1(\tau) \underset{\tau\to+\infty}{=} \mathcal{O}(1)$ (as claimed); and **(2)** for $\arg\big(\widetilde{\Lambda}\big) \underset{\tau\to+\infty}{=} -\frac{\pi}{2} + \mathcal{O}\big(\tau^{-2/3}\big)$,

$$\begin{aligned}
&(\hat{\varrho}_0(1))^2 \underset{\tau\to+\infty}{=} \sec^2(\theta)(1+\mathcal{O}(|\nu(1)+1|)), \qquad (\hat{\varrho}_1(1))^2 \underset{\tau\to+\infty}{=} \frac{\pi}{2}(1+\mathcal{O}(|\nu(1)+1|)),\\
&(\hat{\varrho}_2(1))^2 \underset{\tau\to+\infty}{=} 192\sec^2(\theta)(1+\mathcal{O}(|\nu(1)+1|)), \qquad (\hat{\varrho}_3(1))^2 \underset{\tau\to+\infty}{=} 96\pi(1+\mathcal{O}(|\nu(1)+1|)),\\
&\hat{\rho}_0^2 \underset{\tau\to+\infty}{=} 1+\mathcal{O}(|\nu(1)+1|), \qquad \hat{\rho}_1^2 \underset{\tau\to+\infty}{=} \frac{1}{8\sqrt{3}}(1+\mathcal{O}(|\nu(1)+1|)),\\
&\hat{\rho}_2^2 \underset{\tau\to+\infty}{=} \mathcal{O}\big(|\nu(1)+1|^2\big), \qquad \hat{\rho}_3^2 \underset{\tau\to+\infty}{=} 8\sqrt{3}(1+\mathcal{O}(|\nu(1)+1|)),
\end{aligned}$$

whence $\hat{\mathfrak{c}}^{\sharp}_M \underset{\tau\to+\infty}{=} \mathcal{O}(1)$ and $\hat{\mathfrak{c}}_M \underset{\tau\to+\infty}{=} \mathcal{O}(1) \Rightarrow \mathfrak{c}^{\curlyvee}_1(\tau) \underset{\tau\to+\infty}{=} \mathcal{O}(1)$ (as claimed). The analysis for the case $k = -1$ is analogous.

Remark 3.20. In Lemma 3.17 and hereafter, the function $\Phi_{M,k}(\cdot)$ plays a crucial rôle; therefore, its asymptotics are presented here: for $m \in \{-1,0,1,2\}$ and $k = \pm 1$,

$$\Phi_{M,k}(z) \underset{\substack{\mathbb{C}\ni z\to\infty\\ \arg(z)=\frac{m\pi}{2}+\frac{\pi}{4}-\frac{1}{2}\arg(\mu_k(\tau))}}{=} \left(\mathrm{I}+\sum_{j=1}^{\infty}\hat{\psi}_{j,k}(\tau)z^{-j}\right)\mathrm{e}^{(\frac{1}{2}\mu_k(\tau)z^2-(\nu(k)+1)\ln((2\mu_k(\tau))^{1/2}z))\sigma_3}\mathcal{R}_m(k),$$

where

$$\begin{aligned}
\mathcal{R}_{-1}(k) &:= \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\pi(\nu(k)+1)/2} & 0\\ 0 & -\frac{(2\mu_k(\tau))^{1/2}}{p_k(\tau)} \end{pmatrix},\\
\mathcal{R}_0(k) &:= \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\pi(\nu(k)+1)/2} & 0\\ -\frac{\mathrm{i}\sqrt{2\pi}(2\mu_k(\tau))^{1/2}\mathrm{e}^{-\mathrm{i}\pi(\nu(k)+1)/2}}{p_k(\tau)\Gamma(\nu(k)+1)} & -\frac{(2\mu_k(\tau))^{1/2}}{p_k(\tau)} \end{pmatrix},\\
\mathcal{R}_1(k) &:= \begin{pmatrix} \mathrm{e}^{\mathrm{i}3\pi(\nu(k)+1)/2} & \frac{\sqrt{2\pi}\mathrm{e}^{\mathrm{i}\pi(\nu(k)+1)}}{\Gamma(-\nu(k))}\\ -\frac{\mathrm{i}\sqrt{2\pi}(2\mu_k(\tau))^{1/2}\mathrm{e}^{-\mathrm{i}\pi(\nu(k)+1)/2}}{p_k(\tau)\Gamma(\nu(k)+1)} & -\frac{(2\mu_k(\tau))^{1/2}}{p_k(\tau)} \end{pmatrix},\\
\mathcal{R}_2(k) &:= \begin{pmatrix} \mathrm{e}^{\mathrm{i}3\pi(\nu(k)+1)/2} & \frac{\sqrt{2\pi}\mathrm{e}^{\mathrm{i}\pi(\nu(k)+1)}}{\Gamma(-\nu(k))}\\ 0 & -\frac{(2\mu_k(\tau))^{1/2}\mathrm{e}^{-2\pi\mathrm{i}(\nu(k)+1)}}{p_k(\tau)} \end{pmatrix},
\end{aligned}$$

[29] $-\psi(1) = 0.5772156649015328606065l2\ldots$.

and $\hat{\psi}_{j,k}(\tau)$, $j \in \mathbb{N}$, are off-diagonal (resp., diagonal) $\mathrm{M}_2(\mathbb{C})$-valued functions for j odd (resp., j even); for example,

$$\hat{\psi}_{1,k}(\tau) = -\frac{1}{2\mu_k(\tau)} \begin{pmatrix} 0 & p_k(\tau) \\ -q_k(\tau) & 0 \end{pmatrix},$$
$$\hat{\psi}_{2,k}(\tau) = \frac{(\nu(k)+1)}{4\mu_k(\tau)} \begin{pmatrix} 1+(\nu(k)+1) & 0 \\ 0 & 1-(\nu(k)+1) \end{pmatrix},$$
$$\hat{\psi}_{3,k}(\tau) = \frac{1}{8(\mu_k(\tau))^2} \times \begin{pmatrix} 0 & (1-(\nu(k)+1))(2-(\nu(k)+1))p_k(\tau) \\ (1+(\nu(k)+1))(2+(\nu(k)+1))q_k(\tau) & 0 \end{pmatrix}.$$

These asymptotics are derived from the asymptotics of the parabolic-cylinder functions [27].

3.3 Asymptotic matching

In this subsection, the connection matrix is calculated asymptotically (as $\tau \to +\infty$ with $\varepsilon b > 0$) in terms of the matrix elements of the function $\mathcal{A}(\widetilde{\mu},\tau)$ (cf. equation (3.4)) that are defined in terms of the set of functions $\hat{h}_0(\tau)$, $\tilde{r}_0(\tau)$, $v_0(\tau)$, and $b(\tau)$ concomitant with the conditions (3.17).[30] Thus, the direct monodromy problem for equation (3.3) is solved asymptotically.

Lemma 3.21. *Let $\widetilde{\Psi}_k(\widetilde{\mu},\tau)$, $k=\pm1$, be the fundamental solution of equation (3.3) with asymptotics given in Lemma 3.17, and let $\mathbb{Y}_0^\infty(\widetilde{\mu},\tau)$ be the canonical solution of equation (3.1).[31] Define[32]*

$$\mathfrak{L}_k^\infty(\tau) := \big(\widetilde{\Psi}_k(\widetilde{\mu},\tau)\big)^{-1}\tau^{-\frac{1}{12}\sigma_3}\mathbb{Y}_0^\infty\big(\tau^{-1/6}\widetilde{\mu},\tau\big), \qquad k=\pm1. \tag{3.185}$$

Assume that the parameters $\nu(k)+1$ and δ_k satisfy the restrictions (3.110) and (3.184), respectively, and, additionally, the following conditions are valid[33]

$$p_k(\tau)\mathfrak{B}_k \exp\big(-\mathrm{i}\tau^{2/3}3\sqrt{3}(\varepsilon b)^{1/3}\mathrm{e}^{\mathrm{i}2\pi k/3}\big) \underset{\tau\to+\infty}{=} \mathcal{O}\big((\nu(k)+1)^{\frac{1-k}{2}}\big), \tag{3.186}$$
$$b(\tau)\tau^{\mathrm{i}a/3}\exp\big(\mathrm{i}\tau^{2/3}3(\varepsilon b)^{1/3}\mathrm{e}^{\mathrm{i}2\pi k/3}\big) \underset{\tau\to+\infty}{=} \mathcal{O}(1), \tag{3.187}$$

where $p_k(\tau)$ and $\mathfrak{B}_k$ are defined in Lemma 3.17.[34] Then,

$$\mathfrak{L}_k^\infty(\tau) \underset{\tau\to+\infty}{=} \mathrm{i}(\mathcal{R}_{m_\infty}(k))^{-1}\mathrm{e}^{\tilde{\mathfrak{z}}_k^0(\tau)\sigma_3}\left(\frac{(\varepsilon b)^{1/4}\big(\sqrt{3}+1\big)^{1/2}}{2^{1/4}\sqrt{\mathfrak{B}_k}\sqrt{b(\tau)}}\right)^{\sigma_3}\sigma_2\mathrm{e}^{-\Delta\tilde{\mathfrak{z}}_k(\tau)\sigma_3}$$

---

[30] Equivalently, the set of functions (cf. equations (3.14), (3.15), and (3.16), respectively) $h_0(\tau)$, $\hat{r}_0(\tau)$, and $\hat{u}_0(\tau)$.

[31] See Proposition 1.15.

[32] Since $\tau^{-\frac{1}{12}\sigma_3}\mathbb{Y}_0^\infty\big(\tau^{-1/6}\widetilde{\mu},\tau\big)$ (cf. equations (3.2)) is also a fundamental solution of equation (3.3), it follows, therefore, that $\mathfrak{L}_k^\infty(\tau)$ is independent of $\widetilde{\mu}$.

[33] The conditions (3.17) and (3.184) are consistent with the conditions (3.186) and (3.187).

[34] From the results subsumed in the proof of Lemma 4.1, it will be deduced *a posteriori* that (cf. definition (3.157)) $\mu_k(\tau)$ possesses the asymptotics

$$\mu_k(\tau) \underset{\tau\to+\infty}{=} \mathrm{i}4\sqrt{3} + \sum_{\substack{m_1,m_2,m_3\in\mathbb{Z}_+\\ m_1+m_2+m_3\geqslant2}} c_{m_1,m_2,m_3}(k)(\tilde{r}_0(\tau))^{m_1}(v_0(\tau))^{m_2}\big(\tau^{-1/3}\big)^{m_3} + c_\infty(k)\tau^{-1/3}\mathrm{e}^{-(\beta(\tau)+\mathrm{i}k\vartheta(\tau))}\big(1+\mathcal{O}\big(\tau^{-1/3}\big)\big),$$

$k=\pm1$, where $c_{m_1,m_2,m_3}(k) \in \mathbb{C}$, and $\vartheta(\tau)$ and $\beta(\tau)$ are defined in equations (2.10); via this fact, and the definitions (3.109), (3.130), (3.160), and (3.161), a lengthy and circuitous calculation reveals that the asymptotic expansion of $\nu(k)+1$, $k=\pm1$, can be presented in the form

$$-(\nu(k)+1)$$

$$\times \begin{pmatrix} \hat{\mathbb{B}}_0^{\infty}(\tau) & 0 \\ 0 & \hat{\mathbb{A}}_0^{\infty}(\tau) \end{pmatrix} \left(\mathrm{I} + \mathbb{E}_{\mathcal{N},k}^{\infty}(\tau)\right) \left(\mathrm{I} + \mathcal{O}\left(\mathbb{E}_k^{\infty}(\tau)\right)\right), \tag{3.188}$$

where $\mathrm{M}_2(\mathbb{C}) \ni \mathcal{R}_{m_\infty}(k)$, $m_\infty \in \{-1, 0, 1, 2\}$, *are defined in Remark* 3.20,

$$\tilde{\mathfrak{z}}_k^0(\tau) := -\frac{\mathrm{i}a}{6} \ln \tau + \mathrm{i}\tau^{2/3} 3\big(\sqrt{3}-1\big)\alpha_k^2 + \mathrm{i}(a - \mathrm{i}/2) \ln\big(\big(\sqrt{3}+1\big)\alpha_k/2\big), \tag{3.189}$$

$$\Delta\tilde{\mathfrak{z}}_k(\tau) := -\left(\frac{5-\sqrt{3}}{6\sqrt{3}\alpha_k^2}\right) \mathfrak{p}_k(\tau) + (\nu(k)+1) \ln(2\mu_k(\tau))^{1/2} + \frac{1}{3}(\nu(k)+1) \ln \tau$$
$$+ (\nu(k)+1) \ln\big(6\big(\sqrt{3}+1\big)^{-2} \alpha_k\big), \tag{3.190}$$

with $\mathfrak{p}_k(\tau)$ *defined by equation* (3.57), *and* $\mu_k(\tau)$ *defined in Lemma* 3.17,

$$\hat{\mathbb{A}}_0^{\infty}(\tau) := 1 + \frac{2^{1/4} (\Delta G_k^{\infty}(\tau))_{12}}{(\varepsilon b)^{1/4} \big(\sqrt{3}+1\big)^{1/2}}, \tag{3.191}$$

$$\hat{\mathbb{B}}_0^{\infty}(\tau) := 1 - \frac{(\varepsilon b)^{1/4} \big(\sqrt{3}+1\big)^{1/2}}{2^{1/4}}$$
$$\times \left((\Delta G_k^{\infty}(\tau))_{21} - \frac{\mathfrak{A}_k}{\mathfrak{B}_k} \left(\frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_k}{\chi_k(\tau)} - 1 \right) (\Delta G_k^{\infty}(\tau))_{11} \right), \tag{3.192}$$

with $\mathcal{Z}_k$, $\mathfrak{A}_k$, *and* $\chi_k(\tau)$ *defined in Lemma* 3.17, *and*

$$\Delta G_k^{\infty}(\tau) := \frac{1}{\big(2\sqrt{3}\big(\sqrt{3}+1\big)\big)^{1/2}} \begin{pmatrix} (\Delta G_k^{\infty}(\tau))_{11} & (\Delta G_k^{\infty}(\tau))_{12} \\ (\Delta G_k^{\infty}(\tau))_{21} & (\Delta G_k^{\infty}(\tau))_{22} \end{pmatrix}, \tag{3.193}$$

with

$$(\Delta G_k^{\infty}(\tau))_{11} = \big(\sqrt{3}+1\big)\big(\Delta\mathcal{G}_{0,k}\big)_{22} + (2/\varepsilon b)^{1/2} \big(\Delta\mathcal{G}_{0,k}\big)_{12},$$
$$(\Delta G_k^{\infty}(\tau))_{12} = -\big(\sqrt{3}+1\big)\big(\Delta\mathcal{G}_{0,k}\big)_{12} + (2\varepsilon b)^{1/2} \big(\Delta\mathcal{G}_{0,k}\big)_{22},$$
$$(\Delta G_k^{\infty}(\tau))_{21} = -\big(\sqrt{3}+1\big)\big(\Delta\mathcal{G}_{0,k}\big)_{21} - (2/\varepsilon b)^{1/2} \big(\Delta\mathcal{G}_{0,k}\big)_{11},$$
$$(\Delta G_k^{\infty}(\tau))_{22} = \big(\sqrt{3}+1\big)\big(\Delta\mathcal{G}_{0,k}\big)_{11} - (2\varepsilon b)^{1/2} \big(\Delta\mathcal{G}_{0,k}\big)_{21},$$

where $\big(\Delta\mathcal{G}_{0,k}\big)_{i,j=1,2}$ *are defined by equations* (B.5)–(B.7),

$$\mathbb{E}_{\mathcal{N},k}^{\infty}(\tau) := \left(\frac{\mathrm{e}^{-\tilde{\beta}_k(\tau)}}{\sqrt{b(\tau)}} \right)^{\mathrm{ad}(\sigma_3)} \left(-\frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\ell_{0,k}^{+}}{\chi_k(\tau)} \left(\frac{(\varepsilon b)^{1/2}\big(\sqrt{3}+1\big)(\nu(k)+1)}{\sqrt{2}p_k(\tau)\mathfrak{B}_k} \sigma_+ \right.\right.$$

$$\underset{\tau \to +\infty}{=} \frac{\mathrm{i}}{8\sqrt{3}} \left(\frac{-\alpha_k^2\big(8v_0^2(\tau) + 4v_0(\tau)\tilde{r}_0(\tau) - (\tilde{r}_0(\tau))^2 - v_0(\tau)(\tilde{r}_0(\tau))^2\tau^{-1/3}\big) + 4(a - \mathrm{i}/2)v_0(\tau)\tau^{-1/3}}{1 + v_0(\tau)\tau^{-1/3}} \right)$$
$$+ \frac{2\mathfrak{p}_k(\tau)}{3\sqrt{3}\alpha_k^2} + \sum_{m=2}^{\infty} \hat{\mu}_m^{*}(k)\big(\tau^{-1/3}\big)^m + \hat{c}_{\infty}(k)\tau^{-1/3}\mathrm{e}^{-(\beta(\tau)+\mathrm{i}k\vartheta(\tau))}\big(1 + \mathcal{O}\big(\tau^{-1/3}\big)\big),$$

where $\mathfrak{p}_k(\tau)$ is defined by equation (3.57). From the asymptotics (3.21) and (3.24), and Propositions 3.4 and 3.5, in conjunction with the formulae for the monodromy-data-dependent expansion coefficients A_k, $k = \pm 1$, derived in the proof of Lemma 4.1 (see, in particular, equations (4.71) and (4.92)), the sum of the coefficients of each term $\big(\tau^{-1/3}\big)^j$, $\mathbb{N} \ni j \geqslant 2$, and of the term $\tau^{-1/3}\mathrm{e}^{-(\beta(\tau)+\mathrm{i}k\vartheta(\tau))}$ on the right-hand side of the latter asymptotic expansion for $\nu(k)+1$ are equal to zero (e.g., $\hat{\mu}_2^{*}(k) = -\frac{\mathrm{i}}{24\sqrt{3}\alpha_k^2}\big(\big(a - \mathrm{i}/2\big)^2 - 1/6\big)$), resulting, finally, in the asymptotics $\nu(k) + 1 \underset{\tau \to +\infty}{=} \mathcal{O}\big(\tau^{-2/3}\mathrm{e}^{-\beta(\tau)}\big)$, $k = \pm 1$. The conditions (3.186) and (3.187) will be validated *a posteriori* (see the proof of Lemma 4.1) using the asymptotics $\nu(k) + 1 \underset{\tau \to +\infty}{=} \mathcal{O}\big(\tau^{-2/3}\mathrm{e}^{-\beta(\tau)}\big)$, $k = \pm 1$. Hereafter, whilst reading the text, the reader should be cognizant of the latter asymptotics for $\nu(k)+1$, as all asymptotic expansions, estimates, orderings, etc., rely on this fact.

$$+\frac{p_k(\tau)\mathfrak{B}_k}{\sqrt{2}(\varepsilon b)^{1/2}(\sqrt{3}+1)\mu_k(\tau)}\sigma_-\bigg)\sigma_3+\frac{1}{2\sqrt{3}(\sqrt{3}+1)}$$
$$\times\begin{pmatrix}\frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\ell^+_{0,k}}{\chi_k(\tau)} & -\frac{(\varepsilon b)^{1/2}(\sqrt{3}+1)}{\sqrt{2}\mathfrak{B}_k}\big((\frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k}{\chi_k(\tau)})^2\ell^+_{0,k}-\ell^+_{1,k}-\ell^+_{2,k}\big)\\ \frac{\sqrt{2}\mathfrak{B}_k\ell^+_{0,k}}{(\varepsilon b)^{1/2}(\sqrt{3}+1)} & -\frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\ell^+_{0,k}}{\chi_k(\tau)}\end{pmatrix}$$
$$\times\begin{pmatrix}\sqrt{3}+1 & -(2\varepsilon b)^{1/2}\\ (2/\varepsilon b)^{1/2} & \sqrt{3}+1\end{pmatrix}\begin{pmatrix}\mathbb{T}_{11,k}(1;\tau) & \mathbb{T}_{12,k}(1;\tau)\\ \mathbb{T}_{21,k}(1;\tau) & \mathbb{T}_{22,k}(1;\tau)\end{pmatrix}\Bigg), \tag{3.194}$$

with $\ell^+_{0,k}$, $\ell^+_{1,k}$, and $\ell^+_{2,k}$ defined in Lemma 3.17, $(\mathbb{T}_{ij,k}(1;\tau))_{i,j=1,2}$ defined in Proposition 3.16, and $\tilde{\beta}_k(\tau)$ defined by equation (3.203), and

$$\mathcal{O}\big(\mathbb{E}^\infty_k(\tau)\big)\underset{\tau\to+\infty}{:=}\begin{pmatrix}\mathcal{O}\big(\tau^{-\frac{1}{3}+3\delta_k}\big) & \mathcal{O}\big(\tau^{-\frac{1}{3}(\frac{1+k}{2})-\delta_k}\big)\\ \mathcal{O}\big(\tau^{-\frac{1}{3}(\frac{1-k}{2})-\delta_k}\big) & \mathcal{O}\big(\tau^{-\frac{1}{3}+3\delta_k}\big)\end{pmatrix}. \tag{3.195}$$

Proof. Denote by $\widetilde{\Psi}_{\mathrm{WKB},k}(\widetilde{\mu},\tau)$, $k=\pm1$, the solution of equation (3.3) that has leading-order asymptotics given by equations (3.25)–(3.27) in the canonical domain containing the Stokes curve approaching, for $k=+1$ (resp., $k=-1$), the positive real $\widetilde{\mu}$-axis from above (resp., below) as $\widetilde{\mu}\to+\infty$. Let $\mathfrak{L}^\infty_k(\tau)$, $k=\pm1$, be defined by equation (3.185); rewrite $\mathfrak{L}^\infty_k(\tau)$ in the following form

$$\mathfrak{L}^\infty_k(\tau)=\Big(\big(\widetilde{\Psi}_k(\widetilde{\mu},\tau)\big)^{-1}\widetilde{\Psi}_{\mathrm{WKB},k}(\widetilde{\mu},\tau)\Big)\Big(\big(\widetilde{\Psi}_{\mathrm{WKB},k}(\widetilde{\mu},\tau)\big)^{-1}\tau^{-\frac{1}{12}\sigma_3}\mathbb{Y}^\infty_0\big(\tau^{-1/6}\widetilde{\mu},\tau\big)\Big). \tag{3.196}$$

Taking note of the fact that $\widetilde{\Psi}_k(\widetilde{\mu},\tau)$, $\widetilde{\Psi}_{\mathrm{WKB},k}(\widetilde{\mu},\tau)$, and $\tau^{-\frac{1}{12}\sigma_3}\mathbb{Y}^\infty_0\big(\tau^{-1/6}\widetilde{\mu},\tau\big)$ are all solutions of equation (3.3), it follows that they differ on the right by non-degenerate, $\widetilde{\mu}$-independent, $\mathrm{M}_2(\mathbb{C})$-valued factors: via this observation, one evaluates, asymptotically, each of the factors appearing in equation (3.196) by considering separate limits, namely, $\widetilde{\mu}\to\alpha_k$ and $\widetilde{\mu}\to+\infty$, respectively; more specifically, for $k=\pm1$,

$$\big(\widetilde{\Psi}_k(\widetilde{\mu},\tau)\big)^{-1}\widetilde{\Psi}_{\mathrm{WKB},k}(\widetilde{\mu},\tau)$$
$$\underset{\tau\to+\infty}{:=}\underbrace{\Big((b(\tau))^{-\frac{1}{2}\sigma_3}\mathcal{G}_{0,k}\mathfrak{B}_k^{\frac{1}{2}\sigma_3}\mathbb{F}_k(\tau)\Xi_k\big(\tau;\widetilde{\Lambda}\big)\hat{\chi}_k\big(\widetilde{\Lambda}\big)\Phi_{M,k}\big(\widetilde{\Lambda}\big)\Big)^{-1}T(\widetilde{\mu})\mathrm{e}^{\mathrm{W}_k(\widetilde{\mu},\tau)}}_{\widetilde{\mu}=\widetilde{\mu}_{0,k},\widetilde{\Lambda}\underset{\tau\to+\infty}{\sim}\mathcal{O}(\tau^{\delta_k}),0<\delta<\delta_k<\frac{1}{24},\arg(\widetilde{\Lambda})=\frac{\pi m_\infty}{2}+\frac{\pi}{4}-\frac{1}{2}\arg(\mu_k(\tau)),m_\infty\in\{-1,0,1,2\}}, \tag{3.197}$$

where (cf. Lemma 3.17)

$$\mathbb{F}_k(\tau)=\begin{pmatrix}1 & 0\\ \big(\frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_k}{\chi_k(\tau)}-1\big)\mathfrak{A}_k & 1\end{pmatrix}, \tag{3.198}$$

$$\Xi_k\big(\tau;\widetilde{\Lambda}\big)=\mathrm{I}+\beth_{A,k}(\tau)\widetilde{\Lambda}+\beth_{B,k}(\tau)\widetilde{\Lambda}^2, \tag{3.199}$$

and

$$\hat{\chi}_k\big(\widetilde{\Lambda}\big)\underset{\tau\to+\infty}{=}\mathrm{I}+\mathcal{O}\big(\mathfrak{C}_k(\tau)|\nu(k)+1|^2|p_k(\tau)|^{-2}\tau^{-\epsilon_{\mathrm{TP}}(k)}\big), \tag{3.200}$$

with $\nu(k)+1$, $p_k(\tau)$, $\widetilde{\mu}_{0,k}$, $\mathcal{G}_{0,k}$, $\mathfrak{A}_k$, $\mathfrak{B}_k$, $\mathcal{Z}_k$, $\beth_{A,k}(\tau)$, $\beth_{B,k}(\tau)$, $\mu_k(\tau)$, and $\chi_k(\tau)$ defined in Lemma 3.17,

$$\mathrm{W}_k(\widetilde{\mu},\tau):=-\sigma_3\mathrm{i}\tau^{2/3}\int_{\widetilde{\mu}_{0,k}}^{\widetilde{\mu}}l_k(\xi)\mathrm{d}\xi-\int_{\widetilde{\mu}_{0,k}}^{\widetilde{\mu}}\mathrm{diag}\big((T(\xi))^{-1}\partial_\xi T(\xi)\big)\mathrm{d}\xi,$$

$\epsilon_{\mathrm{TP}}(k) := \frac{1}{3} - 2(3+\mathrm{Re}(\nu(k)+1))\delta_k$ (>0), and $\mathrm{M}_2(\mathbb{C}) \ni \tilde{\mathfrak{C}}_k(\tau) \underset{\tau\to+\infty}{=} \mathcal{O}(1)$, and

$$\begin{aligned}&\big(\widetilde{\Psi}_{\mathrm{WKB},k}(\widetilde{\mu},\tau)\big)^{-1}\tau^{-\frac{1}{12}\sigma_3}\mathbb{Y}_0^{\infty}\big(\tau^{-1/6}\widetilde{\mu},\tau\big)\\ &\quad \underset{\tau\to+\infty}{:=} \lim_{\substack{\Omega_0^{\infty}\ni\widetilde{\mu}\to\infty\\ \arg(\widetilde{\mu})=0}}\Big(\big(T(\widetilde{\mu})\mathrm{e}^{\mathrm{W}_k(\widetilde{\mu},\tau)}\big)^{-1}\tau^{-\frac{1}{12}\sigma_3}\mathbb{Y}_0^{\infty}\big(\tau^{-1/6}\widetilde{\mu},\tau\big)\Big). \end{aligned}\tag{3.201}$$

One commences by considering the asymptotics subsumed in the definition (3.201). From the asymptotics for $\mathbb{Y}_0^{\infty}\big(\tau^{-1/6}\widetilde{\mu},\tau\big)$ stated in Proposition 1.15, equations (3.15), (3.16), (3.18), (3.19), (3.51), (3.52), (3.57), (3.84), (3.85), (3.94), (3.130), and (B.14), one arrives at, via the conditions (3.17) and the asymptotics (3.48) and (3.79),

$$\begin{aligned}&\lim_{\substack{\Omega_0^{\infty}\ni\widetilde{\mu}\to\infty\\ \arg(\widetilde{\mu})=0}}\Big(\big(T(\widetilde{\mu})\mathrm{e}^{\mathrm{W}_k(\widetilde{\mu},\tau)}\big)^{-1}\tau^{-\frac{1}{12}\sigma_3}\mathbb{Y}_0^{\infty}\big(\tau^{-1/6}\widetilde{\mu},\tau\big)\Big) \underset{\tau\to+\infty}{=} \exp\big(\tilde{\beta}_k(\tau)\sigma_3\big),\\ &k=\pm1,\end{aligned}\tag{3.202}$$

where

$$\begin{aligned}\tilde{\beta}_k(\tau) :=\,& \frac{\mathrm{i}a}{6}\ln\tau - \mathrm{i}\tau^{2/3}3\big(\sqrt{3}-1\big)\alpha_k^2 - \mathrm{i}2\sqrt{3}\widetilde{\Lambda}^2 - \mathrm{i}(a-\mathrm{i}/2)\ln\big(\big(\sqrt{3}+1\big)\alpha_k/2\big)\\ &+\frac{\big(5-\sqrt{3}\big)\mathfrak{p}_k(\tau)}{6\sqrt{3}\alpha_k^2} + \left(\frac{\mathrm{i}}{2\sqrt{3}}\big((a-\mathrm{i}/2)+\alpha_k^{-2}\tau^{2/3}\hat{h}_0(\tau)\big) + \frac{2\mathfrak{p}_k(\tau)}{3\sqrt{3}\alpha_k^2}\right)\\ &\quad\times\left(\frac{1}{3}\ln\tau - \ln\widetilde{\Lambda} + \ln\left(\frac{6\alpha_k}{\big(\sqrt{3}+1\big)^2}\right)\right)\\ &-\frac{\big(\sqrt{3}-1\big)\mathfrak{p}_k(\tau)}{\sqrt{3}\alpha_k\tau^{-1/3}\widetilde{\Lambda}} + \mathcal{O}\left(\left(\frac{\mathfrak{c}_{1,k}\tau^{-1/3}+\mathfrak{c}_{2,k}\tilde{r}_0(\tau)}{\widetilde{\Lambda}^2}\right)\right.\\ &\quad\left.\times\big(\mathfrak{c}_{3,k}\tau^{-1/3}+\mathfrak{c}_{4,k}(\tilde{r}_0(\tau)+4v_0(\tau))\big)\right) + \mathcal{O}\big(\tau^{-1/3}\widetilde{\Lambda}^3\big) + \mathcal{O}\big(\tau^{-1/3}\widetilde{\Lambda}\big)\\ &+\mathcal{O}\left(\frac{\tau^{-1/3}}{\widetilde{\Lambda}}\big(\mathfrak{c}_{5,k}+\mathfrak{c}_{6,k}\tau^{2/3}\hat{h}_0(\tau)+\mathfrak{c}_{7,k}\big(\tau^{2/3}\hat{h}_0(\tau)\big)^2\big)\right)\\ &+\mathcal{O}\left(\tau^{-2/3}\hat{d}_{0,k}(\tau)\left(\frac{1}{3}\ln\tau-\ln\widetilde{\Lambda}\right)\right),\end{aligned}\tag{3.203}$$

$\mathfrak{c}_{m,k}$, $m=1,2,\dots,7$, are $\mathcal{O}(1)$, and $\hat{d}_{0,k}(\tau)$ is defined in the proof of Proposition 3.9.

One now derives the asymptotics defined by equation (3.197). From the asymptotics (3.103) for $\varpi=+1$, equation (3.115) for $\Phi_{M,k}\big(\widetilde{\Lambda}\big)$ (in conjunction with its large-$\widetilde{\Lambda}$ asymptotics stated in Remark 3.20), the definitions (3.198) and (3.199) (concomitant with the fact that $\det\big(\Xi_k\big(\tau;\widetilde{\Lambda}\big)\big) = 1$), and the asymptotics (3.200), one shows, via the relation $(\mathrm{W}_k(\widetilde{\mu}_{0,k},\tau))_{i,j=1,2}=0$ and the definition (3.197), that, for $k=\pm1$,

$$\begin{aligned}&\big(\widetilde{\Psi}_k(\widetilde{\mu},\tau)\big)^{-1}\widetilde{\Psi}_{\mathrm{WKB},k}(\widetilde{\mu},\tau)\\ &\quad\underset{\tau\to+\infty}{:=} \Phi_{M,k}^{-1}\big(\widetilde{\Lambda}\big)\hat{\chi}_k^{-1}\big(\widetilde{\Lambda}\big)\Xi_k^{-1}\big(\tau;\widetilde{\Lambda}\big)\mathbb{F}_k^{-1}(\tau)\mathfrak{B}_k^{-\frac{1}{2}\sigma_3}\mathcal{G}_{0,k}^{-1}(b(\tau))^{\frac{1}{2}\sigma_3}T(\widetilde{\mu}_{0,k})\\ &\quad\underset{\tau\to+\infty}{=} (\mathcal{R}_{m_\infty}(k))^{-1}\mathrm{e}^{-\mathcal{P}_0^{*}\sigma_3}\mathfrak{Q}_{\infty,k}(\tau)\left(\mathrm{I}+\frac{1}{\widetilde{\Lambda}}\mathfrak{Q}_{\infty,k}^{-1}(\tau)\hat{\psi}_{1,k}^{-1}(\tau)\mathfrak{Q}_{\infty,k}(\tau)\right.\\ &\qquad\left.+\frac{1}{\widetilde{\Lambda}^2}\mathfrak{Q}_{\infty,k}^{-1}(\tau)\hat{\psi}_{2,k}^{-1}(\tau)\mathfrak{Q}_{\infty,k}(\tau)+\mathcal{O}\left(\frac{1}{\widetilde{\Lambda}^3}\mathfrak{Q}_{\infty,k}^{-1}(\tau)\hat{\psi}_{3,k}^{-1}(\tau)\mathfrak{Q}_{\infty,k}(\tau)\right)\right)\end{aligned}$$

$$\times\big(\mathrm{I}+\mathcal{O}\big(|\nu(k)+1|^2|p_k(\tau)|^{-2}\tau^{-\epsilon_{\mathrm{TP}}(k)}\mathfrak{Q}_{\infty,k}^{-1}(\tau)\tilde{\mathfrak{C}}_k(\tau)\mathfrak{Q}_{\infty,k}(\tau)\big)\big)$$
$$\times\big(\mathrm{I}+\widetilde{\Lambda}\mathfrak{Q}_{\infty,k}^{-1}(\tau)\mathfrak{J}_{A,k}^{-1}(\tau)\mathfrak{Q}_{\infty,k}(\tau)+\widetilde{\Lambda}^2\mathfrak{Q}_{\infty,k}^{-1}(\tau)\mathfrak{J}_{B,k}^{-1}(\tau)\mathfrak{Q}_{\infty,k}(\tau)\big)$$
$$\times\left(\mathrm{I}+\widetilde{\Lambda}\tau^{-1/3}\mathbb{P}_{\infty,k}(\tau)+\frac{1}{\widetilde{\Lambda}}\widehat{\mathbb{E}}_{\infty,k}(\tau)+\mathcal{O}\big(\big(\tau^{-1/3}\widetilde{\Lambda}\big)^2\widetilde{\mathbb{E}}_{\infty,k}(\tau)\big)\right), \tag{3.204}$$

where $\mathrm{M}_2(\mathbb{C})\ni\mathcal{R}_{m_\infty}(k)$, $m_\infty\in\{-1,0,1,2\}$, are defined in Remark 3.20,

$$\mathcal{P}_0^*:=\frac{1}{2}\mu_k(\tau)\widetilde{\Lambda}^2-(\nu(k)+1)\ln\widetilde{\Lambda}-(\nu(k)+1)\ln(2\mu_k(\tau))^{1/2}, \tag{3.205}$$
$$\mathfrak{Q}_{\infty,k}(\tau):=\mathbb{F}_k^{-1}(\tau)\left(\left(\frac{(\varepsilon b)^{1/4}\big(\sqrt{3}+1\big)^{1/2}}{2^{1/4}\sqrt{\mathfrak{B}_k}\sqrt{b(\tau)}}\right)^{\sigma_3}\mathrm{i}\sigma_2+\mathfrak{B}_k^{-\frac{1}{2}\sigma_3}\Delta G_k^{\infty}(\tau)(b(\tau))^{\frac{1}{2}\sigma_3}\right), \tag{3.206}$$

with $\Delta G_k^{\infty}(\tau)$ defined by equation (3.193),

$$\hat{\psi}_{1,k}^{-1}(\tau):=\frac{1}{2\mu_k(\tau)}\begin{pmatrix}0 & p_k(\tau)\\ -q_k(\tau) & 0\end{pmatrix}, \tag{3.207}$$
$$\hat{\psi}_{2,k}^{-1}(\tau):=\frac{(\nu(k)+1)}{4\mu_k(\tau)}\begin{pmatrix}1-(\nu(k)+1) & 0\\ 0 & 1+(\nu(k)+1)\end{pmatrix}, \tag{3.208}$$
$$\hat{\psi}_{3,k}^{-1}(\tau):=-\frac{1}{8(\mu_k(\tau))^2}\times\begin{pmatrix}0 & (1-(\nu(k)+1))(2-(\nu(k)+1))p_k(\tau)\\ (1+(\nu(k)+1))(2+(\nu(k)+1))q_k(\tau) & 0\end{pmatrix}, \tag{3.209}$$
$$\mathbb{P}_{\infty,k}(\tau):=(b(\tau))^{-\frac{1}{2}\operatorname{ad}(\sigma_3)}\begin{pmatrix}0 & -\frac{(\varepsilon b)^{1/2}}{3\sqrt{2}\alpha_k}\\ \frac{(\varepsilon b)^{-1/2}}{3\sqrt{2}\alpha_k} & 0\end{pmatrix}, \tag{3.210}$$
$$\widehat{\mathbb{E}}_{\infty,k}(\tau):=\frac{1}{2\sqrt{3}\big(\sqrt{3}+1\big)}(b(\tau))^{-\frac{1}{2}\operatorname{ad}(\sigma_3)}\begin{pmatrix}\sqrt{3}+1 & -(2\varepsilon b)^{1/2}\\ (2/\varepsilon b)^{1/2} & \sqrt{3}+1\end{pmatrix}\times\begin{pmatrix}\mathbb{T}_{11,k}(1;\tau) & \mathbb{T}_{12,k}(1;\tau)\\ \mathbb{T}_{21,k}(1;\tau) & \mathbb{T}_{22,k}(1;\tau)\end{pmatrix}, \tag{3.211}$$
$$\widetilde{\mathbb{E}}_{\infty,k}(\tau):=\frac{1}{2\sqrt{3}\big(\sqrt{3}+1\big)}(b(\tau))^{-\frac{1}{2}\operatorname{ad}(\sigma_3)}\begin{pmatrix}\sqrt{3}+1 & -(2\varepsilon b)^{1/2}\\ (2/\varepsilon b)^{1/2} & \sqrt{3}+1\end{pmatrix}\tilde{\mathfrak{C}}_k^{\Diamond}, \tag{3.212}$$

$\mathrm{M}_2(\mathbb{C})\ni\tilde{\mathfrak{C}}_k(\tau)\underset{\tau\to+\infty}{=}\mathcal{O}(1)$, $(\mathbb{T}_{ij,k}(1;\tau))_{i,j=1,2}$ defined in Proposition 3.16, and $\mathrm{M}_2(\mathbb{C})\ni\tilde{\mathfrak{C}}_k^{\Diamond}$ is $\mathcal{O}(1)$.

Recalling the definitions (3.197) and (3.201), and substituting the expansions (3.202), (3.203), and (3.204) into equation (3.196), one shows, via the conditions (3.17), the definition (3.109), the restrictions (3.110), the asymptotics (B.1), (B.16), and (B.18), and (cf. step (xi) in the proof of Lemma 3.17) $\arg(\mu_k(\tau))\underset{\tau\to+\infty}{=}\frac{\pi}{2}\big(1+\mathcal{O}\big(\tau^{-2/3}\big)\big)$, and the restriction (3.184), that

$$\mathfrak{L}_k^{\infty}(\tau)\underset{\tau\to+\infty}{=}\mathrm{i}(\mathcal{R}_{m_\infty}(k))^{-1}\mathrm{e}^{\tilde{\mathfrak{z}}_k^0(\tau)\sigma_3}\left(\frac{(\varepsilon b)^{1/4}\big(\sqrt{3}+1\big)^{1/2}}{2^{1/4}\sqrt{\mathfrak{B}_k}\sqrt{b(\tau)}}\right)^{\sigma_3}\sigma_2\mathrm{e}^{-\Delta\tilde{\mathfrak{z}}_k(\tau)\sigma_3}\times\operatorname{diag}\big(\hat{\mathbb{B}}_0^{\infty}(\tau),\hat{\mathbb{A}}_0^{\infty}(\tau)\big)\overset{\curlywedge}{\mathbb{E}}{}^{\lhd}_{\mathfrak{L}_k^{\infty}}(\tau),\qquad k=\pm1, \tag{3.213}$$

where $\tilde{\mathfrak{z}}_k^0(\tau)$, $\Delta\tilde{\mathfrak{z}}_k(\tau)$, $\hat{\mathbb{A}}_0^{\infty}(\tau)$, and $\hat{\mathbb{B}}_0^{\infty}(\tau)$ are defined by equations (3.189)–(3.192), respectively, and

$$\overset{\curlywedge}{\mathbb{E}}{}^{\lhd}_{\mathfrak{L}_k^{\infty}}(\tau)\underset{\tau\to+\infty}{:=}\big(\mathrm{I}+\mathcal{O}\big(\tau^{-1/3}\widetilde{\Lambda}^3\sigma_3\big)\big)$$

$$\times\left(\mathrm{I}+\mathcal{O}\left(\frac{\hat{\mathbb{D}}_0^{\infty}(\tau)}{\hat{\mathbb{B}}_0^{\infty}(\tau)b(\tau)\mathrm{e}^{2\widetilde{\beta}_k^*(\tau)}}\sigma_+\right)+\mathcal{O}\left(\frac{\hat{\mathbb{C}}_0^{\infty}(\tau)b(\tau)\mathrm{e}^{2\widetilde{\beta}_k^*(\tau)}}{\hat{\mathbb{A}}_0^{\infty}(\tau)}\sigma_-\right)\right)$$
$$\times\left(\mathrm{I}+\frac{1}{\widetilde{\Lambda}}\hat{\psi}_{1,k}^{-1,\sharp}(\tau)+\frac{1}{\widetilde{\Lambda}^2}\hat{\psi}_{2,k}^{-1,\sharp}(\tau)+\mathcal{O}\left(\frac{1}{\widetilde{\Lambda}^3}\hat{\psi}_{3,k}^{-1,\sharp}(\tau)\right)\right)$$
$$\times\left(\mathrm{I}+\mathcal{O}\left(\frac{|\nu(k)+1|^2\tau^{-\epsilon_{\mathrm{TP}}(k)}}{|p_k(\tau)|^2}\mathrm{e}^{-\tilde{\beta}_k(\tau)\,\mathrm{ad}(\sigma_3)}\mathfrak{Q}_{\infty,k}^{-1}(\tau)\tilde{\mathfrak{C}}_k(\tau)\mathfrak{Q}_{\infty,k}(\tau)\right)\right)$$
$$\times\left(\mathrm{I}+\widetilde{\Lambda}\mathfrak{I}_{A,k}^{\sharp}(\tau)+\widetilde{\Lambda}^2\mathfrak{I}_{B,k}^{\sharp}(\tau)\right)\Big(\mathrm{I}+\widetilde{\Lambda}\tau^{-1/3}\mathbb{P}_{\infty,k}^{\sharp}(\tau)+\frac{1}{\widetilde{\Lambda}}\widehat{\mathbb{E}}_{\infty,k}^{\sharp}(\tau)$$
$$+\mathcal{O}\big(\big(\tau^{-1/3}\widetilde{\Lambda}\big)^2\widetilde{\mathbb{E}}_{\infty,k}^{\sharp}(\tau)\big)\Big),$$

where $\widetilde{\beta}_k^*(\tau):=\frac{\mathrm{i}a}{6}\ln\tau+\mathrm{i}3\alpha_k^2\tau^{2/3}$,

$$\hat{\mathbb{C}}_0^{\infty}(\tau):=(\Delta G_k^{\infty}(\tau))_{11},\tag{3.214}$$
$$\hat{\mathbb{D}}_0^{\infty}(\tau):=(\Delta G_k^{\infty}(\tau))_{22}-\frac{\mathfrak{A}_k}{\mathfrak{B}_k}\left(\frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_k}{\chi_k(\tau)}-1\right)\left(\frac{(\varepsilon b)^{1/4}\big(\sqrt{3}+1\big)^{1/2}}{2^{1/4}}+(\Delta G_k^{\infty}(\tau))_{12}\right),\tag{3.215}$$
$$\hat{\psi}_{m,k}^{-1,\sharp}(\tau):=\mathrm{e}^{-\tilde{\beta}_k(\tau)\,\mathrm{ad}(\sigma_3)}\mathfrak{Q}_{\infty,k}^{-1}(\tau)\hat{\psi}_{m,k}^{-1}(\tau)\mathfrak{Q}_{\infty,k}(\tau),\qquad m=1,2,3,\tag{3.216}$$
$$\mathfrak{I}_{A,k}^{\sharp}(\tau):=\mathrm{e}^{-\tilde{\beta}_k(\tau)\,\mathrm{ad}(\sigma_3)}\mathfrak{Q}_{\infty,k}^{-1}(\tau)\mathfrak{I}_{A,k}^{-1}(\tau)\mathfrak{Q}_{\infty,k}(\tau),\tag{3.217}$$
$$\mathfrak{I}_{B,k}^{\sharp}(\tau):=\mathrm{e}^{-\tilde{\beta}_k(\tau)\,\mathrm{ad}(\sigma_3)}\mathfrak{Q}_{\infty,k}^{-1}(\tau)\mathfrak{I}_{B,k}^{-1}(\tau)\mathfrak{Q}_{\infty,k}(\tau),\tag{3.218}$$
$$\mathbb{P}_{\infty,k}^{\sharp}(\tau):=\mathrm{e}^{-\tilde{\beta}_k(\tau)\,\mathrm{ad}(\sigma_3)}\mathbb{P}_{\infty,k}(\tau),\qquad\widehat{\mathbb{E}}_{\infty,k}^{\sharp}(\tau):=\mathrm{e}^{-\tilde{\beta}_k(\tau)\,\mathrm{ad}(\sigma_3)}\widehat{\mathbb{E}}_{\infty,k}(\tau),\tag{3.219}$$
$$\widetilde{\mathbb{E}}_{\infty,k}^{\sharp}(\tau):=\mathrm{e}^{-\tilde{\beta}_k(\tau)\,\mathrm{ad}(\sigma_3)}\widetilde{\mathbb{E}}_{\infty,k}(\tau).\tag{3.220}$$

Via the conditions (3.17), the restrictions (3.110) and (3.184), the definitions (3.57), (3.80), (3.109), (3.112), (3.113), (3.160), (3.161), (3.191)–(3.193), (3.198), (3.206)–(3.212), and (3.214)–(3.220), and the asymptotics (3.21), (3.24), (3.79), (B.1), (B.5)–(B.9), (B.14)–(B.19), and (3.203), upon imposing the conditions (3.186) and (3.187), and defining

$$J_k^{\infty}:=\frac{1}{\sqrt{2\sqrt{3}\big(\sqrt{3}+1\big)}}\begin{pmatrix}\sqrt{3}+1 & -(2\varepsilon b)^{1/2}\\ (2/\varepsilon b)^{1/2} & \sqrt{3}+1\end{pmatrix},$$
$$\mathbf{T}_{\infty,k}^{\sharp}:=(\mathbb{T}_{ij,k}(1;\tau))_{i,j=1,2},$$
$$\mathbb{D}_{\infty,k}^{\sharp}:=\mathfrak{B}_k^{-\frac{1}{2}\sigma_3}\begin{pmatrix}0 & -\frac{(\varepsilon b)^{1/4}(\sqrt{3}+1)^{1/2}}{2^{1/4}}\\ \frac{2^{1/4}}{(\varepsilon b)^{1/4}(\sqrt{3}+1)^{1/2}} & 0\end{pmatrix},$$

one shows that, for $k=\pm1$,

$$\overset{\curlywedge}{\hat{\mathbb{E}}}{}_{\mathfrak{L}_k^{\infty}}^{\lhd}(\tau)\underset{\tau\to+\infty}{=}\big(\mathrm{I}+\mathcal{O}\big(\tau^{-1/3}\widetilde{\Lambda}^3\sigma_3\big)\big)$$
$$\times\left(\mathrm{I}+\mathcal{O}\left(\frac{\hat{\mathbb{D}}_0^{\infty}(\tau)}{\hat{\mathbb{B}}_0^{\infty}(\tau)b(\tau)\mathrm{e}^{2\widetilde{\beta}_k^*(\tau)}}\sigma_+\right)+\mathcal{O}\left(\frac{\hat{\mathbb{C}}_0^{\infty}(\tau)b(\tau)\mathrm{e}^{2\widetilde{\beta}_k^*(\tau)}}{\hat{\mathbb{A}}_0^{\infty}(\tau)}\sigma_-\right)\right)$$
$$\times\left(\mathrm{I}+\frac{1}{\widetilde{\Lambda}}\hat{\psi}_{1,k}^{-1,\sharp}(\tau)+\frac{1}{\widetilde{\Lambda}^2}\hat{\psi}_{2,k}^{-1,\sharp}(\tau)+\mathcal{O}\left(\frac{1}{\widetilde{\Lambda}^3}\hat{\psi}_{3,k}^{-1,\sharp}(\tau)\right)\right)$$
$$\times\left(\mathrm{I}+\mathcal{O}\left(\frac{|\nu(k)+1|^2\tau^{-\epsilon_{\mathrm{TP}}(k)}}{|p_k(\tau)|^2}\mathrm{e}^{-\tilde{\beta}_k(\tau)\,\mathrm{ad}(\sigma_3)}\mathfrak{Q}_{\infty,k}^{-1}(\tau)\tilde{\mathfrak{C}}_k(\tau)\mathfrak{Q}_{\infty,k}(\tau)\right)\right)$$

$$
\begin{aligned}
&\times\Big(\mathrm{I}+\beth^{\sharp}_{A,k}(\tau)\widehat{\mathbb{E}}^{\sharp}_{\infty,k}(\tau)+\frac{1}{\widetilde{\Lambda}}\widehat{\mathbb{E}}^{\sharp}_{\infty,k}(\tau)+\widetilde{\Lambda}\big(\tau^{-1/3}\mathbb{P}^{\sharp}_{\infty,k}(\tau)+\beth^{\sharp}_{A,k}(\tau)\\
&\quad+\beth^{\sharp}_{B,k}(\tau)\widehat{\mathbb{E}}^{\sharp}_{\infty,k}(\tau)\big)+\widetilde{\Lambda}^{2}\big(\tau^{-1/3}\beth^{\sharp}_{A,k}(\tau)\mathbb{P}^{\sharp}_{\infty,k}(\tau)+\beth^{\sharp}_{B,k}(\tau)\\
&\quad+\mathcal{O}\big(\tau^{-2/3}\widetilde{\mathbb{E}}^{\sharp}_{\infty,k}(\tau)\big)\big)+\widetilde{\Lambda}^{3}\big(\tau^{-1/3}\beth^{\sharp}_{B,k}(\tau)\mathbb{P}^{\sharp}_{\infty,k}(\tau)\\
&\quad+\mathcal{O}\big(\tau^{-2/3}\beth^{\sharp}_{A,k}(\tau)\widetilde{\mathbb{E}}^{\sharp}_{\infty,k}(\tau)\big)\big)\Big)\\
\underset{\tau\to+\infty}{=}&\big(\mathrm{I}+\mathcal{O}\big(\tau^{-1/3}\widetilde{\Lambda}^{3}\sigma_{3}\big)\big)\left(\mathrm{I}+\mathcal{O}\left(\frac{\hat{\mathbb{D}}^{\infty}_{0}(\tau)}{\hat{\mathbb{B}}^{\infty}_{0}(\tau)b(\tau)\mathrm{e}^{2\widetilde{\beta}^{*}_{k}(\tau)}}\sigma_{+}\right)\right.\\
&\left.+\mathcal{O}\left(\frac{\hat{\mathbb{C}}^{\infty}_{0}(\tau)b(\tau)\mathrm{e}^{2\widetilde{\beta}^{*}_{k}(\tau)}}{\hat{\mathbb{A}}^{\infty}_{0}(\tau)}\sigma_{-}\right)\right)\left(\mathrm{I}+\frac{1}{\widetilde{\Lambda}}\hat{\psi}^{-1,\sharp}_{1,k}(\tau)+\frac{1}{\widetilde{\Lambda}^{2}}\hat{\psi}^{-1,\sharp}_{2,k}(\tau)\right.\\
&\left.+\mathcal{O}\left(\frac{1}{\widetilde{\Lambda}^{3}}\hat{\psi}^{-1,\sharp}_{3,k}(\tau)\right)\right)\left(\mathrm{I}+\beth^{\sharp}_{A,k}(\tau)\widehat{\mathbb{E}}^{\sharp}_{\infty,k}(\tau)+\frac{1}{\widetilde{\Lambda}}\frac{1}{2\sqrt{3}(\sqrt{3}+1)}\right.\\
&\times\left(\frac{\mathrm{e}^{-\tilde{\beta}_{k}(\tau)}}{\sqrt{b(\tau)}}\right)^{\mathrm{ad}(\sigma_{3})}J^{\infty}_{k}\mathbb{T}^{\sharp}_{\infty,k}+\widetilde{\Lambda}\frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_{k}\mathfrak{A}_{k}\ell^{+}_{0,k}}{\chi_{k}(\tau)}\sigma_{3}\\
&\left.+\mathcal{O}\left(\frac{|\nu(k)+1|^{2}\tau^{-\epsilon_{\mathrm{TP}}(k)}}{|p_{k}(\tau)|^{2}}\left(\frac{\mathrm{e}^{-\tilde{\beta}_{k}(\tau)}}{\sqrt{b(\tau)}}\right)^{\mathrm{ad}(\sigma_{3})}\mathbb{D}^{\sharp}_{\infty,k}\tilde{\mathfrak{C}}_{k}(\tau)\big(\mathbb{D}^{\sharp}_{\infty,k}\big)^{-1}\right)\right)\\
\underset{\tau\to+\infty}{=}&\big(\mathrm{I}+\mathcal{O}\big(\tau^{-1/3}\widetilde{\Lambda}^{3}\sigma_{3}\big)\big)\left(\mathrm{I}+\mathcal{O}\left(\frac{\hat{\mathbb{D}}^{\infty}_{0}(\tau)}{\hat{\mathbb{B}}^{\infty}_{0}(\tau)b(\tau)\mathrm{e}^{2\widetilde{\beta}^{*}_{k}(\tau)}}\sigma_{+}\right)\right.\\
&\left.+\mathcal{O}\left(\frac{\hat{\mathbb{C}}^{\infty}_{0}(\tau)b(\tau)\mathrm{e}^{2\widetilde{\beta}^{*}_{k}(\tau)}}{\hat{\mathbb{A}}^{\infty}_{0}(\tau)}\sigma_{-}\right)\right)\\
&\times\left(\mathrm{I}+\beth^{\sharp}_{A,k}(\tau)\widehat{\mathbb{E}}^{\sharp}_{\infty,k}(\tau)+\frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_{k}\mathfrak{A}_{k}\ell^{+}_{0,k}}{\chi_{k}(\tau)}\hat{\psi}^{-1,\sharp}_{1,k}(\tau)\sigma_{3}\right.\\
&+\widetilde{\Lambda}\frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_{k}\mathfrak{A}_{k}\ell^{+}_{0,k}}{\chi_{k}(\tau)}\sigma_{3}+\frac{1}{\widetilde{\Lambda}}\left(\hat{\psi}^{-1,\sharp}_{1,k}(\tau)+\frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_{k}\mathfrak{A}_{k}\ell^{+}_{0,k}}{\chi_{k}(\tau)}\hat{\psi}^{-1,\sharp}_{2,k}(\tau)\sigma_{3}\right.\\
&\left.+\frac{1}{2\sqrt{3}(\sqrt{3}+1)}\left(\frac{\mathrm{e}^{-\tilde{\beta}_{k}(\tau)}}{\sqrt{b(\tau)}}\right)^{\mathrm{ad}(\sigma_{3})}J^{\infty}_{k}\mathbb{T}^{\sharp}_{\infty,k}\right)\\
&+\frac{1}{\widetilde{\Lambda}^{2}}\left(\hat{\psi}^{-1,\sharp}_{2,k}(\tau)+\frac{1}{2\sqrt{3}(\sqrt{3}+1)}\hat{\psi}^{-1,\sharp}_{1,k}(\tau)\left(\frac{\mathrm{e}^{-\tilde{\beta}_{k}(\tau)}}{\sqrt{b(\tau)}}\right)^{\mathrm{ad}(\sigma_{3})}J^{\infty}_{k}\mathbb{T}^{\sharp}_{\infty,k}\right)\\
&+\mathcal{O}\left(\frac{|\nu(k)+1|^{2}\tau^{-\epsilon_{\mathrm{TP}}(k)}}{|p_{k}(\tau)|^{2}}\left(\frac{\mathrm{e}^{-\tilde{\beta}_{k}(\tau)}}{\sqrt{b(\tau)}}\right)^{\mathrm{ad}(\sigma_{3})}\mathbb{D}^{\sharp}_{\infty,k}\tilde{\mathfrak{C}}_{k}(\tau)\big(\mathbb{D}^{\sharp}_{\infty,k}\big)^{-1}\right)\\
&+\mathcal{O}\left(\frac{1}{\widetilde{\Lambda}}\frac{|\nu(k)+1|^{2}\tau^{-\epsilon_{\mathrm{TP}}(k)}}{|p_{k}(\tau)|^{2}}\hat{\psi}^{-1,\sharp}_{1,k}(\tau)\left(\frac{\mathrm{e}^{-\tilde{\beta}_{k}(\tau)}}{\sqrt{b(\tau)}}\right)^{\mathrm{ad}(\sigma_{3})}\right.\\
&\left.\times\mathbb{D}^{\sharp}_{\infty,k}\tilde{\mathfrak{C}}_{k}(\tau)\big(\mathbb{D}^{\sharp}_{\infty,k}\big)^{-1}\right)\\
&+\mathcal{O}\left(\frac{1}{\widetilde{\Lambda}^{2}}\frac{|\nu(k)+1|^{2}\tau^{-\epsilon_{\mathrm{TP}}(k)}}{|p_{k}(\tau)|^{2}}\hat{\psi}^{-1,\sharp}_{2,k}(\tau)\left(\frac{\mathrm{e}^{-\tilde{\beta}_{k}(\tau)}}{\sqrt{b(\tau)}}\right)^{\mathrm{ad}(\sigma_{3})}\right.
\end{aligned}
$$

$$
\begin{aligned}
&\times \mathbb{D}^{\sharp}_{\infty,k}\tilde{\mathfrak{C}}_k(\tau)\big(\mathbb{D}^{\sharp}_{\infty,k}\big)^{-1}\Bigg) + \mathcal{O}\left(\frac{1}{\widetilde{\Lambda}^2}\frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\ell^{+}_{0,k}}{\chi_k(\tau)}\hat{\psi}^{-1,\sharp}_{3,k}(\tau)\sigma_3\right)\Bigg)\\
\underset{\tau\to+\infty}{=}\, &\mathrm{I} + \beth^{\sharp}_{A,k}(\tau)\widehat{\mathbb{E}}^{\sharp}_{\infty,k}(\tau) + \frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\ell^{+}_{0,k}}{\chi_k(\tau)}\hat{\psi}^{-1,\sharp}_{1,k}(\tau)\sigma_3 + \mathcal{O}\big(\tau^{-\frac{1}{3}+3\delta_k}\sigma_3\big)\\
&+ \begin{pmatrix} 0 & \mathcal{O}\big(\tau^{-2/3}\big)\\ \mathcal{O}\big(\tau^{-2/3}\big) & 0\end{pmatrix} + \begin{pmatrix} \mathcal{O}\big(\tau^{-\frac{1}{3}+\delta_k}\big) & 0\\ 0 & \mathcal{O}\big(\tau^{-\frac{1}{3}+\delta_k}\big)\end{pmatrix}\\
&+ \begin{pmatrix} 0 & \mathcal{O}\big(\tau^{-\delta_k}(\nu(k)+1)^{\frac{1+k}{2}}\big)\\ \mathcal{O}\big(\tau^{-\delta_k}(\nu(k)+1)^{\frac{1-k}{2}}\big) & 0\end{pmatrix}\\
&+ \begin{pmatrix} \mathcal{O}\big(\tau^{-\frac{1}{3}-\delta_k}(\nu(k)+1)\big) & 0\\ 0 & \mathcal{O}\big(\tau^{-\frac{1}{3}-\delta_k}(\nu(k)+1)\big)\end{pmatrix}\\
&+ \begin{pmatrix} \mathcal{O}\big(\tau^{-\frac{1}{3}-\delta_k}\big) & \mathcal{O}\big(\tau^{-\frac{1}{3}-\delta_k}\big)\\ \mathcal{O}\big(\tau^{-\frac{1}{3}-\delta_k}\big) & \mathcal{O}\big(\tau^{-\frac{1}{3}-\delta_k}\big)\end{pmatrix}\\
&+ \begin{pmatrix} \mathcal{O}\big(\tau^{-2\delta_k}(\nu(k)+1)\big) & 0\\ 0 & \mathcal{O}\big(\tau^{-2\delta_k}(\nu(k)+1)\big)\end{pmatrix}\\
&+ \begin{pmatrix} \mathcal{O}\big(\tau^{-\frac{1}{3}-2\delta_k}(\nu(k)+1)^{\frac{1+k}{2}}\big) & \mathcal{O}\big(\tau^{-\frac{1}{3}-2\delta_k}(\nu(k)+1)^{\frac{1+k}{2}}\big)\\ \mathcal{O}\big(\tau^{-\frac{1}{3}-2\delta_k}(\nu(k)+1)^{\frac{1-k}{2}}\big) & \mathcal{O}\big(\tau^{-\frac{1}{3}-2\delta_k}(\nu(k)+1)^{\frac{1-k}{2}}\big)\end{pmatrix}\\
&+ \begin{pmatrix} \mathcal{O}\big(\tau^{-2-\epsilon_{\mathrm{TP}}(k)}\big) & \mathcal{O}\big(\tau^{-1-\epsilon_{\mathrm{TP}}(k)}(\nu(k)+1)^{\frac{1+k}{2}}\big)\\ \mathcal{O}\big(\tau^{-3-\epsilon_{\mathrm{TP}}(k)}(\nu(k)+1)^{\frac{1-k}{2}}\big) & \mathcal{O}\big(\tau^{-2-\epsilon_{\mathrm{TP}}(k)}\big)\end{pmatrix}\\
&+ \begin{pmatrix} \mathcal{O}\big(\tau^{-3-\delta_k-\epsilon_{\mathrm{TP}}(k)}(\nu(k)+1)\big) & \mathcal{O}\big(\tau^{-2-\delta_k-\epsilon_{\mathrm{TP}}(k)}(\nu(k)+1)^{\frac{1+k}{2}}\big)\\ \mathcal{O}\big(\tau^{-2-\delta_k-\epsilon_{\mathrm{TP}}(k)}(\nu(k)+1)^{\frac{1-k}{2}}\big) & \mathcal{O}\big(\tau^{-1-\delta_k-\epsilon_{\mathrm{TP}}(k)}(\nu(k)+1)\big)\end{pmatrix}\\
&+ \begin{pmatrix} \mathcal{O}\big(\tau^{-2-2\delta_k-\epsilon_{\mathrm{TP}}(k)}(\nu(k)+1)\big) & \mathcal{O}\big(\tau^{-1-2\delta_k-\epsilon_{\mathrm{TP}}(k)}(\nu(k)+1)^{\frac{3+k}{2}}\big)\\ \mathcal{O}\big(\tau^{-3-2\delta_k-\epsilon_{\mathrm{TP}}(k)}(\nu(k)+1)^{\frac{3-k}{2}}\big) & \mathcal{O}\big(\tau^{-2-2\delta_k-\epsilon_{\mathrm{TP}}(k)}(\nu(k)+1)\big)\end{pmatrix}\\
&+ \begin{pmatrix} 0 & \mathcal{O}\big(\tau^{-\frac{1}{3}-2\delta_k}(\nu(k)+1)^{\frac{1+k}{2}}\big)\\ \mathcal{O}\big(\tau^{-\frac{1}{3}-2\delta_k}(\nu(k)+1)^{\frac{1-k}{2}}\big) & 0\end{pmatrix}\\
\underset{\tau\to+\infty}{=}\, &\mathrm{I} + \beth^{\sharp}_{A,k}(\tau)\widehat{\mathbb{E}}^{\sharp}_{\infty,k}(\tau) + \frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\ell^{+}_{0,k}}{\chi_k(\tau)}\hat{\psi}^{-1,\sharp}_{1,k}(\tau)\sigma_3\\
&+ \begin{pmatrix} \mathcal{O}\big(\tau^{-\frac{1}{3}+3\delta_k}\big) & \mathcal{O}\big(\tau^{-2/3}\big)\\ \mathcal{O}\big(\tau^{-2/3}\big) & \mathcal{O}\big(\tau^{-\frac{1}{3}+3\delta_k}\big)\end{pmatrix} + \begin{pmatrix} \mathcal{O}\big(\tau^{-\frac{1}{3}+\delta_k}\big) & \mathcal{O}\big(\tau^{-\frac{1}{3}(\frac{1+k}{2})-\delta_k}\big)\\ \mathcal{O}\big(\tau^{-\frac{1}{3}(\frac{1-k}{2})-\delta_k}\big) & \mathcal{O}\big(\tau^{-\frac{1}{3}+\delta_k}\big)\end{pmatrix}\\
\underset{\tau\to+\infty}{=}\, &\mathrm{I} + \underbrace{\beth^{\sharp}_{A,k}(\tau)\widehat{\mathbb{E}}^{\sharp}_{\infty,k}(\tau) + \frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\ell^{+}_{0,k}}{\chi_k(\tau)}\hat{\psi}^{-1,\sharp}_{1,k}(\tau)\sigma_3}_{=:\mathbb{E}^{\infty}_{\mathcal{N},k}(\tau)}\\
&+ \underbrace{\begin{pmatrix} \mathcal{O}\big(\tau^{-\frac{1}{3}+3\delta_k}\big) & \mathcal{O}\big(\tau^{-\frac{1}{3}(\frac{1+k}{2})-\delta_k}\big)\\ \mathcal{O}\big(\tau^{-\frac{1}{3}(\frac{1-k}{2})-\delta_k}\big) & \mathcal{O}\big(\tau^{-\frac{1}{3}+3\delta_k}\big)\end{pmatrix}}_{=:\mathcal{O}(\mathbb{E}^{\infty}_{k}(\tau))}\\
\underset{\tau\to+\infty}{=}\, &(\mathrm{I}+\mathbb{E}^{\infty}_{\mathcal{N},k}(\tau))\Bigg(\mathrm{I} + \underbrace{(\mathrm{I}+\mathbb{E}^{\infty}_{\mathcal{N},k}(\tau))^{-1}}_{=\mathcal{O}(1)}\mathcal{O}(\mathbb{E}^{\infty}_{k}(\tau))\Bigg) \qquad \Rightarrow
\end{aligned}
$$

$$
\overset{\curlywedge}{\mathbb{E}}^{\lhd}_{\mathfrak{L}^{\infty}_{k}}(\tau) \underset{\tau\to+\infty}{=} \big(\mathrm{I}+\mathbb{E}^{\infty}_{\mathcal{N},k}(\tau)\big)\big(\mathrm{I}+\mathcal{O}(\mathbb{E}^{\infty}_{k}(\tau))\big), \tag{3.221}
$$

where $\mathbb{E}^{\infty}_{\mathcal{N},k}(\tau)$ and $\mathcal{O}(\mathbb{E}^{\infty}_{k}(\tau))$ are defined by equations (3.194) and (3.195), respectively.[35] Thus, via the asymptotics (3.213) and (3.221), one arrives at the results stated in the lemma. ■

Lemma 3.22. *Let $\widetilde{\Psi}_k(\widetilde{\mu},\tau)$, $k=\pm 1$, be the fundamental solution of equation (3.3) with asymptotics given in Lemma 3.17, and let $\mathbb{X}^0_{1-k}(\widetilde{\mu},\tau)$ be the canonical solution of equation (3.1).[36] Define[37]*

$$\mathfrak{L}^0_k(\tau) := \big(\widetilde{\Psi}_k(\widetilde{\mu},\tau)\big)^{-1}\tau^{-\frac{1}{12}\sigma_3}\mathbb{X}^0_{1-k}\big(\tau^{-1/6}\widetilde{\mu},\tau\big), \qquad k=\pm 1. \tag{3.222}$$

Assume that the parameters $\nu(k)+1$ and δ_k satisfy the restrictions (3.110) and (3.184), respectively, and, additionally, the conditions (3.186) and (3.187) are valid. Then,

$$\begin{aligned}\mathfrak{L}^0_k(\tau) \underset{\tau\to+\infty}{=} (\mathcal{R}_{m_0}(k))^{-1}\mathrm{e}^{\hat{\mathfrak{z}}^0_k(\tau)\sigma_3}\left(\frac{\mathrm{i}2^{1/4}}{\big(\sqrt{3}-1\big)^{1/2}\sqrt{\mathfrak{B}_k}}\right)^{\sigma_3}\mathrm{e}^{\Delta\hat{\mathfrak{z}}_k(\tau)\sigma_3}\begin{pmatrix}\hat{\mathbb{A}}^0_0(\tau) & 0\\ 0 & \hat{\mathbb{B}}^0_0(\tau)\end{pmatrix}\\ \times\big(\mathrm{I}+\mathbb{E}^0_{\mathcal{N},k}(\tau)\big)\,\mathbb{S}^*_k\,\big(\mathrm{I}+\mathcal{O}\big(\mathbb{E}^0_k(\tau)\big)\big),\end{aligned} \tag{3.223}$$

where $\mathrm{M}_2(\mathbb{C})\ni\mathcal{R}_{m_0}(k)$, $m_0\in\{-1,0,1,2\}$, are defined in Remark 3.20,

$$\hat{\mathfrak{z}}^0_k(\tau) := \mathrm{i}\tau^{2/3}3\sqrt{3}\alpha_k^2+\mathrm{i}(a-\mathrm{i}/2)\ln\big(2^{-1/2}\big(\sqrt{3}+1\big)\big), \tag{3.224}$$

$$\begin{aligned}\Delta\hat{\mathfrak{z}}_k(\tau) := -\left(\frac{5+9\sqrt{3}}{6\sqrt{3}\alpha_k^2}\right)\mathfrak{p}_k(\tau)+(\nu(k)+1)\ln(2\mu_k(\tau))^{1/2}+\frac{1}{3}(\nu(k)+1)\ln\tau\\ -(\nu(k)+1)\ln\big(\mathrm{e}^{\mathrm{i}k\pi}/3\alpha_k\big),\end{aligned} \tag{3.225}$$

with $\mathfrak{p}_k(\tau)$ defined by equation (3.57), and $\mathfrak{B}_k$ and $\mu_k(\tau)$ defined in Lemma 3.17,

$$\hat{\mathbb{A}}^0_0(\tau) := 1+\frac{(\varepsilon b)^{1/4}\big(\sqrt{3}-1\big)^{1/2}\big(\Delta G^0_k(\tau)\big)_{11}}{2^{1/4}}, \tag{3.226}$$

$$\begin{aligned}\hat{\mathbb{B}}^0_0(\tau) := 1+\frac{2^{1/4}}{(\varepsilon b)^{1/4}\big(\sqrt{3}-1\big)^{1/2}}\\ \times\left(\big(\Delta G^0_k(\tau)\big)_{22}-\frac{\mathfrak{A}_k}{\mathfrak{B}_k}\left(\frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_k}{\chi_k(\tau)}-1\right)\big(\Delta G^0_k(\tau)\big)_{12}\right),\end{aligned} \tag{3.227}$$

with $\mathcal{Z}_k$, $\mathfrak{A}_k$, and $\chi_k(\tau)$ defined in Lemma 3.17, and

$$\Delta G^0_k(\tau) := \frac{1}{\big(2\sqrt{3}\big(\sqrt{3}-1\big)\big)^{1/2}}\begin{pmatrix}\big(\Delta G^0_k(\tau)\big)_{11} & \big(\Delta G^0_k(\tau)\big)_{12}\\ \big(\Delta G^0_k(\tau)\big)_{21} & \big(\Delta G^0_k(\tau)\big)_{22}\end{pmatrix}, \tag{3.228}$$

with

$$\begin{aligned}\big(\Delta G^0_k(\tau)\big)_{11} &= \big(\sqrt{3}-1\big)\big(\Delta\mathcal{G}_{0,k}\big)_{22}-(2/\varepsilon b)^{1/2}\big(\Delta\mathcal{G}_{0,k}\big)_{12},\\ \big(\Delta G^0_k(\tau)\big)_{12} &= -\big(\sqrt{3}-1\big)\big(\Delta\mathcal{G}_{0,k}\big)_{12}-(2\varepsilon b)^{1/2}\big(\Delta\mathcal{G}_{0,k}\big)_{22},\\ \big(\Delta G^0_k(\tau)\big)_{21} &= -\big(\sqrt{3}-1\big)\big(\Delta\mathcal{G}_{0,k}\big)_{21}+(2/\varepsilon b)^{1/2}\big(\Delta\mathcal{G}_{0,k}\big)_{11},\\ \big(\Delta G^0_k(\tau)\big)_{22} &= \big(\sqrt{3}-1\big)\big(\Delta\mathcal{G}_{0,k}\big)_{11}+(2\varepsilon b)^{1/2}\big(\Delta\mathcal{G}_{0,k}\big)_{21},\end{aligned}$$

[35] The asymptotics for the function $\mathbb{E}^{\infty}_{\mathcal{N},k}(\tau)$ is presented in the proof of Lemma 4.1 (see Section 4).

[36] See Proposition 1.15.

[37] Since (cf. equations (3.2)) $\tau^{-\frac{1}{12}\sigma_3}\mathbb{X}^0_{1-k}\big(\tau^{-1/6}\widetilde{\mu},\tau\big)$, $k=\pm 1$, is also a fundamental solution of equation (3.3), it follows, therefore, that $\mathfrak{L}^0_k(\tau)$ is independent of $\widetilde{\mu}$.

where $\big(\Delta\mathcal{G}_{0,k}\big)_{i,j=1,2}$ *are defined by equations* (B.5)–(B.7),

$$\mathbb{S}_k^{*} := \begin{pmatrix} 1 & -(1+k)s_0^0/2 \\ (1-k)s_0^0/2 & 1 \end{pmatrix}, \tag{3.229}$$

$$\begin{aligned}\mathbb{E}^0_{\mathcal{N},k}(\tau) &:= \mathrm{e}^{-\hat{\beta}_k(\tau)\,\mathrm{ad}(\sigma_3)} \\ &\quad\times\Bigg(\frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\ell^{+}_{0,k}}{\chi_k(\tau)}\left(\frac{(\sqrt{3}-1)p_k(\tau)\mathfrak{B}_k}{2^{3/2}\mu_k(\tau)}\sigma_+ + \frac{\sqrt{2}(\nu(k)+1)}{(\sqrt{3}-1)p_k(\tau)\mathfrak{B}_k}\sigma_-\right)\sigma_3 \\ &\quad+\frac{1}{2\sqrt{3}(\sqrt{3}-1)}\begin{pmatrix} -\frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\ell^{+}_{0,k}}{\chi_k(\tau)} & \frac{(\sqrt{3}-1)\mathfrak{B}_k\ell^{+}_{0,k}}{\sqrt{2}} \\ -\frac{\sqrt{2}}{(\sqrt{3}-1)\mathfrak{B}_k}\big(\big(\frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k}{\chi_k(\tau)}\big)^2\ell^{+}_{0,k}-\ell^{+}_{1,k}-\ell^{+}_{2,k}\big) & \frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\ell^{+}_{0,k}}{\chi_k(\tau)}\end{pmatrix} \\ &\quad\times\begin{pmatrix}\sqrt{3}-1 & (2\varepsilon b)^{1/2} \\ -(2/\varepsilon b)^{1/2} & \sqrt{3}-1\end{pmatrix}\begin{pmatrix}\mathbb{T}_{11,k}(-1;\tau) & \mathbb{T}_{12,k}(-1;\tau) \\ \mathbb{T}_{21,k}(-1;\tau) & \mathbb{T}_{22,k}(-1;\tau)\end{pmatrix}\Bigg),\end{aligned} \tag{3.230}$$

with $\ell^{+}_{0,k}$, $\ell^{+}_{1,k}$, *and* $\ell^{+}_{2,k}$ *defined in Lemma* 3.17, $(\mathbb{T}_{ij,k}(-1;\tau))_{i,j=1,2}$ *defined in Proposition* 3.16, *and* $\hat{\beta}_k(\tau)$ *defined by equation* (3.236), *and*

$$\mathcal{O}\big(\mathbb{E}_k^0(\tau)\big) \underset{\tau\to+\infty}{:=} \begin{pmatrix}\mathcal{O}\big(\tau^{-\frac{1}{3}+3\delta_k}\big) & \mathcal{O}\big(\tau^{-\frac{1}{3}(\frac{1-k}{2})-\delta_k}\big) \\ \mathcal{O}\big(\tau^{-\frac{1}{3}(\frac{1+k}{2})-\delta_k}\big) & \mathcal{O}\big(\tau^{-\frac{1}{3}+3\delta_k}\big)\end{pmatrix}. \tag{3.231}$$

Proof. Denote by $\widetilde{\Psi}_{\mathrm{WKB},k}(\widetilde{\mu},\tau)$, $k=\pm1$, the solution of equation (3.3) that has leading-order asymptotics given by equations (3.25)–(3.27) in the canonical domain containing the Stokes curve approaching, for $k=+1$ (resp., $k=-1$), the real $\widetilde{\mu}$-axis from above (resp., below) as $\widetilde{\mu}\to0$. Let $\mathfrak{L}_k^0(\tau)$, $k=\pm1$, be defined by equation (3.222); rewrite $\mathfrak{L}_k^0(\tau)$ in the following form:

$$\mathfrak{L}_k^0(\tau) = \Big(\big(\widetilde{\Psi}_k(\widetilde{\mu},\tau)\big)^{-1}\widetilde{\Psi}_{\mathrm{WKB},k}(\widetilde{\mu},\tau)\Big)\Big(\big(\widetilde{\Psi}_{\mathrm{WKB},k}(\widetilde{\mu},\tau)\big)^{-1}\tau^{-\frac{1}{12}\sigma_3}\mathbb{X}_1^0\big(\tau^{-1/6}\widetilde{\mu},\tau\big)\Big)\mathbb{S}_k^{*}, \tag{3.232}$$

where $\mathbb{S}_k^{*}$ is defined by equation (3.229). Since $\widetilde{\Psi}_k(\widetilde{\mu},\tau)$, $\widetilde{\Psi}_{\mathrm{WKB},k}(\widetilde{\mu},\tau)$, and $\tau^{-\frac{1}{12}\sigma_3}\mathbb{X}_1^0\big(\tau^{-1/6}\widetilde{\mu},\tau\big)$ are all solutions of equation (3.3), it follows that they differ on the right by non-degenerate, $\widetilde{\mu}$-independent, $\mathrm{M}_2(\mathbb{C})$-valued factors: via this observation, one evaluates, asymptotically, each of the factors appearing in equation (3.232) by considering separate limits, namely, $\widetilde{\mu}\to\alpha_k$ and $\widetilde{\mu}\to0$, respectively; more precisely, for $k=\pm1$,

$$\begin{aligned}&\big(\widetilde{\Psi}_k(\widetilde{\mu},\tau)\big)^{-1}\widetilde{\Psi}_{\mathrm{WKB},k}(\widetilde{\mu},\tau) \\ &\underset{\tau\to+\infty}{:=} \underbrace{\Big((b(\tau))^{-\frac{1}{2}\sigma_3}\mathcal{G}_{0,k}\mathfrak{B}_k^{\frac{1}{2}\sigma_3}\mathbb{F}_k(\tau)\Xi_k(\tau;\widetilde{\Lambda})\hat{\chi}_k\big(\widetilde{\Lambda}\big)\Phi_{M,k}\big(\widetilde{\Lambda}\big)\Big)^{-1}T(\widetilde{\mu})\mathrm{e}^{\mathrm{W}_k(\widetilde{\mu},\tau)}}_{\widetilde{\mu}=\widetilde{\mu}_{0,k},\widetilde{\Lambda}\underset{\tau\to+\infty}{\sim}\mathcal{O}(\tau^{\delta_k}),0<\delta<\delta_k<\frac{1}{24},\arg(\widetilde{\Lambda})=\frac{\pi m_0}{2}+\frac{\pi}{4}-\frac{1}{2}\arg(\mu_k(\tau)),m_0\in\{-1,0,1,2\}},\end{aligned} \tag{3.233}$$

where (cf. Lemma 3.21) $\mathbb{F}_k(\tau)$ and $\Xi_k\big(\tau;\widetilde{\Lambda}\big)$ are given in equations (3.198) and (3.199), respectively, $\mathrm{W}_k(\widetilde{\mu},\tau):=-\sigma_3\mathrm{i}\tau^{2/3}\int_{\widetilde{\mu}_{0,k}}^{\widetilde{\mu}}l_k(\xi)\mathrm{d}\xi-\int_{\widetilde{\mu}_{0,k}}^{\widetilde{\mu}}\mathrm{diag}((T(\xi))^{-1}\partial_\xi T(\xi))\mathrm{d}\xi$, and $\hat{\chi}_k\big(\widetilde{\Lambda}\big)$ has the asymptotics (3.200), and

$$\begin{aligned}&\big(\widetilde{\Psi}_{\mathrm{WKB},k}(\widetilde{\mu},\tau)\big)^{-1}\tau^{-\frac{1}{12}\sigma_3}\mathbb{X}_1^0\big(\tau^{-1/6}\widetilde{\mu},\tau\big) \\ &\underset{\tau\to+\infty}{:=} \lim_{\substack{\Omega_1^0\ni\widetilde{\mu}\to0 \\ \arg(\widetilde{\mu})=\pi}}\Big(\big(T(\widetilde{\mu})\mathrm{e}^{\mathrm{W}_k(\widetilde{\mu},\tau)}\big)^{-1}\tau^{-\frac{1}{12}\sigma_3}\mathbb{X}_1^0\big(\tau^{-1/6}\widetilde{\mu},\tau\big)\Big).\end{aligned} \tag{3.234}$$

One commences by considering the asymptotics subsumed in the definition (3.234). From the asymptotics for $\mathbb{X}_1^0\big(\tau^{-1/6}\widetilde{\mu},\tau\big)$ stated in Proposition 1.15, equations (3.15), (3.16), (3.18), (3.19), (3.53), (3.54), (3.57), (3.86), (3.87), (3.95), and (3.130), one arrives at, via the conditions (3.17) and the asymptotics (3.48), (3.79), and (B.14),

$$\lim_{\substack{\Omega_1^0\ni\widetilde{\mu}\to 0\\ \arg(\widetilde{\mu})=\pi}}\Big(\big(T(\widetilde{\mu})\mathrm{e}^{\mathrm{W}_k(\widetilde{\mu},\tau)}\big)^{-1}\tau^{-\frac{1}{12}\sigma_3}\mathbb{X}_1^0\big(\tau^{-1/6}\widetilde{\mu},\tau\big)\Big) \underset{\tau\to+\infty}{=} \left(\frac{\mathrm{i}(\varepsilon b)^{1/4}}{\sqrt{b(\tau)}}\right)^{\sigma_3}\exp\big(\hat{\beta}_k(\tau)\sigma_3\big),$$
$$k=\pm 1, \tag{3.235}$$

where

$$\begin{aligned}\hat{\beta}_k(\tau) :={}& \mathrm{i}\tau^{2/3}3\sqrt{3}\alpha_k^2+\mathrm{i}2\sqrt{3}\widetilde{\Lambda}^2+\mathrm{i}(a-\mathrm{i}/2)\ln\big((\sqrt{3}+1)/\sqrt{2}\big)-\frac{(5+9\sqrt{3})\mathfrak{p}_k(\tau)}{6\sqrt{3}\alpha_k^2}\\ &+\left(\frac{\mathrm{i}}{2\sqrt{3}}\big((a-\mathrm{i}/2)+\alpha_k^{-2}\tau^{2/3}\hat{h}_0(\tau)\big)+\frac{2\mathfrak{p}_k(\tau)}{3\sqrt{3}\alpha_k^2}\right)\\ &\times\left(-\frac{1}{3}\ln\tau+\ln\widetilde{\Lambda}+\ln\big(\mathrm{e}^{\mathrm{i}k\pi}/3\alpha_k\big)\right)-\frac{(\sqrt{3}+1)\mathfrak{p}_k(\tau)}{\sqrt{3}\alpha_k\tau^{-1/3}\widetilde{\Lambda}}\\ &+\mathcal{O}\left(\left(\frac{\tilde{\mathfrak{c}}_{1,k}\tau^{-1/3}+\tilde{\mathfrak{c}}_{2,k}\tilde{r}_0(\tau)}{\widetilde{\Lambda}^2}\right)\big(\tilde{\mathfrak{c}}_{3,k}\tau^{-1/3}+\tilde{\mathfrak{c}}_{4,k}(\tilde{r}_0(\tau)+4v_0(\tau))\big)\right)\\ &+\mathcal{O}\big(\tau^{-1/3}\widetilde{\Lambda}^3\big)+\mathcal{O}\big(\tau^{-1/3}\widetilde{\Lambda}\big)\\ &+\mathcal{O}\left(\frac{\tau^{-1/3}}{\widetilde{\Lambda}}\big(\tilde{\mathfrak{c}}_{5,k}+\tilde{\mathfrak{c}}_{6,k}\tau^{2/3}\hat{h}_0(\tau)+\tilde{\mathfrak{c}}_{7,k}\big(\tau^{2/3}\hat{h}_0(\tau)\big)^2\big)\right)\\ &+\mathcal{O}\left(\tau^{-2/3}\hat{d}_{0,k}(\tau)\left(-\frac{1}{3}\ln\tau+\ln\widetilde{\Lambda}\right)\right),\end{aligned} \tag{3.236}$$

$\tilde{\mathfrak{c}}_{m,k}$, $m=1,2,\dots,7$, are $\mathcal{O}(1)$, and $\hat{d}_{0,k}(\tau)$ is defined in the proof of Proposition 3.9.

One now derives the asymptotics defined by equation (3.233). From the asymptotics (3.103) for $\varpi=-1$, equation (3.115) for $\Phi_{M,k}(\widetilde{\Lambda})$ (in conjunction with its large-$\widetilde{\Lambda}$ asymptotics stated in Remark 3.20), the definitions (3.198) and (3.199) (concomitant with the fact that $\det\big(\Xi_k(\tau;\widetilde{\Lambda})\big)=1$), and the asymptotics (3.200), one shows, via the relation $(\mathrm{W}_k(\widetilde{\mu}_{0,k},\tau))_{i,j=1,2}=0$ and the definition (3.233), that, for $k=\pm1$,

$$\begin{aligned}&\big(\widetilde{\Psi}_k(\widetilde{\mu},\tau)\big)^{-1}\widetilde{\Psi}_{\mathrm{WKB},k}(\widetilde{\mu},\tau)\\ &\underset{\tau\to+\infty}{:=}\Phi_{M,k}^{-1}\big(\widetilde{\Lambda}\big)\hat{\chi}_k^{-1}\big(\widetilde{\Lambda}\big)\Xi_k^{-1}\big(\tau;\widetilde{\Lambda}\big)\mathbb{F}_k^{-1}(\tau)\mathfrak{B}_k^{-\frac{1}{2}\sigma_3}\mathcal{G}_{0,k}^{-1}(b(\tau))^{\frac{1}{2}\sigma_3}T(\widetilde{\mu}_{0,k})\\ &\underset{\tau\to+\infty}{=}(\mathcal{R}_{m_0}(k))^{-1}\mathrm{e}^{-\mathcal{P}_0^*\sigma_3}\mathfrak{Q}_{0,k}(\tau)\left(\mathrm{I}+\frac{1}{\widetilde{\Lambda}}\mathfrak{Q}_{0,k}^{-1}(\tau)\hat{\psi}_{1,k}^{-1}(\tau)\mathfrak{Q}_{0,k}(\tau)\right.\\ &\qquad\left.+\frac{1}{\widetilde{\Lambda}^2}\mathfrak{Q}_{0,k}^{-1}(\tau)\hat{\psi}_{2,k}^{-1}(\tau)\mathfrak{Q}_{0,k}(\tau)+\mathcal{O}\left(\frac{1}{\widetilde{\Lambda}^3}\mathfrak{Q}_{0,k}^{-1}(\tau)\hat{\psi}_{3,k}^{-1}(\tau)\mathfrak{Q}_{0,k}(\tau)\right)\right)\\ &\qquad\times\big(\mathrm{I}+\mathcal{O}\big(|\nu(k)+1|^2|p_k(\tau)|^{-2}\tau^{-\epsilon_{\mathrm{TP}}(k)}\mathfrak{Q}_{0,k}^{-1}(\tau)\tilde{\mathfrak{C}}_k(\tau)\mathfrak{Q}_{0,k}(\tau)\big)\big)\\ &\qquad\times\big(\mathrm{I}+\widetilde{\Lambda}\mathfrak{Q}_{0,k}^{-1}(\tau)\mathfrak{I}_{A,k}^{-1}(\tau)\mathfrak{Q}_{0,k}(\tau)+\widetilde{\Lambda}^2\mathfrak{Q}_{0,k}^{-1}(\tau)\mathfrak{I}_{B,k}^{-1}(\tau)\mathfrak{Q}_{0,k}(\tau)\big)\\ &\qquad\times\left(\mathrm{I}+\widetilde{\Lambda}\tau^{-1/3}\mathbb{P}_{0,k}(\tau)+\frac{1}{\widetilde{\Lambda}}\widehat{\mathbb{E}}_{0,k}(\tau)+\mathcal{O}\big(\big(\tau^{-1/3}\widetilde{\Lambda}\big)^2\widetilde{\mathbb{E}}_{0,k}(\tau)\big)\right),\end{aligned} \tag{3.237}$$

where $\mathrm{M}_2(\mathbb{C})\ni\mathcal{R}_{m_0}(k)$, $m_0\in\{-1,0,1,2\}$, are defined in Remark 3.20, $\mathcal{P}_0^*$, $\hat{\psi}_{1,k}^{-1}(\tau)$, $\hat{\psi}_{2,k}^{-1}(\tau)$, and $\hat{\psi}_{3,k}^{-1}(\tau)$ are defined by equations (3.205), (3.207), (3.208), and (3.209), respectively,

$$\mathfrak{Q}_{0,k}(\tau):=\mathbb{F}_k^{-1}(\tau)\left(\left(\frac{2^{1/4}\sqrt{b(\tau)}}{(\varepsilon b)^{1/4}\big(\sqrt{3}-1\big)^{1/2}\sqrt{\mathfrak{B}_k}}\right)^{\sigma_3}+\mathfrak{B}_k^{-\frac{1}{2}\sigma_3}\Delta G_k^0(\tau)(b(\tau))^{\frac{1}{2}\sigma_3}\right), \tag{3.238}$$

with $\Delta G_k^0(\tau)$ defined by equation (3.228),

$$\mathbb{P}_{0,k}(\tau) := (b(\tau))^{-\frac{1}{2}\,\mathrm{ad}(\sigma_3)} \begin{pmatrix} 0 & -\frac{(\varepsilon b)^{1/2}}{3\sqrt{2}\alpha_k} \\ \frac{(\varepsilon b)^{-1/2}}{3\sqrt{2}\alpha_k} & 0 \end{pmatrix}, \tag{3.239}$$

$$\widehat{\mathbb{E}}_{0,k}(\tau) := \frac{1}{2\sqrt{3}\big(\sqrt{3}-1\big)}(b(\tau))^{-\frac{1}{2}\,\mathrm{ad}(\sigma_3)} \begin{pmatrix} \sqrt{3}-1 & (2\varepsilon b)^{1/2} \\ -(2/\varepsilon b)^{1/2} & \sqrt{3}-1 \end{pmatrix} \times \begin{pmatrix} \mathbb{T}_{11,k}(-1;\tau) & \mathbb{T}_{12,k}(-1;\tau) \\ \mathbb{T}_{21,k}(-1;\tau) & \mathbb{T}_{22,k}(-1;\tau) \end{pmatrix}, \tag{3.240}$$

$$\widetilde{\mathbb{E}}_{0,k}(\tau) := \frac{1}{2\sqrt{3}\big(\sqrt{3}-1\big)}(b(\tau))^{-\frac{1}{2}\,\mathrm{ad}(\sigma_3)} \begin{pmatrix} \sqrt{3}-1 & (2\varepsilon b)^{1/2} \\ -(2/\varepsilon b)^{1/2} & \sqrt{3}-1 \end{pmatrix} \tilde{\mathfrak{C}}_k^{\diamondsuit}, \tag{3.241}$$

$\mathrm{M}_2(\mathbb{C}) \ni \tilde{\mathfrak{C}}_k(\tau) \underset{\tau\to+\infty}{=} \mathcal{O}(1)$, $(\mathbb{T}_{ij,k}(-1;\tau))_{i,j=1,2}$ defined in Proposition 3.16, and $\mathrm{M}_2(\mathbb{C}) \ni \tilde{\mathfrak{C}}_k^{\diamondsuit}$ is $\mathcal{O}(1)$.

Recalling the definitions (3.233) and (3.234), and substituting the expansions (3.235), (3.236), and (3.237) into equation (3.232), one shows, via the conditions (3.17), the definition (3.109), the restrictions (3.110), the asymptotics (B.1), (B.16), and (B.18), and (cf. step (xi) in the proof of Lemma 3.17) $\arg(\mu_k(\tau)) \underset{\tau\to+\infty}{=} \frac{\pi}{2}\big(1+\mathcal{O}\big(\tau^{-2/3}\big)\big)$, and the restriction (3.184), that

$$\mathfrak{L}_k^0(\tau) \underset{\tau\to+\infty}{=} (\mathcal{R}_{m_0}(k))^{-1} \mathrm{e}^{\hat{\mathfrak{z}}_k^0(\tau)\sigma_3} \left(\frac{\mathrm{i}2^{1/4}}{\big(\sqrt{3}-1\big)^{1/2}\sqrt{\mathfrak{B}_k}}\right)^{\sigma_3} \mathrm{e}^{\Delta\hat{\mathfrak{z}}_k(\tau)\sigma_3}\,\mathrm{diag}\big(\hat{\mathbb{A}}_0^0(\tau),\hat{\mathbb{B}}_0^0(\tau)\big) \times \overset{\curlyvee}{\mathbb{E}}{}^{\lhd}_{\mathfrak{L}_k^0}(\tau)\mathbb{S}_k^*, \qquad k=\pm1, \tag{3.242}$$

where $\hat{\mathfrak{z}}_k^0(\tau)$, $\Delta\hat{\mathfrak{z}}_k(\tau)$, $\hat{\mathbb{A}}_0^0(\tau)$, and $\hat{\mathbb{B}}_0^0(\tau)$ are defined by equations (3.224)–(3.227), respectively, and

$$\begin{aligned} \overset{\curlyvee}{\mathbb{E}}{}^{\lhd}_{\mathfrak{L}_k^0}(\tau) \underset{\tau\to+\infty}{:=} &\big(\mathrm{I}+\mathcal{O}\big(\tau^{-1/3}\widetilde{\Lambda}^3\sigma_3\big)\big)\left(\mathrm{I}+\mathcal{O}\left(\frac{\hat{\mathbb{C}}_0^0(\tau)\sqrt{b(\tau)}}{\hat{\mathbb{A}}_0^0(\tau)\mathrm{e}^{\widehat{\beta}_k^*(\tau)}}\sigma_+\right)+\mathcal{O}\left(\frac{\hat{\mathbb{D}}_0^0(\tau)\mathrm{e}^{\widehat{\beta}_k^*(\tau)}}{\hat{\mathbb{B}}_0^0(\tau)\sqrt{b(\tau)}}\sigma_-\right)\right) \\ &\times\left(\mathrm{I}+\frac{1}{\widetilde{\Lambda}}\hat{\psi}_{1,k}^{-1,\natural}(\tau)+\frac{1}{\widetilde{\Lambda}^2}\hat{\psi}_{2,k}^{-1,\natural}(\tau)+\mathcal{O}\left(\frac{1}{\widetilde{\Lambda}^3}\hat{\psi}_{3,k}^{-1,\natural}(\tau)\right)\right) \\ &\times\left(\mathrm{I}+\mathcal{O}\left(\frac{|\nu(k)+1|^2\tau^{-\epsilon_{\mathrm{TP}}(k)}}{|p_k(\tau)|^2}\mathfrak{Q}_{*,k}^{-1}(\tau)\tilde{\mathfrak{C}}_k(\tau)\mathfrak{Q}_{*,k}(\tau)\right)\right) \\ &\times\big(\mathrm{I}+\widetilde{\Lambda}\mathfrak{I}_{A,k}^{\natural}(\tau)+\widetilde{\Lambda}^2\mathfrak{I}_{B,k}^{\natural}(\tau)\big) \\ &\times\left(\mathrm{I}+\widetilde{\Lambda}\tau^{-1/3}\mathbb{P}_{0,k}^{\natural}(\tau)+\frac{1}{\widetilde{\Lambda}}\widehat{\mathbb{E}}_{0,k}^{\natural}(\tau)+\mathcal{O}\big(\big(\tau^{-1/3}\widetilde{\Lambda}\big)^2\widetilde{\mathbb{E}}_{0,k}^{\natural}(\tau)\big)\right), \end{aligned} \tag{3.243}$$

where $-\widehat{\beta}_k^*(\tau) := \frac{\mathrm{i}a}{6}\ln\tau+\mathrm{i}3\alpha_k^2\tau^{2/3}$,

$$\hat{\mathbb{C}}_0^0(\tau) := -\mathrm{i}(\varepsilon b)^{-1/4}\big(\Delta G_k^0(\tau)\big)_{12}, \tag{3.244}$$

$$\hat{\mathbb{D}}_0^0(\tau) := \mathrm{i}(\varepsilon b)^{1/4}\big(\Delta G_k^0(\tau)\big)_{21}-\frac{\mathfrak{A}_k}{\mathfrak{B}_k}\left(\frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_k}{\chi_k(\tau)}-1\right) \times\left(\frac{\mathrm{i}2^{1/4}}{\big(\sqrt{3}-1\big)^{1/2}}+\mathrm{i}(\varepsilon b)^{1/4}\big(\Delta G_k^0(\tau)\big)_{11}\right), \tag{3.245}$$

$$\hat{\psi}_{m,k}^{-1,\natural}(\tau) := \mathfrak{Q}_{*,k}^{-1}(\tau)\hat{\psi}_{m,k}^{-1}(\tau)\mathfrak{Q}_{*,k}(\tau), \qquad m=1,2,3, \tag{3.246}$$

$$\mathfrak{Q}_{*,k}(\tau) := \mathfrak{Q}_{0,k}(\tau)\big(\mathrm{i}(\varepsilon b)^{1/4}\big)^{\sigma_3}(b(\tau))^{-\frac{1}{2}\sigma_3}\mathrm{e}^{\hat{\beta}_k(\tau)\sigma_3}, \tag{3.247}$$

$$\mathfrak{I}^{\natural}_{A,k}(\tau) := \mathfrak{Q}^{-1}_{*,k}(\tau)\mathfrak{I}^{-1}_{A,k}(\tau)\mathfrak{Q}_{*,k}(\tau), \qquad \mathfrak{I}^{\natural}_{B,k}(\tau) := \mathfrak{Q}^{-1}_{*,k}(\tau)\mathfrak{I}^{-1}_{B,k}(\tau)\mathfrak{Q}_{*,k}(\tau), \tag{3.248}$$

$$\mathbb{P}^{\natural}_{0,k}(\tau) := \big(\mathrm{i}(\varepsilon b)^{1/4}\big)^{-\operatorname{ad}(\sigma_3)}(b(\tau))^{\frac{1}{2}\operatorname{ad}(\sigma_3)}\mathrm{e}^{-\hat{\beta}_k(\tau)\operatorname{ad}(\sigma_3)}\mathbb{P}_{0,k}(\tau), \tag{3.249}$$

$$\widehat{\mathbb{E}}^{\natural}_{0,k}(\tau) := \big(\mathrm{i}(\varepsilon b)^{1/4}\big)^{-\operatorname{ad}(\sigma_3)}(b(\tau))^{\frac{1}{2}\operatorname{ad}(\sigma_3)}\mathrm{e}^{-\hat{\beta}_k(\tau)\operatorname{ad}(\sigma_3)}\widehat{\mathbb{E}}_{0,k}(\tau), \tag{3.250}$$

$$\widetilde{\mathbb{E}}^{\natural}_{0,k}(\tau) := \big(\mathrm{i}(\varepsilon b)^{1/4}\big)^{-\operatorname{ad}(\sigma_3)}(b(\tau))^{\frac{1}{2}\operatorname{ad}(\sigma_3)}\mathrm{e}^{-\hat{\beta}_k(\tau)\operatorname{ad}(\sigma_3)}\widetilde{\mathbb{E}}_{0,k}(\tau). \tag{3.251}$$

The calculations for the asymptotics (as $\tau \to +\infty$ with $\varepsilon b > 0$) of the error function $\overset{\curlyvee}{\mathbb{E}}^{\lhd}_{\mathfrak{L}^0_k}(\tau)$ (cf. definition (3.243)) are similar to those for the error function $\overset{\curlywedge}{\mathbb{E}}^{\lhd}_{\mathfrak{L}^\infty_k}(\tau)$ presented in the proof of Lemma 3.21; therefore, via the conditions (3.17), the restrictions (3.110) and (3.184), the definitions (3.57), (3.80), (3.109), (3.112), (3.113), (3.160), (3.161), (3.198), (3.207)–(3.209), (3.226)–(3.228), (3.238)–(3.241), and (3.244)–(3.251), and the asymptotics (3.21), (3.24), (3.79), (B.1), (B.5)–(B.9), (B.14)–(B.19), and (3.236), upon imposing the conditions (3.186) and (3.187) and proceeding as in the proof of Lemma 3.21, one shows that, for $k = \pm 1$,

$$\overset{\curlyvee}{\mathbb{E}}^{\lhd}_{\mathfrak{L}^0_k}(\tau) \underset{\tau\to+\infty}{=} \big(\mathrm{I} + \mathbb{E}^0_{\mathcal{N},k}(\tau)\big)\big(\mathrm{I} + \mathcal{O}\big(\mathbb{E}^0_k(\tau)\big)\big), \tag{3.252}$$

where $\mathbb{E}^0_{\mathcal{N},k}(\tau)$ and $\mathcal{O}\big(\mathbb{E}^0_k(\tau)\big)$ are defined by equations (3.230) and (3.231), respectively.[38] Thus, via the asymptotics (3.242) and (3.252), one arrives at the results stated in the lemma. ∎

Theorem 3.23. *Assume that the conditions* (3.17), (3.110), (3.184), (3.186), *and* (3.187) *are valid; then, the connection matrix has the following asymptotics:*

$$G_k \underset{\tau\to+\infty}{=} \widetilde{G}(k)\widehat{\mathbb{G}}(k)\big(\mathrm{I} + \mathcal{O}\big(\mathbb{E}^{G_k}_k(\tau)\big)\big), \qquad k = \pm 1, \tag{3.253}$$

where

$$\widetilde{G}(k) := (\mathbb{S}^*_k)^{-1}\mathrm{G}^*(k), \tag{3.254}$$

$$\widehat{\mathbb{G}}(k) := (\mathrm{G}^*(k))^{-1}\big(\mathrm{I} + \mathbb{E}^0_{\mathcal{N},k}(\tau)\big)^{-1}\mathrm{G}^*(k)\big(\mathrm{I} + \mathbb{E}^\infty_{\mathcal{N},k}(\tau)\big), \tag{3.255}$$

with $\mathbb{E}^\infty_{\mathcal{N},k}(\tau)$, $\mathbb{S}^*_k$, *and* $\mathbb{E}^0_{\mathcal{N},k}(\tau)$ *defined by equations* (3.194), (3.229), *and* (3.230), *respectively, and*

$$\mathrm{G}^*(k) = \begin{pmatrix} \frac{\hat{\mathbb{G}}_{11}(k)\hat{\mathbb{B}}^\infty_0(\tau)}{\hat{\mathbb{A}}^0_0(\tau)}\mathrm{e}^{-\Delta\tilde{\mathfrak{z}}_k(\tau)-\Delta\hat{\mathfrak{z}}_k(\tau)} & \frac{\hat{\mathbb{G}}_{12}(k)\hat{\mathbb{A}}^\infty_0(\tau)}{\hat{\mathbb{A}}^0_0(\tau)}\mathrm{e}^{\Delta\tilde{\mathfrak{z}}_k(\tau)-\Delta\hat{\mathfrak{z}}_k(\tau)} \\ \frac{\hat{\mathbb{G}}_{21}(k)\hat{\mathbb{B}}^\infty_0(\tau)}{\hat{\mathbb{B}}^0_0(\tau)}\mathrm{e}^{-\Delta\tilde{\mathfrak{z}}_k(\tau)+\Delta\hat{\mathfrak{z}}_k(\tau)} & \frac{\hat{\mathbb{G}}_{22}(k)\hat{\mathbb{A}}^\infty_0(\tau)}{\hat{\mathbb{B}}^0_0(\tau)}\mathrm{e}^{\Delta\tilde{\mathfrak{z}}_k(\tau)+\Delta\hat{\mathfrak{z}}_k(\tau)} \end{pmatrix}, \tag{3.256}$$

where

$$\hat{\mathbb{G}}_{11}(k) := -\frac{\mathrm{i}\sqrt{2\pi}p_k(\tau)\mathfrak{B}_k\sqrt{b(\tau)}\mathrm{e}^{\mathrm{i}\pi(\nu(k)+1)}}{(\varepsilon b)^{1/4}\big(2+\sqrt{3}\big)^{1/2}(2\mu_k(\tau))^{1/2}\Gamma(-\nu(k))}\exp\big(-\tilde{\mathfrak{z}}^0_k(\tau) - \hat{\mathfrak{z}}^0_k(\tau)\big), \tag{3.257}$$

$$\hat{\mathbb{G}}_{12}(k) := -\frac{\mathrm{i}(\varepsilon b)^{1/4}}{\sqrt{b(\tau)}}\exp\big(\tilde{\mathfrak{z}}^0_k(\tau) - \hat{\mathfrak{z}}^0_k(\tau)\big), \tag{3.258}$$

$$\hat{\mathbb{G}}_{21}(k) := -\frac{\mathrm{i}\sqrt{b(\tau)}\mathrm{e}^{-2\pi\mathrm{i}(\nu(k)+1)}}{(\varepsilon b)^{1/4}}\exp\big(-\tilde{\mathfrak{z}}^0_k(\tau) + \hat{\mathfrak{z}}^0_k(\tau)\big), \tag{3.259}$$

[38] Note that

$$\mathcal{O}\left(\frac{\hat{\mathbb{C}}^0_0(\tau)\sqrt{b(\tau)}}{\hat{\mathbb{A}}^0_0(\tau)\mathrm{e}^{\widehat{\beta}^*_k(\tau)}}\right) \underset{\tau\to+\infty}{=} \mathcal{O}\big(\tau^{-2/3}\big) \quad \text{and} \quad \mathcal{O}\left(\frac{\hat{\mathbb{D}}^0_0(\tau)\mathrm{e}^{\widehat{\beta}^*_k(\tau)}}{\hat{\mathbb{B}}^0_0(\tau)\sqrt{b(\tau)}}\right) \underset{\tau\to+\infty}{=} \mathcal{O}\big(\tau^{-2/3}\big).$$

The asymptotics for the function $\mathbb{E}^0_{\mathcal{N},k}(\tau)$ is presented in the proof of Lemma 4.1 (see Section 4).

$$\hat{\mathbb{G}}_{22}(k) := -\frac{\sqrt{2\pi}(\varepsilon b)^{1/4}\big(2+\sqrt{3}\big)^{1/2}(2\mu_k(\tau))^{1/2}\mathrm{e}^{-2\pi\mathrm{i}(\nu(k)+1)}}{p_k(\tau)\mathfrak{B}_k\sqrt{b(\tau)}\Gamma(\nu(k)+1)}\exp\big(\tilde{\mathfrak{z}}_k^0(\tau)+\hat{\mathfrak{z}}_k^0(\tau)\big), \tag{3.260}$$

with $\tilde{\mathfrak{z}}_k^0(\tau)$, $\Delta\tilde{\mathfrak{z}}_k(\tau)$, $\hat{\mathbb{A}}_0^{\infty}(\tau)$, $\hat{\mathbb{B}}_0^{\infty}(\tau)$, $\hat{\mathfrak{z}}_k^0(\tau)$, $\Delta\hat{\mathfrak{z}}_k(\tau)$, $\hat{\mathbb{A}}_0^0(\tau)$, *and* $\hat{\mathbb{B}}_0^0(\tau)$ *defined by equations* (3.189), (3.190), (3.191), (3.192), (3.224), (3.225), (3.226), *and* (3.227), *respectively, and*

$$\mathcal{O}(\mathbb{E}_k^{G_k}(\tau)) \underset{\tau\to+\infty}{:=} \mathcal{O}\big(\mathbb{E}_k^{\infty}(\tau)\big)+\mathcal{O}\Big(\big(\widetilde{G}(k)\widehat{\mathcal{G}}(k)\big)^{-1}\mathbb{E}_k^0(\tau)\widetilde{G}(k)\widehat{\mathcal{G}}(k)\Big), \tag{3.261}$$

with the asymptotics $\mathcal{O}(\mathbb{E}_k^{\infty}(\tau))$ *and* $\mathcal{O}\big(\mathbb{E}_k^0(\tau)\big)$ *defined by equations* (3.195) *and* (3.231), *respectively.*

Proof. Mimicking the calculations subsumed in the proof of [57, Theorem 3.4.1], one shows that

$$G_k = \big(\mathfrak{L}_k^0(\tau)\big)^{-1}\mathfrak{L}_k^{\infty}(\tau), \qquad k=\pm1. \tag{3.262}$$

From equations (3.188)–(3.195), (3.223)–(3.231), and (3.262), one arrives at

$$\begin{aligned} G_k \underset{\tau\to+\infty}{=} & \big(\mathrm{I}+\mathcal{O}\big(\mathbb{E}_k^0(\tau)\big)\big)(\mathbb{S}_k^*)^{-1}\big(\mathrm{I}+\mathbb{E}_{\mathcal{N},k}^0(\tau)\big)^{-1}\mathrm{e}^{-\Delta\hat{\mathfrak{z}}_k(\tau)\sigma_3}\operatorname{diag}\Big(\big(\hat{\mathbb{A}}_0^0(\tau)\big)^{-1},\big(\hat{\mathbb{B}}_0^0(\tau)\big)^{-1}\Big) \\ & \times\left(\frac{\mathrm{i}2^{1/4}}{\big(\sqrt{3}-1\big)^{1/2}\sqrt{\mathfrak{B}_k}}\right)^{-\sigma_3}\mathrm{e}^{-\hat{\mathfrak{z}}_k^0(\tau)\sigma_3}\mathcal{R}_{m_0}(k)(\mathcal{R}_{m_\infty}(k))^{-1}\mathrm{e}^{\tilde{\mathfrak{z}}_k^0(\tau)\sigma_3} \\ & \times\left(\frac{(\varepsilon b)^{1/4}\big(\sqrt{3}+1\big)^{1/2}}{2^{1/4}\sqrt{\mathfrak{B}_k}\sqrt{b(\tau)}}\right)^{\sigma_3}\mathrm{i}\sigma_2\mathrm{e}^{-\Delta\tilde{\mathfrak{z}}_k(\tau)\sigma_3}\operatorname{diag}\big(\hat{\mathbb{B}}_0^{\infty}(\tau),\hat{\mathbb{A}}_0^{\infty}(\tau)\big)\big(\mathrm{I}+\mathbb{E}_{\mathcal{N},k}^{\infty}(\tau)\big) \\ & \times\big(\mathrm{I}+\mathcal{O}(\mathbb{E}_k^{\infty}(\tau))\big), \end{aligned} \tag{3.263}$$

taking $(m_\infty,m_0)=(0,2)$, that is, $\Delta\arg\big(\widetilde{\Lambda}\big):=\pi(m_0-m_\infty)/2=\pi$, and using the definitions of $\mathcal{R}_0(k)$ and $\mathcal{R}_2(k)$ given in Remark 3.20, one arrives at, via equation (3.263) and the reflection formula $\Gamma(z)\Gamma(1-z)=\pi/\sin(\pi z)$, the result stated in the theorem. ■

4 The inverse monodromy problem: Asymptotic solution

In Section 3.3, the corresponding connection matrices, G_k, $k\in\{\pm1\}$, were calculated asymptotically (as $\tau\to+\infty$ with $\varepsilon b>0$) under the assumption of the validity of the conditions (3.17), (3.110), (3.184), (3.186), and (3.187). Using these conditions, one can derive the τ-dependent class(es) of functions G_k belongs to: this, most general, approach will not be adopted here; rather, the isomonodromy condition will be evoked on G_k, that is, $g_{ij}:=(G_k)_{ij}$, $i,j\in\{1,2\}$, are $\mathcal{O}(1)$ constants, and then the formula for G_k will be inverted in order to derive the coefficient functions of equation (3.3), after which, it will be verified that they satisfy all of the imposed conditions for this isomonodromy case. The latter procedure gives rise to explicit asymptotic formulae for the coefficient functions of equation (3.3), leading to asymptotics of the solution of the system of isomonodromy deformations (1.36),[39] and, in turn, defines asymptotics of the solution $u(\tau)$ of the DP3E (1.1) and the related, auxiliary functions $\mathcal{H}(\tau)$, $f_\pm(\tau)$, $\sigma(\tau)$,[40] and $\hat{\varphi}(\tau)$.

Lemma 4.1. *Let* $g_{ij}:=(G_k)_{ij}$, $i,j\in\{1,2\}$, $k=\pm1$, *denote the matrix elements of the corresponding connection matrices. Assume that all the conditions stated in Theorem* 3.23 *are*

[39] Via the definitions (1.31), also the asymptotics of the solution of the (original) system of isomonodromy deformations (1.22).

[40] See the definitions (1.7), (1.41), (1.42), and (1.10), respectively.

valid. For $k=+1$*, let* $g_{11}g_{12}g_{21}\neq 0$ *and* $g_{22}=0$*, and, for* $k=-1$*, let* $g_{12}g_{21}g_{22}\neq 0$ *and* $g_{11}=0$*. Then, for* $0<\delta<\delta_k<1/24$*,* $k=\pm 1$*, the functions* $v_0(\tau)$*,* $\tilde{r}_0(\tau)$*,*[41] *and* $b(\tau)$ *have the following asymptotics:*

$$v_0(\tau):=v_{0,k}(\tau)\underset{\tau\to+\infty}{=}\sum_{m=0}^{\infty}\frac{\mathfrak{u}_m(k)}{\left(\tau^{1/3}\right)^{m+1}}+\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\pi k/4}\mathrm{e}^{-\mathrm{i}\pi k/3}(\mathcal{P}_a)^k\left(s_0^0-\mathrm{i}\mathrm{e}^{-\pi a}\right)}{\sqrt{2\pi}3^{1/4}(\varepsilon b)^{1/6}}\mathrm{e}^{-(\beta(\tau)+\mathrm{i}k\vartheta(\tau))}$$
$$\times\left(1+\mathcal{O}\left(\tau^{-1/3}\right)\right), \tag{4.1}$$

$$\tilde{r}_0(\tau):=\tilde{r}_{0,k}(\tau)\underset{\tau\to+\infty}{=}\sum_{m=0}^{\infty}\frac{\mathfrak{r}_m(k)}{\left(\tau^{1/3}\right)^{m+1}}+\frac{\mathrm{i}k\left(\sqrt{3}+1\right)^k\mathrm{e}^{\mathrm{i}\pi k/4}\mathrm{e}^{-\mathrm{i}\pi k/3}(\mathcal{P}_a)^k\left(s_0^0-\mathrm{i}\mathrm{e}^{-\pi a}\right)}{\sqrt{\pi}2^{(k-2)/2}3^{1/4}(\varepsilon b)^{1/6}}$$
$$\times\,\mathrm{e}^{-(\beta(\tau)+\mathrm{i}k\vartheta(\tau))}\left(1+\mathcal{O}\left(\tau^{-1/3}\right)\right), \tag{4.2}$$

and

$$\sqrt{b(\tau)}\underset{\tau\to+\infty}{=}\mathfrak{b}(k)(\varepsilon b)^{1/4}\exp\Bigg(\mathrm{i}(a-\mathrm{i}/2)\ln\left(\alpha_k/\sqrt{2}\right)-\frac{\mathrm{i}a}{6}\ln\tau+\frac{3k}{4}\left(\sqrt{3}+\mathrm{i}k\right)(\varepsilon b)^{1/3}\tau^{2/3}$$
$$+\,\mathcal{O}\left(\tau^{-\delta_k}\right)\Bigg), \tag{4.3}$$

where $\vartheta(\tau)$ *and* $\beta(\tau)$ *are defined in equations* (2.10)*,*

$$\mathcal{P}_a:=\left(2+\sqrt{3}\right)^{\mathrm{i}a}, \tag{4.4}$$

$$\mathfrak{b}(k)=\begin{cases}g_{11}\mathrm{e}^{\pi a}, & k=+1,\\ -\left(g_{22}\mathrm{e}^{\pi a}\right)^{-1}, & k=-1,\end{cases} \tag{4.5}$$

and the expansion coefficients $\mathfrak{u}_m(k)$ *(resp.,* $\mathfrak{r}_m(k)$*),* $m\in\mathbb{Z}_+$*, are given in equations* (2.2)*–*(2.9) *(resp.,* (2.13) *and* (2.14)*).*[42]

Proof. The scheme of the proof is, *mutatis mutandis*, similar for both cases ($k=\pm 1$); therefore, without loss of generality, the proof for the case $k=+1$ is presented: the case $k=-1$ is proved analogously.

It follows from the asymptotics (3.21), (3.24), and (B.9), the conditions (3.186) and (3.187), and the definitions (3.189) and (3.224) that $p_1(\tau)\underset{\tau\to+\infty}{=}\mathcal{O}\left(\tau^{1/3}\mathrm{e}^{-\beta(\tau)}\right)$ and $\sqrt{b(\tau)}\underset{\tau\to+\infty}{=}\mathcal{O}\big(\tau^{-\frac{\mathrm{i}a}{6}}\times\exp\big(\frac{3\sqrt{3}}{4}(\varepsilon b)^{1/3}\tau^{2/3}\big)\big)$, where $\vartheta(\tau)$ and $\beta(\tau)$ are defined in equations (2.10). From the definitions (3.109), (3.149), (3.157), (3.160), and (3.161), and the asymptotics (3.21), (3.24), (B.8), (B.14), and (B.16)–(B.19), it follows, via a linearisation and inversion argument,[43] in conjunction with the latter asymptotics for $p_1(\tau)$, that, for $k=+1$,

$$\mathfrak{r}_0(1)\tau^{-1/3}+\mathcal{O}\left(\tau^{-2/3}\right)\underset{\tau\to+\infty}{=}\frac{1}{2\sqrt{3}}\Bigg(\frac{2(a-\mathrm{i}/2)\tau^{-1/3}}{\sqrt{3}\alpha_1^2}-\frac{48\sqrt{3}(p_1(\tau)-1)(\nu(1)+1)}{p_1(\tau)\tau^{-1/3}}$$
$$-\frac{\mathrm{i}p_1(\tau)\tau^{-1/3}}{3\alpha_1^2(p_1(\tau)-1)}\Bigg), \tag{4.6}$$

$$\mathfrak{u}_0(1)\tau^{-1/3}+\mathcal{O}\left(\tau^{-2/3}\right)\underset{\tau\to+\infty}{=}\frac{1}{8\sqrt{3}}\Bigg(\frac{4(a-\mathrm{i}/2)\tau^{-1/3}}{\sqrt{3}\alpha_1^2}$$

[41] See the asymptotics (3.21) and (3.24), respectively.

[42] Trans-series asymptotics (as $\tau\to+\infty$ with $\varepsilon b>0$) for $b(\tau)$ are given in the proof of Theorem E.3.

[43] That is, retaining only those terms that are $\mathcal{O}\left(\tau^{-1/3}\right)$.

$$+\frac{48\sqrt{3}\big(\sqrt{3}+1\big)(p_1(\tau)-1)(\nu(1)+1)}{p_1(\tau)\tau^{-1/3}}$$
$$+\frac{\mathrm{i}\tau^{-1/3}}{3\alpha_1^2}\left(\sqrt{3}+1-\frac{\big(\sqrt{3}-1\big)}{p_1(\tau)-1}\right)\Bigg), \tag{4.7}$$

where

$$-\frac{(\nu(1)+1)}{p_1(\tau)}=\frac{q_1(\tau)}{2\mu_1(\tau)}, \tag{4.8}$$

with

$$q_1(\tau)\underset{\tau\to+\infty}{=}c_q^*(1)\tau^{-2/3}+\mathcal{O}\big(\tau^{-1}\big), \tag{4.9}$$
$$2\mu_1(\tau)\underset{\tau\to+\infty}{=}\mathrm{i}8\sqrt{3}\big(1+\mathcal{O}\big(\tau^{-2/3}\big)\big), \tag{4.10}$$

where $c_q^*(1)$ is some to-be-determined coefficient. Recalling from Propositions 3.4 and 3.5, respectively, that $\mathfrak{u}_0(1)=a/6\alpha_1^2$ and $\mathfrak{r}_0(1)=(a-\mathrm{i}/2)/3\alpha_1^2$, it follows via the asymptotic relations (4.6) and (4.7), equation (4.8), the asymptotics (4.9) and (4.10), and the asymptotics for $p_1(\tau)$ stated above that

$$\frac{(a-\mathrm{i}/2)\tau^{-1/3}}{3\alpha_1^2}+\mathcal{O}\big(\tau^{-2/3}\big)\underset{\tau\to+\infty}{=}\frac{\tau^{-1/3}}{2\sqrt{3}}\left(\frac{2(a-\mathrm{i}/2)}{\sqrt{3}\alpha_1^2}+\mathrm{i}6c_q^*(1)\right)+\mathcal{O}\big(\tau^{-2/3}\big),$$
$$\frac{a\tau^{-1/3}}{6\alpha_1^2}+\mathcal{O}\big(\tau^{-2/3}\big)\underset{\tau\to+\infty}{=}\frac{\tau^{-1/3}}{8\sqrt{3}}\left(\frac{4a}{\sqrt{3}\alpha_1^2}-\mathrm{i}6\big(\sqrt{3}+1\big)c_q^*(1)\right)+\mathcal{O}\big(\tau^{-2/3}\big),$$

whence

$$c_q^*(1)=0. \tag{4.11}$$

Thus, from equation (4.8), the asymptotics (4.9) and (4.10), the relation (4.11), and the asymptotics (see above) $p_1(\tau)\underset{\tau\to+\infty}{=}\mathcal{O}\big(\tau^{1/3}\mathrm{e}^{-\beta(\tau)}\big)$, one deduces that, for $k=+1$,[44]

$$\nu(1)+1\underset{\tau\to+\infty}{=}\mathcal{O}\big(\tau^{-2/3}\mathrm{e}^{-\beta(\tau)}\big). \tag{4.12}$$

From the corresponding ($k=+1$) asymptotics (3.21) and (3.24), the definitions (3.57), (3.190), and (3.225), the expansion $\mathrm{e}^z=\sum_{m=0}^{\infty}\frac{z^m}{m!}$, and the leading-order asymptotics (4.10) and (4.12), one shows that, for $k=+1$,

$$\mathrm{e}^{\pm\Delta\tilde{\mathfrak{z}}_1(\tau)}\underset{\tau\to+\infty}{=}1+\tau^{-2/3}\sum_{m=0}^{\infty}\tilde{\zeta}_m^{\pm}(1)\big(\tau^{-1/3}\big)^m+\mathcal{O}\big(\tau^{-1/3}\mathrm{e}^{-\beta(\tau)}\big), \tag{4.13}$$
$$\mathrm{e}^{\pm\Delta\hat{\mathfrak{z}}_1(\tau)}\underset{\tau\to+\infty}{=}1+\tau^{-2/3}\sum_{m=0}^{\infty}\hat{\zeta}_m^{\pm}(1)\big(\tau^{-1/3}\big)^m+\mathcal{O}\big(\tau^{-1/3}\mathrm{e}^{-\beta(\tau)}\big), \tag{4.14}$$

[44]Even though this realisation is not exploited in this work, it turns out that $\nu(k)+1$ has the asymptotic trans-series expansion

$$\nu(k)+1\underset{\tau\to+\infty}{=}\sum_{j\in\mathbb{Z}_+}\sum_{m\in\mathbb{N}}\hat{\mathfrak{s}}_{j,k}(m)\big(\tau^{-1/3}\big)^j\big(\mathrm{e}^{-(\beta(\tau)+\mathrm{i}k\vartheta(\tau))}\big)^m, \qquad k=\pm1,$$

for certain coefficients $\hat{\mathfrak{s}}_{j,k}(m)\colon\mathbb{Z}_+\times\{\pm1\}\times\mathbb{N}\to\mathbb{C}$, where, in particular, $\hat{\mathfrak{s}}_{0,k}(1)=\hat{\mathfrak{s}}_{1,k}(1)=0$.

for $\mathcal{O}(1)$ coefficients $\tilde{\zeta}_m^{\pm}(1)$ and $\hat{\zeta}_m^{\pm}(1)$. From the corresponding ($k=+1$) asymptotics (3.21), (3.24), (B.1), (B.14), (B.16), and (B.18), the definition (3.160), and $p_1(\tau) \underset{\tau\to+\infty}{=} \mathcal{O}\big(\tau^{1/3}\mathrm{e}^{-\beta(\tau)}\big)$, it follows that, for $k=+1$,

$$\frac{1}{(2\mu_1(\tau))^{1/2}} \underset{\tau\to+\infty}{=} \frac{\mathrm{e}^{-\mathrm{i}\pi/4}}{2^{3/2}3^{1/4}}\left(1+\tau^{-2/3}\sum_{m=0}^{\infty}\alpha_m^{\sharp}(1)\big(\tau^{-1/3}\big)^m+\mathcal{O}\big(\tau^{-1/3}\mathrm{e}^{-\beta(\tau)}\big)\right), \tag{4.15}$$

for $\mathcal{O}(1)$ coefficients $\alpha_m^{\sharp}(1)$. From the corresponding ($k=+1$) asymptotics (3.21), (3.24), (B.1), (B.5)–(B.9), (B.14), and (B.16), and the definitions (3.160), (3.191)–(3.193), and (3.226)–(3.228), one shows that (cf. Lemmata 3.21 and 3.22), for $k=\pm1$, to leading order,

$$\mathbb{E}_{\mathcal{N},k}^{\infty}(\tau) \underset{\tau\to+\infty}{=} \begin{pmatrix} \mathcal{O}\big(\tau^{-2/3}\big) & \mathcal{O}\Big(\tau^{-1/3}\big(\mathrm{e}^{-\beta(\tau)}\big)^{\frac{1+k}{2}}\Big) \\ \mathcal{O}\Big(\tau^{-1/3}\big(\mathrm{e}^{-\beta(\tau)}\big)^{\frac{1-k}{2}}\Big) & \mathcal{O}\big(\tau^{-2/3}\big) \end{pmatrix}, \tag{4.16}$$

$$\mathbb{E}_{\mathcal{N},k}^{0}(\tau) \underset{\tau\to+\infty}{=} \begin{pmatrix} \mathcal{O}\big(\tau^{-2/3}\big) & \mathcal{O}\Big(\tau^{-1/3}\big(\mathrm{e}^{-\beta(\tau)}\big)^{\frac{1-k}{2}}\Big) \\ \mathcal{O}\Big(\tau^{-1/3}\big(\mathrm{e}^{-\beta(\tau)}\big)^{\frac{1+k}{2}}\Big) & \mathcal{O}\big(\tau^{-2/3}\big) \end{pmatrix}, \tag{4.17}$$

whence, via the asymptotics (4.12), (4.16), and (4.17), and the above asymptotics for $p_1(\tau)$, it follows via the relation $\det(\mathrm{I}+\mathbb{J})=1+\operatorname{tr}(\mathbb{J})+\det(\mathbb{J})$, $\mathbb{J}\in\mathrm{M}_2(\mathbb{C})$, that, for $k=+1$, to all orders,

$$\mathrm{I}+\mathbb{E}_{\mathcal{N},1}^{\infty}(\tau) \underset{\tau\to+\infty}{=} \mathrm{I}+\sum_{m=1}^{\infty}\zeta_m^{\flat}(1)\big(\tau^{-1/3}\big)^m+\mathcal{O}\big(\tau^{-1/3}\mathrm{e}^{-\beta(\tau)}\sigma_+\big), \tag{4.18}$$

$$\big(\mathrm{I}+\mathbb{E}_{\mathcal{N},1}^{0}(\tau)\big)^{-1} \underset{\tau\to+\infty}{=} \mathrm{I}+\sum_{m=1}^{\infty}\zeta_m^{\natural}(1)\big(\tau^{-1/3}\big)^m+\mathcal{O}\big(\tau^{-1/3}\mathrm{e}^{-\beta(\tau)}\sigma_-\big), \tag{4.19}$$

for $\mathrm{M}_2(\mathbb{C})$-valued, $\mathcal{O}(1)$ coefficients $\zeta_m^{\flat}(1)$ and $\zeta_m^{\natural}(1)$. It now follows from the corresponding ($k=+1$) conditions (3.186) and (3.187), that is,

$$p_1(\tau)\mathfrak{B}_1 \underset{\tau\to+\infty}{=} \mathcal{O}\big(\mathrm{e}^{2\hat{\mathfrak{z}}_1^0(\tau)}\big) \qquad \text{and} \qquad \sqrt{b(\tau)} \underset{\tau\to+\infty}{=} \mathcal{O}\big(\mathrm{e}^{\tilde{\mathfrak{z}}_1^0(\tau)-\hat{\mathfrak{z}}_1^0(\tau)}\big),$$

respectively, where $\tilde{\mathfrak{z}}_1^0(\tau)$ and $\hat{\mathfrak{z}}_1^0(\tau)$ are defined by equations (3.189) and (3.224), respectively, the expansion $\mathrm{e}^z=\sum_{m=0}^{\infty}\frac{z^m}{m!}$, the reflection formula $\Gamma(z)\Gamma(1-z)=\pi/\sin\pi z$, the definitions (3.257)–(3.260), and the asymptotics (4.12) and (4.15), that, for $k=+1$,

$$\hat{\mathbb{G}}(1) := \begin{pmatrix} \hat{\mathbb{G}}_{11}(1) & \hat{\mathbb{G}}_{12}(1) \\ \hat{\mathbb{G}}_{21}(1) & \hat{\mathbb{G}}_{22}(1) \end{pmatrix} \underset{\tau\to+\infty}{=} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(\nu(1)+1) \end{pmatrix}, \tag{4.20}$$

and, from equation (3.256), the definitions (3.191), (3.192), (3.226), and (3.227), and the asymptotics (4.13), (4.14), and (4.20),

$$\mathrm{G}^{*}(1) \underset{\tau\to+\infty}{=} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(\nu(1)+1) \end{pmatrix},$$

whence, via the definitions (3.229), (3.254), and (3.255), and the asymptotics (4.18) and (4.19),

$$\widetilde{G}(1) \underset{\tau\to+\infty}{=} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(\nu(1)+1) \end{pmatrix}, \tag{4.21}$$

$$\widehat{\mathfrak{G}}(1) \underset{\tau\to+\infty}{=} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix}. \tag{4.22}$$

From the asymptotics (3.195) and (3.231), the definition (3.261), the asymptotics (4.21) and (4.22), and the relations $\max\{z_1,z_2\}=(z_1+z_2+|z_1-z_2|)/2$, $\min\{z_1,z_2\}=(z_1+z_2-|z_1-z_2|)/2$,

$z_1, z_2 \in \mathbb{R}$, and $\max_{k=\pm 1}\{3\delta_k - 1/3, -\delta_k - (1+k)/6, -\delta_k - (1-k)/6\} = -\delta_k$, it follows that, for $k=+1$,

$$\mathbb{E}_1^{G_1}(\tau) \underset{\tau\to+\infty}{=} \mathcal{O}\big(\tau^{-\delta_1}\big). \tag{4.23}$$

Finally, from the asymptotics (3.253) and (4.21)–(4.23), one arrives at $(G_1)_{i,j=1,2} \underset{\tau\to+\infty}{=} \mathcal{O}(1)$ (for $k=+1$), which is, in fact, the isomonodromy condition for the corresponding connection matrix.

From the definition (3.229), the asymptotics (3.253), the definitions (3.254) and (3.255), equation (3.256), the definitions (3.257)–(3.260), the asymptotics (4.18), (4.19), and (4.23), and the isomonodromy condition for the corresponding connection matrix G_1, it follows that, for $k=+1$, upon setting $g_{ij} := (G_1)_{ij}$, $i,j \in \{1,2\}$,

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \underset{\tau\to+\infty}{=} \begin{pmatrix} 1 & s_0^0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{G}_{11}^*(1) & \mathrm{G}_{12}^*(1) \\ \mathrm{G}_{21}^*(1) & \mathrm{G}_{22}^*(1) \end{pmatrix} \begin{pmatrix} 1+\eta_{11}(\tau) & \eta_{12}(\tau) \\ \eta_{21}(\tau) & 1+\eta_{22}(\tau) \end{pmatrix} \\ \times \big(\mathrm{I} + \mathcal{O}\big(\tau^{-\delta_1}\big)\big), \tag{4.24}$$

where

$$\eta_{ij}(\tau) \underset{\tau\to+\infty}{:=} \sum_{m=1}^{\infty} (\mathbb{H}_m(1))_{ij} \big(\tau^{-1/3}\big)^m + \mathcal{O}\big(\tau^{-1/3}\mathrm{e}^{-\beta(\tau)}\big), \qquad i,j \in \{1,2\}, \tag{4.25}$$

for $\mathcal{O}(1)$ coefficients $(\mathbb{H}_m(1))_{ij}$. It follows from the asymptotics (4.24) that

$$g_{12}g_{21} \underset{\tau\to\infty}{=} \big(\mathrm{G}_{21}^*(1)(1+\eta_{11}(\tau)) + \mathrm{G}_{22}^*(1)\eta_{21}(\tau)\big)\big(\mathrm{G}_{12}^*(1) + s_0^0\mathrm{G}_{22}^*(1) + \big(\mathrm{G}_{12}^*(1) \\ + s_0^0\mathrm{G}_{22}^*(1)\big)\eta_{22}(\tau) + \big(\mathrm{G}_{11}^*(1) + s_0^0\mathrm{G}_{21}^*(1)\big)\eta_{12}(\tau)\big)\big(1+\mathcal{O}\big(\tau^{-\delta_1}\big)\big). \tag{4.26}$$

From the corresponding ($k=+1$) conditions (3.186) and (3.187), that is, $p_1(\tau)\mathfrak{B}_1 \underset{\tau\to+\infty}{=} \mathcal{O}\big(\mathrm{e}^{2\hat{\mathfrak{z}}_1^0(\tau)}\big)$ and $\sqrt{b(\tau)} \underset{\tau\to+\infty}{=} \mathcal{O}\big(\mathrm{e}^{\tilde{\mathfrak{z}}_1^0(\tau)-\hat{\mathfrak{z}}_1^0(\tau)}\big)$, respectively, where $\tilde{\mathfrak{z}}_1^0(\tau)$ and $\hat{\mathfrak{z}}_1^0(\tau)$ are defined by equations (3.189) and (3.224), respectively, equation (3.256), the definitions (3.257)–(3.260), the expansion $\mathrm{e}^z = \sum_{m=0}^{\infty}\frac{z^m}{m!}$, the asymptotics (4.12)–(4.15), and the definitions (3.191), (3.192), (3.226), and (3.227), one shows that, for $k=+1$,

$$\mathrm{G}_{21}^*(1)\eta_{11}(\tau) = \eta_{11}(\tau)\frac{\hat{\mathbb{G}}_{21}(1)\hat{\mathbb{B}}_0^{\infty}(\tau)}{\hat{\mathbb{B}}_0^0(\tau)}\mathrm{e}^{-\Delta\tilde{\mathfrak{z}}_1(\tau)+\Delta\hat{\mathfrak{z}}_1(\tau)} \underset{\tau\to+\infty}{=} \mathcal{O}\big(\tau^{-1/3}\big),$$

$$\mathrm{G}_{22}^*(1)\eta_{21}(\tau) = \eta_{21}(\tau)\frac{\hat{\mathbb{G}}_{22}(1)\hat{\mathbb{A}}_0^{\infty}(\tau)}{\hat{\mathbb{B}}_0^0(\tau)}\mathrm{e}^{\Delta\tilde{\mathfrak{z}}_1(\tau)+\Delta\hat{\mathfrak{z}}_1(\tau)} \underset{\tau\to+\infty}{=} \mathcal{O}\big(\tau^{-1}\mathrm{e}^{-\beta(\tau)}\big),$$

$$\big(\mathrm{G}_{12}^*(1) + s_0^0\mathrm{G}_{22}^*(1)\big)\eta_{22}(\tau) \\ = \eta_{22}(\tau)\left(\frac{\hat{\mathbb{G}}_{12}(1)\hat{\mathbb{A}}_0^{\infty}(\tau)}{\hat{\mathbb{A}}_0^0(\tau)}\mathrm{e}^{\Delta\tilde{\mathfrak{z}}_1(\tau)-\Delta\hat{\mathfrak{z}}_1(\tau)} + s_0^0\frac{\hat{\mathbb{G}}_{22}(1)\hat{\mathbb{A}}_0^{\infty}(\tau)}{\hat{\mathbb{B}}_0^0(\tau)}\mathrm{e}^{\Delta\tilde{\mathfrak{z}}_1(\tau)+\Delta\hat{\mathfrak{z}}_1(\tau)}\right) \\ \underset{\tau\to+\infty}{=} \mathcal{O}\big(\tau^{-1/3}\big)\big(\mathcal{O}(1) + \mathcal{O}\big(\tau^{-2/3}\mathrm{e}^{-\beta(\tau)}\big)\big) \underset{\tau\to+\infty}{=} \mathcal{O}\big(\tau^{-1/3}\big),$$

$$\big(\mathrm{G}_{11}^*(1) + s_0^0\mathrm{G}_{21}^*(1)\big)\eta_{12}(\tau) \\ = \eta_{12}(\tau)\left(\frac{\hat{\mathbb{G}}_{11}(1)\hat{\mathbb{B}}_0^{\infty}(\tau)}{\hat{\mathbb{A}}_0^0(\tau)}\mathrm{e}^{-\Delta\tilde{\mathfrak{z}}_1(\tau)-\Delta\hat{\mathfrak{z}}_1(\tau)} + s_0^0\frac{\hat{\mathbb{G}}_{21}(1)\hat{\mathbb{B}}_0^{\infty}(\tau)}{\hat{\mathbb{B}}_0^0(\tau)}\mathrm{e}^{-\Delta\tilde{\mathfrak{z}}_1(\tau)+\Delta\hat{\mathfrak{z}}_1(\tau)}\right) \\ \underset{\tau\to+\infty}{=} \mathcal{O}\big(\tau^{-1/3}\big)\big(\mathcal{O}(1) + \mathcal{O}(1)\big) \underset{\tau\to+\infty}{=} \mathcal{O}\big(\tau^{-1/3}\big),$$

whence (cf. asymptotics (4.26))

$$g_{12}g_{21} \underset{\tau\to+\infty}{=} \big(\mathrm{G}_{21}^*(1) + \mathcal{O}\big(\tau^{-1/3}\big) + \mathcal{O}\big(\tau^{-1}\mathrm{e}^{-\beta(\tau)}\big)\big)\big(1+\mathcal{O}\big(\tau^{-\delta_1}\big)\big)$$

$$
\begin{aligned}
&\times\big(\mathrm{G}_{12}^{*}(1)+\mathcal{O}\big(\tau^{-1/3}\big)+\mathcal{O}\big(\tau^{-2/3}\mathrm{e}^{-\beta(\tau)}\big)\big) \\
&\underset{\tau\to+\infty}{=} \mathrm{G}_{12}^{*}(1)\mathrm{G}_{21}^{*}(1)\big(1+\mathcal{O}\big(\tau^{-\delta_1}\big)\big) \underset{\tau\to+\infty}{=} \hat{\mathbb{G}}_{12}(1)\hat{\mathbb{G}}_{21}(1)\frac{\hat{\mathbb{A}}_0^{\infty}(\tau)\hat{\mathbb{B}}_0^{\infty}(\tau)}{\hat{\mathbb{A}}_0^{0}(\tau)\hat{\mathbb{B}}_0^{0}(\tau)}\big(1+\mathcal{O}\big(\tau^{-\delta_1}\big)\big) \\
&\underset{\tau\to+\infty}{=} -\mathrm{e}^{-2\pi\mathrm{i}(\nu(1)+1)}\big(1+\mathcal{O}\big(\tau^{-2/3}\big)\big)\big(1+\mathcal{O}\big(\tau^{-\delta_1}\big)\big) \underset{\tau\to+\infty}{=} -(1+\mathcal{O}(\nu(1)+1)) \\
&\times\big(1+\mathcal{O}\big(\tau^{-\delta_1}\big)\big) \underset{\tau\to+\infty}{=} -\big(1+\mathcal{O}\big(\tau^{-2/3}\mathrm{e}^{-\beta(\tau)}\big)\big)\big(1+\mathcal{O}\big(\tau^{-\delta_1}\big)\big) \quad\Rightarrow \\
-g_{12}g_{21} &\underset{\tau\to+\infty}{=} 1+\mathcal{O}\big(\tau^{-\delta_1}\big); \qquad (4.27)
\end{aligned}
$$

analogously,

$$
\begin{aligned}
g_{21} &\underset{\tau\to+\infty}{=} (\mathrm{G}_{21}^{*}(1)(1+\eta_{11}(\tau))+\mathrm{G}_{22}^{*}(1)\eta_{21}(\tau))\big(1+\mathcal{O}\big(\tau^{-\delta_1}\big)\big) \\
&\underset{\tau\to+\infty}{=} \big(\mathrm{G}_{21}^{*}(1)+\mathcal{O}\big(\tau^{-1/3}\big)+\mathcal{O}\big(\tau^{-1}\mathrm{e}^{-\beta(\tau)}\big)\big)\big(1+\mathcal{O}\big(\tau^{-\delta_1}\big)\big) \\
&\underset{\tau\to+\infty}{=} \mathrm{G}_{21}^{*}(1)\big(1+\mathcal{O}\big(\tau^{-\delta_1}\big)\big) \underset{\tau\to+\infty}{=} \hat{\mathbb{G}}_{21}(1)\frac{\hat{\mathbb{B}}_0^{\infty}(\tau)}{\hat{\mathbb{B}}_0^{0}(\tau)}\mathrm{e}^{-\Delta\tilde{\mathfrak{z}}_1(\tau)+\Delta\hat{\mathfrak{z}}_1(\tau)}\big(1+\mathcal{O}\big(\tau^{-\delta_1}\big)\big) \\
&\underset{\tau\to+\infty}{=} -\frac{\mathrm{i}\sqrt{b(\tau)}}{(\varepsilon b)^{1/4}}\mathrm{e}^{-\tilde{\mathfrak{z}}_1^0(\tau)+\hat{\mathfrak{z}}_1^0(\tau)}\mathrm{e}^{-2\pi\mathrm{i}(\nu(1)+1)}\big(1+\mathcal{O}\big(\tau^{-2/3}\big)\big)\big(1+\mathcal{O}\big(\tau^{-2/3}\big)\big)\big(1+\mathcal{O}\big(\tau^{-\delta_1}\big)\big) \\
&\underset{\tau\to+\infty}{=} -\frac{\mathrm{i}\sqrt{b(\tau)}}{(\varepsilon b)^{1/4}}\mathrm{e}^{-\tilde{\mathfrak{z}}_1^0(\tau)+\hat{\mathfrak{z}}_1^0(\tau)}(1+\mathcal{O}(\nu(1)+1))\big(1+\mathcal{O}\big(\tau^{-\delta_1}\big)\big) \\
&\underset{\tau\to+\infty}{=} -\frac{\mathrm{i}\sqrt{b(\tau)}}{(\varepsilon b)^{1/4}}\mathrm{e}^{-\tilde{\mathfrak{z}}_1^0(\tau)+\hat{\mathfrak{z}}_1^0(\tau)}\big(1+\mathcal{O}\big(\tau^{-2/3}\mathrm{e}^{-\beta(\tau)}\big)\big)\big(1+\mathcal{O}\big(\tau^{-\delta_1}\big)\big) \quad\Rightarrow \\
g_{21} &\underset{\tau\to+\infty}{=} -\frac{\mathrm{i}\sqrt{b(\tau)}}{(\varepsilon b)^{1/4}}\mathrm{e}^{-\tilde{\mathfrak{z}}_1^0(\tau)+\hat{\mathfrak{z}}_1^0(\tau)}\big(1+\mathcal{O}\big(\tau^{-\delta_1}\big)\big). \qquad (4.28)
\end{aligned}
$$

It follows, upon inversion, from the asymptotics (4.27) and (4.28) that, for $k=+1$,

$$
\begin{aligned}
\sqrt{b(\tau)} &\underset{\tau\to+\infty}{=} \mathrm{i}g_{21}(\varepsilon b)^{1/4}\mathrm{e}^{\tilde{\mathfrak{z}}_1^0(\tau)-\hat{\mathfrak{z}}_1^0(\tau)}\big(1+\mathcal{O}\big(\tau^{-\delta_1}\big)\big) \\
&\underset{\tau\to+\infty}{=} -\mathrm{i}g_{12}^{-1}(\varepsilon b)^{1/4}\mathrm{e}^{\tilde{\mathfrak{z}}_1^0(\tau)-\hat{\mathfrak{z}}_1^0(\tau)}\big(1+\mathcal{O}\big(\tau^{-\delta_1}\big)\big), \qquad (4.29)
\end{aligned}
$$

whence, via equations (1.52) and the definitions (3.189) and (3.224), one arrives at the corresponding ($k=+1$) asymptotics for $\sqrt{b(\tau)}$ stated in equation (4.3) of the lemma.[45]

Recall the following formula (cf. equations (1.51)), which is one of the defining relations for the manifold of the monodromy data $\mathcal{M}$,

$$
g_{21}g_{22}-g_{11}g_{12}+s_0^0g_{11}g_{22}=\mathrm{i}\mathrm{e}^{-\pi a}. \qquad (4.30)
$$

Let

$$
\begin{aligned}
x &:= \frac{\sqrt{2\pi}p_1(\tau)\mathfrak{B}_1\mathrm{e}^{-2\hat{\mathfrak{z}}_1^0(\tau)}\mathrm{e}^{\mathrm{i}\pi(\nu(1)+1)}}{\big(2+\sqrt{3}\big)^{1/2}(2\mu_1(\tau))^{1/2}\Gamma(-\nu(1))}\frac{\hat{\mathbb{A}}_0^{\infty}(\tau)\hat{\mathbb{B}}_0^{\infty}(\tau)}{\hat{\mathbb{A}}_0^{0}(\tau)\hat{\mathbb{A}}_0^{0}(\tau)}\mathrm{e}^{-2\Delta\hat{\mathfrak{z}}_1(\tau)} \\
&\times(1+\eta_{11}(\tau))(1+\eta_{22}(\tau)), \qquad (4.31)
\end{aligned}
$$

substituting equation (3.256), the definitions (3.257)–(3.260), and the asymptotics (4.24) into equation (4.30), an algebraic exercise reveals that, in terms of the newly-defined variable x, it can be recast in the form

$$
y_1x^{-2}+(y_2+y_3+y_4)x^{-1}+(1+y_5+y_6)x+y_7x^2+y_8+y_9+y_{10}+y_{11}-\mathrm{i}\mathrm{e}^{-\pi a}
$$

[45] Note that the asymptotics (4.29) is consistent with the corresponding ($k=+1$) condition (3.187).

$$+\mathcal{O}\big(\tau^{-\delta_1}\big) \underset{\tau\to+\infty}{=} 0, \tag{4.32}$$

where

$$y_1 := \left(\mathrm{i}2g_{21}^{-1}\sin(\pi(\nu(1)+1))\mathrm{e}^{-\mathrm{i}\pi(\nu(1)+1)}\frac{\hat{\mathbb{A}}_0^{\infty}(\tau)\hat{\mathbb{B}}_0^{\infty}(\tau)}{\hat{\mathbb{A}}_0^{0}(\tau)\hat{\mathbb{A}}_0^{0}(\tau)}\right)^2\left(\frac{\hat{\mathbb{A}}_0^{\infty}(\tau)}{\hat{\mathbb{B}}_0^{0}(\tau)}\right)^2\mathrm{e}^{2(\Delta\tilde{\mathfrak{z}}_1(\tau)-\Delta\hat{\mathfrak{z}}_1(\tau))}$$
$$\times(1+\eta_{11}(\tau))^2(1+\eta_{22}(\tau))^3\eta_{21}(\tau)\big(1+\mathcal{O}\big(\tau^{-\delta_1}\big)\big), \tag{4.33}$$

$$y_2 := \mathrm{i}2\sin(\pi(\nu(1)+1))\mathrm{e}^{-\mathrm{i}3\pi(\nu(1)+1)}\left(\frac{\hat{\mathbb{A}}_0^{\infty}(\tau)\hat{\mathbb{B}}_0^{\infty}(\tau)}{\hat{\mathbb{A}}_0^{0}(\tau)\hat{\mathbb{B}}_0^{0}(\tau)}(1+\eta_{11}(\tau))(1+\eta_{22}(\tau))\right)^2, \tag{4.34}$$

$$y_3 := \mathrm{i}2s_0^0g_{21}^{-2}\sin(\pi(\nu(1)+1))\mathrm{e}^{-\mathrm{i}\pi(\nu(1)+1)}\left(\frac{\hat{\mathbb{A}}_0^{\infty}(\tau)}{\hat{\mathbb{A}}_0^{0}(\tau)}\right)^3\frac{\hat{\mathbb{B}}_0^{\infty}(\tau)}{\hat{\mathbb{B}}_0^{0}(\tau)}\mathrm{e}^{2(\Delta\tilde{\mathfrak{z}}_1(\tau)-\Delta\hat{\mathfrak{z}}_1(\tau))}$$
$$\times(1+\eta_{11}(\tau))(1+\eta_{22}(\tau))^2\eta_{21}(\tau), \tag{4.35}$$

$$y_4 := \mathrm{i}2\sin(\pi(\nu(1)+1))\mathrm{e}^{-\mathrm{i}3\pi(\nu(1)+1)}\left(\frac{\hat{\mathbb{A}}_0^{\infty}(\tau)\hat{\mathbb{B}}_0^{\infty}(\tau)}{\hat{\mathbb{A}}_0^{0}(\tau)\hat{\mathbb{B}}_0^{0}(\tau)}\right)^2$$
$$\times(1+\eta_{11}(\tau))(1+\eta_{22}(\tau))\eta_{12}(\tau)\eta_{21}(\tau), \tag{4.36}$$

$$y_5 := -s_0^0g_{21}^2\frac{\hat{\mathbb{A}}_0^{0}(\tau)\hat{\mathbb{B}}_0^{\infty}(\tau)}{\hat{\mathbb{A}}_0^{\infty}(\tau)\hat{\mathbb{B}}_0^{0}(\tau)}\mathrm{e}^{-2(\Delta\tilde{\mathfrak{z}}_1(\tau)-\Delta\hat{\mathfrak{z}}_1(\tau))}\frac{\eta_{12}(\tau)}{1+\eta_{22}(\tau)}\big(1+\mathcal{O}\big(\tau^{-\delta_1}\big)\big), \tag{4.37}$$

$$y_6 := \frac{\eta_{12}(\tau)\eta_{21}(\tau)}{(1+\eta_{11}(\tau))(1+\eta_{22}(\tau))}, \tag{4.38}$$

$$y_7 := -g_{21}^2\left(\frac{\hat{\mathbb{A}}_0^{0}(\tau)}{\hat{\mathbb{A}}_0^{\infty}(\tau)}\right)^2\mathrm{e}^{-2(\Delta\tilde{\mathfrak{z}}_1(\tau)-\Delta\hat{\mathfrak{z}}_1(\tau))}\frac{\eta_{12}(\tau)}{(1+\eta_{11}(\tau))(1+\eta_{22}(\tau))^2}\big(1+\mathcal{O}\big(\tau^{-\delta_1}\big)\big), \tag{4.39}$$

$$y_8 := s_0^0\mathrm{e}^{-\mathrm{i}2\pi(\nu(1)+1)}\frac{\hat{\mathbb{A}}_0^{\infty}(\tau)\hat{\mathbb{B}}_0^{\infty}(\tau)}{\hat{\mathbb{A}}_0^{0}(\tau)\hat{\mathbb{B}}_0^{0}(\tau)}(1+\eta_{11}(\tau))(1+\eta_{22}(\tau)), \tag{4.40}$$

$$y_9 := g_{21}^2\left(\frac{\hat{\mathbb{B}}_0^{\infty}(\tau)}{\hat{\mathbb{B}}_0^{0}(\tau)}\right)^2\mathrm{e}^{-2(\Delta\tilde{\mathfrak{z}}_1(\tau)-\Delta\hat{\mathfrak{z}}_1(\tau))}(1+\eta_{11}(\tau))\eta_{12}(\tau)\big(1+\mathcal{O}\big(\tau^{-\delta_1}\big)\big), \tag{4.41}$$

$$y_{10} := -g_{21}^{-2}\left(\frac{\hat{\mathbb{A}}_0^{\infty}(\tau)}{\hat{\mathbb{A}}_0^{0}(\tau)}\right)^2\mathrm{e}^{2(\Delta\tilde{\mathfrak{z}}_1(\tau)-\Delta\hat{\mathfrak{z}}_1(\tau))}(1+\eta_{22}(\tau))\eta_{21}(\tau)\big(1+\mathcal{O}\big(\tau^{-\delta_1}\big)\big), \tag{4.42}$$

$$y_{11} := -\mathrm{i}2s_0^0\sin(\pi(\nu(1)+1))\mathrm{e}^{-\mathrm{i}\pi(\nu(1)+1)}\frac{\hat{\mathbb{A}}_0^{\infty}(\tau)\hat{\mathbb{B}}_0^{\infty}(\tau)}{\hat{\mathbb{A}}_0^{0}(\tau)\hat{\mathbb{B}}_0^{0}(\tau)}\eta_{12}(\tau)\eta_{21}(\tau). \tag{4.43}$$

Via the asymptotics (4.12)–(4.15) and (4.25), the definitions (3.191), (3.192), (3.226), and (3.227), and the expansion $\mathrm{e}^z = \sum_{m=0}^{\infty}\frac{z^m}{m!}$, it follows from the definitions (4.33)–(4.43) that

$$y_1 \underset{\tau\to+\infty}{=} \mathcal{O}\big(\tau^{-5/3}\mathrm{e}^{-2\beta(\tau)}\big), \qquad y_2 \underset{\tau\to+\infty}{=} \mathcal{O}\big(\tau^{-2/3}\mathrm{e}^{-\beta(\tau)}\big), \tag{4.44}$$

$$y_3 \underset{\tau\to+\infty}{=} \mathcal{O}\big(\tau^{-1}\mathrm{e}^{-\beta(\tau)}\big), \qquad y_4 \underset{\tau\to+\infty}{=} \mathcal{O}\big(\tau^{-4/3}\mathrm{e}^{-\beta(\tau)}\big), \qquad y_5 \underset{\tau\to+\infty}{=} \mathcal{O}\big(\tau^{-1/3}\big), \tag{4.45}$$

$$y_6 \underset{\tau\to+\infty}{=} \mathcal{O}\big(\tau^{-2/3}\big), \qquad y_7 \underset{\tau\to+\infty}{=} \mathcal{O}\big(\tau^{-1/3}\big), \qquad y_8 \underset{\tau\to+\infty}{=} s_0^0\big(1+\mathcal{O}\big(\tau^{-1/3}\big)\big), \tag{4.46}$$

$$y_9 \underset{\tau\to+\infty}{=} \mathcal{O}\big(\tau^{-1/3}\big), \qquad y_{10} \underset{\tau\to+\infty}{=} \mathcal{O}\big(\tau^{-1/3}\big), \qquad y_{11} \underset{\tau\to+\infty}{=} \mathcal{O}\big(\tau^{-4/3}\mathrm{e}^{-\beta(\tau)}\big). \tag{4.47}$$

One notes that the asymptotic equation (4.32) is a quartic equation for the indeterminate x, which can be solved explicitly: via a study of the four solutions of the quartic equation (see, for example, [48]), in conjunction with the asymptotics (4.44)–(4.47), it can be shown that the sought-after solution, that is, the one for which $x \underset{\tau\to+\infty}{=} \mathcal{O}(1)$, can be extracted as one of the two solutions of the quadratic equation

$$(1+\upsilon_1^*)x^2 + \big(y_8 + \upsilon_2^* - \mathrm{i}\mathrm{e}^{-\pi a} + \mathcal{O}\big(\tau^{-\delta_1}\big)\big)x + \upsilon_3^* \underset{\tau\to+\infty}{=} 0, \tag{4.48}$$

where

$$\upsilon_1^* := y_5 + y_6 \underset{\tau\to+\infty}{=} \mathcal{O}\big(\tau^{-1/3}\big), \qquad \upsilon_2^* := y_9 + y_{10} + y_{11} \underset{\tau\to+\infty}{=} \mathcal{O}\big(\tau^{-1/3}\big), \tag{4.49}$$

$$\upsilon_3^* := y_2 + y_3 + y_4 \underset{\tau\to+\infty}{=} \mathcal{O}\big(\tau^{-2/3}\mathrm{e}^{-\beta(\tau)}\big). \tag{4.50}$$

The roots of the quadratic equation (4.48) are

$$x \underset{\tau\to+\infty}{=} \frac{-\big(y_8 + \upsilon_2^* - \mathrm{i}\mathrm{e}^{-\pi a} + \mathcal{O}\big(\tau^{-\delta_1}\big)\big)}{2(1+\upsilon_1^*)} \pm \frac{\sqrt{\big(y_8 + \upsilon_2^* - \mathrm{i}\mathrm{e}^{-\pi a} + \mathcal{O}\big(\tau^{-\delta_1}\big)\big)^2 - 4(1+\upsilon_1^*)\upsilon_3^*}}{2(1+\upsilon_1^*)}; \tag{4.51}$$

of the two solutions given by equation (4.51), the one that is consistent with the corresponding ($k=+1$) condition (3.186) reads

$$x \underset{\tau\to+\infty}{=} \frac{-\big(y_8 + \upsilon_2^* - \mathrm{i}\mathrm{e}^{-\pi a} + \mathcal{O}\big(\tau^{-\delta_1}\big)\big)}{2(1+\upsilon_1^*)} - \frac{\sqrt{\big(y_8 + \upsilon_2^* - \mathrm{i}\mathrm{e}^{-\pi a} + \mathcal{O}\big(\tau^{-\delta_1}\big)\big)^2 - 4(1+\upsilon_1^*)\upsilon_3^*}}{2(1+\upsilon_1^*)}: \tag{4.52}$$

via the definition (4.31), and the asymptotics (4.44), (4.49), and (4.50), it follows from equation (4.52) and an application of the binomial theorem that, for $s_0^0 \neq \mathrm{i}\mathrm{e}^{-\pi a}$,

$$\frac{\sqrt{2\pi}p_1(\tau)\mathfrak{B}_1\mathrm{e}^{-2\hat{\mathfrak{z}}_1^0(\tau)}\mathrm{e}^{\mathrm{i}\pi(\nu(1)+1)}}{\big(2+\sqrt{3}\big)^{1/2}(2\mu_1(\tau))^{1/2}\Gamma(-\nu(1))}\frac{\hat{\mathbb{A}}_0^{\infty}(\tau)\hat{\mathbb{B}}_0^{\infty}(\tau)}{\hat{\mathbb{A}}_0^{0}(\tau)\hat{\mathbb{A}}_0^{0}(\tau)}\mathrm{e}^{-2\Delta\hat{\mathfrak{z}}_1(\tau)}(1+\eta_{11}(\tau))(1+\eta_{22}(\tau)) \underset{\tau\to+\infty}{=} -\big(s_0^0 - \mathrm{i}\mathrm{e}^{-\pi a}\big) + \mathcal{O}\big(\tau^{-\delta_1}\big). \tag{4.53}$$

From the asymptotics (3.21), (3.24), (4.12), (4.14), and (4.25), the definitions (3.191), (3.192), (3.226), and (3.227), the reflection formula $\Gamma(z)\Gamma(1-z) = \pi/\sin\pi z$, the expansion $\mathrm{e}^z = \sum_{m=0}^{\infty}\frac{z^m}{m!}$, and the asymptotics $\frac{1}{\Gamma(-\nu(1))} \underset{\tau\to+\infty}{=} 1+\mathcal{O}(\nu(1)+1) \underset{\tau\to+\infty}{=} 1+\mathcal{O}\big(\tau^{-2/3}\mathrm{e}^{-\beta(\tau)}\big)$, one shows that, for $k=+1$,

$$\frac{\mathrm{e}^{\mathrm{i}\pi(\nu(1)+1)}}{\Gamma(-\nu(1))}\frac{\hat{\mathbb{A}}_0^{\infty}(\tau)\hat{\mathbb{B}}_0^{\infty}(\tau)}{\hat{\mathbb{A}}_0^{0}(\tau)\hat{\mathbb{A}}_0^{0}(\tau)} \underset{\tau\to+\infty}{:=} 1 + \tau^{-2/3}\sum_{m=0}^{\infty}\alpha_m(1)\big(\tau^{-1/3}\big)^m + \mathcal{O}\big(\tau^{-1/3}\mathrm{e}^{-\beta(\tau)}\big), \tag{4.54}$$

$$\mathrm{e}^{-2\Delta\hat{\mathfrak{z}}_1(\tau)} \underset{\tau\to+\infty}{:=} 1 + \tau^{-2/3}\sum_{m=0}^{\infty}\alpha_m^{\natural}(1)\big(\tau^{-1/3}\big)^m + \mathcal{O}\big(\tau^{-1/3}\mathrm{e}^{-\beta(\tau)}\big), \tag{4.55}$$

$$(1+\eta_{11}(\tau))(1+\eta_{22}(\tau)) \underset{\tau\to+\infty}{:=} 1 + \tau^{-1/3}\sum_{m=0}^{\infty}\alpha_m^{\flat}(1)\big(\tau^{-1/3}\big)^m + \mathcal{O}\big(\tau^{-1/3}\mathrm{e}^{-\beta(\tau)}\big), \tag{4.56}$$

for $\mathcal{O}(1)$ coefficients $\alpha_m(1)$, $\alpha_m^{\natural}(1)$, and $\alpha_m^{\flat}(1)$. Via the asymptotics (4.15) and (4.54)–(4.56), upon defining

$$\left(1 + \sum_{m_1=0}^{\infty}\frac{\alpha_{m_1}^{\flat}(1)}{\big(\tau^{1/3}\big)^{m_1+1}} + \mathcal{O}\big(\tau^{-1/3}\mathrm{e}^{-\beta(\tau)}\big)\right)\left(1 + \sum_{m_2=0}^{\infty}\frac{\alpha_{m_2}(1)}{\big(\tau^{1/3}\big)^{m_2+2}} + \mathcal{O}\big(\tau^{-1/3}\mathrm{e}^{-\beta(\tau)}\big)\right)$$

$$\times\left(1+\sum_{m_3=0}^{\infty}\frac{\alpha^{\natural}_{m_3}(1)}{(\tau^{1/3})^{m_3+2}}+\mathcal{O}\big(\tau^{-1/3}\mathrm{e}^{-\beta(\tau)}\big)\right)$$
$$\times\left(1+\sum_{m_4=0}^{\infty}\frac{\alpha^{\sharp}_{m_4}(1)}{(\tau^{1/3})^{m_4+2}}+\mathcal{O}\big(\tau^{-1/3}\mathrm{e}^{-\beta(\tau)}\big)\right)$$
$$\underset{\tau\to+\infty}{=:}1+\sum_{m=0}^{\infty}\frac{\hat{\epsilon}^{\sharp}_{m}(1)}{(\tau^{1/3})^{m+1}}+\mathcal{O}\big(\tau^{-1/3}\mathrm{e}^{-\beta(\tau)}\big),\tag{4.57}$$

it follows from the corresponding ($k=+1$) definition (3.224) and the asymptotics (4.53) and (4.57) that, for $s_0^0\neq\mathrm{i}\mathrm{e}^{-\pi a}$,

$$p_1(\tau)\mathfrak{B}_1\left(1+\sum_{m=0}^{\infty}\frac{\hat{\epsilon}^{\sharp}_{m}(1)}{(\tau^{1/3})^{m+1}}+\mathcal{O}\big(\tau^{-1/3}\mathrm{e}^{-\beta(\tau)}\big)\right)$$
$$\underset{\tau\to+\infty}{=}-\frac{2^{3/2}3^{1/4}\mathrm{e}^{\mathrm{i}\pi/4}(2+\sqrt{3})\mathcal{P}_a\big(s_0^0-\mathrm{i}\mathrm{e}^{-\pi a}\big)}{\sqrt{2\pi}}\mathrm{e}^{-(\beta(\tau)+\mathrm{i}\vartheta(\tau))}\big(1+\mathcal{O}(\tau^{-\delta_1})\big),\tag{4.58}$$

where $\mathcal{P}_a$ is defined by equation (4.4).[46] Via the asymptotics (B.15) and the definition (3.160), a multiplication argument shows that

$$p_1(\tau)\mathfrak{B}_1\underset{\tau\to+\infty}{=}-\frac{\mathrm{i}\mathfrak{B}^{\sharp}_{0,1}}{8\sqrt{3}}+\mathfrak{B}_1\big(1+\hat{\mathbb{L}}_1(\tau)\big)-\frac{\mathrm{i}\tilde{r}_{0,1}(\tau)\tau^{-1/3}}{96\sqrt{3}}\big(1+\mathcal{O}\big((\tilde{r}_{0,1}(\tau)\tau^{-1/3})^2\big)\big)\mathfrak{B}^{\sharp}_{0,1}$$
$$+\frac{\mathrm{i}\omega_{0,1}^2}{(8\sqrt{3})^3}\left(1+\frac{\tilde{r}_{0,1}(\tau)\tau^{-1/3}}{12}+\mathcal{O}\big((\tilde{r}_{0,1}(\tau)\tau^{-1/3})^3\big)\right)^3\left(\frac{\mathfrak{B}^{\sharp}_{0,1}}{\mathfrak{B}_1}\right)^2\mathfrak{B}_1$$
$$+\mathcal{O}\left(\omega_{0,1}^4\left(1+\frac{\tilde{r}_{0,1}(\tau)\tau^{-1/3}}{12}+\mathcal{O}\big((\tilde{r}_{0,1}(\tau)\tau^{-1/3})^3\big)\right)^5\left(\frac{\mathfrak{B}^{\sharp}_{0,1}}{\mathfrak{B}_1}\right)^3\mathfrak{B}_1\right);\tag{4.59}$$

from the corresponding ($k=+1$) asymptotics (3.21), (3.24), (B.9), (B.12), and (B.14), the various terms appearing in the asymptotics (4.59) can be presented as follows[47]

$$-\frac{\mathrm{i}\mathfrak{B}^{\sharp}_{0,1}}{8\sqrt{3}}\underset{\tau\to+\infty}{=}\frac{(\sqrt{3}+1)\tau^{-1/3}}{6\alpha_1}+\sum_{m=0}^{\infty}\frac{\mathfrak{b}^{\flat}_{m}(1)}{(\tau^{1/3})^{m+3}}+\mathcal{O}\big(\tau^{-2/3}\mathrm{e}^{-\beta(\tau)}\big),$$
$$-\frac{\mathrm{i}\tilde{r}_{0,1}(\tau)\tau^{-1/3}}{96\sqrt{3}}\big(1+\mathcal{O}\big((\tilde{r}_{0,1}(\tau)\tau^{-1/3})^2\big)\big)\mathfrak{B}^{\sharp}_{0,1}\underset{\tau\to+\infty}{=}\sum_{m=0}^{\infty}\frac{\mathfrak{b}^{\natural}_{m}(1)}{(\tau^{1/3})^{m+3}}+\mathcal{O}\big(\tau^{-2/3}\mathrm{e}^{-\beta(\tau)}\big),$$
$$\frac{\mathrm{i}\omega_{0,1}^2}{(8\sqrt{3})^3}\left(1+\frac{\tilde{r}_{0,1}(\tau)\tau^{-1/3}}{12}+\mathcal{O}\big((\tilde{r}_{0,1}(\tau)\tau^{-1/3})^3\big)\right)^3\left(\frac{\mathfrak{B}^{\sharp}_{0,1}}{\mathfrak{B}_1}\right)^2\mathfrak{B}_1$$
$$+\mathcal{O}\left(\omega_{0,1}^4\left(1+\frac{\tilde{r}_{0,1}(\tau)\tau^{-1/3}}{12}+\mathcal{O}\big((\tilde{r}_{0,1}(\tau)\tau^{-1/3})^3\big)\right)^5\left(\frac{\mathfrak{B}^{\sharp}_{0,1}}{\mathfrak{B}_1}\right)^3\mathfrak{B}_1\right)$$
$$\underset{\tau\to+\infty}{=}\sum_{m=0}^{\infty}\frac{\mathfrak{b}^{\sharp}_{m}(1)}{(\tau^{1/3})^{m+3}}+\mathcal{O}\big(\tau^{-2/3}\mathrm{e}^{-\beta(\tau)}\big),$$

[46] From the leading term of asymptotics for $\mathfrak{B}_1$ given in equation (B.9), that is, $\mathfrak{B}_1\underset{\tau\to+\infty}{=}-\frac{(\sqrt{3}+1)}{6\alpha_1\tau^{1/3}}+\mathcal{O}(\tau^{-1})$, and the asymptotics (4.58), it follows that $p_1(\tau)\underset{\tau\to+\infty}{=}\mathfrak{D}_1\tau^{1/3}\mathrm{e}^{-(\beta(\tau)+\mathrm{i}\vartheta(\tau))}\big(1+\mathcal{O}(\tau^{-\delta_1})\big)$, where $\mathfrak{D}_1:=6(\sqrt{3}+1)3^{1/4}\mathrm{e}^{\mathrm{i}\pi/4}\alpha_1\mathcal{P}_a\big(s_0^0-\mathrm{i}\mathrm{e}^{-\pi a}\big)/\sqrt{\pi}$, whence $p_1(\tau)\mathfrak{B}_1\underset{\tau\to+\infty}{=}\mathcal{O}(\mathrm{e}^{-\beta(\tau)})$, which is consistent with the corresponding ($k=+1$) condition (3.186).

[47] Note, in particular, that $\mathfrak{B}^{\sharp}_{0,1}/\mathfrak{B}_1\underset{\tau\to+\infty}{=}-\mathrm{i}8\sqrt{3}(1+o(1))$.

for $\mathcal{O}(1)$ coefficients $\mathfrak{b}_m^{\flat}(1)$, $\mathfrak{b}_m^{\natural}(1)$, and $\mathfrak{b}_m^{\sharp}(1)$, whence (cf. asymptotics (4.59))

$$p_1(\tau)\mathfrak{B}_1 \underset{\tau\to+\infty}{=} \mathfrak{B}_1\big(1+\hat{\mathbb{L}}_1(\tau)\big) + \frac{\big(\sqrt{3}+1\big)\tau^{-1/3}}{6\alpha_1} + \sum_{m=0}^{\infty}\frac{\mathfrak{b}_m^{\dagger}(1)}{\big(\tau^{1/3}\big)^{m+3}} + \mathcal{O}\big(\tau^{-2/3}\mathrm{e}^{-\beta(\tau)}\big), \tag{4.60}$$

for $\mathcal{O}(1)$ coefficients $\mathfrak{b}_m^{\dagger}(1)$; for example,

$$\mathfrak{b}_0^{\dagger}(1) = \frac{\mathrm{i}\big(\sqrt{3}+1\big)}{48\sqrt{3}\alpha_1}\big(6\mathrm{i}\mathfrak{r}_0(1)+4(a-\mathrm{i}/2)\mathfrak{u}_0(1)-\alpha_1^2\big(8\mathfrak{u}_0^2(1)+4\mathfrak{u}_0(1)\mathfrak{r}_0(1)-\mathfrak{r}_0^2(1)\big)\big).$$

One shows from the corresponding ($k=+1$) asymptotics (3.21), (3.24), and (B.9) that

$$\mathfrak{B}_1 \underset{\tau\to+\infty}{=} \llbracket\mathfrak{B}_1\rrbracket + \frac{\mathrm{i}\big(\sqrt{3}+1\big)\alpha_1}{2}\big(4\mathrm{A}_1+\big(\sqrt{3}+1\big)\mathrm{B}_1\big)\mathrm{e}^{-(\beta(\tau)+\mathrm{i}\vartheta(\tau))}\big(1+\mathcal{O}\big(\tau^{-1/3}\big)\big), \tag{4.61}$$

where

$$\mathrm{B}_1 := 2\big(1+\sqrt{3}\big)\mathrm{A}_1, \tag{4.62}$$

and

$$\llbracket\mathfrak{B}_1\rrbracket := -\frac{\big(\sqrt{3}+1\big)\tau^{-1/3}}{6\alpha_1} + \sum_{m=0}^{\infty}\frac{b_m(1)}{\big(\tau^{1/3}\big)^{m+3}}, \tag{4.63}$$

for $\mathcal{O}(1)$ coefficients $b_m(1)$; for example,

$$b_0(1) = \frac{\mathrm{i}\big(\sqrt{3}+1\big)^2}{2}\Bigg(\alpha_1\mathfrak{r}_2(1)+\frac{1}{2\sqrt{3}}\Bigg(-\frac{\alpha_1}{2}\big(\mathfrak{r}_0^2(1)+2\big(\sqrt{3}+1\big)\mathfrak{r}_0(1)\mathfrak{u}_0(1)+8\mathfrak{u}_0^2(1)\big) + \frac{(a-\mathrm{i}/2)}{6\alpha_1}\big(12\mathfrak{u}_0(1)+\big(2\sqrt{3}-1\big)\mathfrak{r}_0(1)\big)\Bigg)\Bigg), \tag{4.64}$$

$$b_1(1) = 0. \tag{4.65}$$

From the expansions (4.60) and (4.61), and the definition (4.63), it follows that

$$p_1(\tau)\mathfrak{B}_1 \underset{\tau\to+\infty}{=} \tau^{-1}\sum_{m=0}^{\infty}\frac{d_m^{*}(1)}{\big(\tau^{1/3}\big)^m} + \hat{\mathbb{L}}_1(\tau)\big(\llbracket\mathfrak{B}_1\rrbracket+\mathcal{O}\big(\mathrm{e}^{-\beta(\tau)}\big)\big) + \frac{\mathrm{i}\big(\sqrt{3}+1\big)\alpha_1}{2}\big(4\mathrm{A}_1+\big(\sqrt{3}+1\big)\mathrm{B}_1\big)\mathrm{e}^{-(\beta(\tau)+\mathrm{i}\vartheta(\tau))}\big(1+\mathcal{O}\big(\tau^{-1/3}\big)\big), \tag{4.66}$$

for $\mathcal{O}(1)$ coefficients $d_m^{*}(1) := \mathfrak{b}_m^{\dagger}(1)+b_m(1)$, $m\in\mathbb{Z}_+$; for example,

$$d_0^{*}(1) = \frac{\mathrm{i}\big(\sqrt{3}+1\big)}{48\sqrt{3}\alpha_1}\big(6\mathrm{i}\mathfrak{r}_0(1)+4(a-\mathrm{i}/2)\mathfrak{u}_0(1)-\alpha_1^2\big(8\mathfrak{u}_0^2(1)+4\mathfrak{u}_0(1)\mathfrak{r}_0(1)-\mathfrak{r}_0^2(1)\big)\big) + \frac{\mathrm{i}\big(\sqrt{3}+1\big)^2}{2}\Bigg(\alpha_1\mathfrak{r}_2(1)+\frac{1}{2\sqrt{3}}\Bigg(-\frac{\alpha_1}{2}\big(\mathfrak{r}_0^2(1)+2\big(\sqrt{3}+1\big)\mathfrak{r}_0(1)\mathfrak{u}_0(1)+8\mathfrak{u}_0^2(1)\big) + \frac{(a-\mathrm{i}/2)}{6\alpha_1}\big(12\mathfrak{u}_0(1)+\big(2\sqrt{3}-1\big)\mathfrak{r}_0(1)\big)\Bigg)\Bigg).$$

Thus, via the asymptotics (4.58) and (4.66), one arrives at

$$\Bigg(\sum_{m=0}^{\infty}\frac{d_m^{*}(1)}{\big(\tau^{1/3}\big)^{m+3}} + \hat{\mathbb{L}}_1(\tau)\big(\llbracket\mathfrak{B}_1\rrbracket+\mathcal{O}\big(\mathrm{e}^{-\beta(\tau)}\big)\big) + \frac{\mathrm{i}\big(\sqrt{3}+1\big)\alpha_1}{2}\big(4\mathrm{A}_1+\big(\sqrt{3}+1\big)\mathrm{B}_1\big)$$

$$\times \mathrm{e}^{-(\beta(\tau)+\mathrm{i}\vartheta(\tau))}\big(1+\mathcal{O}\big(\tau^{-1/3}\big)\big)\Big)\left(1+\sum_{m=0}^{\infty}\frac{\hat{\epsilon}_m^{\sharp}(1)}{\big(\tau^{1/3}\big)^{m+1}}+\mathcal{O}\big(\tau^{-1/3}\mathrm{e}^{-\beta(\tau)}\big)\right)$$
$$\underset{\tau\to+\infty}{=} -\mathcal{Q}_1\mathrm{e}^{-(\beta(\tau)+\mathrm{i}\vartheta(\tau))}\big(1+\mathcal{O}\big(\tau^{-\delta_1}\big)\big), \tag{4.67}$$

where

$$\mathcal{Q}_1 := \frac{2^{3/2}3^{1/4}\mathrm{e}^{\mathrm{i}\pi/4}\big(2+\sqrt{3}\big)\mathcal{P}_a\big(s_0^0-\mathrm{i}\mathrm{e}^{-\pi a}\big)}{\sqrt{2\pi}}. \tag{4.68}$$

One now chooses $\hat{\mathbb{L}}_1(\tau)$ so that the (divergent) power series on the left-hand side of equation (4.67) is identically equal to zero

$$\left(\tau^{-1}\sum_{m=0}^{\infty}\frac{d_m^{*}(1)}{\big(\tau^{1/3}\big)^m}+\hat{\mathbb{L}}_1(\tau)[\![\mathfrak{B}_1]\!]\right)\left(1+\tau^{-1/3}\sum_{m=0}^{\infty}\frac{\hat{\epsilon}_m^{\sharp}(1)}{\big(\tau^{1/3}\big)^m}\right)=0; \tag{4.69}$$

via the definition (4.63), one solves equation (4.69) for $\hat{\mathbb{L}}_1(\tau)$ to arrive at

$$\hat{\mathbb{L}}_1(\tau)=\tau^{-2/3}\sum_{m=0}^{\infty}\frac{\hat{\mathfrak{l}}_{m+2}(1)}{\big(\tau^{1/3}\big)^m}, \tag{4.70}$$

where the coefficients $\hat{\mathfrak{l}}_{m'}(1)$, $m'\in\mathbb{Z}_+$, are determined according to the recursion relation

$$\hat{\mathfrak{l}}_0(1)=\hat{\mathfrak{l}}_1(1)=0, \qquad \hat{\mathfrak{l}}_2(1)=\frac{6\alpha_1 d_0^{*}(1)}{\sqrt{3}+1},$$
$$\hat{\mathfrak{l}}_{m+3}(1)=\frac{6\alpha_1}{\sqrt{3}+1}\left(d_{m+1}^{*}(1)+\sum_{p=0}^{m}d_p^{*}(1)\hat{\epsilon}_{m-p}^{\sharp}(1)+\sum_{j=0}^{m+2}\hat{\mathfrak{l}}_j(1)\hat{d}_{m+4-j}(1)\right), \qquad m\in\mathbb{Z}_+,$$

with

$$\hat{d}_0(1)=0, \qquad \hat{d}_1(1)=-\frac{\big(\sqrt{3}+1\big)}{6\alpha_1}, \qquad \hat{d}_2(1)=-\frac{\big(\sqrt{3}+1\big)\hat{\epsilon}_0^{\sharp}(1)}{6\alpha_1},$$
$$\hat{d}_3(1)=b_0(1)-\frac{\big(\sqrt{3}+1\big)\hat{\epsilon}_1^{\sharp}(1)}{6\alpha_1},$$
$$\hat{d}_{m+4}(1)=b_{m+1}(1)-\frac{\big(\sqrt{3}+1\big)\hat{\epsilon}_{m+2}^{\sharp}(1)}{6\alpha_1}+\sum_{p=0}^{m}b_p(1)\hat{\epsilon}_{m-p}^{\sharp}(1), \qquad m\in\mathbb{Z}_+.$$

From the condition (4.69), equation (4.70), and the asymptotics (4.67), it follows that

$$\frac{\mathrm{i}\big(\sqrt{3}+1\big)\alpha_1}{2}\big(4\mathrm{A}_1+\big(\sqrt{3}+1\big)\mathrm{B}_1\big)\mathrm{e}^{-(\beta(\tau)+\mathrm{i}\vartheta(\tau))}\underset{\tau\to+\infty}{=}-\mathcal{Q}_1\mathrm{e}^{-(\beta(\tau)+\mathrm{i}\vartheta(\tau))}\big(1+\mathcal{O}\big(\tau^{-\delta_1}\big)\big),$$

whence, via the definitions (4.4), (4.62), and (4.68), one arrives at

$$\mathrm{A}_1=\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\pi/4}\mathrm{e}^{-\mathrm{i}\pi/3}\big(2+\sqrt{3}\big)^{\mathrm{i}a}\big(s_0^0-\mathrm{i}\mathrm{e}^{-\pi a}\big)}{\sqrt{2\pi}3^{1/4}(\varepsilon b)^{1/6}}. \tag{4.71}$$

Alternatively, one may proceed as follows. Substituting the asymptotics (4.60) and (4.61) into equation (4.58), one shows, via the definition (4.63) and the definition $d_m^{*}(1):=\mathfrak{b}_m^{\dagger}(1)+b_m(1)$, $m\in\mathbb{Z}_+$, that

$$\mathfrak{B}_1+\frac{\big(\sqrt{3}+1\big)\tau^{-1/3}}{6\alpha_1}+\tau^{-1}\sum_{m=0}^{\infty}\frac{d_m(1)}{\big(\tau^{1/3}\big)^m}+\hat{\mathbb{L}}_1(\tau)\mathfrak{B}_1\Bigg(1+\tau^{-1/3}\sum_{m=0}^{\infty}\frac{\hat{\epsilon}_m^{\sharp}(1)}{\big(\tau^{1/3}\big)^m}$$

$$+\mathcal{O}\big(\tau^{-1/3}\mathrm{e}^{-\beta(\tau)}\big)\bigg)+\mathcal{O}\big(\tau^{-1/3}\mathrm{e}^{-\beta(\tau)}\big)\underset{\tau\to+\infty}{=}-\mathcal{Q}_1\mathrm{e}^{-(\beta(\tau)+\mathrm{i}\vartheta(\tau))}\big(1+\mathcal{O}\big(\tau^{-\delta_1}\big)\big), \quad (4.72)$$

where $\mathcal{Q}_1$ is defined by equation (4.68),

$$d_0(1)=\mathfrak{b}_0^{\dagger}(1), \qquad d_{m+1}(1)=\mathfrak{b}_{m+1}^{\dagger}(1)+\sum_{p=0}^{m}d_p^{*}(1)\hat{\epsilon}_{m-p}^{\sharp}(1), \qquad m\in\mathbb{Z}_+. \quad (4.73)$$

From the condition (4.69), equation (4.70), the asymptotics (4.72), the definition $d_m^{*}(1):=\mathfrak{b}_m^{\dagger}(1)+b_m(1)$, $m\in\mathbb{Z}_+$, and equations (4.73), it follows that

$$\mathfrak{B}_1\underset{\tau\to+\infty}{=}-\frac{\big(\sqrt{3}+1\big)\tau^{-1/3}}{6\alpha_1}+\tau^{-1}\sum_{m=0}^{\infty}\frac{b_m(1)}{\big(\tau^{1/3}\big)^m}-\mathcal{Q}_1\mathrm{e}^{-(\beta(\tau)+\mathrm{i}\vartheta(\tau))}\big(1+\mathcal{O}\big(\tau^{-\delta_1}\big)\big). \quad (4.74)$$

It follows from the corresponding ($k=+1$) asymptotics (3.21), (3.24), and (B.9) that the function $\mathfrak{B}_1$ can also be presented in the form

$$\mathfrak{B}_1\underset{\tau\to+\infty}{=}\mathrm{i}\big(\sqrt{3}+1\big)\left(\frac{\alpha_1}{2}\big(4v_{0,1}(\tau)+\big(\sqrt{3}+1\big)\tilde{r}_{0,1}(\tau)\big)-\frac{\big(\sqrt{3}+1\big)(a-\mathrm{i}/2)}{2\sqrt{3}\alpha_1\tau^{1/3}}\right)$$
$$+\sum_{m=0}^{\infty}\frac{\hat{b}_m^{*}(1)}{\big(\tau^{1/3}\big)^{m+3}}+\mathcal{O}\big(\tau^{-2/3}\mathrm{e}^{-\beta(\tau)}\big), \quad (4.75)$$

for $\mathcal{O}(1)$ coefficients $\hat{b}_m^{*}(1)$ (see, for example, equations (4.90) and (4.91)); hence, from the asymptotics (4.74) and (4.75), one deduces that

$$4v_{0,1}(\tau)+\big(\sqrt{3}+1\big)\tilde{r}_{0,1}(\tau)\underset{\tau\to+\infty}{=}\frac{\big(\sqrt{3}+1\big)\big(\sqrt{3}a-\mathrm{i}/2\big)}{3\alpha_1^2\tau^{1/3}}+\sum_{m=0}^{\infty}\frac{\iota_m^{*}(1)}{\big(\tau^{1/3}\big)^{m+3}}$$
$$+\frac{2\mathrm{i}\mathcal{Q}_1\mathrm{e}^{-(\beta(\tau)+\mathrm{i}\vartheta(\tau))}}{\big(\sqrt{3}+1\big)\alpha_1}\big(1+\mathcal{O}\big(\tau^{-\delta_1}\big)\big), \quad (4.76)$$

where

$$\iota_m^{*}(1):=-\frac{\mathrm{i}2\big(b_m(1)-\hat{b}_m^{*}(1)\big)}{\big(\sqrt{3}+1\big)\alpha_1}, \qquad m\in\mathbb{Z}_+. \quad (4.77)$$

Combining the corresponding ($k=+1$) equations (3.20) and (3.23), it follows that, in terms of the corresponding ($k=+1$) solution of the DP3E (1.1),

$$4v_{0,1}(\tau)+\big(\sqrt{3}+1\big)\tilde{r}_{0,1}(\tau)=\frac{8\mathrm{e}^{2\pi\mathrm{i}/3}u(\tau)}{\varepsilon(\varepsilon b)^{2/3}}-\frac{\mathrm{i}\big(\sqrt{3}+1\big)\mathrm{e}^{-\mathrm{i}2\pi/3}\tau^{2/3}}{(\varepsilon b)^{1/3}}\left(\frac{u'(\tau)-\mathrm{i}b}{u(\tau)}\right)$$
$$+2\big(\sqrt{3}-1\big)\tau^{1/3}; \quad (4.78)$$

finally, from the asymptotics (4.76) and equation (4.78), one arrives at the (asymptotic) Riccati differential equation

$$u'(\tau)\underset{\tau\to+\infty}{=}\tilde{\mathfrak{a}}(\tau)+\tilde{\mathfrak{b}}(\tau)u(\tau)+\tilde{\mathfrak{c}}(\tau)(u(\tau))^2, \quad (4.79)$$

where

$$\tilde{\mathfrak{a}}(\tau):=\mathrm{i}b, \qquad \tilde{\mathfrak{c}}(\tau):=\frac{\mathrm{i}8\sqrt{2}\varepsilon\alpha_1\tau^{-2/3}}{\big(\sqrt{3}+1\big)(\varepsilon b)^{1/2}},$$

$$\tilde{\mathfrak{b}}(\tau) := -\frac{8\mathrm{i}\alpha_1^2\tau^{-1/3}}{\big(\sqrt{3}+1\big)^2} + \frac{2\mathrm{i}\big(\sqrt{3}a-\mathrm{i}/2\big)}{3\tau} + \frac{2\mathrm{i}\alpha_1^2}{\big(\sqrt{3}+1\big)} \sum_{m=0}^{\infty} \frac{\iota_m^*(1)}{\big(\tau^{1/3}\big)^{m+5}} \\ - \frac{4\alpha_1\mathcal{Q}_1\mathrm{e}^{-(\beta(\tau)+\mathrm{i}\vartheta(\tau))}}{\big(\sqrt{3}+1\big)^2\tau^{2/3}}\big(1+\mathcal{O}\big(\tau^{-\delta_1}\big)\big).$$

Incidentally, changing the dependent variable according to $w(\tau) = \frac{1}{2}\tilde{\mathfrak{b}}(\tau) + \frac{1}{2}\frac{\tilde{\mathfrak{c}}'(\tau)}{\tilde{\mathfrak{c}}(\tau)} + \tilde{\mathfrak{c}}(\tau)u(\tau)$,[48] it follows that the Riccati differential equation (4.79) transforms into

$$w'(\tau) \underset{\tau\to+\infty}{=} \Xi(\tau) + (w(\tau))^2, \tag{4.80}$$

where

$$-\Xi(\tau) := -\tilde{\mathfrak{a}}(\tau)\tilde{\mathfrak{c}}(\tau) + \frac{1}{4}(\tilde{\mathfrak{b}}(\tau))^2 - \frac{1}{2}\tilde{\mathfrak{b}}'(\tau) + \frac{1}{2}\frac{\tilde{\mathfrak{b}}(\tau)\tilde{\mathfrak{c}}'(\tau)}{\tilde{\mathfrak{c}}(\tau)} - \frac{1}{2}\frac{\tilde{\mathfrak{c}}''(\tau)}{\tilde{\mathfrak{c}}(\tau)} + \frac{3}{4}\left(\frac{\tilde{\mathfrak{c}}'(\tau)}{\tilde{\mathfrak{c}}(\tau)}\right)^2.$$

Substituting the corresponding ($k=+1$) differentiable asymptotics (3.22) into either the Riccati differential equation (4.79) or its dependent-variable-transformed variant (4.80), and recalling that $c_{0,1} = \frac{1}{2}\varepsilon(\varepsilon b)^{2/3}\mathrm{e}^{-\mathrm{i}2\pi/3}$, one shows that

$$\begin{aligned}
&\frac{8\varepsilon\mathrm{e}^{\mathrm{i}2\pi/3}}{(\varepsilon b)^{2/3}}\Bigg(c_{0,1}^2\tau^{2/3} + 2c_{0,1}^2\sum_{m=0}^{\infty}\frac{\mathfrak{u}_m(1)}{\big(\tau^{1/3}\big)^m} + c_{0,1}^2\tau^{-2/3}\sum_{m=0}^{\infty}\sum_{m_1=0}^{m}\mathfrak{u}_{m_1}(1)\mathfrak{u}_{m-m_1}(1)\big(\tau^{-1/3}\big)^m \\
&\quad + 2c_{0,1}\mathbb{P}\tau^{1/3}\mathrm{e}^{-(\beta(\tau)+\mathrm{i}\vartheta(\tau))}\big(1+\mathcal{O}\big(\tau^{-1/3}\big)\big)\Bigg) - \frac{\mathrm{i}\big(\sqrt{3}+1\big)\mathrm{e}^{-\mathrm{i}2\pi/3}\tau^{2/3}}{(\varepsilon b)^{1/3}}\Bigg(-\mathrm{i}b + \frac{c_{0,1}}{3}\tau^{-2/3} \\
&\quad - \frac{c_{0,1}}{3}\sum_{m=0}^{\infty}\frac{(m+1)\mathfrak{u}_m(1)}{\big(\tau^{1/3}\big)^{m+4}} + \mathrm{i}2\sqrt{3}(\varepsilon b)^{1/3}\mathrm{e}^{\mathrm{i}2\pi/3}\mathbb{P}\tau^{-1/3}\mathrm{e}^{-(\beta(\tau)+\mathrm{i}\vartheta(\tau))}\big(1+\mathcal{O}\big(\tau^{-1/3}\big)\big)\Bigg) \\
&\quad +2\big(\sqrt{3}-1\big)\tau^{1/3}\Bigg(c_{0,1}\tau^{1/3} + c_{0,1}\sum_{m=0}^{\infty}\frac{\mathfrak{u}_m(1)}{\big(\tau^{1/3}\big)^{m+1}} + \mathbb{P}\mathrm{e}^{-(\beta(\tau)+\mathrm{i}\vartheta(\tau))}\big(1+\mathcal{O}\big(\tau^{-1/3}\big)\big)\Bigg) \\
&\quad \underset{\tau\to+\infty}{=} \Bigg(\frac{\big(\sqrt{3}+1\big)\big(\sqrt{3}a-\mathrm{i}/2\big)}{3\alpha_1^2\tau^{1/3}} + \sum_{m=0}^{\infty}\frac{\iota_m^*(1)}{\big(\tau^{1/3}\big)^{m+3}} \\
&\qquad + \frac{2\mathrm{i}\mathcal{Q}_1\mathrm{e}^{-(\beta(\tau)+\mathrm{i}\vartheta(\tau))}}{\big(\sqrt{3}+1\big)\alpha_1}\big(1+\mathcal{O}\big(\tau^{-\delta_1}\big)\big)\Bigg) \\
&\qquad \times\Bigg(c_{0,1}\tau^{1/3} + c_{0,1}\sum_{m=0}^{\infty}\frac{\mathfrak{u}_m(1)}{\big(\tau^{1/3}\big)^{m+1}} + \mathbb{P}\mathrm{e}^{-(\beta(\tau)+\mathrm{i}\vartheta(\tau))}\big(1+\mathcal{O}\big(\tau^{-1/3}\big)\big)\Bigg),
\end{aligned} \tag{4.81}$$

where

$$\mathbb{P} := c_{0,1}\mathrm{A}_1. \tag{4.82}$$

Equating the coefficients of terms of order $\mathcal{O}\big(\tau^{\frac{1}{3}}\mathrm{e}^{-(\beta(\tau)+\mathrm{i}\vartheta(\tau))}\big)$, $\mathcal{O}\big(\tau^{\frac{2}{3}}\big)$, $\mathcal{O}(1)$, $\mathcal{O}\big(\tau^{-\frac{1}{3}}\big)$, $\mathcal{O}\big(\tau^{-\frac{2}{3}}\big)$, and $\mathcal{O}\big(\tau^{-1}\big)$, respectively, in equation (4.81), one arrives at, in the indicated order,

$$\left(\frac{16\mathrm{e}^{\mathrm{i}2\pi/3}c_{0,1}}{\varepsilon(\varepsilon b)^{2/3}} + 2\sqrt{3}\big(\sqrt{3}+1\big) + 2\big(\sqrt{3}-1\big)\right)\mathbb{P} = \frac{2\mathrm{i}\mathcal{Q}_1c_{0,1}}{\big(\sqrt{3}+1\big)\alpha_1}, \tag{4.83}$$

$$\frac{8\mathrm{e}^{\mathrm{i}2\pi/3}c_{0,1}^2}{\varepsilon(\varepsilon b)^{2/3}} - \frac{\big(\sqrt{3}+1\big)b\mathrm{e}^{-\mathrm{i}2\pi/3}}{(\varepsilon b)^{1/3}} + 2\big(\sqrt{3}-1\big)c_{0,1} = 0, \tag{4.84}$$

[48]See [40, Section 4.6]; see also [74, Chapter 5].

$$\frac{16\mathrm{e}^{\mathrm{i}2\pi/3}c_{0,1}\mathfrak{u}_0(1)}{\varepsilon(\varepsilon b)^{2/3}}-\frac{\mathrm{i}\left(\sqrt{3}+1\right)\mathrm{e}^{-\mathrm{i}2\pi/3}}{3(\varepsilon b)^{1/3}}+2\left(\sqrt{3}-1\right)\mathfrak{u}_0(1)=\frac{\left(\sqrt{3}+1\right)\left(\sqrt{3}a-\mathrm{i}/2\right)}{3\alpha_1^2},\tag{4.85}$$

$$\left(\frac{16\mathrm{e}^{\mathrm{i}2\pi/3}c_{0,1}}{\varepsilon(\varepsilon b)^{2/3}}+2\left(\sqrt{3}-1\right)\right)\mathfrak{u}_1(1)=0,\tag{4.86}$$

$$\begin{aligned}&\frac{8\mathrm{e}^{\mathrm{i}2\pi/3}c_{0,1}}{\varepsilon(\varepsilon b)^{2/3}}\left(2\mathfrak{u}_2(1)+\mathfrak{u}_0^2(1)\right)+\frac{\mathrm{i}\left(\sqrt{3}+1\right)\mathrm{e}^{-\mathrm{i}2\pi/3}\mathfrak{u}_0(1)}{3(\varepsilon b)^{1/3}}+2\left(\sqrt{3}-1\right)\mathfrak{u}_2(1)\\&\quad=\frac{\left(\sqrt{3}+1\right)\left(\sqrt{3}a-\mathrm{i}/2\right)}{3\alpha_1^2}+\iota_0^*(1),\end{aligned}\tag{4.87}$$

$$\begin{aligned}&\frac{16\mathrm{e}^{\mathrm{i}2\pi/3}c_{0,1}}{\varepsilon(\varepsilon b)^{2/3}}\left(\mathfrak{u}_3(1)+\mathfrak{u}_0(1)\mathfrak{u}_1(1)\right)+\frac{2\mathrm{i}\left(\sqrt{3}+1\right)\mathrm{e}^{-\mathrm{i}2\pi/3}\mathfrak{u}_1(1)}{3(\varepsilon b)^{1/3}}+2\left(\sqrt{3}-1\right)\mathfrak{u}_3(1)\\&\quad=\frac{\left(\sqrt{3}+1\right)\left(\sqrt{3}a-\mathrm{i}/2\right)\mathfrak{u}_1(1)}{3\alpha_1^2}+\iota_1^*(1).\end{aligned}\tag{4.88}$$

Using the corresponding ($k=+1$) coefficients (2.3), in particular, $\mathfrak{u}_0(1)=a/6\alpha_1^2$ and $\mathfrak{u}_1(1)=\mathfrak{u}_2(1)=\mathfrak{u}_3(1)=0$, one analyses equations (4.83)–(4.88), in the indicated order, in order to arrive at the following conclusions: (i) solving equation (4.83) for $\mathbb{P}$, one gets that

$$\mathbb{P}=-\frac{\mathrm{i}\varepsilon(\varepsilon b)^{1/2}\mathrm{e}^{\mathrm{i}\pi/4}\mathcal{P}_a\left(s_0^0-\mathrm{i}\mathrm{e}^{-\pi a}\right)}{\sqrt{\pi}2^{3/2}3^{1/4}},$$

whence, from the definition (4.82), one arrives, again, at equation (4.71); (ii) equations (4.84)–(4.86) are identically true; and (iii) solving equations (4.87) and (4.88) for $\iota_0^*(1)$ and $\iota_1^*(1)$, respectively, one concludes that

$$\iota_0^*(1)=\frac{\mathrm{i}a(1+\mathrm{i}a)\left(\sqrt{3}+1\right)}{18\alpha_1^4}\qquad\text{and}\qquad\iota_1^*(1)=0;\tag{4.89}$$

moreover, from equations (4.64) and (4.65), the definition (4.77), and equations (4.89), it also follows that

$$\begin{aligned}\hat{b}_0^*(1)&=\frac{\mathrm{i}\left(\sqrt{3}+1\right)^2}{4\sqrt{3}}\left(-\frac{\alpha_1}{2}\left(\mathfrak{r}_0^2(1)+2\left(\sqrt{3}+1\right)\mathfrak{r}_0(1)\mathfrak{u}_0(1)+8\mathfrak{u}_0^2(1)\right)\right.\\&\qquad\left.+\frac{(a-\mathrm{i}/2)}{6\alpha_1}\left(12\mathfrak{u}_0(1)+\left(2\sqrt{3}-1\right)\mathfrak{r}_0(1)\right)\right),\end{aligned}\tag{4.90}$$

$$\hat{b}_1^*(1)=0.\tag{4.91}$$

Finally, from the asymptotics (3.21) and (3.24) (for $k=+1$) and equation (4.71), one arrives at the corresponding asymptotics for $v_0(\tau):=v_{0,1}(\tau)$ and $\tilde{r}_0(\tau):=\tilde{r}_{0,1}(\tau)$ stated in equations (4.1) and (4.2), respectively, of the lemma.

Similarly, proceeding as delineated above, one shows that, for $k=-1$,

$$\mathrm{A}_{-1}=\frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}\pi/4}\mathrm{e}^{\mathrm{i}\pi/3}\left(2+\sqrt{3}\right)^{-\mathrm{i}a}\left(s_0^0-\mathrm{i}\mathrm{e}^{-\pi a}\right)}{\sqrt{2\pi}3^{1/4}(\varepsilon b)^{1/6}};\tag{4.92}$$

thus, from the asymptotics (3.21) and (3.24) (for $k=-1$) and equation (4.92), one arrives at the corresponding asymptotics for $v_0(\tau):=v_{0,-1}(\tau)$ and $\tilde{r}_0(\tau):=\tilde{r}_{0,-1}(\tau)$ stated in equations (4.1) and (4.2), respectively, of the lemma. ∎

From equation (3.20), the asymptotics (4.1), the definition (4.4), and recalling that (cf. equation (2.2)) $c_{0,k}=\frac{1}{2}\varepsilon(\varepsilon b)^{2/3}\mathrm{e}^{-\mathrm{i}2\pi k/3}$, $k=\pm1$, one arrives at the corresponding $(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)=$

$(0,0,0|0)$ asymptotics (as $\tau \to +\infty$ with $\varepsilon b > 0$) for the solution $u(\tau)$ of the DP3E (1.1) stated in Theorem 2.4.

Via the definitions (1.41) and (1.42) and equations (1.43) and (3.23), one deduces that, for $k = \pm 1$,

$$2f_{-}(\tau) = -\mathrm{i}(a-\mathrm{i}/2) + \frac{\mathrm{i}(\varepsilon b)^{1/3}\mathrm{e}^{\mathrm{i}2\pi k/3}}{2}\tau^{2/3}\big(-2+\tilde{r}_0(\tau)\tau^{-1/3}\big), \tag{4.93}$$

$$\frac{4\mathrm{i}}{\varepsilon b}f_{+}(\tau) = \mathrm{i}(a+\mathrm{i}/2) + \frac{\mathrm{i}(\varepsilon b)^{1/3}\mathrm{e}^{\mathrm{i}2\pi k/3}}{2}\tau^{2/3}\big(-2+\tilde{r}_0(\tau)\tau^{-1/3}\big) + \frac{ib\tau}{u(\tau)}; \tag{4.94}$$

thus, from the asymptotics (4.1) and (4.2), the definition (4.4), and equations (4.93) and (4.94), one arrives at the corresponding $(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell) = (0,0,0|0)$ asymptotics (as $\tau \to +\infty$ with $\varepsilon b > 0$) for the principal auxiliary functions $f_{\pm}(\tau)$ (corresponding to $u(\tau)$) stated in Theorem 2.4.

It was shown in [57, equation (4.25)] that, in terms of the function $h_0(\tau)$, the Hamiltonian function $\mathcal{H}(\tau)$ (corresponding to $u(\tau)$) defined by equation (1.7) is given by

$$\mathcal{H}(\tau) = 3(\varepsilon b)^{2/3}\tau^{1/3} + \frac{1}{2\tau}(a-\mathrm{i}/2)^2 - 4\tau^{-1/3}h_0(\tau), \tag{4.95}$$

via the definition (3.14), and equation (4.95), it follows that, in terms of the function $\hat{h}_0(\tau) := \hat{h}_{0,k}(\tau)$,

$$\mathcal{H}(\tau) = 3(\varepsilon b)^{2/3}\mathrm{e}^{-\mathrm{i}2\pi k/3}\tau^{1/3} + \frac{1}{2\tau}(a-\mathrm{i}/2)^2 - 4\tau^{1/3}\hat{h}_{0,k}(\tau), \qquad k=\pm 1; \tag{4.96}$$

consequently, from equation (3.18), the third relation of equations (3.19), and equation (4.96), upon recalling that $v_0(\tau) := v_{0,k}(\tau)$ and $\tilde{r}_0(\tau) := \tilde{r}_{0,k}(\tau)$, one shows that the Hamiltonian function is given by

$$\begin{aligned}\mathcal{H}(\tau) = {} & 3(\varepsilon b)^{2/3}\mathrm{e}^{-\mathrm{i}2\pi k/3}\tau^{1/3} + \frac{1}{2\tau}(a-\mathrm{i}/2)^2 + \frac{\alpha_k^2\tau^{-1/3}}{1+\tau^{-1/3}v_{0,k}(\tau)}\big(\alpha_k^2\big(8v_{0,k}^2(\tau) + (4v_{0,k}(\tau) \\ & - \tilde{r}_{0,k}(\tau))\tilde{r}_{0,k}(\tau) - \tau^{-1/3}v_{0,k}(\tau)(\tilde{r}_{0,k}(\tau))^2\big) + 4(a-\mathrm{i}/2)\big), \qquad k=\pm 1.\end{aligned} \tag{4.97}$$

Finally, from the asymptotics (4.1) and (4.2), the definition (4.4), and equation (4.97), one arrives at, after a lengthy calculation, the corresponding $(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell) = (0,0,0|0)$ asymptotics (as $\tau \to +\infty$ with $\varepsilon b > 0$) for the Hamiltonian function, $\mathcal{H}(\tau)$, stated in Theorem 2.4.

Via the definition (1.10) and the asymptotics (as $\tau \to +\infty$ with $\varepsilon b > 0$) for $f_{-}(\tau)$ and $\mathcal{H}(\tau)$ stated above, one arrives at the corresponding $(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell) = (0,0,0|0)$ asymptotics for the function $\sigma(\tau)$ stated in Theorem 2.4.

Proposition 4.2. *Under the conditions of Lemma 4.1, the functions $a(\tau)$, $b(\tau)$, $c(\tau)$, and $d(\tau)$, defining, via equations (3.2), the solution of the corresponding system of isomonodromy deformations (1.36), have the following asymptotic representations, for $k=\pm 1$:*

$$\begin{aligned}\sqrt{-a(\tau)b(\tau)} \underset{\tau\to+\infty}{=} {} & \frac{(\varepsilon b)^{2/3}\mathrm{e}^{-\mathrm{i}2\pi k/3}}{2}\left(1 + \sum_{m=0}^{\infty}\frac{\mathfrak{u}_m(k)}{\left(\tau^{1/3}\right)^{m+2}}\right) \\ & - \frac{\mathrm{i}(\varepsilon b)^{1/2}\mathrm{e}^{\mathrm{i}\pi k/4}(\mathcal{P}_a)^k\big(s_0^0 - \mathrm{i}\mathrm{e}^{-\pi a}\big)}{\sqrt{\pi}2^{3/2}3^{1/4}\tau^{1/3}}\mathrm{e}^{-(\beta(\tau)+\mathrm{i}k\vartheta(\tau))}\big(1+\mathcal{O}\big(\tau^{-1/3}\big)\big),\end{aligned} \tag{4.98}$$

$$a(\tau)d(\tau) \underset{\tau\to+\infty}{=} -\frac{\mathrm{i}(\varepsilon b)}{4} - \frac{\mathrm{i}(\varepsilon b)^{2/3}\mathrm{e}^{-\mathrm{i}2\pi k/3}}{4}(a-\mathrm{i}/3)\tau^{-2/3} + \frac{\mathrm{i}(\varepsilon b)}{8}(\mathfrak{r}_1(k) - 2\mathfrak{u}_1(k))\tau^{-1}$$

$$
\begin{aligned}
&+\left(\tau^{-1/3}\right)^4\sum_{m=0}^{\infty}\left(\frac{\mathrm{i}(\varepsilon b)}{8}(\mathfrak{r}_{m+2}(k)-2\mathfrak{u}_{m+2}(k))\right.\\
&\left.-\frac{\mathrm{i}(\varepsilon b)^{2/3}\mathrm{e}^{-\mathrm{i}2\pi k/3}}{4}(a-\mathrm{i}/2)\mathfrak{u}_m(k)+\frac{\mathrm{i}(\varepsilon b)}{8}\sum_{p=0}^{m}\mathfrak{u}_p(k)\mathfrak{r}_{m-p}(k)\right)\left(\tau^{-1/3}\right)^m\\
&-\frac{k(\varepsilon b)^{5/6}3^{1/4}\mathrm{e}^{\mathrm{i}\pi k/4}(\mathcal{P}_a)^k\left(s_0^0-\mathrm{i}\mathrm{e}^{-\pi a}\right)}{4\sqrt{2\pi}\mathrm{e}^{\mathrm{i}\pi k/3}\tau^{1/3}}\mathrm{e}^{-(\beta(\tau)+\mathrm{i}k\vartheta(\tau))}\left(1+\mathcal{O}\left(\tau^{-1/3}\right)\right), \quad (4.99)
\end{aligned}
$$

$$
\begin{aligned}
b(\tau)c(\tau)\underset{\tau\to+\infty}{=}&-\frac{\mathrm{i}(\varepsilon b)}{4}-\frac{\mathrm{i}(\varepsilon b)^{2/3}\mathrm{e}^{-\mathrm{i}2\pi k/3}}{4}(a+\mathrm{i}/3)\tau^{-2/3}-\frac{\mathrm{i}(\varepsilon b)}{8}(\mathfrak{r}_1(k)-2\mathfrak{u}_1(k))\tau^{-1}\\
&+\left(\tau^{-1/3}\right)^4\sum_{m=0}^{\infty}\left(-\frac{\mathrm{i}(\varepsilon b)}{8}(\mathfrak{r}_{m+2}(k)-2\mathfrak{u}_{m+2}(k))\right.\\
&\left.-\frac{\mathrm{i}(\varepsilon b)^{2/3}\mathrm{e}^{-\mathrm{i}2\pi k/3}}{4}(a+\mathrm{i}/2)\mathfrak{u}_m(k)-\frac{\mathrm{i}(\varepsilon b)}{8}\sum_{p=0}^{m}\mathfrak{u}_p(k)\mathfrak{r}_{m-p}(k)\right)\left(\tau^{-1/3}\right)^m\\
&+\frac{k(\varepsilon b)^{5/6}3^{1/4}\mathrm{e}^{\mathrm{i}\pi k/4}(\mathcal{P}_a)^k\left(s_0^0-\mathrm{i}\mathrm{e}^{-\pi a}\right)}{4\sqrt{2\pi}\mathrm{e}^{\mathrm{i}\pi k/3}\tau^{1/3}}\mathrm{e}^{-(\beta(\tau)+\mathrm{i}k\vartheta(\tau))}\left(1+\mathcal{O}\left(\tau^{-1/3}\right)\right), \quad (4.100)
\end{aligned}
$$

$$
\begin{aligned}
-c(\tau)d(\tau)\underset{\tau\to+\infty}{=}&\frac{(\varepsilon b)^{2/3}\mathrm{e}^{\mathrm{i}\pi k/3}}{4}-\frac{a(\varepsilon b)^{1/3}\mathrm{e}^{\mathrm{i}2\pi k/3}}{3}\tau^{-2/3}-\frac{(\varepsilon b)^{2/3}\mathrm{e}^{\mathrm{i}\pi k/3}}{2}\mathfrak{u}_1(k)\tau^{-1}\\
&-\left(\frac{1}{6}(a^2+1/6)+\frac{(\varepsilon b)^{2/3}\mathrm{e}^{\mathrm{i}\pi k/3}}{2}\mathfrak{u}_2(k)\right)\left(\tau^{-1/3}\right)^4\\
&+\left(\tau^{-1/3}\right)^4\sum_{m=1}^{\infty}\left(-\frac{(\varepsilon b)^{2/3}\mathrm{e}^{\mathrm{i}\pi k/3}}{2}\mathfrak{u}_{m+2}(k)+\frac{\mathrm{i}(\varepsilon b)^{1/3}\mathrm{e}^{\mathrm{i}2\pi k/3}}{8}\mathfrak{r}_m(k)\right.\\
&-\frac{(\varepsilon b)^{1/3}\mathrm{e}^{\mathrm{i}2\pi k/3}}{2}(a-\mathrm{i}/2)\mathfrak{w}_m(k)-\frac{(\varepsilon b)^{2/3}\mathrm{e}^{\mathrm{i}\pi k/3}}{2}\\
&\left.\times\sum_{p=0}^{m}\left(\left(\mathfrak{u}_p(k)+\frac{1}{2}\mathfrak{r}_p(k)\right)\mathfrak{w}_{m-p}(k)+\frac{1}{8}\mathfrak{r}_p(k)\mathfrak{r}_{m-p}(k)\right)\right)\left(\tau^{-1/3}\right)^m\\
&-\frac{\mathrm{i}(\varepsilon b)^{1/2}\mathrm{e}^{\mathrm{i}\pi k/4}(\mathcal{P}_a)^k\left(s_0^0-\mathrm{i}\mathrm{e}^{-\pi a}\right)}{\sqrt{\pi}2^{3/2}3^{1/4}\tau^{1/3}}\mathrm{e}^{-(\beta(\tau)+\mathrm{i}k\vartheta(\tau))}\left(1+\mathcal{O}\left(\tau^{-1/3}\right)\right), \quad (4.101)
\end{aligned}
$$

where the expansion coefficients $\mathfrak{u}_m(k)$ *(resp.,* $\mathfrak{r}_m(k)$*),* $m\in\mathbb{Z}_+$*, are given in equations* (2.2)–(2.9) *(resp.,* (2.13) *and* (2.14)*).*

Proof. If, for $k=\pm1$, g_{ij}, $i,j\in\{1,2\}$, are τ dependent, then, functions whose asymptotics (as $\tau\to+\infty$ with $\varepsilon b>0$) are given by equations (4.1)–(4.3) satisfy the conditions (3.17), (3.110), (3.184), (3.186), and (3.187); therefore, one can use the justification scheme suggested in [54] (see also [42]). From equations (3.8), (3.10), (3.11), and (3.13), respectively, one shows, via the definitions (3.15) and (3.16), that, for $k=\pm1$,[49]

$$
\sqrt{-a(\tau)b(\tau)}=\frac{(\varepsilon b)^{2/3}\mathrm{e}^{-\mathrm{i}2\pi k/3}}{2}\left(1+\tau^{-1/3}v_{0,k}(\tau)\right), \quad (4.102)
$$

$$
\begin{aligned}
a(\tau)d(\tau)=&\frac{\mathrm{i}(\varepsilon b)}{8}\left(1+\tau^{-1/3}v_{0,k}(\tau)\right)\left(-2+\tau^{-1/3}\tilde{r}_{0,k}(\tau)\right)\\
&-\frac{\mathrm{i}(\varepsilon b)^{2/3}\mathrm{e}^{-\mathrm{i}2\pi k/3}}{4}(a-\mathrm{i}/2)\left(1+\tau^{-1/3}v_{0,k}(\tau)\right)\tau^{-2/3}, \quad (4.103)
\end{aligned}
$$

[49] Recall that (cf. Lemma 4.1) $v_0(\tau):=v_{0,k}(\tau)$ and $\tilde{r}_0(\tau):=\tilde{r}_{0,k}(\tau)$, $k=\pm1$.

$$\begin{aligned} b(\tau)c(\tau) &= -\frac{\mathrm{i}(\varepsilon b)}{2} - \frac{\mathrm{i}(\varepsilon b)}{8}\big(1+\tau^{-1/3}v_{0,k}(\tau)\big)\big(-2+\tau^{-1/3}\tilde{r}_{0,k}(\tau)\big) \\ &\quad - \frac{\mathrm{i}(\varepsilon b)^{2/3}\mathrm{e}^{-\mathrm{i}2\pi k/3}}{4}(a+\mathrm{i}/2)\big(1+\tau^{-1/3}v_{0,k}(\tau)\big)\tau^{-2/3}, \end{aligned} \tag{4.104}$$

$$\begin{aligned} -c(\tau)d(\tau) &= -\frac{(\varepsilon b)^{2/3}\mathrm{e}^{\mathrm{i}\pi k/3}}{4}\left(\frac{-2+\tau^{-1/3}\tilde{r}_{0,k}(\tau)}{1+\tau^{-1/3}v_{0,k}(\tau)}\right) - \frac{(\varepsilon b)^{2/3}\mathrm{e}^{\mathrm{i}\pi k/3}}{16}\big(-2+\tau^{-1/3}\tilde{r}_{0,k}(\tau)\big)^2 \\ &\quad - \frac{1}{4}(a-\mathrm{i}/2)(a+\mathrm{i}/2)\tau^{-4/3} + \frac{(\varepsilon b)^{1/3}\mathrm{e}^{\mathrm{i}2\pi k/3}}{2}\bigg(\mathrm{i}\big(-2+\tau^{-1/3}\tilde{r}_{0,k}(\tau)\big)/4 \\ &\quad - \frac{(a-\mathrm{i}/2)}{1+\tau^{-1/3}v_{0,k}(\tau)}\bigg)\tau^{-2/3}. \end{aligned} \tag{4.105}$$

Via the asymptotics (4.1) and (4.2), and equations (4.102)–(4.105), one arrives at the asymptotics (as $\tau \to +\infty$ with $\varepsilon b > 0$) for the functions $\sqrt{-a(\tau)b(\tau)}$, $a(\tau)d(\tau)$, $b(\tau)c(\tau)$, and $-c(\tau)d(\tau)$ stated in equations (4.98)–(4.101), respectively. ∎

Remark 4.3. It is important to note that the asymptotics (4.98)–(4.101) are consistent with equation (3.9); moreover, via the definitions (1.31), equations (3.2), and the asymptotics (4.3) and (4.98)–(4.101), one arrives at the asymptotics (as $\tau \to +\infty$ with $\varepsilon b > 0$) for the solution of the (original) system of isomonodromy deformations (1.22).

A Proof of Proposition 3.4

Proof. As the exponentially small correction term does not contribute to the algebraic determination of the coefficients $\mathfrak{u}_m(k)$, $m \in \mathbb{Z}_+$, $k = \pm 1$, hereafter, only the following 'truncated' (and differentiable) asymptotics of $u(\tau)$ will be considered (with abuse of notation, also denoted as $u(\tau)$)

$$u(\tau) \underset{\tau\to+\infty}{=} c_{0,k}\tau^{1/3}\left(1+\tau^{-2/3}\sum_{m=0}^{\infty}\frac{\mathfrak{u}_m(k)}{(\tau^{1/3})^m}\right), \qquad k=\pm 1. \tag{A.1}$$

Via the asymptotics (A.1), one shows that

$$\frac{1}{u(\tau)} \underset{\tau\to+\infty}{=} \frac{\tau^{-1/3}}{c_{0,k}}\left(1+\tau^{-2/3}\sum_{m=0}^{\infty}\frac{\mathfrak{w}_m(k)}{(\tau^{1/3})^m}\right), \qquad k=\pm 1, \tag{A.2}$$

where $\mathfrak{w}_m(k)$, $m \in \mathbb{Z}_+$, are determined iteratively from equations (2.8); in particular (this will be required for the ensuing proof), for $k = \pm 1$,

$$\mathfrak{w}_0(k) = -\mathfrak{u}_0(k), \tag{A.3}$$

$$\mathfrak{w}_1(k) = -\mathfrak{u}_1(k), \qquad \mathfrak{w}_2(k) = -\mathfrak{u}_2(k) + \mathfrak{u}_0^2(k), \qquad \mathfrak{w}_3(k) = -\mathfrak{u}_3(k) + 2\mathfrak{u}_0(k)\mathfrak{u}_1(k), \tag{A.4}$$

$$\mathfrak{w}_4(k) = -\mathfrak{u}_4(k) + 2\mathfrak{u}_0(k)\mathfrak{u}_2(k) + \mathfrak{u}_1^2(k) - \mathfrak{u}_0^3(k), \tag{A.5}$$

$$\mathfrak{w}_5(k) = -\mathfrak{u}_5(k) + 2\mathfrak{u}_0(k)\mathfrak{u}_3(k) + 2\mathfrak{u}_1(k)\mathfrak{u}_2(k) - 3\mathfrak{u}_0^2(k)\mathfrak{u}_1(k), \tag{A.6}$$

$$\begin{aligned} \mathfrak{w}_6(k) &= -\mathfrak{u}_6(k) + 2\mathfrak{u}_0(k)\mathfrak{u}_4(k) + 2\mathfrak{u}_1(k)\mathfrak{u}_3(k) + \mathfrak{u}_2^2(k) - 3\mathfrak{u}_0^2(k)\mathfrak{u}_2(k) - 3\mathfrak{u}_0(k)\mathfrak{u}_1^2(k) \\ &\quad + \mathfrak{u}_0^4(k), \end{aligned} \tag{A.7}$$

$$\begin{aligned} \mathfrak{w}_7(k) &= -\mathfrak{u}_7(k) + 2\mathfrak{u}_0(k)\mathfrak{u}_5(k) + 2\mathfrak{u}_1(k)\mathfrak{u}_4(k) + 2\mathfrak{u}_2(k)\mathfrak{u}_3(k) - 3\mathfrak{u}_3(k)\mathfrak{u}_0^2(k) \\ &\quad - 6\mathfrak{u}_0(k)\mathfrak{u}_1(k)\mathfrak{u}_2(k) + 4\mathfrak{u}_1(k)\mathfrak{u}_0^3(k) - \mathfrak{u}_1^3(k). \end{aligned} \tag{A.8}$$

From equations (2.8) and the asymptotics (A.1) and (A.2), one shows that (cf. DP3E (1.1)), for $k=\pm 1$,

$$\frac{b^2}{u(\tau)} \underset{\tau\to+\infty}{=} \frac{b^2\tau^{-1/3}}{c_{0,k}} \times\left(1-\mathfrak{u}_0(k)\tau^{-2/3}-\mathfrak{u}_1(k)\big(\tau^{-1/3}\big)^3-\big(\tau^{-1/3}\big)^4\sum_{m=0}^{\infty}\lambda_m(k)\big(\tau^{-1/3}\big)^m\right), \tag{A.9}$$

where $\lambda_j(k) := -\mathfrak{w}_{j+2}(k)$, $j\in\mathbb{Z}_+$,

$$\begin{aligned}\frac{1}{\tau}\big(-8\varepsilon u^2(\tau)+2ab\big) &\underset{\tau\to+\infty}{=} -8\varepsilon c_{0,k}^2\tau^{-1/3}+\big(2ab-16\varepsilon c_{0,k}^2\mathfrak{u}_0(k)\big)\big(\tau^{-1/3}\big)^3-16\varepsilon c_{0,k}^2\mathfrak{u}_1(k)\big(\tau^{-1/3}\big)^4\\ &\quad -8\varepsilon c_{0,k}^2\big(\tau^{-1/3}\big)^5\sum_{m=0}^{\infty}\left(2\mathfrak{u}_{m+2}(k)+\sum_{p=0}^{m}\mathfrak{u}_p(k)\mathfrak{u}_{m-p}(k)\right)\big(\tau^{-1/3}\big)^m,\end{aligned} \tag{A.10}$$

$$\frac{u'(\tau)}{\tau} \underset{\tau\to+\infty}{=} \frac{1}{3}c_{0,k}\big(\tau^{-1/3}\big)^5\left(1-\tau^{-2/3}\sum_{m=0}^{\infty}(m+1)\mathfrak{u}_m(k)\big(\tau^{-1/3}\big)^m\right), \tag{A.11}$$

$$\begin{aligned}\frac{(u'(\tau))^2}{u(\tau)} \underset{\tau\to+\infty}{=} \frac{1}{9}c_{0,k}\big(\tau^{-1/3}\big)^5\bigg(&1-3\mathfrak{u}_0(k)\tau^{-2/3}-5\mathfrak{u}_1(k)\big(\tau^{-1/3}\big)^3\\ &+\big(2\mathfrak{u}_0^2(k)-\lambda_0(k)+\eta_0(k)\big)\big(\tau^{-1/3}\big)^4+\big(6\mathfrak{u}_0(k)\mathfrak{u}_1(k)-\lambda_1(k)\\ &+\eta_1(k)\big)\big(\tau^{-1/3}\big)^5+\big(4\mathfrak{u}_1^2(k)-\lambda_2(k)+2\mathfrak{u}_0(k)\lambda_0(k)+\eta_2(k)\\ &-\mathfrak{u}_0(k)\eta_0(k)\big)\big(\tau^{-1/3}\big)^6+\big(-\lambda_3(k)+2\mathfrak{u}_0(k)\lambda_1(k)+4\mathfrak{u}_1(k)\lambda_0(k)+\eta_3(k)\\ &-\mathfrak{u}_0(k)\eta_1(k)-\mathfrak{u}_1(k)\eta_0(k)\big)\big(\tau^{-1/3}\big)^7+\big(\tau^{-1/3}\big)^8\sum_{m=0}^{\infty}\bigg(-\lambda_{m+4}(k)\\ &+2\mathfrak{u}_0(k)\lambda_{m+2}(k)+4\mathfrak{u}_1(k)\lambda_{m+1}(k)+\eta_{m+4}(k)-\mathfrak{u}_0(k)\eta_{m+2}(k)\\ &-\mathfrak{u}_1(k)\eta_{m+1}(k)-\sum_{p=0}^{m}\eta_p(k)\lambda_{m-p}(k)\bigg)\big(\tau^{-1/3}\big)^m\bigg),\end{aligned} \tag{A.12}$$

where $\eta_m(k)$ is defined by equation (2.9), and

$$u''(\tau) \underset{\tau\to+\infty}{=} -\frac{2}{9}c_{0,k}\big(\tau^{-1/3}\big)^5\left(1-\tau^{-2/3}\sum_{m=0}^{\infty}\frac{(m+1)(m+4)}{2}\mathfrak{u}_m(k)\big(\tau^{-1/3}\big)^m\right). \tag{A.13}$$

Substituting, now, the expansions (A.9)–(A.13) into the DP3E (1.1), and equating coefficients of like powers of $\big(\tau^{-1/3}\big)^m$, $m\in\mathbb{N}$, one arrives at, for $k=\pm1$, the following system of recurrence relations for the expansion coefficients $\mathfrak{u}_{m'}(k)$, $m'\in\mathbb{Z}_+$:

$$\mathcal{O}\big(\tau^{-1/3}\big):\ 0=-8\varepsilon c_{0,k}^2+b^2c_{0,k}^{-1}, \tag{A.14}$$

$$\mathcal{O}\big(\big(\tau^{-1/3}\big)^3\big):\ 0=-16\varepsilon c_{0,k}^2\mathfrak{u}_0(k)+2ab-b^2c_{0,k}^{-1}\mathfrak{u}_0(k), \tag{A.15}$$

$$\mathcal{O}\big(\big(\tau^{-1/3}\big)^4\big):\ 0=-16\varepsilon c_{0,k}^2\mathfrak{u}_1(k)-b^2c_{0,k}^{-1}\mathfrak{u}_1(k), \tag{A.16}$$

$$\mathcal{O}\big((\tau^{-1/3})^{5}\big):\ 0=\mathfrak{t}_k(2,0),\qquad \mathcal{O}\big((\tau^{-1/3})^{6}\big):\ 0=\mathfrak{t}_k(3,1),\tag{A.17}$$

$$\mathcal{O}\big((\tau^{-1/3})^{7}\big):\ \frac{4}{9}c_{0,k}\mathfrak{u}_0(k)=\mathfrak{t}_k(4,2),\qquad \mathcal{O}\big((\tau^{-1/3})^{8}\big):\ c_{0,k}\mathfrak{u}_1(k)=\mathfrak{t}_k(5,3),\tag{A.18}$$

$$\mathcal{O}\big((\tau^{-1/3})^{9}\big):\ c_{0,k}\mathfrak{u}_2(k)=\frac{1}{9}c_{0,k}\big(2\mathfrak{u}_0^2(k)-\lambda_0(k)+\eta_0(k)\big)+\mathfrak{t}_k(6,4),\tag{A.19}$$

$$\mathcal{O}\big((\tau^{-1/3})^{10}\big):\ \left(\frac{4}{3}\right)^2c_{0,k}\mathfrak{u}_3(k)=\frac{1}{9}c_{0,k}\big(6\mathfrak{u}_0(k)\mathfrak{u}_1(k)-\lambda_1(k)+\eta_1(k)\big)+\mathfrak{t}_k(7,5),\tag{A.20}$$

$$\begin{aligned}\mathcal{O}\big((\tau^{-1/3})^{11}\big):\ \left(\frac{5}{3}\right)^2c_{0,k}\mathfrak{u}_4(k)=\frac{1}{9}c_{0,k}\big(&4\mathfrak{u}_1^2(k)-\lambda_2(k)+2\mathfrak{u}_0(k)\lambda_0(k)\\&+\eta_2(k)-\mathfrak{u}_0(k)\eta_0(k)\big)+\mathfrak{t}_k(8,6),\end{aligned}\tag{A.21}$$

$$\begin{aligned}\mathcal{O}\big((\tau^{-1/3})^{12}\big):\ \left(\frac{6}{3}\right)^2c_{0,k}\mathfrak{u}_5(k)=\frac{1}{9}c_{0,k}\big(&-\lambda_3(k)+2\mathfrak{u}_0(k)\lambda_1(k)+4\mathfrak{u}_1(k)\lambda_0(k)+\eta_3(k)\\&-\mathfrak{u}_0(k)\eta_1(k)-\mathfrak{u}_1(k)\eta_0(k)\big)+\mathfrak{t}_k(9,7),\end{aligned}\tag{A.22}$$

$$\begin{aligned}\mathcal{O}\big((\tau^{-1/3})^{m+13}\big):\ \left(\frac{m+7}{3}\right)^2c_{0,k}\mathfrak{u}_{m+6}(k)=\frac{1}{9}c_{0,k}\Bigg(&-\lambda_{m+4}(k)+2\mathfrak{u}_0(k)\lambda_{m+2}(k)\\&+4\mathfrak{u}_1(k)\lambda_{m+1}(k)+\eta_{m+4}(k)-\mathfrak{u}_0(k)\eta_{m+2}(k)\\&-\mathfrak{u}_1(k)\eta_{m+1}(k)-\sum_{p=0}^{m}\eta_p(k)\lambda_{m-p}(k)\Bigg)\\&+\mathfrak{t}_k(m+10,m+8),\qquad m\in\mathbb{Z}_+,\end{aligned}\tag{A.23}$$

where

$$\mathfrak{t}_k(j,l):=-8\varepsilon c_{0,k}^2\left(2\mathfrak{u}_j(k)+\sum_{p=0}^{l}\mathfrak{u}_p(k)\mathfrak{u}_{l-p}(k)\right)-b^2c_{0,k}^{-1}\lambda_l(k).$$

Noting that (cf. definition (2.2)) equation (A.14) is identically true, the algorithm, hereafter, is as follows: (i) one solves equation (A.15) for $\mathfrak{u}_0(k)$ in order to arrive at the first of equations (2.3); (ii) via the formula for $\mathfrak{u}_0(k)$, the definitions of $c_{0,k}$, $\lambda_i(k)$, and $\eta_m(k)$ given heretofore, and equations (A.3)–(A.8), one solves equations (A.16)–(A.22), in the indicated order, to arrive at the expressions for the coefficients $\mathfrak{u}_j(k)$, $j=1,2,\dots,9$, given in equations (2.3) and (2.4); and (iii) using the fact that $\mathfrak{u}_1(k)=0$ (cf. equations (2.3)), and the definition of $\lambda_i(k)$, one solves equation (A.23) for $\mathfrak{u}_{m+10}(k)$, $m\in\mathbb{Z}_+$, and, after a lengthy induction argument, arrives at equations (2.6) and (2.7). ■

B Asymptotics as $\tau\to+\infty$ for $\mathcal{Z}_k$, $\mathcal{G}_{0,k}$, $\mathfrak{A}_k$, $\mathfrak{B}_k$, $\mathfrak{C}_k$, $\mathfrak{A}^{\sharp}_{0,k}$, $\mathfrak{B}^{\sharp}_{0,k}$, $\mathfrak{C}^{\sharp}_{0,k}$, $\omega^2_{0,k}$, $\ell^+_{0,k}$, $\chi_k(\tau)$, $\ell^+_{1,k}$, $\mu_k(\tau)$, and $\ell^+_{2,k}$, $k=\pm1$

For the requisite estimates in step (xi) of the proof of Lemma 3.17, the $\tau\to+\infty$ asymptotics for $\mathcal{Z}_k$, $\mathcal{G}_{0,k}$, $\mathfrak{A}_k$, $\mathfrak{B}_k$, $\mathfrak{C}_k$, $\mathfrak{A}^{\sharp}_{0,k}$, $\mathfrak{B}^{\sharp}_{0,k}$, $\mathfrak{C}^{\sharp}_{0,k}$, $\omega^2_{0,k}$, $\ell^+_{0,k}$, $\chi_k(\tau)$, $\ell^+_{1,k}$, $\mu_k(\tau)$, and $\ell^+_{2,k}$, $k=\pm1$, are necessary. From the conditions (3.17), the asymptotics (3.21) and (3.24), the definitions (3.117), (3.120), (3.121), (3.124), (3.130), (3.136), (3.137), (3.138), (3.143), (3.148), (3.149), (3.156), and (3.157), and equations (3.124)–(3.129), a lengthy, but otherwise straightforward, algebraic calculation shows that, in the indicated order,

$$\mathcal{Z}_k \underset{\tau\to+\infty}{=} 1-\frac{\tilde{r}_0(\tau)\tau^{-1/3}}{12}+\left(\frac{\tilde{r}_0(\tau)\tau^{-1/3}}{12}\right)^2+\mathcal{O}\big(\big(\tilde{r}_0(\tau)\tau^{-1/3}\big)^3\big), \tag{B.1}$$

$$\mathcal{G}_{0,k} \underset{\tau\to+\infty}{=} \mathcal{G}_{0,k}^{\infty}+\Delta\mathcal{G}_{0,k}, \qquad k=\pm1, \tag{B.2}$$

where

$$(6\varepsilon b)^{1/4}\mathcal{G}_{0,k}^{\infty}=\begin{pmatrix}\frac{(\varepsilon b)^{1/2}(\sqrt{3}-1)}{\sqrt{2}} & -\frac{(\varepsilon b)^{1/2}(\sqrt{3}+1)}{\sqrt{2}}\\ 1 & 1\end{pmatrix}, \tag{B.3}$$

and

$$\Delta\mathcal{G}_{0,k}:=\mathcal{G}_{0,k}-\mathcal{G}_{0,k}^{\infty}=\begin{pmatrix}\big(\Delta\mathcal{G}_{0,k}\big)_{11} & \big(\Delta\mathcal{G}_{0,k}\big)_{12}\\ \big(\Delta\mathcal{G}_{0,k}\big)_{21} & \big(\Delta\mathcal{G}_{0,k}\big)_{22}\end{pmatrix}, \tag{B.4}$$

with

$$\begin{aligned}
&(6\varepsilon b)^{1/4}\big(\Delta\mathcal{G}_{0,k}\big)_{11}\\
&\quad=\frac{(\varepsilon b)^{1/2}}{4\sqrt{2}}\left(\frac{\big(\sqrt{3}-1\big)\big(2\sqrt{3}+1\big)}{6}\tilde{r}_0(\tau)\tau^{-1/3}+\frac{1}{12\sqrt{3}}\left(1+\frac{\big(\sqrt{3}-1\big)\big(4\sqrt{3}-1\big)}{8\sqrt{3}}\right)\right.\\
&\qquad\left.\times\big(\tilde{r}_0(\tau)\tau^{-1/3}\big)^2+\mathcal{O}\big(\big(\tilde{r}_0(\tau)\tau^{-1/3}\big)^3\big)\right),
\end{aligned} \tag{B.5}$$

$$\begin{aligned}
&(6\varepsilon b)^{1/4}\big(\Delta\mathcal{G}_{0,k}\big)_{12}\\
&\quad=\frac{(\varepsilon b)^{1/2}}{4\sqrt{2}}\left(\frac{\big(\sqrt{3}+1\big)\big(2\sqrt{3}-1\big)}{6}\tilde{r}_0(\tau)\tau^{-1/3}+\frac{1}{12\sqrt{3}}\left(-1+\frac{\big(\sqrt{3}+1\big)\big(4\sqrt{3}+1\big)}{8\sqrt{3}}\right)\right.\\
&\qquad\left.\times\big(\tilde{r}_0(\tau)\tau^{-1/3}\big)^2+\mathcal{O}\big(\big(\tilde{r}_0(\tau)\tau^{-1/3}\big)^3\big)\right),
\end{aligned} \tag{B.6}$$

$$\begin{aligned}
&(6\varepsilon b)^{1/4}\big(\Delta\mathcal{G}_{0,k}\big)_{21}\\
&\quad=(6\varepsilon b)^{1/4}\big(\Delta\mathcal{G}_{0,k}\big)_{22}=\frac{1}{24}\tilde{r}_0(\tau)\tau^{-1/3}-\frac{1}{2(24)^2}\big(\tilde{r}_0(\tau)\tau^{-1/3}\big)^2\\
&\qquad+\mathcal{O}\big(\big(\tilde{r}_0(\tau)\tau^{-1/3}\big)^3\big),
\end{aligned} \tag{B.7}$$

$$\begin{aligned}
\mathfrak{A}_k \underset{\tau\to+\infty}{=}& \frac{\mathrm{i}(a-\mathrm{i}/2)\tau^{-1/3}}{\sqrt{3}\alpha_k}+\frac{\mathrm{i}\tau^{-1/3}}{4\sqrt{3}}\left(\alpha_k\big(4v_0(\tau)(\tilde{r}_0(\tau)+2v_0(\tau))-(\tilde{r}_0(\tau))^2\big)\right.\\
&\left.-\frac{(a-\mathrm{i}/2)(12v_0(\tau)-\tilde{r}_0(\tau))\tau^{-1/3}}{3\alpha_k}\right)+\mathcal{O}\left((6\varepsilon b)^{-1/2}\Big(-\mathrm{i}(\varepsilon b)^{1/3}\big((\varepsilon b)^{1/3}\right.\\
&\times\mathrm{e}^{\mathrm{i}\pi k/3}(\tilde{r}_0(\tau)+2v_0(\tau))+2(a-\mathrm{i}/2)\mathrm{e}^{\mathrm{i}2\pi k/3}\tau^{-1/3}\big)\big(v_0(\tau)\tau^{-1/3}\big)^2\\
&+\frac{\mathrm{i}(\varepsilon b)^{1/3}}{12}\left(-\frac{(\varepsilon b)^{1/3}\mathrm{e}^{\mathrm{i}\pi k/3}}{4}(\tilde{r}_0(\tau))^2\tau^{-1/3}+\big((\varepsilon b)^{1/3}\mathrm{e}^{\mathrm{i}\pi k/3}(\tilde{r}_0(\tau)+2v_0(\tau))\right.\\
&\left.\left.\left.+2(a-\mathrm{i}/2)\mathrm{e}^{\mathrm{i}2\pi k/3}\tau^{-1/3}\big)v_0(\tau)\tau^{-1/3}\right)\tilde{r}_0(\tau)\tau^{-1/3}\Big)\right),
\end{aligned} \tag{B.8}$$

$$\mathfrak{B}_k \underset{\tau\to+\infty}{=} \mathrm{i}\big(\sqrt{3}+1\big)\left(\frac{\alpha_k}{2}(4v_0(\tau)+\big(\sqrt{3}+1\big)\tilde{r}_0(\tau))-\frac{\big(\sqrt{3}+1\big)(a-\mathrm{i}/2)\tau^{-1/3}}{2\sqrt{3}\alpha_k}\right)$$

$$
\begin{aligned}
&+\frac{\mathrm{i}\big(\sqrt{3}+1\big)^2\tau^{-1/3}}{4\sqrt{3}}\Bigg(-\frac{\alpha_k}{2}\big((\tilde{r}_0(\tau))^2+2\big(\sqrt{3}+1\big)v_0(\tau)\tilde{r}_0(\tau)+8v_0^2(\tau)\big)\\
&+\frac{(a-\mathrm{i}/2)(12v_0(\tau)+\big(2\sqrt{3}-1\big)\tilde{r}_0(\tau))\tau^{-1/3}}{6\alpha_k}\Bigg)\\
&+\mathcal{O}\Bigg((6\varepsilon b)^{-1/2}\Bigg(-\frac{\mathrm{i}\big(\sqrt{3}+1\big)^2(a-\mathrm{i}/2)(\varepsilon b)^{1/3}\mathrm{e}^{\mathrm{i}2\pi k/3}}{12}\tilde{r}_0(\tau)\big(\tau^{-1/3}\big)^3\\
&\times\big(v_0(\tau)+\tilde{r}_0(\tau)/2\sqrt{3}\big)-\frac{\mathrm{i}\big(\sqrt{3}+1\big)(\varepsilon b)^{2/3}\mathrm{e}^{\mathrm{i}\pi k/3}}{48\sqrt{3}}\\
&\times\tilde{r}_0(\tau)\big(\tau^{-1/3}\big)^2\big((\tilde{r}_0(\tau))^2+\big(\sqrt{3}+1\big)(\tilde{r}_0(\tau)+2\sqrt{3}v_0(\tau))(\tilde{r}_0(\tau)+2v_0(\tau))\big)\\
&+\frac{\mathrm{i}\alpha_k(\varepsilon b)^{1/2}}{24\sqrt{6}}(\tilde{r}_0(\tau))^3\big(\tau^{-1/3}\big)^2+\Bigg(\frac{\mathrm{i}(\varepsilon b)^{1/3}\big(\sqrt{3}+1\big)^2}{2}\big(v_0(\tau)\tau^{-1/3}\big)^2\\
&+\frac{\mathrm{i}(\varepsilon b)^{1/3}\big(3\sqrt{3}+4\big)}{48\sqrt{3}}\big(\tilde{r}_0(\tau)\tau^{-1/3}\big)^2+\frac{\mathrm{i}(\varepsilon b)^{1/3}\big(2+\sqrt{3}\big)}{2\sqrt{3}}v_0(\tau)\tilde{r}_0(\tau)\big(\tau^{-1/3}\big)^2\Bigg)\\
&\times\big((\varepsilon b)^{1/3}\mathrm{e}^{\mathrm{i}\pi k/3}(\tilde{r}_0(\tau)+2v_0(\tau))+2(a-\mathrm{i}/2)\mathrm{e}^{\mathrm{i}2\pi k/3}\tau^{-1/3}\big)\Bigg)\Bigg),
\end{aligned}
\tag{B.9}
$$

$$
\begin{aligned}
\mathfrak{C}_k \underset{\tau\to+\infty}{=}& -\mathrm{i}\big(\sqrt{3}-1\big)\left(\frac{\alpha_k}{2}\big(4v_0(\tau)-\big(\sqrt{3}-1\big)\tilde{r}_0(\tau)\big)-\frac{\big(\sqrt{3}-1\big)(a-\mathrm{i}/2)\tau^{-1/3}}{2\sqrt{3}\alpha_k}\right)\\
&+\frac{\mathrm{i}\big(\sqrt{3}-1\big)^2\tau^{-1/3}}{4\sqrt{3}}\Bigg(\frac{\alpha_k}{2}\big((\tilde{r}_0(\tau))^2-2\big(\sqrt{3}-1\big)v_0(\tau)\tilde{r}_0(\tau)+8v_0^2(\tau)\big)\\
&-\frac{(a-\mathrm{i}/2)(12v_0(\tau)-\big(2\sqrt{3}+1\big)\tilde{r}_0(\tau))\tau^{-1/3}}{6\alpha_k}\Bigg)\\
&+\mathcal{O}\Bigg((6\varepsilon b)^{-1/2}\Bigg(\frac{\mathrm{i}\big(\sqrt{3}-1\big)^2(a-\mathrm{i}/2)(\varepsilon b)^{1/3}\mathrm{e}^{\mathrm{i}2\pi k/3}}{12}\tilde{r}_0(\tau)\big(\tau^{-1/3}\big)^3\\
&\times\big(v_0(\tau)-\tilde{r}_0(\tau)/2\sqrt{3}\big)+\frac{\mathrm{i}\big(\sqrt{3}-1\big)(\varepsilon b)^{2/3}\mathrm{e}^{\mathrm{i}\pi k/3}}{48\sqrt{3}}\tilde{r}_0(\tau)\big(\tau^{-1/3}\big)^2\\
&\times\big((\tilde{r}_0(\tau))^2+\big(\sqrt{3}-1\big)\big(2\sqrt{3}v_0(\tau)-\tilde{r}_0(\tau)\big)(\tilde{r}_0(\tau)+2v_0(\tau))\big)+\frac{\mathrm{i}\alpha_k(\varepsilon b)^{1/2}}{24\sqrt{6}}\\
&\times(\tilde{r}_0(\tau))^3\big(\tau^{-1/3}\big)^2-\Bigg(\frac{\mathrm{i}(\varepsilon b)^{1/3}\big(\sqrt{3}-1\big)^2}{2}\big(v_0(\tau)\tau^{-1/3}\big)^2+\frac{\mathrm{i}(\varepsilon b)^{1/3}(3\sqrt{3}-4)}{48\sqrt{3}}\\
&\times\big(\tilde{r}_0(\tau)\tau^{-1/3}\big)^2-\frac{\mathrm{i}(\varepsilon b)^{1/3}\big(2-\sqrt{3}\big)}{2\sqrt{3}}v_0(\tau)\tilde{r}_0(\tau)\big(\tau^{-1/3}\big)^2\Bigg)\\
&\times\big((\varepsilon b)^{1/3}\mathrm{e}^{\mathrm{i}\pi k/3}(\tilde{r}_0(\tau)+2v_0(\tau))+2(a-\mathrm{i}/2)\mathrm{e}^{\mathrm{i}2\pi k/3}\tau^{-1/3}\big)\Bigg)\Bigg),
\end{aligned}
\tag{B.10}
$$

$$
\mathfrak{A}^{\sharp}_{0,k} \underset{\tau\to+\infty}{=} -\frac{14\mathrm{i}\tau^{-1/3}}{\sqrt{3}\alpha_k}-\frac{\mathrm{i}\tilde{r}_0(\tau)\big(\tau^{-1/3}\big)^2}{4\sqrt{3}\alpha_k}\left(-\frac{4}{3}+\frac{1}{2}\tilde{r}_0(\tau)\tau^{-1/3}+\mathcal{O}\big(\big(\tilde{r}_0(\tau)\tau^{-1/3}\big)^2\big)\right), \tag{B.11}
$$

$$
\mathfrak{B}^{\sharp}_{0,k} \underset{\tau\to+\infty}{=} \frac{4\mathrm{i}\big(\sqrt{3}+1\big)\tau^{-1/3}}{\sqrt{3}\alpha_k}+\frac{\mathrm{i}\tilde{r}_0(\tau)\big(\tau^{-1/3}\big)^2}{4\sqrt{3}\alpha_k}
$$

$$\times\left(-\frac{2(3\sqrt{3}+7)}{3}+\mathcal{O}\big(\big(\tilde{r}_0(\tau)\tau^{-1/3}\big)^2\big)\right), \tag{B.12}$$

$$\begin{aligned}\mathfrak{C}^{\sharp}_{0,k} \underset{\tau\to+\infty}{=} & \frac{4\mathrm{i}\big(\sqrt{3}-1\big)\tau^{-1/3}}{\sqrt{3}\alpha_k} \\ &+\frac{\mathrm{i}\tilde{r}_0(\tau)\big(\tau^{-1/3}\big)^2}{4\sqrt{3}\alpha_k}\left(-\frac{2\big(3\sqrt{3}-7\big)}{3}+\mathcal{O}\big(\big(\tilde{r}_0(\tau)\tau^{-1/3}\big)^2\big)\right),\end{aligned} \tag{B.13}$$

$$\begin{aligned}\omega^2_{0,k} \underset{\tau\to+\infty}{=} & -\alpha_k^2\big(8v_0^2(\tau)+4v_0(\tau)\tilde{r}_0(\tau)-(\tilde{r}_0(\tau))^2\big)+4(a-\mathrm{i}/2)v_0(\tau)\tau^{-1/3} \\ &+\big(4\alpha_k^2v_0(\tau)(\tilde{r}_0(\tau)+2v_0(\tau))-4(a-\mathrm{i}/2)v_0(\tau)\tau^{-1/3}\big)v_0(\tau)\tau^{-1/3} \\ &+\mathcal{O}\big(\big(-4\alpha_k^2v_0(\tau)(\tilde{r}_0(\tau)+2v_0(\tau))+4(a-\mathrm{i}/2)v_0(\tau)\tau^{-1/3}\big) \\ &\quad\times\big(v_0(\tau)\tau^{-1/3}\big)^2\big),\end{aligned} \tag{B.14}$$

$$\begin{aligned}\ell^+_{0,k} \underset{\tau\to+\infty}{=} & \frac{\mathrm{i}}{8\sqrt{3}}\left(1+\frac{\tilde{r}_0(\tau)\tau^{-1/3}}{12}+\mathcal{O}\big(\big(\tilde{r}_0(\tau)\tau^{-1/3}\big)^3\big)\right)\frac{\mathfrak{B}^{\sharp}_{0,k}}{\mathfrak{B}_k} \\ &-\frac{\mathrm{i}\omega^2_{0,k}}{\big(8\sqrt{3}\big)^3}\left(1+\frac{\tilde{r}_0(\tau)\tau^{-1/3}}{12}+\mathcal{O}\big(\big(\tilde{r}_0(\tau)\tau^{-1/3}\big)^3\big)\right)^3\left(\frac{\mathfrak{B}^{\sharp}_{0,k}}{\mathfrak{B}_k}\right)^2 \\ &+\mathcal{O}\left(\omega^4_{0,k}\left(1+\frac{\tilde{r}_0(\tau)\tau^{-1/3}}{12}+\mathcal{O}\big(\big(\tilde{r}_0(\tau)\tau^{-1/3}\big)^3\big)\right)^5\left(\frac{\mathfrak{B}^{\sharp}_{0,k}}{\mathfrak{B}_k}\right)^3\right),\end{aligned} \tag{B.15}$$

$$\begin{aligned}\chi_k(\tau) \underset{\tau\to+\infty}{=} & \mathrm{i}4\sqrt{3}\mathcal{Z}_k+\omega^2_{0,k}\ell^+_{0,k}+\frac{\mathfrak{R}^*_{0,k}\big(-\ell^+_{0,k}+1\big)}{2\big(\mathrm{i}4\sqrt{3}\mathcal{Z}_k+\omega^2_{0,k}\ell^+_{0,k}\big)}-\frac{\big(\mathfrak{R}^*_{0,k}\big(-\ell^+_{0,k}+1\big)\big)^2}{8\big(\mathrm{i}4\sqrt{3}\mathcal{Z}_k+\omega^2_{0,k}\ell^+_{0,k}\big)^3} \\ &+\mathcal{O}\left(\frac{\big(\mathfrak{R}^*_{0,k}\big(-\ell^+_{0,k}+1\big)\big)^3}{\big(\mathrm{i}4\sqrt{3}\mathcal{Z}_k+\omega^2_{0,k}\ell^+_{0,k}\big)^5}\right),\end{aligned} \tag{B.16}$$

$$\begin{aligned}\ell^+_{1,k} \underset{\tau\to+\infty}{=} & \frac{\mathfrak{R}^*_{0,k}}{2\big(\mathrm{i}4\sqrt{3}\mathcal{Z}_k+\omega^2_{0,k}\ell^+_{0,k}\big)}-\frac{(\mathfrak{R}^*_{0,k})^2\big(-\ell^+_{0,k}+1\big)}{8\big(\mathrm{i}4\sqrt{3}\mathcal{Z}_k+\omega^2_{0,k}\ell^+_{0,k}\big)^3} \\ &+\mathcal{O}\left(\frac{(\mathfrak{R}^*_{0,k})^3\big(-\ell^+_{0,k}+1\big)^2}{\big(\mathrm{i}4\sqrt{3}\mathcal{Z}_k+\omega^2_{0,k}\ell^+_{0,k}\big)^5}\right),\end{aligned} \tag{B.17}$$

$$\begin{aligned}\mu_k(\tau) \underset{\tau\to+\infty}{=} & \chi_k(\tau)-\frac{\mathrm{i}8\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\big(-\ell^+_{0,k}+1\big)\big(\mathfrak{P}^*_{0,k}-\mathrm{i}8\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\ell^+_{0,k}\big)}{2\chi_k^2(\tau)} \\ &-\frac{\big(\mathrm{i}8\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\big(-\ell^+_{0,k}+1\big)\big(\mathfrak{P}^*_{0,k}-\mathrm{i}8\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\ell^+_{0,k}\big)\big)^2}{8\chi_k^5(\tau)} \\ &+\mathcal{O}\left(\frac{\big(\mathrm{i}8\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\big(-\ell^+_{0,k}+1\big)\big(\mathfrak{P}^*_{0,k}-\mathrm{i}8\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\ell^+_{0,k}\big)\big)^3}{\chi_k^8(\tau)}\right),\end{aligned} \tag{B.18}$$

and

$$\begin{aligned}\ell^+_{2,k} \underset{\tau\to+\infty}{=} & -\frac{\mathrm{i}4\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\big(\mathfrak{P}^*_{0,k}-\mathrm{i}8\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\ell^+_{0,k}\big)}{\chi_k^2(\tau)} \\ &-\frac{\big(-\ell^+_{0,k}+1\big)\big(\mathrm{i}8\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\big(\mathfrak{P}^*_{0,k}-\mathrm{i}8\sqrt{3}\mathcal{Z}_k\mathfrak{A}_k\ell^+_{0,k}\big)\big)^2}{8\chi_k^5(\tau)}\end{aligned}$$

$$+\mathcal{O}\left(\frac{\big(-\ell_{0,k}^{+}+1\big)^{2}\big(\mathrm{i}8\sqrt{3}\mathcal{Z}_{k}\mathfrak{A}_{k}\big(\mathfrak{P}_{0,k}^{*}-\mathrm{i}8\sqrt{3}\mathcal{Z}_{k}\mathfrak{A}_{k}\ell_{0,k}^{+}\big)\big)^{3}}{\chi_{k}^{8}(\tau)}\right). \tag{B.19}$$

C Asymptotic estimates as $\tau\to+\infty$ for $|(\Phi_{M,k}(\xi))_{ij}|$, $k=\pm1$, $i,j=1,2$, on the Stokes rays

Asymptotic estimates as $\tau\to+\infty$ for the moduli $|(\Phi_{M,k}(\xi))_{ij}|$, $k=\pm1$, $i,j=1,2$, on the Stokes rays $\widehat{\mathcal{S}}$ are: **(a)** for $\arg(\xi)\underset{\tau\to+\infty}{=}0+\mathcal{O}\big(\tau^{-2/3}\big)$,[50]

$$|(\Phi_{M,k}(\xi))_{11}|\underset{\tau\to+\infty}{\leqslant}\left(\frac{2^{3/2}\mathrm{e}^{\pi\,\mathrm{Im}(\nu(k)+1)/2}2^{\mathrm{Re}(\nu(k))/2}\cosh^{3}\big(\frac{\pi}{2}\,\mathrm{Im}(\nu(k)+1)\big)\Gamma(-\mathrm{Re}(\nu(k)))}{\Gamma\big(\frac{1}{2}-\frac{\mathrm{Re}(\nu(k))}{2}\big)\sin\big(-\frac{\pi}{2}\,\mathrm{Re}(\nu(k))\big)}\right.$$
$$\left.+\frac{\sqrt{\pi}\mathrm{e}^{\pi\,\mathrm{Im}(\nu(k)+1)}2^{-\mathrm{Re}(\nu(k)+1)/2}\big|\sin\big(\frac{\pi}{2}(\nu(k)+1)\big)\big|}{\Gamma\big(\frac{1}{2}+\frac{\mathrm{Re}(\nu(k)+1)}{2}\big)\sin\big(\frac{\pi}{2}\,\mathrm{Re}(\nu(k)+1)\big)}\right)\big(1+\mathcal{O}\big(\tau^{-2/3}\big)\big),$$

$$|(\Phi_{M,k}(\xi))_{12}|\underset{\tau\to+\infty}{\leqslant}\frac{\sqrt{\pi}2^{\mathrm{Re}(\nu(k))/2}\cosh\big(\frac{\pi}{2}\,\mathrm{Im}(\nu(k)+1)\big)}{\Gamma\big(\frac{1}{2}-\frac{\mathrm{Re}(\nu(k))}{2}\big)\sin\big(-\frac{\pi}{2}\,\mathrm{Re}(\nu(k))\big)}\big(1+\mathcal{O}\big(\tau^{-2/3}\big)\big),$$

$$|(\Phi_{M,k}(\xi))_{21}|$$
$$\underset{\tau\to+\infty}{\leqslant}\frac{4\sqrt{3}|\xi|\,\mathrm{Re}(\nu(k)+1)}{|p_{k}(\tau)|}$$
$$\times\left(\frac{\mathrm{e}^{\pi\,\mathrm{Im}(\nu(k)+1)}2^{\mathrm{Re}(\nu(k)+1)/2}\big|\sin\big(\frac{\pi}{2}(\nu(k)+1)\big)\big|\Gamma\big(\frac{\mathrm{Re}(\nu(k)+1)}{2}\big)}{\sin\big(\frac{\pi}{2}\,\mathrm{Re}(\nu(k)+1)\big)\Gamma(\mathrm{Re}(\nu(k)+1))}\right.$$
$$\left.+\frac{2^{3/2}\mathrm{e}^{\pi\,\mathrm{Im}(\nu(k)+1)/2}2^{-\mathrm{Re}(\nu(k))/2}\cosh^{3}\big(\frac{\pi}{2}\,\mathrm{Im}(\nu(k)+1)\big)\Gamma\big(-\frac{\mathrm{Re}(\nu(k))}{2}\big)}{\sqrt{\pi}\sin\big(-\frac{\pi}{2}\,\mathrm{Re}(\nu(k))\big)}\right)$$
$$\times\big(1+\mathcal{O}\big(\tau^{-2/3}\big)\big),$$

$$|(\Phi_{M,k}(\xi))_{22}|\underset{\tau\to+\infty}{\leqslant}\frac{4\sqrt{3}|\xi|\,\mathrm{Re}(\nu(k)+1)2^{-\mathrm{Re}(\nu(k))/2}\cosh\big(\frac{\pi}{2}\,\mathrm{Im}(\nu(k)+1)\big)\Gamma\big(-\frac{\mathrm{Re}(\nu(k))}{2}\big)}{|p_{k}(\tau)|\sin\big(-\frac{\pi}{2}\,\mathrm{Re}(\nu(k))\big)\Gamma(-\mathrm{Re}(\nu(k)))}$$
$$\times\big(1+\mathcal{O}\big(\tau^{-2/3}\big)\big);$$

(b) for $\arg(\xi)\underset{\tau\to+\infty}{=}-\pi/2+\mathcal{O}\big(\tau^{-2/3}\big)$,

$$|(\Phi_{M,k}(\xi))_{11}|\underset{\tau\to+\infty}{\leqslant}\frac{\sqrt{\pi}2^{-\mathrm{Re}(\nu(k)+1)/2}\big|\sin\big(\frac{\pi}{2}(\nu(k)+1)\big)\big|}{\Gamma\big(\frac{1}{2}+\frac{\mathrm{Re}(\nu(k)+1)}{2}\big)\sin\big(\frac{\pi}{2}\,\mathrm{Re}(\nu(k)+1)\big)}\big(1+\mathcal{O}\big(\tau^{-2/3}\big)\big)$$
$$=:\hat{\varrho}_{0}(k)\big(1+\mathcal{O}\big(\tau^{-2/3}\big)\big), \tag{C.1}$$

$$|(\Phi_{M,k}(\xi))_{12}|\underset{\tau\to+\infty}{\leqslant}\frac{\sqrt{\pi}2^{\mathrm{Re}(\nu(k))/2}\cosh\big(\frac{\pi}{2}\,\mathrm{Im}(\nu(k)+1)\big)}{\Gamma\big(\frac{1}{2}-\frac{\mathrm{Re}(\nu(k))}{2}\big)\sin\big(-\frac{\pi}{2}\,\mathrm{Re}(\nu(k))\big)}\big(1+\mathcal{O}\big(\tau^{-2/3}\big)\big)$$
$$=:\hat{\varrho}_{1}(k)\left(1+\mathcal{O}\big(\tau^{-2/3}\big)\right), \tag{C.2}$$

$$|(\Phi_{M,k}(\xi))_{21}|\underset{\tau\to+\infty}{\leqslant}\frac{4\sqrt{3}|\xi|\,\mathrm{Re}(\nu(k)+1)2^{\mathrm{Re}(\nu(k)+1)/2}\Gamma\big(\frac{\mathrm{Re}(\nu(k)+1)}{2}\big)\big|\sin\big(\frac{\pi}{2}(\nu(k)+1)\big)\big|}{|p_{k}(\tau)|\Gamma(\mathrm{Re}(\nu(k)+1))\sin\big(\frac{\pi}{2}\,\mathrm{Re}(\nu(k)+1)\big)}$$

[50]The asymptotic estimate $\mathcal{O}\big(\tau^{-2/3}\big)$ appears on the Stokes rays because of the factor $(2\mu_{k}(\tau))^{1/2}$ in the arguments of the various parabolic-cylinder functions in equation (3.115) and the fact that (cf. expansions (B.1), (B.16), and (B.18)) $\arg(\mu_{k}(\tau))\underset{\tau\to+\infty}{=}\frac{\pi}{2}\big(1+\mathcal{O}\big(\tau^{-2/3}\big)\big)$.

$$\times \left(1+\mathcal{O}\left(\tau^{-2/3}\right)\right) =: \hat{\varrho}_2(k)\frac{|\xi|\operatorname{Re}(\nu(k)+1)}{|p_k(\tau)|}\left(1+\mathcal{O}\left(\tau^{-2/3}\right)\right), \tag{C.3}$$

$$|(\Phi_{M,k}(\xi))_{22}| \underset{\tau\to+\infty}{\leqslant} \frac{4\sqrt{3}|\xi|\operatorname{Re}(\nu(k)+1)2^{-\operatorname{Re}(\nu(k))/2}\cosh\left(\frac{\pi}{2}\operatorname{Im}(\nu(k)+1)\right)\Gamma\left(-\frac{\operatorname{Re}(\nu(k))}{2}\right)}{|p_k(\tau)|\sin\left(-\frac{\pi}{2}\operatorname{Re}(\nu(k))\right)\Gamma(-\operatorname{Re}(\nu(k)))}$$
$$\times \left(1+\mathcal{O}\left(\tau^{-2/3}\right)\right) =: \hat{\varrho}_3(k)\frac{|\xi|\operatorname{Re}(\nu(k)+1)}{|p_k|}\left(1+\mathcal{O}\left(\tau^{-2/3}\right)\right); \tag{C.4}$$

(c) for $\arg(\xi) \underset{\tau\to+\infty}{=} -\pi+\mathcal{O}\left(\tau^{-2/3}\right)$,

$$|(\Phi_{M,k}(\xi))_{11}| \underset{\tau\to+\infty}{\leqslant} \frac{\sqrt{\pi}2^{-\operatorname{Re}(\nu(k)+1)/2}\left|\sin\left(\frac{\pi}{2}(\nu(k)+1)\right)\right|}{\Gamma\left(\frac{1}{2}+\frac{\operatorname{Re}(\nu(k)+1)}{2}\right)\sin\left(\frac{\pi}{2}\operatorname{Re}(\nu(k)+1)\right)}\left(1+\mathcal{O}\left(\tau^{-2/3}\right)\right),$$

$$|(\Phi_{M,k}(\xi))_{12}|$$
$$\underset{\tau\to+\infty}{\leqslant} \left(\frac{2^{3/2}\mathrm{e}^{\pi\operatorname{Im}(\nu(k)+1)/2}\left|\cos\left(\frac{\pi}{2}(\nu(k)+1)\right)\right|\left|\sin\left(\frac{\pi}{2}(\nu(k)+1)\right)\right|^2\Gamma(\operatorname{Re}(\nu(k)+1))}{2^{\operatorname{Re}(\nu(k)+1)/2}\Gamma\left(\frac{1}{2}+\frac{\operatorname{Re}(\nu(k)+1)}{2}\right)\sin\left(\frac{\pi}{2}\operatorname{Re}(\nu(k)+1)\right)}\right.$$
$$\left.+\frac{\sqrt{\pi}\mathrm{e}^{\pi\operatorname{Im}(\nu(k)+1)}2^{\operatorname{Re}(\nu(k))/2}\cosh\left(\frac{\pi}{2}\operatorname{Im}(\nu(k)+1)\right)}{\Gamma\left(\frac{1}{2}-\frac{\operatorname{Re}(\nu(k))}{2}\right)\sin\left(-\frac{\pi}{2}\operatorname{Re}(\nu(k))\right)}\right)\left(1+\mathcal{O}\left(\tau^{-2/3}\right)\right),$$

$$|(\Phi_{M,k}(\xi))_{21}| \underset{\tau\to+\infty}{\leqslant} \frac{4\sqrt{3}|\xi|\operatorname{Re}(\nu(k)+1)2^{\operatorname{Re}(\nu(k)+1)/2}\left|\sin\left(\frac{\pi}{2}(\nu(k)+1)\right)\right|\Gamma\left(\frac{\operatorname{Re}(\nu(k)+1)}{2}\right)}{|p_k(\tau)|\sin\left(\frac{\pi}{2}\operatorname{Re}(\nu(k)+1)\right)\Gamma(\operatorname{Re}(\nu(k)+1))}$$
$$\times\left(1+\mathcal{O}\left(\tau^{-2/3}\right)\right),$$

$$|(\Phi_{M,k}(\xi))_{22}|$$
$$\underset{\tau\to+\infty}{\leqslant} \frac{4\sqrt{3}|\xi|\operatorname{Re}(\nu(k)+1)}{|p_k(\tau)|}\left(\frac{\mathrm{e}^{\pi\operatorname{Im}(\nu(k)+1)}\cosh\left(\frac{\pi}{2}\operatorname{Im}(\nu(k)+1)\right)\Gamma\left(-\frac{\operatorname{Re}(\nu(k))}{2}\right)}{2^{\operatorname{Re}(\nu(k))/2}\sin\left(-\frac{\pi}{2}\operatorname{Re}(\nu(k))\right)\Gamma(-\operatorname{Re}(\nu(k)))}\right.$$
$$\left.+\frac{2^{3/2}\mathrm{e}^{\pi\operatorname{Im}(\nu(k)+1)/2}|\cos\left(\frac{\pi}{2}(\nu(k)+1)\right)||\sin\left(\frac{\pi}{2}(\nu(k)+1)\right)|^2\Gamma\left(\frac{\operatorname{Re}(\nu(k)+1)}{2}\right)}{\sqrt{\pi}2^{-\operatorname{Re}(\nu(k)+1)/2}\sin\left(\frac{\pi}{2}\operatorname{Re}(\nu(k)+1)\right)}\right)$$
$$\times\left(1+\mathcal{O}\left(\tau^{-2/3}\right)\right);$$

and **(d)** for $\arg(\xi) \underset{\tau\to+\infty}{=} -3\pi/2+\mathcal{O}\left(\tau^{-2/3}\right)$,

$$|(\Phi_{M,k}(\xi))_{11}| \underset{\tau\to+\infty}{\leqslant} \left(\frac{2^{3/2}\mathrm{e}^{-\pi\operatorname{Im}(\nu(k)+1)/2}2^{\operatorname{Re}(\nu(k))/2}\cosh^3\left(\frac{\pi}{2}\operatorname{Im}(\nu(k)+1)\right)\Gamma(-\operatorname{Re}(\nu(k)))}{\Gamma\left(\frac{1}{2}-\frac{\operatorname{Re}(\nu(k))}{2}\right)\sin\left(-\frac{\pi}{2}\operatorname{Re}(\nu(k))\right)}\right.$$
$$\left.+\frac{\sqrt{\pi}\mathrm{e}^{-\pi\operatorname{Im}(\nu(k)+1)}2^{-\operatorname{Re}(\nu(k)+1)/2}\left|\sin\left(\frac{\pi}{2}(\nu(k)+1)\right)\right|}{\Gamma\left(\frac{1}{2}+\frac{\operatorname{Re}(\nu(k)+1)}{2}\right)\sin\left(\frac{\pi}{2}\operatorname{Re}(\nu(k)+1)\right)}\right)\left(1+\mathcal{O}\left(\tau^{-2/3}\right)\right)$$
$$=: \tilde{\varrho}_0(k)\left(1+\mathcal{O}\left(\tau^{-2/3}\right)\right), \tag{C.5}$$

$$|(\Phi_{M,k}(\xi))_{12}|$$
$$\underset{\tau\to+\infty}{\leqslant} \left(\frac{2^{3/2}\mathrm{e}^{-\pi\operatorname{Im}(\nu(k)+1)/2}\left|\cos\left(\frac{\pi}{2}(\nu(k)+1)\right)\right|\left|\sin\left(\frac{\pi}{2}(\nu(k)+1)\right)\right|^2\Gamma(\operatorname{Re}(\nu(k)+1))}{2^{\operatorname{Re}(\nu(k)+1)/2}\Gamma\left(\frac{1}{2}+\frac{\operatorname{Re}(\nu(k)+1)}{2}\right)\sin\left(\frac{\pi}{2}\operatorname{Re}(\nu(k)+1)\right)}\right.$$
$$\left.+\frac{\sqrt{\pi}\mathrm{e}^{-\pi\operatorname{Im}(\nu(k)+1)}2^{\operatorname{Re}(\nu(k))/2}\cosh\left(\frac{\pi}{2}\operatorname{Im}(\nu(k)+1)\right)}{\Gamma\left(\frac{1}{2}-\frac{\operatorname{Re}(\nu(k))}{2}\right)\sin\left(-\frac{\pi}{2}\operatorname{Re}(\nu(k))\right)}\right)\left(1+\mathcal{O}\left(\tau^{-2/3}\right)\right)$$
$$=: \tilde{\varrho}_1(k)\left(1+\mathcal{O}\left(\tau^{-2/3}\right)\right), \tag{C.6}$$

$$|(\Phi_{M,k}(\xi))_{21}|$$

$$\underset{\tau\to+\infty}{\leqslant} \frac{4\sqrt{3}|\xi|\operatorname{Re}(\nu(k)+1)}{|p_k(\tau)|}\left(\frac{\mathrm{e}^{-\pi\operatorname{Im}(\nu(k)+1)/2}|\sin\bigl(\frac{\pi}{2}(\nu(k)+1)\bigr)|\Gamma\bigl(\frac{\operatorname{Re}(\nu(k)+1)}{2}\bigr)}{2^{-\operatorname{Re}(\nu(k)+1)/2}\sin\bigl(\frac{\pi}{2}\operatorname{Re}(\nu(k)+1)\bigr)\Gamma(\operatorname{Re}(\nu(k)+1))}\right.$$
$$\left.+\frac{2^{3/2}\mathrm{e}^{-\pi\operatorname{Im}(\nu(k)+1)/2}\cosh^3\bigl(\frac{\pi}{2}\operatorname{Im}(\nu(k)+1)\bigr)\Gamma\bigl(-\frac{\operatorname{Re}(\nu(k))}{2}\bigr)}{2^{\operatorname{Re}(\nu(k))/2}\sin\bigl(-\frac{\pi}{2}\operatorname{Re}(\nu(k))\bigr)}\right)\bigl(1+\mathcal{O}\bigl(\tau^{-2/3}\bigr)\bigr)$$
$$=: \tilde{\varrho}_2(k)\frac{|\xi|\operatorname{Re}(\nu(k)+1)}{|p_k(\tau)|}\bigl(1+\mathcal{O}\bigl(\tau^{-2/3}\bigr)\bigr), \tag{C.7}$$

$$|(\Phi_{M,k}(\xi))_{22}|$$
$$\underset{\tau\to+\infty}{\leqslant} \frac{4\sqrt{3}|\xi|\operatorname{Re}(\nu(k)+1)}{|p_k(\tau)|}\left(\frac{\mathrm{e}^{-\pi\operatorname{Im}(\nu(k)+1)}\cosh\bigl(\frac{\pi}{2}\operatorname{Im}(\nu(k)+1)\bigr)\Gamma\bigl(-\frac{\operatorname{Re}(\nu(k))}{2}\bigr)}{2^{\operatorname{Re}(\nu(k))/2}\sin\bigl(-\frac{\pi}{2}\operatorname{Re}(\nu(k))\bigr)\Gamma(-\operatorname{Re}(\nu(k)))}\right.$$
$$\left.+\frac{2^{3/2}\mathrm{e}^{-\pi\operatorname{Im}(\nu(k)+1)/2}\bigl|\cos\bigl(\frac{\pi}{2}(\nu(k)+1)\bigr)\bigr|\bigl|\sin\bigl(\frac{\pi}{2}(\nu(k)+1)\bigr)\bigr|^2\Gamma\bigl(\frac{\operatorname{Re}(\nu(k)+1)}{2}\bigr)}{\sqrt{\pi}2^{-\operatorname{Re}(\nu(k)+1)/2}\sin\bigl(\frac{\pi}{2}\operatorname{Re}(\nu(k)+1)\bigr)}\right)$$
$$\times\bigl(1+\mathcal{O}\bigl(\tau^{-2/3}\bigr)\bigr) =: \tilde{\varrho}_3(k)\frac{|\xi|\operatorname{Re}(\nu(k)+1)}{|p_k(\tau)|}\bigl(1+\mathcal{O}\bigl(\tau^{-2/3}\bigr)\bigr). \tag{C.8}$$

D Symmetries and transformations

It was shown in Proposition 1.5 that (cf. system (1.23)), given any solution $\hat{u}(\tau)$ of the DP3E (1.1), the function $\hat{\varphi}(\tau)$ is defined as the general solution of the ODE $\hat{\varphi}'(\tau)=2a\tau^{-1}+b(\hat{u}(\tau))^{-1}$. From the latter ODE, it is clear that, given $\hat{u}(\tau)$, the function $\hat{\varphi}(\tau)$ is defined up to a τ-independent "additive parameter", that is, $\hat{\varphi}(\tau)\to\hat{\varphi}(\tau)+\hat{\varphi}_0$, where $\hat{\varphi}_0\in\mathbb{C}$.[51] As the principal focus of the symmetry transformations derived in [61, Section 6] was on the function $\hat{u}(\tau)$ and not the function $\hat{\varphi}(\tau)$, it must be noted that the additive parameter, $\hat{\varphi}_0$, appears non-uniformly (though correctly!) in those symmetries; for example, for the transformation 6.2.1 changing $\tau\to-\tau$, $\hat{\varphi}_0=-\pi\epsilon_1^*$, $\epsilon_1^*\in\{\pm1\}$, whilst for the transformation 6.2.3 changing $\tau\to\mathrm{i}\tau$, $\hat{\varphi}_0=0$. In order to, with abuse of nomenclature, "uniformize" the presentation of the final asymptotic results of the present work, this appendix considers the concomitant actions (see the brief discussion below) of the Lie-point symmetries for the DP3E (1.1) and the systems of isomonodromy deformations (1.22) and (1.36) on the fundamental solutions of the systems (1.18) and (1.32) and the manifold of the monodromy data $\mathcal{M}$,[52] under the strict caveat that, for every symmetry, the additive parameter is equal to zero; en route, novel sets of symmetry transformations not identified in [61] are obtained.

Before proceeding, however, some preamble regarding group actions on sets is necessary (see, for example, [12]). The terms 'function' and 'transformation' will be used interchangeably throughout the following discussion. Let G be a group and X denote a set. An *action* of G on X is a function from $\mathrm{G}\times\mathrm{X}$ to X if, for every pair $(\mathfrak{g},\mathfrak{x})\in\mathrm{G}\times\mathrm{X}$, there is an element $\mathfrak{g}\mathfrak{x}\in\mathrm{X}$ such that $(\mathfrak{g}_1\mathfrak{g}_2)\mathfrak{x}=\mathfrak{g}_1(\mathfrak{g}_2\mathfrak{x})$ and $\mathfrak{e}\mathfrak{x}=\mathfrak{x}$ ($\mathfrak{e}$ is the identity in G). For fixed $\mathfrak{g}\in\mathrm{G}$, there is a function (transformation) $\aleph_{\mathfrak{g}}\colon\mathrm{X}\mapsto\mathfrak{g}\mathfrak{x}$ for $\mathfrak{x}\in\mathrm{X}$, that is, $\mathrm{Act}(\mathrm{G})_{\mathrm{X}}\colon\mathrm{G}\times\mathrm{X}\to\mathrm{X}$, $(\mathfrak{g},\mathfrak{x})\mapsto\aleph_{\mathfrak{g}}(\mathfrak{x}):=\mathfrak{g}\mathfrak{x}$. As $\aleph_{\mathfrak{g}_1}\circ\aleph_{\mathfrak{g}_2}=\aleph_{\mathfrak{g}_1\mathfrak{g}_2}$ and $\aleph_{\mathfrak{e}}=\mathbf{id}_{\mathrm{X}}$ (the identity mapping on X), it follows that $\aleph_{\mathfrak{g}}$ is a bijection on X, since $\aleph_{\mathfrak{g}}\circ\aleph_{\mathfrak{g}^{-1}}=\aleph_{\mathfrak{g}\mathfrak{g}^{-1}}=\aleph_{\mathfrak{e}}=\aleph_{\mathfrak{g}^{-1}\mathfrak{g}}=\aleph_{\mathfrak{g}^{-1}}\circ\aleph_{\mathfrak{g}}$, where $\aleph_{\mathfrak{g}^{-1}}$ denotes the inverse function of $\aleph_{\mathfrak{g}}$. All bijective functions $\aleph\colon\mathrm{X}\to\mathrm{X}$ form a group under composition of functions (the composition of functions is associative, the identity is the identity function $\mathbf{id}(\mathfrak{x})=\mathfrak{x}$ for $\mathfrak{x}\in\mathrm{X}$,

[51] Of course, it also follows from the definitions (1.24) and (1.25) that $\hat{\varphi}(\tau)$ is defined $\mathrm{mod}(2\pi)$: similar statements apply, *mutatis mutandis*, for the pair of functions $(u(\tau),\varphi(\tau))$ that solve the system (1.37), where, in particular, $\varphi(\tau)$ is also defined $\mathrm{mod}(2\pi)$ (cf. definitions (1.38) and (1.39)).

[52] The group of symmetries derived in this section preserve, in particular, the invariance of the system (1.51) defining $\mathcal{M}$.

and the inverse of $\aleph$ is the inverse function $\aleph^{-1}$). Denoting by $\mathbb{B}(\mathfrak{x})$ the group of all bijections on X, one defines a transformation group of X as any subgroup of $\mathbb{B}(\mathfrak{x})$.[53] Any action of a group G on a set X defines a homomorphism from G to the transformation group $\mathbb{B}(\mathfrak{x})$ such that $\mathfrak{g} \in \mathrm{G}$ maps onto the transformation $\aleph_{\mathfrak{g}}$. Denoting such a homomorphism by $\daleth\colon \mathrm{G} \to \mathbb{B}(\mathfrak{x})$, it follows that $\daleth(\mathfrak{g}) = \aleph_{\mathfrak{g}}$; conversely, any homomorphism $\daleth\colon \mathrm{G} \to \mathbb{B}(\mathfrak{x})$ defines an action of G on X if one defines $\mathfrak{g}\mathfrak{x} := \daleth(\mathfrak{g})(\mathfrak{x})$.[54] For a group G acting on a set X, the *orbit* of $\mathfrak{x} \in \mathrm{X}$, denoted by $\mathrm{G}\mathfrak{x}$, is defined as $\mathrm{G}\mathfrak{x} := \{\mathfrak{g}\mathfrak{x}, \forall \mathfrak{g} \in \mathrm{G}\}$ (the set of all images of $\mathfrak{x}$ under the elements of G).

Remark D.1. In this work (see Appendix D.5 below for complete details), the group G of all (Lie-point) symmetries of interest is written as the disjoint union of two subgroups, $\mathrm{G} = \widetilde{\boldsymbol{\mathcal{W}}} \cup \widehat{\boldsymbol{\mathcal{W}}}$, where the elements of the subgroup $\widetilde{\boldsymbol{\mathcal{W}}}$ are denoted by $\mathcal{F}^{\{\ell\}}_{\varepsilon_1,\varepsilon_2,m(\varepsilon_2)}$, with

$$\varepsilon_1 \in \{0, \pm 1\}, \qquad \varepsilon_2 \in \{0, \pm 1\}, \qquad m(\varepsilon_2) = \begin{cases} 0, & \varepsilon_2 = 0, \\ \pm\varepsilon_2, & \varepsilon_2 \in \{\pm 1\}, \end{cases}$$

and $\ell \in \{0, 1\}$, and the elements of the subgroup $\widehat{\boldsymbol{\mathcal{W}}}$ are denoted by $\hat{\mathcal{F}}^{\{\hat{\ell}\}}_{\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)}$, with

$$\hat{\varepsilon}_1 \in \{\pm 1\}, \qquad \hat{\varepsilon}_2 \in \{0, \pm 1\}, \qquad \hat{m}(\hat{\varepsilon}_2) = \begin{cases} 0, & \hat{\varepsilon}_2 \in \{\pm 1\}, \\ \pm\hat{\varepsilon}_1, & \hat{\varepsilon}_2 = 0, \end{cases}$$

and $\hat{\ell} \in \{0, 1\}$, and the action of the group elements $\mathcal{F}^{\{\ell\}}_{\varepsilon_1,\varepsilon_2,m(\varepsilon_2)}$ on $\mathcal{M}$,

$$\begin{aligned} \mathcal{F}^{\{\ell\}}_{\varepsilon_1,\varepsilon_2,m(\varepsilon_2)}\mathcal{M} := \big(&\mathcal{F}^{\{\ell\}}_{\varepsilon_1,\varepsilon_2,m(\varepsilon_2)}a, \mathcal{F}^{\{\ell\}}_{\varepsilon_1,\varepsilon_2,m(\varepsilon_2)}s_0^0, \mathcal{F}^{\{\ell\}}_{\varepsilon_1,\varepsilon_2,m(\varepsilon_2)}s_0^{\infty}, \mathcal{F}^{\{\ell\}}_{\varepsilon_1,\varepsilon_2,m(\varepsilon_2)}s_1^{\infty}, \mathcal{F}^{\{\ell\}}_{\varepsilon_1,\varepsilon_2,m(\varepsilon_2)}g_{11}, \\ &\mathcal{F}^{\{\ell\}}_{\varepsilon_1,\varepsilon_2,m(\varepsilon_2)}g_{12}, \mathcal{F}^{\{\ell\}}_{\varepsilon_1,\varepsilon_2,m(\varepsilon_2)}g_{21}, \mathcal{F}^{\{\ell\}}_{\varepsilon_1,\varepsilon_2,m(\varepsilon_2)}g_{22}\big), \end{aligned}$$

is given in equations (D.47)–(D.61) and (D.71)–(D.85) below, whilst the action of the group elements $\hat{\mathcal{F}}^{\{\hat{\ell}\}}_{\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)}$ on $\mathcal{M}$,

$$\begin{aligned} \hat{\mathcal{F}}^{\{\hat{\ell}\}}_{\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)}\mathcal{M} := \big(&\hat{\mathcal{F}}^{\{\hat{\ell}\}}_{\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)}a, \hat{\mathcal{F}}^{\{\hat{\ell}\}}_{\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)}s_0^0, \hat{\mathcal{F}}^{\{\hat{\ell}\}}_{\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)}s_0^{\infty}, \hat{\mathcal{F}}^{\{\hat{\ell}\}}_{\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)}s_1^{\infty}, \hat{\mathcal{F}}^{\{\hat{\ell}\}}_{\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)}g_{11}, \\ &\hat{\mathcal{F}}^{\{\hat{\ell}\}}_{\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)}g_{12}, \hat{\mathcal{F}}^{\{\hat{\ell}\}}_{\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)}g_{21}, \hat{\mathcal{F}}^{\{\hat{\ell}\}}_{\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)}g_{22}\big), \end{aligned}$$

is given in equations (D.62)–(D.70) and (D.86)–(D.93) below. The orbit of G on $\mathcal{M}$ considered in this work reads

$$\mathrm{G}\mathcal{M} = \bigcup_{\mathfrak{g}\in\mathrm{G}} \bigcup_{\mathfrak{x}\in\mathcal{M}} \mathfrak{g}\mathfrak{x} = \bigcup_{\varepsilon_1,\varepsilon_2,m(\varepsilon_2),\ell} \bigcup_{\mathfrak{x}\in\mathcal{M}} \{\mathcal{F}^{\{\ell\}}_{\varepsilon_1,\varepsilon_2,m(\varepsilon_2)}\mathfrak{x}\} \bigcup \bigcup_{\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2),\hat{\ell}} \bigcup_{\mathfrak{x}\in\mathcal{M}} \{\hat{\mathcal{F}}^{\{\hat{\ell}\}}_{\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)}\mathfrak{x}\}.$$

Remark D.2. Throughout this appendix, let o denote "old" (or original) variables and let n denote "new" (or transformed) variables, respectively.

D.1 The transformation $\boldsymbol{\tau \to -\tau}$

Let $(\hat{u}_o(\tau_o), \hat{\varphi}_o(\tau_o))$ solve the system (1.23) for $\tau = \tau_o$, $\varepsilon = \varepsilon_o \in \{\pm 1\}$, $a = a_o$, and $b = b_o$, and let the 4-tuple of functions $\big(\hat{A}_o(\tau_o), \hat{B}_o(\tau_o), \hat{C}_o(\tau_o), \hat{D}_o(\tau_o)\big)$, defined via equations (1.24) for $\hat{u}(\tau) = \hat{u}_o(\tau_o)$, $\hat{\varphi}(\tau) = \hat{\varphi}_o(\tau_o)$, $\tau = \tau_o$, and $\varepsilon = \varepsilon_o$, solve the system of isomonodromy

[53] In this work, the transformation group is a disjoint union of two subgroups of Lie-point symmetries for the DP3E (1.1) and the systems of isomonodromy deformations (1.22) and (1.36), and, in particular, the actions (symmetry transformations) of these subgroups on $\mathcal{M}$ is studied.

[54] For $\mathfrak{g}_1, \mathfrak{g}_2 \in \mathrm{G}$ and $\mathfrak{x} \in \mathrm{X}$, the properties $\daleth(\mathfrak{g}_1\mathfrak{g}_2) = \daleth(\mathfrak{g}_1)\daleth(\mathfrak{g}_2)$ and $\daleth(\mathfrak{e}) = \mathbf{id}$ imply that $(\mathfrak{g}_1\mathfrak{g}_2)\mathfrak{x} = \mathfrak{g}_1(\mathfrak{g}_2\mathfrak{x})$ and $\mathfrak{e}\mathfrak{x} = \mathfrak{x}$.

deformations (1.22) for $\tau=\tau_o$ and $a=a_o$. Set $\hat{u}_o(\tau_o)=-\hat{u}_n(\tau_n)$, $\hat{\varphi}_o(\tau_o)=\hat{\varphi}_n(\tau_n)$, $\tau_o=\tau_n\mathrm{e}^{-\mathrm{i}\pi\varepsilon_1}$, $\varepsilon_1\in\{\pm1\}$, $a_o=a_n$, $\varepsilon_o=\varepsilon_n$, $b_o=b_n$ (that is, $\varepsilon_o b_o=\varepsilon_n b_n$), and

$$\big(\hat{A}_o(\tau_o),\hat{B}_o(\tau_o),\hat{C}_o(\tau_o),\hat{D}_o(\tau_o)\big)=\big(\hat{A}_n(\tau_n),\hat{B}_n(\tau_n),-\hat{C}_n(\tau_n),-\hat{D}_n(\tau_n)\big);$$

then, $(\hat{u}_n(\tau_n),\hat{\varphi}_n(\tau_n))$ solves the system (1.23) for $\tau=\tau_n$, $\varepsilon=\varepsilon_n\in\{\pm1\}$, $a=a_n$, and $b=b_n$, and the 4-tuple of functions $\big(\hat{A}_n(\tau_n),\hat{B}_n(\tau_n),\hat{C}_n(\tau_n),\hat{D}_n(\tau_n)\big)$, defined via equations (1.24) for $\hat{u}(\tau)=\hat{u}_n(\tau_n)$, $\hat{\varphi}(\tau)=\hat{\varphi}_n(\tau_n)$, $\tau=\tau_n$, and $\varepsilon=\varepsilon_n$, solve the system (1.22) for $\tau=\tau_n$, $a=a_n$, and

$$\sqrt{-\hat{A}_o(\tau_o)\hat{B}_o(\tau_o)}=\sqrt{-\hat{A}_n(\tau_n)\hat{B}_n(\tau_n)}.$$

Furthermore, let the functions $\hat{A}_o(\tau_o)$, $\hat{B}_o(\tau_o)$, $\hat{C}_o(\tau_o)$, and $\hat{D}_o(\tau_o)$ be the ones appearing in the definition (1.21) of $\hat{\alpha}(\tau)$ for $\tau=\tau_o$ and $a=a_o$, and in the first integral (cf. Remark 1.3) for $\varepsilon=\varepsilon_o\in\{\pm1\}$ and $b=b_o$; then, under the above symmetry transformations, $\hat{\alpha}_o(\tau_o)=\hat{\alpha}_n(\tau_n)$, where

$$\hat{\alpha}_n(\tau_n):=-2\big(\hat{B}_n(\tau_n)\big)^{-1}\Big(\mathrm{i}a_n\sqrt{-\hat{A}_n(\tau_n)\hat{B}_n(\tau_n)}+\tau_n\big(\hat{A}_n(\tau_n)\hat{D}_n(\tau_n)+\hat{B}_n(\tau_n)\hat{C}_n(\tau_n)\big)\Big),$$

and $-\mathrm{i}\hat{\alpha}_n(\tau_n)\hat{B}_n(\tau_n)=\varepsilon_n b_n$, $\varepsilon_n\in\{\pm1\}$. On the corresponding fundamental solution of the system (1.18) (cf. equations (1.19) and (1.20)), the aforementioned transformations act as follows $\mu_o=\mu_n\mathrm{e}^{\mathrm{i}\pi l/2}$, $l\in\{\pm1\}$, and $\widehat{\Psi}_o(\mu_o,\tau_o)=\mathrm{e}^{-\frac{\mathrm{i}\pi l}{4}\sigma_3}\widehat{\Psi}_n(\mu_n,\tau_n)$.

Let $(u_o(\tau_o),\varphi_o(\tau_o))$ solve the system (1.37) for $\tau=\tau_o$, $\varepsilon=\varepsilon_o\in\{\pm1\}$, $a=a_o$, and $b=b_o$, and let the 4-tuple of functions $(A_o(\tau_o),B_o(\tau_o),C_o(\tau_o),D_o(\tau_o))$, defined via equations (1.38) for $u(\tau)=u_o(\tau_o)$, $\varphi(\tau)=\varphi_o(\tau_o)$, $\tau=\tau_o$, and $\varepsilon=\varepsilon_o$, solve the corresponding system of isomonodromy deformations (1.36) for $\tau=\tau_o$ and $a=a_o$. Set $u_o(\tau_o)=-u_n(\tau_n)$, $\varphi_o(\tau_o)=\varphi_n(\tau_n)$, $\tau_o=\tau_n\mathrm{e}^{-\mathrm{i}\pi\varepsilon_1}$, $\varepsilon_1\in\{\pm1\}$, $a_o=a_n$, $\varepsilon_o=\varepsilon_n$, $b_o=b_n$ (that is, $\varepsilon_o b_o=\varepsilon_n b_n$), and

$$(A_o(\tau_o),B_o(\tau_o),C_o(\tau_o),D_o(\tau_o))=(A_n(\tau_n),B_n(\tau_n),-C_n(\tau_n),-D_n(\tau_n));$$

then, $(u_n(\tau_n),\varphi_n(\tau_n))$ solves the system (1.37) for $\tau=\tau_n$, $\varepsilon=\varepsilon_n\in\{\pm1\}$, $a=a_n$, and $b=b_n$, and the 4-tuple of functions $(A_n(\tau_n),B_n(\tau_n),C_n(\tau_n),D_n(\tau_n))$, defined via equations (1.38) for $u(\tau)=u_n(\tau_n)$, $\varphi(\tau)=\varphi_n(\tau_n)$, $\tau=\tau_n$, and $\varepsilon=\varepsilon_n$, solve the system (1.36) for $\tau=\tau_n$, $a=a_n$, and

$$\sqrt{-A_o(\tau_o)B_o(\tau_o)}=\sqrt{-A_n(\tau_n)B_n(\tau_n)}.$$

Furthermore, let the functions $A_o(\tau_o)$, $B_o(\tau_o)$, $C_o(\tau_o)$, and $D_o(\tau_o)$ be the ones appearing in the definition (1.35) of $\alpha(\tau)$ for $\tau=\tau_o$ and $a=a_o$, and in the first integral (cf. Remark 1.8) for $\varepsilon=\varepsilon_o\in\{\pm1\}$ and $b=b_o$; then, under the above transformations, $\alpha_o(\tau_o)=\alpha_n(\tau_n)$, where

$$\alpha_n(\tau_n):=-2(B_n(\tau_n))^{-1}\Big(\mathrm{i}a_n\sqrt{-A_n(\tau_n)B_n(\tau_n)}+\tau_n(A_n(\tau_n)D_n(\tau_n)+B_n(\tau_n)C_n(\tau_n))\Big),$$

and $-\mathrm{i}\alpha_n(\tau_n)B_n(\tau_n)=\varepsilon_n b_n$, $\varepsilon_n\in\{\pm1\}$. On the corresponding fundamental solution of the system (1.32) (cf. equations (1.33) and (1.34)), the aforementioned symmetry transformations act as follows

$$\mu_o=\mu_n\mathrm{e}^{\mathrm{i}\pi l/2},\qquad l\in\{\pm1\},\qquad\text{and}\qquad\Psi_o(\mu_o,\tau_o)=\mathrm{e}^{-\frac{\mathrm{i}\pi l}{4}\sigma_3}\Psi_n(\mu_n,\tau_n).\tag{D.1}$$

In terms of the canonical solutions of the system (1.32), the actions (D.1) read, for $k\in\mathbb{Z}$ and $\varepsilon_1,l\in\{\pm1\}$,

$$\mathbb{Y}^{\infty}_{o,k}(\mu_o)=\mathrm{e}^{-\frac{\mathrm{i}\pi l}{4}\sigma_3}\mathbb{Y}^{\infty}_{n,k-l+\varepsilon_1}(\mu_n)\mathrm{e}^{\frac{\pi l a_n}{2}\sigma_3},\tag{D.2}$$

and

$$\mathbb{X}^0_{o,k}(\mu_o) = \begin{cases} \mathrm{e}^{-\frac{\mathrm{i}\pi l}{4}\sigma_3}\mathbb{X}^0_{n,k}(\mu_n), & \varepsilon_1 = -l, \\ \mathrm{i}l\mathrm{e}^{-\frac{\mathrm{i}\pi l}{4}\sigma_3}\mathbb{X}^0_{n,k-l}(\mu_n)\sigma_1, & \varepsilon_1 = l. \end{cases} \tag{D.3}$$

The transformations (D.2) and (D.3) for the canonical solutions of the system (1.32) imply the following action on $\mathcal{M}$, for $k \in \mathbb{Z}$ and $\varepsilon_1, l \in \{\pm 1\}$,

$$S^\infty_{o,k} = \mathrm{e}^{-\frac{\pi l a_n}{2}\sigma_3} S^\infty_{n,k-l+\varepsilon_1} \mathrm{e}^{\frac{\pi l a_n}{2}\sigma_3}, \tag{D.4}$$

$$S^0_{o,k} = \begin{cases} S^0_{n,k}, & \varepsilon_1 = -l, \\ \sigma_1 S^0_{n,k-l}\sigma_1, & \varepsilon_1 = l, \end{cases} \tag{D.5}$$

$$G_o = \begin{cases} -\mathrm{i}S^0_{n,0}\sigma_1 G_n \mathrm{e}^{\frac{\pi a_n}{2}\sigma_3}, & \varepsilon_1 = 1, \\ \mathrm{i}\sigma_1 (S^0_{n,0})^{-1} G_n \mathrm{e}^{-\frac{\pi a_n}{2}\sigma_3}, & \varepsilon_1 = -1. \end{cases} \tag{D.6}$$

The actions (D.4)–(D.6) on $\mathcal{M}$ can be expressed in terms of an intermediate auxiliary mapping $\mathcal{F}^{\rhd}_{\mathcal{M}}(\varepsilon_1)\colon \mathbb{C}^8 \to \mathbb{C}^8$, $\varepsilon_1 \in \{\pm 1\}$, which is an isomorphism on $\mathcal{M}$

$$\begin{aligned} \mathcal{F}^{\rhd}_{\mathcal{M}}(\varepsilon_1)\colon\ & \mathcal{M} \to \mathcal{M}, \big(a, s^0_0, s^\infty_0, s^\infty_1, g_{11}, g_{12}, g_{21}, g_{22}\big) \\ & \mapsto \big(a, s^0_0(\varepsilon_1), s^\infty_0(\varepsilon_1), s^\infty_1(\varepsilon_1), g_{11}(\varepsilon_1), g_{12}(\varepsilon_1), g_{21}(\varepsilon_1), g_{22}(\varepsilon_1)\big), \end{aligned}$$

where, for $\varepsilon_1 = -1$,

$$\begin{aligned} & s^0_0(-1) = s^0_0, \qquad s^\infty_0(-1) = s^\infty_0 \mathrm{e}^{\pi a}, \qquad s^\infty_1(-1) = s^\infty_1 \mathrm{e}^{-\pi a}, \\ & g_{11}(-1) = -\mathrm{i}\big(g_{21} + s^0_0 g_{11}\big)\mathrm{e}^{\pi a/2}, \qquad g_{12}(-1) = -\mathrm{i}\big(g_{22} + s^0_0 g_{12}\big)\mathrm{e}^{-\pi a/2}, \\ & g_{21}(-1) = -\mathrm{i}g_{11}\mathrm{e}^{\pi a/2}, \qquad g_{22}(-1) = -\mathrm{i}g_{12}\mathrm{e}^{-\pi a/2}, \end{aligned} \tag{D.7}$$

and, for $\varepsilon_1 = 1$,

$$\begin{aligned} & s^0_0(1) = s^0_0, \qquad s^\infty_0(1) = s^\infty_0 \mathrm{e}^{-\pi a}, \qquad s^\infty_1(1) = s^\infty_1 \mathrm{e}^{\pi a}, \qquad g_{11}(1) = \mathrm{i}g_{21}\mathrm{e}^{-\pi a/2}, \\ & g_{12}(1) = \mathrm{i}g_{22}\mathrm{e}^{\pi a/2}, \qquad g_{21}(1) = \mathrm{i}\big(g_{11} - s^0_0 g_{21}\big)\mathrm{e}^{-\pi a/2}, \\ & g_{22}(1) = \mathrm{i}\big(g_{12} - s^0_0 g_{22}\big)\mathrm{e}^{\pi a/2}. \end{aligned} \tag{D.8}$$

One uses this transformation in order to arrive at asymptotics for $\tau < 0$ by using those for $\tau > 0$.[55]

D.2 The transformation $\tau \to \tau$

Let $(\hat{u}_o(\tau_o), \hat{\varphi}_o(\tau_o))$ solve the system (1.23) for $\tau = \tau_o$, $\varepsilon = \varepsilon_o \in \{\pm 1\}$, $a = a_o$, and $b = b_o$, and let the 4-tuple of functions $\big(\hat{A}_o(\tau_o), \hat{B}_o(\tau_o), \hat{C}_o(\tau_o), \hat{D}_o(\tau_o)\big)$, defined via equations (1.24) for $\hat{u}(\tau) = \hat{u}_o(\tau_o)$, $\hat{\varphi}(\tau) = \hat{\varphi}_o(\tau_o)$, $\tau = \tau_o$, and $\varepsilon = \varepsilon_o$, solve the system of isomonodromy deformations (1.22) for $\tau = \tau_o$ and $a = a_o$. Set $\hat{u}_o(\tau_o) = -\hat{u}_n(\tau_n)$, $\hat{\varphi}_o(\tau_o) = \hat{\varphi}_n(\tau_n)$, $\tau_o = \tau_n$, $a_o = a_n$, $\varepsilon_o = -\varepsilon_n$, $b_o = -b_n$ (that is, $\varepsilon_o b_o = \varepsilon_n b_n$), and

$$\big(\hat{A}_o(\tau_o), \hat{B}_o(\tau_o), \hat{C}_o(\tau_o), \hat{D}_o(\tau_o)\big) = \big(-\hat{A}_n(\tau_n), -\hat{B}_n(\tau_n), -\hat{C}_n(\tau_n), -\hat{D}_n(\tau_n)\big);$$

[55] In [56, Section 7, p. 45], it is stated that the Lie-point symmetry $\tau \to -\tau$ in [61, Section 6.2.1] requires correction. Keeping in mind the mod(2π) arbitrariness inherent in the definition of the function $\hat{\varphi}(\tau)$ discussed in the Introduction to this appendix, the Lie-point symmetry $\tau \to -\tau$ alluded to in [56, Section 7, p. 45] is the one for which the "additive parameter", denoted by $\hat{\varphi}_0$, is equal to zero: the transformation changing $\tau \to -\tau$ for which $\hat{\varphi}_0 = 0$ is presented *here*, in Appendix D.1, and *not* in [61, Section 6.2.1] wherein the Transformation 6.2.1 changing $\tau \to -\tau$ was derived under the condition $\hat{\varphi}_o(\tau_o) \to \hat{\varphi}_o(\tau_o) - \pi\epsilon^*_1 =: \hat{\varphi}_n(\tau_n)$, $\epsilon^*_1 \in \{\pm 1\}$, that is, the additive parameter is equal to $-\pi\epsilon^*_1$ (unfortunately, the action of the symmetry $\tau \to -\tau$ on the function $\hat{\varphi}(\tau)$ was not emphasized in [61]).

then, $(\hat{u}_n(\tau_n),\hat{\varphi}_n(\tau_n))$ solves the system (1.23) for $\tau=\tau_n$, $\varepsilon=\varepsilon_n\in\{\pm1\}$, $a=a_n$, and $b=b_n$, and the 4-tuple of functions $\big(\hat{A}_n(\tau_n),\hat{B}_n(\tau_n),\hat{C}_n(\tau_n),\hat{D}_n(\tau_n)\big)$, defined via equations (1.24) for $\hat{u}(\tau)=\hat{u}_n(\tau_n)$, $\hat{\varphi}(\tau)=\hat{\varphi}_n(\tau_n)$, $\tau=\tau_n$, and $\varepsilon=\varepsilon_n$, solve the system (1.22) for $\tau=\tau_n$, $a=a_n$, and

$$\sqrt{-\hat{A}_o(\tau_o)\hat{B}_o(\tau_o)}=\sqrt{-\hat{A}_n(\tau_n)\hat{B}_n(\tau_n)}.$$

Moreover, let the functions $\hat{A}_o(\tau_o)$, $\hat{B}_o(\tau_o)$, $\hat{C}_o(\tau_o)$, and $\hat{D}_o(\tau_o)$ be the ones appearing in the definition (1.21) of $\hat{\alpha}(\tau)$ for $\tau=\tau_o$ and $a=a_o$, and in the first integral (cf. Remark 1.3) for $\varepsilon=\varepsilon_o\in\{\pm1\}$ and $b=b_o$; then, under the above transformations, $\hat{\alpha}_o(\tau_o)=-\hat{\alpha}_n(\tau_n)$, where

$$\hat{\alpha}_n(\tau_n):=-2\big(\hat{B}_n(\tau_n)\big)^{-1}\Big(\mathrm{i}a_n\sqrt{-\hat{A}_n(\tau_n)\hat{B}_n(\tau_n)}+\tau_n(\hat{A}_n(\tau_n)\hat{D}_n(\tau_n)+\hat{B}_n(\tau_n)\hat{C}_n(\tau_n))\Big),$$

and $-\mathrm{i}\hat{\alpha}_n(\tau_n)\hat{B}_n(\tau_n)=\varepsilon_nb_n$, $\varepsilon_n\in\{\pm1\}$. On the corresponding fundamental solution of the system (1.18) (cf. equations (1.19) and (1.20)), the aforementioned symmetry transformations act as follows:

$$\mu_o=\mu_n\mathrm{e}^{\mathrm{i}\pi m},\qquad m\in\{0,1\},\qquad\text{and}\qquad \widehat{\Psi}_o(\mu_o,\tau_o)=\mathrm{e}^{\frac{\mathrm{i}\pi}{2}(m-1)\sigma_3}\widehat{\Psi}_n(\mu_n,\tau_n).$$

Let $(u_o(\tau_o),\varphi_o(\tau_o))$ solve the system (1.37) for $\tau=\tau_o$, $\varepsilon=\varepsilon_o\in\{\pm1\}$, $a=a_o$, and $b=b_o$, and let the 4-tuple of functions $(A_o(\tau_o),B_o(\tau_o),C_o(\tau_o),D_o(\tau_o))$, defined via equations (1.38) for $u(\tau)=u_o(\tau_o)$, $\varphi(\tau)=\varphi_o(\tau_o)$, $\tau=\tau_o$, and $\varepsilon=\varepsilon_o$, solve the corresponding system of isomonodromy deformations (1.36) for $\tau=\tau_o$ and $a=a_o$. Set $u_o(\tau_o)=-u_n(\tau_n)$, $\varphi_o(\tau_o)=\varphi_n(\tau_n)$, $\tau_o=\tau_n$, $a_o=a_n$, $\varepsilon_o=-\varepsilon_n$, $b_o=-b_n$ (that is, $\varepsilon_ob_o=\varepsilon_nb_n$), and

$$(A_o(\tau_o),B_o(\tau_o),C_o(\tau_o),D_o(\tau_o))=(-A_n(\tau_n),-B_n(\tau_n),-C_n(\tau_n),-D_n(\tau_n));$$

then, $(u_n(\tau_n),\varphi_n(\tau_n))$ solves the system (1.37) for $\tau=\tau_n$, $\varepsilon=\varepsilon_n\in\{\pm1\}$, $a=a_n$, and $b=b_n$, and the 4-tuple of functions $(A_n(\tau_n),B_n(\tau_n),C_n(\tau_n),D_n(\tau_n))$, defined via equations (1.38) for $u(\tau)=u_n(\tau_n)$, $\varphi(\tau)=\varphi_n(\tau_n)$, $\tau=\tau_n$, and $\varepsilon=\varepsilon_n$, solve the system (1.36) for $\tau=\tau_n$, $a=a_n$, and

$$\sqrt{-A_o(\tau_o)B_o(\tau_o)}=\sqrt{-A_n(\tau_n)B_n(\tau_n)}.$$

Furthermore, let the functions $A_o(\tau_o)$, $B_o(\tau_o)$, $C_o(\tau_o)$, and $D_o(\tau_o)$ be the ones appearing in the definition (1.35) of $\alpha(\tau)$ for $\tau=\tau_o$ and $a=a_o$, and in the first integral (cf. Remark 1.8) for $\varepsilon=\varepsilon_o\in\{\pm1\}$ and $b=b_o$; then, under the above transformations, $\alpha_o(\tau_o)=-\alpha_n(\tau_n)$, where

$$\alpha_n(\tau_n):=-2(B_n(\tau_n))^{-1}\Big(\mathrm{i}a_n\sqrt{-A_n(\tau_n)B_n(\tau_n)}+\tau_n(A_n(\tau_n)D_n(\tau_n)+B_n(\tau_n)C_n(\tau_n))\Big),$$

and $-\mathrm{i}\alpha_n(\tau_n)B_n(\tau_n)=\varepsilon_nb_n$, $\varepsilon_n\in\{\pm1\}$. On the corresponding fundamental solution of the system (1.32) (cf. equations (1.33) and (1.34)), the aforementioned symmetry transformations act as follows:

$$\mu_o=\mu_n\mathrm{e}^{\mathrm{i}\pi m},\qquad m\in\{0,1\},\qquad\text{and}\qquad \Psi_o(\mu_o,\tau_o)=\mathrm{e}^{\frac{\mathrm{i}\pi}{2}(m-1)\sigma_3}\Psi_n(\mu_n,\tau_n).\tag{D.9}$$

In terms of the canonical solutions of the system (1.32), the actions (D.9) read, for $k\in\mathbb{Z}$, $m\in\{0,1\}$, and $\tilde{l}\in\{\pm1\}$,[56]

$$\mathbb{Y}^{\infty}_{o,k}(\mu_o)=\mathrm{e}^{\frac{\mathrm{i}\pi}{2}(m-1)\sigma_3}\mathbb{Y}^{\infty}_{n,k-2m}(\mu_n)\mathrm{e}^{-\frac{\mathrm{i}\pi}{2}(m-1)\sigma_3}\mathrm{e}^{\pi m(a_n-\mathrm{i}/2)\sigma_3},\tag{D.10}$$

[56]As discussed in Remarks 1.16 and 1.17, since the canonical solutions $\mathbb{X}^0_k(\mu)$, $k\in\mathbb{Z}$, are defined uniquely provided the branch of $(B(\tau))^{1/2}$ is fixed, it follows that, since the branch of $(B(\tau))^{1/2}$ is not fixed, the canonical solutions $\mathbb{X}^0_k(\mu)$, $k\in\mathbb{Z}$, are defined up to a sign (plus or minus), thus the appearance of the 'sign parameter' $\tilde{l}$: this comment applies, *mutatis mutandis*, throughout the remaining sub-appendices.

and

$$\mathbb{X}^0_{o,k}(\mu_o) = \begin{cases} -\tilde{l}\mathrm{e}^{-\frac{\mathrm{i}\pi}{2}\sigma_3}\mathbb{X}^0_{n,k}(\mu_n), & m=0, \\ \mathrm{i}\tilde{l}\mathbb{X}^0_{n,k-1}(\mu_n)\sigma_1, & m=1. \end{cases} \tag{D.11}$$

The transformations (D.10) and (D.11) for the canonical solutions of the system (1.32) imply the following action on $\mathcal{M}$: for $k \in \mathbb{Z}$, $m \in \{0,1\}$, and $\tilde{l} \in \{\pm 1\}$,

$$S^\infty_{o,k} = \mathrm{e}^{\frac{\mathrm{i}\pi}{2}(m-1)\sigma_3}\mathrm{e}^{-\pi m(a_n-\mathrm{i}/2)\sigma_3}S^\infty_{n,k-2m}\mathrm{e}^{\pi m(a_n-\mathrm{i}/2)\sigma_3}\mathrm{e}^{-\frac{\mathrm{i}\pi}{2}(m-1)\sigma_3}, \tag{D.12}$$

$$S^0_{o,k} = \begin{cases} S^0_{n,k}, & m=0, \\ \sigma_1 S^0_{n,k-1}\sigma_1, & m=1, \end{cases} \tag{D.13}$$

$$G_o = -\tilde{l}G_n\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\sigma_3}. \tag{D.14}$$

The actions (D.12)–(D.14) on $\mathcal{M}$ can be expressed in terms of an intermediate auxiliary mapping $\mathcal{F}^{\triangleleft}_{\mathcal{M}}(\tilde{l})\colon \mathbb{C}^8 \to \mathbb{C}^8$, $\tilde{l} \in \{\pm 1\}$, which is an isomorphism on $\mathcal{M}$,

$$\begin{aligned} \mathcal{F}^{\triangleleft}_{\mathcal{M}}(\tilde{l})\colon \mathcal{M} \to \mathcal{M}, & \big(a, s^0_0, s^\infty_0, s^\infty_1, g_{11}, g_{12}, g_{21}, g_{22}\big) \\ & \mapsto \big(a, s^0_0(\tilde{l}), s^\infty_0(\tilde{l}), s^\infty_1(\tilde{l}), g_{11}(\tilde{l}), g_{12}(\tilde{l}), g_{21}(\tilde{l}), g_{22}(\tilde{l})\big), \end{aligned}$$

where

$$\begin{aligned} & s^0_0(\tilde{l}) = s^0_0, \qquad s^\infty_0(\tilde{l}) = -s^\infty_0, \qquad s^\infty_1(\tilde{l}) = -s^\infty_1, \qquad g_{11}(\tilde{l}) = \mathrm{i}\tilde{l}g_{11}, \\ & g_{12}(\tilde{l}) = -\mathrm{i}\tilde{l}g_{12}, \qquad g_{21}(\tilde{l}) = \mathrm{i}\tilde{l}g_{21}, \qquad g_{22}(\tilde{l}) = -\mathrm{i}\tilde{l}g_{22}. \end{aligned} \tag{D.15}$$

One uses this transformation in order to define an analogue of the identity map; see, in particular, Appendix D.5, definitions (D.43) and (D.44).

D.3 The transformation $a \to -a$

Let $(\hat{u}_o(\tau_o), \hat{\varphi}_o(\tau_o))$ solve the system (1.23) for $\tau = \tau_o$, $\varepsilon = \varepsilon_o \in \{\pm 1\}$, $a = a_o$, and $b = b_o$, and let the 4-tuple of functions $\big(\hat{A}_o(\tau_o), \hat{B}_o(\tau_o), \hat{C}_o(\tau_o), \hat{D}_o(\tau_o)\big)$, defined via equations (1.24) for $\hat{u}(\tau) = \hat{u}_o(\tau_o)$, $\hat{\varphi}(\tau) = \hat{\varphi}_o(\tau_o)$, $\tau = \tau_o$, and $\varepsilon = \varepsilon_o$, solve the system of isomonodromy deformations (1.22) for $\tau = \tau_o$ and $a = a_o$. Set $\hat{u}_o(\tau_o) = -\hat{u}_n(\tau_n)$, $\hat{\varphi}_o(\tau_o) = -\hat{\varphi}_n(\tau_n)$, $\tau_o = \tau_n$, $a_o = -a_n$, $\varepsilon_o = \varepsilon_n\mathrm{e}^{-\mathrm{i}\pi\varepsilon_2}$, $\varepsilon_2 \in \{\pm 1\}$, $b_o = b_n$ (that is, $\varepsilon_o b_o = \varepsilon_n b_n \mathrm{e}^{-\mathrm{i}\pi\varepsilon_2}$), and

$$\big(\hat{A}_o(\tau_o), \hat{B}_o(\tau_o), \hat{C}_o(\tau_o), \hat{D}_o(\tau_o)\big) = \big(\hat{B}_n(\tau_n), \hat{A}_n(\tau_n), -\hat{D}_n(\tau_n), -\hat{C}_n(\tau_n)\big);$$

then, $(\hat{u}_n(\tau_n), \hat{\varphi}_n(\tau_n))$ solves the system (1.23) for $\tau = \tau_n$, $\varepsilon = \varepsilon_n \in \{\pm 1\}$, $a = a_n$, and $b = b_n$, and the 4-tuple of functions $\big(\hat{A}_n(\tau_n), \hat{B}_n(\tau_n), \hat{C}_n(\tau_n), \hat{D}_n(\tau_n)\big)$, defined via equations (1.24) for $\hat{u}(\tau) = \hat{u}_n(\tau_n)$, $\hat{\varphi}(\tau) = \hat{\varphi}_n(\tau_n)$, $\tau = \tau_n$, and $\varepsilon = \varepsilon_n$, solve the system (1.22) for $\tau = \tau_n$, $a = a_n$, and

$$\sqrt{-\hat{A}_o(\tau_o)\hat{B}_o(\tau_o)} = \sqrt{-\hat{A}_n(\tau_n)\hat{B}_n(\tau_n)}.$$

Furthermore, let the functions $\hat{A}_o(\tau_o)$, $\hat{B}_o(\tau_o)$, $\hat{C}_o(\tau_o)$, and $\hat{D}_o(\tau_o)$ be the ones appearing in the definition (1.21) of $\hat{\alpha}(\tau)$ for $\tau = \tau_o$ and $a = a_o$, and in the first integral (cf. Remark 1.3) for $\varepsilon = \varepsilon_o \in \{\pm 1\}$ and $b = b_o$; then, under the above symmetry transformations,

$$\hat{\alpha}_o(\tau_o) = -\hat{B}_n(\tau_n)\big(\hat{A}_n(\tau_n)\big)^{-1}\hat{\alpha}_n(\tau_n),$$

where

$$\hat{\alpha}_n(\tau_n) := -2\big(\hat{B}_n(\tau_n)\big)^{-1}\Big(\mathrm{i}a_n\sqrt{-\hat{A}_n(\tau_n)\hat{B}_n(\tau_n)} + \tau_n\big(\hat{A}_n(\tau_n)\hat{D}_n(\tau_n) + \hat{B}_n(\tau_n)\hat{C}_n(\tau_n)\big)\Big),$$

and $-\mathrm{i}\hat{\alpha}_n(\tau_n)\hat{B}_n(\tau_n) = \varepsilon_n b_n$, $\varepsilon_n \in \{\pm 1\}$. On the corresponding fundamental solution of the system (1.18) (cf. equations (1.19) and (1.20)), the aforementioned transformations act as follows:

$$\mu_o = \mu_n \mathrm{e}^{\mathrm{i}\pi m/2}, \qquad m \in \{\pm 1\}, \qquad \text{and} \qquad \widehat{\Psi}_o(\mu_o,\tau_o) = \widehat{\mathcal{Q}}(\mu_n,\tau_n)\widehat{\Psi}_n(\mu_n,\tau_n),$$

where

$$\widehat{\mathcal{Q}}(\mu_n,\tau_n) := \left(\frac{\hat{B}_n(\tau_n)\mathrm{e}^{-\mathrm{i}\pi m/4}}{\sqrt{-\hat{A}_n(\tau_n)\hat{B}_n(\tau_n)}}\right)^{\sigma_3} + \mu_n \mathrm{e}^{\mathrm{i}\pi m/4}\sigma_-.$$

Let $(u_o(\tau_o),\varphi_o(\tau_o))$ solve the system (1.37) for $\tau=\tau_o$, $\varepsilon=\varepsilon_o\in\{\pm 1\}$, $a=a_o$, and $b=b_o$, and let the 4-tuple of functions $(A_o(\tau_o),B_o(\tau_o),C_o(\tau_o),D_o(\tau_o))$, defined via equations (1.38) for $u(\tau)=u_o(\tau_o)$, $\varphi(\tau)=\varphi_o(\tau_o)$, $\tau=\tau_o$, and $\varepsilon=\varepsilon_o$, solve the corresponding system of isomonodromy deformations (1.36) for $\tau=\tau_o$ and $a=a_o$. Set $u_o(\tau_o)=-u_n(\tau_n)$, $\varphi_o(\tau_o)=-\varphi_n(\tau_n)$, $\tau_o=\tau_n$, $a_o=-a_n$, $\varepsilon_o=\varepsilon_n\mathrm{e}^{-\mathrm{i}\pi\varepsilon_2}$, $\varepsilon_2\in\{\pm 1\}$, $b_o=b_n$ $\big($that is, $\varepsilon_o b_o=\varepsilon_n b_n \mathrm{e}^{-\mathrm{i}\pi\varepsilon_2}\big)$, and

$$(A_o(\tau_o),B_o(\tau_o),C_o(\tau_o),D_o(\tau_o)) = (B_n(\tau_n),A_n(\tau_n),-D_n(\tau_n),-C_n(\tau_n));$$

then, $(u_n(\tau_n),\varphi_n(\tau_n))$ solves the system (1.37) for $\tau=\tau_n$, $\varepsilon=\varepsilon_n\in\{\pm 1\}$, $a=a_n$, and $b=b_n$, and the 4-tuple of functions $(A_n(\tau_n),B_n(\tau_n),C_n(\tau_n),D_n(\tau_n))$, defined via equations (1.38) for $u(\tau)=u_n(\tau_n)$, $\varphi(\tau)=\varphi_n(\tau_n)$, $\tau=\tau_n$, and $\varepsilon=\varepsilon_n$, solve the system (1.36) for $\tau=\tau_n$, $a=a_n$, and

$$\sqrt{-A_o(\tau_o)B_o(\tau_o)} = \sqrt{-A_n(\tau_n)B_n(\tau_n)}.$$

Furthermore, let the functions $A_o(\tau_o)$, $B_o(\tau_o)$, $C_o(\tau_o)$, and $D_o(\tau_o)$ be the ones appearing in the definition (1.35) of $\alpha(\tau)$ for $\tau=\tau_o$ and $a=a_o$, and in the first integral (cf. Remark 1.8) for $\varepsilon=\varepsilon_o\in\{\pm 1\}$ and $b=b_o$; then, under the above transformations,

$$\alpha_o(\tau_o) = -B_n(\tau_n)(A_n(\tau_n))^{-1}\alpha_n(\tau_n),$$

where

$$\alpha_n(\tau_n) := -2(B_n(\tau_n))^{-1}\big(\mathrm{i}a_n\sqrt{-A_n(\tau_n)B_n(\tau_n)} + \tau_n(A_n(\tau_n)D_n(\tau_n)+B_n(\tau_n)C_n(\tau_n))\big),$$

and $-\mathrm{i}\alpha_n(\tau_n)B_n(\tau_n)=\varepsilon_n b_n$, $\varepsilon_n\in\{\pm 1\}$. On the corresponding fundamental solution of the system (1.32) (cf. equations (1.33) and (1.34)), the aforementioned symmetry transformations act as follows:

$$\mu_o = \mu_n \mathrm{e}^{\mathrm{i}\pi m/2}, \qquad m \in \{\pm 1\}, \qquad \text{and} \qquad \Psi_o(\mu_o,\tau_o) = \mathcal{Q}(\mu_n,\tau_n)\Psi_n(\mu_n,\tau_n), \tag{D.16}$$

where

$$\mathcal{Q}(\mu_n,\tau_n) := \left(\frac{B_n(\tau_n)\mathrm{e}^{-\mathrm{i}\pi m/4}}{\sqrt{-A_n(\tau_n)B_n(\tau_n)}}\right)^{\sigma_3} + \mu_n \mathrm{e}^{\mathrm{i}\pi m/4}\sigma_-.$$

In terms of the canonical solutions of the system (1.32), the actions (D.16) read, for $k\in\mathbb{Z}$ and $m,\varepsilon_2,l\in\{\pm 1\}$,

$$\mathbb{Y}^{\infty}_{o,k}(\mu_o) = \mathcal{Q}(\mu_n,\tau_n)\mathbb{Y}^{\infty}_{n,k-m}(\mu_n)\mathrm{e}^{\frac{\pi m a_n}{2}\sigma_3}\sigma_3\sigma_1, \tag{D.17}$$

and

$$\mathbb{X}^{0}_{o,k}(\mu_o) = \begin{cases} l\mathcal{Q}(\mu_n,\tau_n)\mathbb{X}^{0}_{n,k}(\mu_n), & m=-\varepsilon_2, \\ \mathrm{i}l\mathcal{Q}(\mu_n,\tau_n)\mathbb{X}^{0}_{n,k-m}(\mu_n)\sigma_1, & m=\varepsilon_2. \end{cases} \tag{D.18}$$

The transformations (D.17) and (D.18) for the canonical solutions of the system (1.32) imply the following action on $\mathcal{M}$, for $k \in \mathbb{Z}$ and $m, \varepsilon_2, l \in \{\pm 1\}$,

$$S^{\infty}_{o,k} = \sigma_1\sigma_3 \mathrm{e}^{-\frac{\pi m a_n}{2}\sigma_3} S^{\infty}_{n,k-m} \mathrm{e}^{\frac{\pi m a_n}{2}\sigma_3}\sigma_3\sigma_1, \tag{D.19}$$

$$S^0_{o,k} = \begin{cases} S^0_{n,k}, & m = -\varepsilon_2, \\ \sigma_1 S^0_{n,k-m}\sigma_1, & m = \varepsilon_2, \end{cases} \tag{D.20}$$

$$G_o = \begin{cases} -\mathrm{i}lS^0_{o,0}\sigma_1 G_n \mathrm{e}^{\pi(a_n-\mathrm{i}/2)\sigma_3}\sigma_3 (S^{\infty}_{n,1})^{-1}\sigma_3 \mathrm{e}^{-\pi(a_n-\mathrm{i}/2)\sigma_3} & \\ \qquad \times \mathrm{e}^{\frac{\pi a_n}{2}\sigma_3}\sigma_3\sigma_1, & (m,\varepsilon_2) = (1,1), \\ lG_n \mathrm{e}^{\pi(a_n-\mathrm{i}/2)\sigma_3}\sigma_3 (S^{\infty}_{n,1})^{-1}\sigma_3 \mathrm{e}^{-\pi(a_n-\mathrm{i}/2)\sigma_3} \mathrm{e}^{\frac{\pi a_n}{2}\sigma_3}\sigma_3\sigma_1, & (m,\varepsilon_2) = (1,-1), \\ lG_n S^{\infty}_{n,0} \mathrm{e}^{-\frac{\pi a_n}{2}\sigma_3}\sigma_3\sigma_1, & (m,\varepsilon_2) = (-1,1), \\ -\mathrm{i}l\sigma_1 \big(S^0_{o,0}\big)^{-1} G_n S^{\infty}_{n,0} \mathrm{e}^{-\frac{\pi a_n}{2}\sigma_3}\sigma_3\sigma_1, & (m,\varepsilon_2) = (-1,-1). \end{cases} \tag{D.21}$$

The actions (D.19)–(D.21) on $\mathcal{M}$ can be expressed in terms of an intermediate auxiliary mapping $\mathcal{F}^{\leftrightarrow}_{\mathcal{M}}(m,\varepsilon_2)\colon \mathbb{C}^8 \to \mathbb{C}^8$, $m, \varepsilon_2 \in \{\pm 1\}$, which is an isomorphism on $\mathcal{M}$, for $l \in \{\pm 1\}$,

$$\begin{aligned} \mathcal{F}^{\leftrightarrow}_{\mathcal{M}}(m,\varepsilon_2)\colon \mathcal{M} \to \mathcal{M}, \big(a, s^0_0, s^{\infty}_0, s^{\infty}_1, g_{11}, g_{12}, g_{21}, g_{22}\big) \\ \mapsto \big(-a, s^0_0(m,\varepsilon_2), s^{\infty}_0(m,\varepsilon_2), s^{\infty}_1(m,\varepsilon_2), g_{11}(m,\varepsilon_2), g_{12}(m,\varepsilon_2), \\ g_{21}(m,\varepsilon_2), g_{22}(m,\varepsilon_2)\big), \end{aligned}$$

where, for $(m,\varepsilon_2) = (1,1)$,

$$\begin{aligned} & s^0_0(1,1) = s^0_0, \qquad s^{\infty}_0(1,1) = -s^{\infty}_1 \mathrm{e}^{\pi a}, \qquad s^{\infty}_1(1,1) = -s^{\infty}_0 \mathrm{e}^{\pi a}, \qquad g_{11}(1,1) = \mathrm{i}lg_{22}\mathrm{e}^{\pi a/2}, \\ & g_{12}(1,1) = -\mathrm{i}l(g_{21} + s^{\infty}_0 g_{22})\mathrm{e}^{-\pi a/2}, \qquad g_{21}(1,1) = \mathrm{i}l\big(g_{12} - s^0_0 g_{22}\big)\mathrm{e}^{\pi a/2}, \\ & g_{22}(1,1) = \mathrm{i}l\big(-g_{11} - s^{\infty}_0 g_{12} + s^0_0(g_{21} + s^{\infty}_0 g_{22})\big)\mathrm{e}^{-\pi a/2}, \end{aligned} \tag{D.22}$$

for $(m,\varepsilon_2) = (1,-1)$,

$$\begin{aligned} & s^0_0(1,-1) = s^0_0, \qquad s^{\infty}_0(1,-1) = -s^{\infty}_1 \mathrm{e}^{\pi a}, \qquad s^{\infty}_1(1,-1) = -s^{\infty}_0 \mathrm{e}^{\pi a}, \\ & g_{11}(1,-1) = lg_{12}\mathrm{e}^{\pi a/2}, \qquad g_{12}(1,-1) = -l(g_{11} + s^{\infty}_0 g_{12})\mathrm{e}^{-\pi a/2}, \\ & g_{21}(1,-1) = lg_{22}\mathrm{e}^{\pi a/2}, \qquad g_{22}(1,-1) = -l(g_{21} + s^{\infty}_0 g_{22})\mathrm{e}^{-\pi a/2}, \end{aligned} \tag{D.23}$$

for $(m,\varepsilon_2) = (-1,1)$,

$$\begin{aligned} & s^0_0(-1,1) = s^0_0, \qquad s^{\infty}_0(-1,1) = -s^{\infty}_1 \mathrm{e}^{\pi a}, \qquad s^{\infty}_1(-1,1) = -s^{\infty}_0 \mathrm{e}^{\pi a}, \\ & g_{11}(-1,1) = l\big(g_{12} - s^{\infty}_1 g_{11}\mathrm{e}^{2\pi a}\big)\mathrm{e}^{-\pi a/2}, \qquad g_{12}(-1,1) = -lg_{11}\mathrm{e}^{\pi a/2}, \\ & g_{21}(-1,1) = l\big(g_{22} - s^{\infty}_1 g_{21}\mathrm{e}^{2\pi a}\big)\mathrm{e}^{-\pi a/2}, \qquad g_{22}(-1,1) = -lg_{21}\mathrm{e}^{\pi a/2}, \end{aligned} \tag{D.24}$$

and, for $(m,\varepsilon_2) = (-1,-1)$,

$$\begin{aligned} & s^0_0(-1,-1) = s^0_0, \qquad s^{\infty}_0(-1,-1) = -s^{\infty}_1 \mathrm{e}^{\pi a}, \qquad s^{\infty}_1(-1,-1) = -s^{\infty}_0 \mathrm{e}^{\pi a}, \\ & g_{11}(-1,-1) = \mathrm{i}l\big(g_{22} - s^{\infty}_1 g_{21}\mathrm{e}^{2\pi a} + s^0_0\big(g_{12} - s^{\infty}_1 g_{11}\mathrm{e}^{2\pi a}\big)\big)\mathrm{e}^{-\pi a/2}, \\ & g_{12}(-1,-1) = -\mathrm{i}l\big(g_{21} + s^0_0 g_{11}\big)\mathrm{e}^{\pi a/2}, \qquad g_{21}(-1,-1) = \mathrm{i}l\big(g_{12} - s^{\infty}_1 g_{11}\mathrm{e}^{2\pi a}\big)\mathrm{e}^{-\pi a/2}, \\ & g_{22}(-1,-1) = -\mathrm{i}lg_{11}\mathrm{e}^{\pi a/2}. \end{aligned} \tag{D.25}$$

One uses this transformation in order to arrive at asymptotics for $\varepsilon b < 0$ by using those for $\varepsilon b > 0$.

D.4 The transformation $\tau \to \pm\mathrm{i}\tau$

Let $(\hat{u}_o(\tau_o), \hat{\varphi}_o(\tau_o))$ solve the system (1.23) for $\tau = \tau_o$, $\varepsilon = \varepsilon_o \in \{\pm 1\}$, $a = a_o$, and $b = b_o$, and let the 4-tuple of functions $\big(\hat{A}_o(\tau_o), \hat{B}_o(\tau_o), \hat{C}_o(\tau_o), \hat{D}_o(\tau_o)\big)$, defined via equations (1.24) for $\hat{u}(\tau) = \hat{u}_o(\tau_o)$, $\hat{\varphi}(\tau) = \hat{\varphi}_o(\tau_o)$, $\tau = \tau_o$, and $\varepsilon = \varepsilon_o$, solve the system of isomonodromy deformations (1.22) for $\tau = \tau_o$ and $a = a_o$. Set $\hat{u}_o(\tau_o) = \hat{u}_n(\tau_n)\mathrm{e}^{\mathrm{i}\pi\tilde{\varepsilon}_1/2}$, $\tilde{\varepsilon}_1 \in \{\pm 1\}$, $\hat{\varphi}_o(\tau_o) = \hat{\varphi}_n(\tau_n)$, $\tau_o = \tau_n\mathrm{e}^{-\mathrm{i}\pi\tilde{\varepsilon}_1/2}$, $a_o = a_n$, $\varepsilon_o = \varepsilon_n$, and $b_o = b_n\mathrm{e}^{-\mathrm{i}\pi\tilde{\varepsilon}_2}$, $\tilde{\varepsilon}_2 \in \{\pm 1\}$ $\big($that is, $\varepsilon_o b_o = \varepsilon_n b_n \mathrm{e}^{-\mathrm{i}\pi\tilde{\varepsilon}_2}\big)$, and

$$\big(\hat{A}_o(\tau_o), \hat{B}_o(\tau_o), \hat{C}_o(\tau_o), \hat{D}_o(\tau_o)\big) = \big(\hat{A}_n(\tau_n)\mathrm{e}^{\mathrm{i}\pi\tilde{\varepsilon}_1}, \hat{B}_n(\tau_n)\mathrm{e}^{\mathrm{i}\pi\tilde{\varepsilon}_1}, \hat{C}_n(\tau_n)\mathrm{e}^{\mathrm{i}\pi\tilde{\varepsilon}_1/2}, \hat{D}_n(\tau_n)\mathrm{e}^{\mathrm{i}\pi\tilde{\varepsilon}_1/2}\big);$$

then, $(\hat{u}_n(\tau_n), \hat{\varphi}_n(\tau_n))$ solves the system (1.23) for $\tau = \tau_n$, $\varepsilon = \varepsilon_n \in \{\pm 1\}$, $a = a_n$, and $b = b_n$, and the 4-tuple of functions $\big(\hat{A}_n(\tau_n), \hat{B}_n(\tau_n), \hat{C}_n(\tau_n), \hat{D}_n(\tau_n)\big)$, defined via equations (1.24) for $\hat{u}(\tau) = \hat{u}_n(\tau_n)$, $\hat{\varphi}(\tau) = \hat{\varphi}_n(\tau_n)$, $\tau = \tau_n$, and $\varepsilon = \varepsilon_n$, solve the system (1.22) for $\tau = \tau_n$, $a = a_n$, and

$$\sqrt{-\hat{A}_o(\tau_o)\hat{B}_o(\tau_o)} = \mathrm{e}^{\mathrm{i}\pi\tilde{\varepsilon}_1}\sqrt{-\hat{A}_n(\tau_n)\hat{B}_n(\tau_n)}.$$

Moreover, let the functions $\hat{A}_o(\tau_o)$, $\hat{B}_o(\tau_o)$, $\hat{C}_o(\tau_o)$, and $\hat{D}_o(\tau_o)$ be the ones appearing in the definition (1.21) of $\hat{\alpha}(\tau)$ for $\tau = \tau_o$ and $a = a_o$, and in the first integral (cf. Remark 1.3) for $\varepsilon = \varepsilon_o \in \{\pm 1\}$ and $b = b_o$; then, under the above symmetry transformations, $\hat{\alpha}_o(\tau_o) = \hat{\alpha}_n(\tau_n)$, where

$$\hat{\alpha}_n(\tau_n) := -2\big(\hat{B}_n(\tau_n)\big)^{-1}\Big(\mathrm{i}a_n\sqrt{-\hat{A}_n(\tau_n)\hat{B}_n(\tau_n)} + \tau_n\big(\hat{A}_n(\tau_n)\hat{D}_n(\tau_n) + \hat{B}_n(\tau_n)\hat{C}_n(\tau_n)\big)\Big),$$

and $-\mathrm{i}\hat{\alpha}_n(\tau_n)\hat{B}_n(\tau_n) = \varepsilon_n b_n$, $\varepsilon_n \in \{\pm 1\}$. On the corresponding fundamental solution of the system (1.18) (cf. equations (1.19) and (1.20)), the aforementioned transformations act as follows

$$\mu_o = \mu_n\mathrm{e}^{\mathrm{i}\pi\tilde{\varepsilon}_1/4}, \qquad \tilde{\varepsilon}_1 \in \{\pm 1\}, \qquad \text{and} \qquad \widehat{\Psi}_o(\mu_o, \tau_o) = \mathrm{e}^{-\frac{\mathrm{i}\pi\tilde{\varepsilon}_1}{8}\sigma_3}\widehat{\Psi}_n(\mu_n, \tau_n).$$

Let $(u_o(\tau_o), \varphi_o(\tau_o))$ solve the system (1.37) for $\tau = \tau_o$, $\varepsilon = \varepsilon_o \in \{\pm 1\}$, $a = a_o$, and $b = b_o$, and let the 4-tuple of functions $(A_o(\tau_o), B_o(\tau_o), C_o(\tau_o), D_o(\tau_o))$, defined via equations (1.38) for $u(\tau) = u_o(\tau_o)$, $\varphi(\tau) = \varphi_o(\tau_o)$, $\tau = \tau_o$, and $\varepsilon = \varepsilon_o$, solve the corresponding system of isomonodromy deformations (1.36) for $\tau = \tau_o$ and $a = a_o$. Set $u_o(\tau_o) = u_n(\tau_n)\mathrm{e}^{\mathrm{i}\pi\tilde{\varepsilon}_1/2}$, $\tilde{\varepsilon}_1 \in \{\pm 1\}$, $\varphi_o(\tau_o) = \varphi_n(\tau_n)$, $\tau_o = \tau_n\mathrm{e}^{-\mathrm{i}\pi\tilde{\varepsilon}_1/2}$, $a_o = a_n$, $\varepsilon_o = \varepsilon_n$, and $b_o = b_n\mathrm{e}^{-\mathrm{i}\pi\tilde{\varepsilon}_2}$, $\tilde{\varepsilon}_2 \in \{\pm 1\}$ $\big($that is, $\varepsilon_o b_o = \varepsilon_n b_n\mathrm{e}^{-\mathrm{i}\pi\tilde{\varepsilon}_2}\big)$, and

$$(A_o(\tau_o), B_o(\tau_o), C_o(\tau_o), D_o(\tau_o)) = \big(A_n(\tau_n)\mathrm{e}^{\mathrm{i}\pi\tilde{\varepsilon}_1}, B_n(\tau_n)\mathrm{e}^{\mathrm{i}\pi\tilde{\varepsilon}_1}, C_n(\tau_n)\mathrm{e}^{\mathrm{i}\pi\tilde{\varepsilon}_1/2}, D_n(\tau_n)\mathrm{e}^{\mathrm{i}\pi\tilde{\varepsilon}_1/2}\big);$$

then, $(u_n(\tau_n), \varphi_n(\tau_n))$ solves the system (1.37) for $\tau = \tau_n$, $\varepsilon = \varepsilon_n \in \{\pm 1\}$, $a = a_n$, and $b = b_n$, and the 4-tuple of functions $(A_n(\tau_n), B_n(\tau_n), C_n(\tau_n), D_n(\tau_n))$, defined via equations (1.38) for $u(\tau) = u_n(\tau_n)$, $\varphi(\tau) = \varphi_n(\tau_n)$, $\tau = \tau_n$, and $\varepsilon = \varepsilon_n$, solve the system (1.36) for $\tau = \tau_n$, $a = a_n$, and

$$\sqrt{-A_o(\tau_o)B_o(\tau_o)} = \mathrm{e}^{\mathrm{i}\pi\tilde{\varepsilon}_1}\sqrt{-A_n(\tau_n)B_n(\tau_n)}.$$

Furthermore, let the functions $A_o(\tau_o)$, $B_o(\tau_o)$, $C_o(\tau_o)$, and $D_o(\tau_o)$ be the ones appearing in the definition (1.35) of $\alpha(\tau)$ for $\tau = \tau_o$ and $a = a_o$, and in the first integral (cf. Remark 1.8) for $\varepsilon = \varepsilon_o \in \{\pm 1\}$ and $b = b_o$; then, under the above transformations, $\alpha_o(\tau_o) = \alpha_n(\tau_n)$, where

$$\alpha_n(\tau_n) := -2(B_n(\tau_n))^{-1}\Big(\mathrm{i}a_n\sqrt{-A_n(\tau_n)B_n(\tau_n)} + \tau_n(A_n(\tau_n)D_n(\tau_n) + B_n(\tau_n)C_n(\tau_n))\Big),$$

and $-\mathrm{i}\alpha_n(\tau_n)B_n(\tau_n) = \varepsilon_n b_n$, $\varepsilon_n \in \{\pm 1\}$. On the corresponding fundamental solution of the system (1.32) (cf. equations (1.33) and (1.34)), the aforementioned symmetry transformations act as follows:

$$\mu_o = \mu_n\mathrm{e}^{\mathrm{i}\pi\tilde{\varepsilon}_1/4}, \qquad \tilde{\varepsilon}_1 \in \{\pm 1\}, \qquad \text{and} \qquad \Psi_o(\mu_o, \tau_o) = \mathrm{e}^{-\frac{\mathrm{i}\pi\tilde{\varepsilon}_1}{8}\sigma_3}\Psi_n(\mu_n, \tau_n). \tag{D.26}$$

In terms of the canonical solutions of the system (1.32), the actions (D.26) read: for $k \in \mathbb{Z}$ and $\tilde{\varepsilon}_1, \tilde{\varepsilon}_2 \in \{\pm 1\}$,

$$\mathbb{Y}^{\infty}_{o,k}(\mu_o) = \mathrm{e}^{-\frac{\mathrm{i}\pi\tilde{\varepsilon}_1}{8}\sigma_3}\mathbb{Y}^{\infty}_{n,k}(\mu_n)\mathrm{e}^{\frac{a_n\pi\tilde{\varepsilon}_1}{4}\sigma_3}, \tag{D.27}$$

and

$$\mathbb{X}^{0}_{o,k}(\mu_o) = \begin{cases} \mathrm{e}^{-\frac{\mathrm{i}\pi\tilde{\varepsilon}_1}{8}\sigma_3}\mathbb{X}^{0}_{n,k}(\mu_n), & \tilde{\varepsilon}_1 = -\tilde{\varepsilon}_2, \\ -\mathrm{i}\tilde{\varepsilon}_1\mathrm{e}^{-\frac{\mathrm{i}\pi\tilde{\varepsilon}_1}{8}\sigma_3}\mathbb{X}^{0}_{n,k-\tilde{\varepsilon}_1}(\mu_n)\sigma_1, & \tilde{\varepsilon}_1 = \tilde{\varepsilon}_2. \end{cases} \tag{D.28}$$

The transformations (D.27) and (D.28) for the canonical solutions of the system (1.32) imply the following action on $\mathcal{M}$: for $k \in \mathbb{Z}$ and $\tilde{\varepsilon}_1, \tilde{\varepsilon}_2 \in \{\pm 1\}$,

$$S^{\infty}_{o,k} = \mathrm{e}^{-\frac{a_n\pi\tilde{\varepsilon}_1}{4}\sigma_3}S^{\infty}_{n,k}\mathrm{e}^{\frac{a_n\pi\tilde{\varepsilon}_1}{4}\sigma_3}, \tag{D.29}$$

$$S^{0}_{o,k} = \begin{cases} S^{0}_{n,k}, & \tilde{\varepsilon}_1 = -\tilde{\varepsilon}_2, \\ \sigma_1 S^{0}_{n,k-\tilde{\varepsilon}_1}\sigma_1, & \tilde{\varepsilon}_1 = \tilde{\varepsilon}_2, \end{cases} \tag{D.30}$$

$$G_o = \begin{cases} \mathrm{i}S^{0}_{o,0}\sigma_1 G_n\mathrm{e}^{\frac{a_n\pi}{4}\sigma_3}, & (\tilde{\varepsilon}_1, \tilde{\varepsilon}_2) = (1,1), \\ G_n\mathrm{e}^{\frac{a_n\pi}{4}\sigma_3}, & (\tilde{\varepsilon}_1, \tilde{\varepsilon}_2) = (1,-1), \\ G_n\mathrm{e}^{-\frac{a_n\pi}{4}\sigma_3}, & (\tilde{\varepsilon}_1, \tilde{\varepsilon}_2) = (-1,1), \\ -\mathrm{i}\sigma_1\big(S^{0}_{o,0}\big)^{-1}G_n\mathrm{e}^{-\frac{a_n\pi}{4}\sigma_3}, & (\tilde{\varepsilon}_1, \tilde{\varepsilon}_2) = (-1,-1). \end{cases} \tag{D.31}$$

The actions (D.29)–(D.31) on $\mathcal{M}$ can be expressed in terms of an intermediate auxiliary mapping $\mathcal{F}^{\leftrightsquigarrow}_{\mathcal{M}}(\tilde{\varepsilon}_1, \tilde{\varepsilon}_2)\colon \mathbb{C}^8 \to \mathbb{C}^8$, $\tilde{\varepsilon}_1, \tilde{\varepsilon}_2 \in \{\pm 1\}$, which is an isomorphism on $\mathcal{M}$

$$\begin{aligned} \mathcal{F}^{\leftrightsquigarrow}_{\mathcal{M}}(\tilde{\varepsilon}_1, \tilde{\varepsilon}_2)\colon \mathcal{M} \to \mathcal{M},\ & \big(a, s^0_0, s^{\infty}_0, s^{\infty}_1, g_{11}, g_{12}, g_{21}, g_{22}\big) \\ & \mapsto \big(a, s^0_0(\tilde{\varepsilon}_1, \tilde{\varepsilon}_2), s^{\infty}_0(\tilde{\varepsilon}_1, \tilde{\varepsilon}_2), s^{\infty}_1(\tilde{\varepsilon}_1, \tilde{\varepsilon}_2), g_{11}(\tilde{\varepsilon}_1, \tilde{\varepsilon}_2), g_{12}(\tilde{\varepsilon}_1, \tilde{\varepsilon}_2), \\ & \quad g_{21}(\tilde{\varepsilon}_1, \tilde{\varepsilon}_2), g_{22}(\tilde{\varepsilon}_1, \tilde{\varepsilon}_2)\big), \end{aligned}$$

where, for $(\tilde{\varepsilon}_1, \tilde{\varepsilon}_2) = (1,1)$,

$$\begin{aligned} & s^0_0(1,1) = s^0_0, \qquad s^{\infty}_0(1,1) = s^{\infty}_0\mathrm{e}^{-\pi a/2}, \qquad s^{\infty}_1(1,1) = s^{\infty}_1\mathrm{e}^{\pi a/2}, \\ & g_{11}(1,1) = -\mathrm{i}g_{21}\mathrm{e}^{-\pi a/4}, \qquad g_{12}(1,1) = -\mathrm{i}g_{22}\mathrm{e}^{\pi a/4}, \\ & g_{21}(1,1) = -\mathrm{i}\big(g_{11} - s^0_0 g_{21}\big)\mathrm{e}^{-\pi a/4}, \qquad g_{22}(1,1) = -\mathrm{i}\big(g_{12} - s^0_0 g_{22}\big)\mathrm{e}^{\pi a/4}, \end{aligned} \tag{D.32}$$

for $(\tilde{\varepsilon}_1, \tilde{\varepsilon}_2) = (1,-1)$,

$$\begin{aligned} & s^0_0(1,-1) = s^0_0, \qquad s^{\infty}_0(1,-1) = s^{\infty}_0\mathrm{e}^{-\pi a/2}, \qquad s^{\infty}_1(1,-1) = s^{\infty}_1\mathrm{e}^{\pi a/2}, \\ & g_{11}(1,-1) = g_{11}\mathrm{e}^{-\pi a/4}, \qquad g_{12}(1,-1) = g_{12}\mathrm{e}^{\pi a/4}, \qquad g_{21}(1,-1) = g_{21}\mathrm{e}^{-\pi a/4}, \\ & g_{22}(1,-1) = g_{22}\mathrm{e}^{\pi a/4}, \end{aligned} \tag{D.33}$$

for $(\tilde{\varepsilon}_1, \tilde{\varepsilon}_2) = (-1,1)$,

$$\begin{aligned} & s^0_0(-1,1) = s^0_0, \qquad s^{\infty}_0(-1,1) = s^{\infty}_0\mathrm{e}^{\pi a/2}, \qquad s^{\infty}_1(-1,1) = s^{\infty}_1\mathrm{e}^{-\pi a/2}, \\ & g_{11}(-1,1) = g_{11}\mathrm{e}^{\pi a/4}, \qquad g_{12}(-1,1) = g_{12}\mathrm{e}^{-\pi a/4}, \qquad g_{21}(-1,1) = g_{21}\mathrm{e}^{\pi a/4}, \\ & g_{22}(-1,1) = g_{22}\mathrm{e}^{-\pi a/4}, \end{aligned} \tag{D.34}$$

and, for $(\tilde{\varepsilon}_1, \tilde{\varepsilon}_2) = (-1,-1)$,

$$\begin{aligned} & s^0_0(-1,-1) = s^0_0, \qquad s^{\infty}_0(-1,-1) = s^{\infty}_0\mathrm{e}^{\pi a/2}, \qquad s^{\infty}_1(-1,-1) = s^{\infty}_1\mathrm{e}^{-\pi a/2}, \\ & g_{11}(-1,-1) = \mathrm{i}\big(g_{21} + s^0_0 g_{11}\big)\mathrm{e}^{\pi a/4}, \qquad g_{12}(-1,-1) = \mathrm{i}\big(g_{22} + s^0_0 g_{12}\big)\mathrm{e}^{-\pi a/4}, \\ & g_{21}(-1,-1) = \mathrm{i}g_{11}\mathrm{e}^{\pi a/4}, \qquad g_{22}(-1,-1) = \mathrm{i}g_{12}\mathrm{e}^{-\pi a/4}. \end{aligned} \tag{D.35}$$

One uses this transformation in order to arrive at asymptotics for pure-imaginary τ by using those for real τ.

D.5 Composed symmetries and asymptotics

In order to derive the complete set of requisite transformations, one considers the actions (D.7), (D.8), (D.15), (D.22)–(D.25), and (D.32)–(D.35) as a group of basis symmetries, the compositions of whose elements yield the remaining isomorphisms on $\mathcal{M}$.

In order to do so, however, additional notation is necessary. For symmetries related to real τ, introduce the auxiliary parameters

$$\varepsilon_1 \in \{0, \pm 1\}, \qquad \varepsilon_2 \in \{0, \pm 1\}, \qquad m(\varepsilon_2) = \begin{cases} 0, & \varepsilon_2 = 0, \\ \pm\varepsilon_2, & \varepsilon_2 \in \{\pm 1\}, \end{cases}$$

and $\ell \in \{0, 1\}$, and consider the 4-tuple $(\varepsilon_1, \varepsilon_2, m(\varepsilon_2)|\ell)$ concomitant with its associated isomorphism(s) on $\mathcal{M}$ denoted by $\mathcal{F}^{\{\ell\}}_{\varepsilon_1,\varepsilon_2,m(\varepsilon_2)} \colon \mathbb{C}^8 \to \mathbb{C}^8$, where

$$\begin{aligned} \mathcal{F}^{\{\ell\}}_{\varepsilon_1,\varepsilon_2,m(\varepsilon_2)} \colon\ & \mathcal{M} \to \mathcal{M}, \big(a, s_0^0, s_0^\infty, s_1^\infty, g_{11}, g_{12}, g_{21}, g_{22}\big) \\ & \mapsto \big((-1)^{\varepsilon_2} a, s_0^0(\varepsilon_1, \varepsilon_2, m(\varepsilon_2)|\ell), s_0^\infty(\varepsilon_1, \varepsilon_2, m(\varepsilon_2)|\ell), s_1^\infty(\varepsilon_1, \varepsilon_2, m(\varepsilon_2)|\ell), \\ & \quad g_{11}(\varepsilon_1, \varepsilon_2, m(\varepsilon_2)|\ell), g_{12}(\varepsilon_1, \varepsilon_2, m(\varepsilon_2)|\ell), g_{21}(\varepsilon_1, \varepsilon_2, m(\varepsilon_2)|\ell), \\ & \quad g_{22}(\varepsilon_1, \varepsilon_2, m(\varepsilon_2)|\ell)\big); \end{aligned} \tag{D.36}$$

and, for symmetries related to pure-imaginary τ, introduce the auxiliary parameters

$$\hat{\varepsilon}_1 \in \{\pm 1\}, \qquad \hat{\varepsilon}_2 \in \{0, \pm 1\}, \qquad \hat{m}(\hat{\varepsilon}_2) = \begin{cases} 0, & \hat{\varepsilon}_2 \in \{\pm 1\}, \\ \pm\hat{\varepsilon}_1, & \hat{\varepsilon}_2 = 0, \end{cases}$$

and $\hat{\ell} \in \{0, 1\}$, and consider the 4-tuple $\big(\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\big)$ concomitant with its associated isomorphism(s) on $\mathcal{M}$ denoted by $\hat{\mathcal{F}}^{\{\hat{\ell}\}}_{\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)} \colon \mathbb{C}^8 \to \mathbb{C}^8$, where

$$\begin{aligned} \hat{\mathcal{F}}^{\{\hat{\ell}\}}_{\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)} \colon\ & \mathcal{M} \to \mathcal{M}, \big(a, s_0^0, s_0^\infty, s_1^\infty, g_{11}, g_{12}, g_{21}, g_{22}\big) \\ & \mapsto \big((-1)^{1+\hat{\varepsilon}_2} a, \hat{s}_0^0\big(\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\big), \hat{s}_0^\infty\big(\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\big), \\ & \quad \hat{s}_1^\infty\big(\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\big), \hat{g}_{11}\big(\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\big), \hat{g}_{12}\big(\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\big), \\ & \quad \hat{g}_{21}\big(\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\big), \hat{g}_{22}\big(\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\big)\big). \end{aligned} \tag{D.37}$$

Let

$$\mathcal{F}^{\{0\}}_{0,0,0} \colon\ \mathcal{M} \to \mathcal{M}, \big(a, s_0^0, s_0^\infty, s_1^\infty, g_{11}, g_{12}, g_{21}, g_{22}\big) \mapsto \big(a, s_0^0, s_0^\infty, s_1^\infty, g_{11}, g_{12}, g_{21}, g_{22}\big) \tag{D.38}$$

denote the *identity map*,[57] and, for $\ell = 0$, set

$$\mathcal{F}^{\{0\}}_{\varepsilon_1,\varepsilon_2,m(\varepsilon_2)} := \begin{cases} \mathcal{F}^{\rhd}_{\mathcal{M}}(1), & (\varepsilon_1, \varepsilon_2, m(\varepsilon_2)|\ell) = (1, 0, 0|0), \\ \mathcal{F}^{\rhd}_{\mathcal{M}}(-1), & (\varepsilon_1, \varepsilon_2, m(\varepsilon_2)|\ell) = (-1, 0, 0|0), \\ \mathcal{F}^{\leftrightarrow}_{\mathcal{M}}(1, 1), & (\varepsilon_1, \varepsilon_2, m(\varepsilon_2)|\ell) = (0, 1, 1|0), \\ \mathcal{F}^{\leftrightarrow}_{\mathcal{M}}(1, -1), & (\varepsilon_1, \varepsilon_2, m(\varepsilon_2)|\ell) = (0, -1, 1|0), \\ \mathcal{F}^{\leftrightarrow}_{\mathcal{M}}(-1, 1), & (\varepsilon_1, \varepsilon_2, m(\varepsilon_2)|\ell) = (0, 1, -1|0), \\ \mathcal{F}^{\leftrightarrow}_{\mathcal{M}}(-1, -1), & (\varepsilon_1, \varepsilon_2, m(\varepsilon_2)|\ell) = (0, -1, -1|0), \end{cases} \tag{D.39}$$

[57] That is, $s_0^0(0,0,0|0) = s_0^0$, $s_0^\infty(0,0,0|0) = s_0^\infty$, $s_1^\infty(0,0,0|0) = s_1^\infty$, and $g_{ij}(0,0,0|0) = g_{ij}$, $i, j \in \{1, 2\}$.

and, for $\hat{\ell}=0$, set

$$\hat{\mathcal{F}}^{\{0\}}_{\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)} := \begin{cases} \mathcal{F}^{\leftrightsquigarrow}_{\mathcal{M}}(1,1), & \big(\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\big)=(1,1,0|0),\\ \mathcal{F}^{\leftrightsquigarrow}_{\mathcal{M}}(1,-1), & \big(\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\big)=(1,-1,0|0),\\ \mathcal{F}^{\leftrightsquigarrow}_{\mathcal{M}}(-1,1), & \big(\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\big)=(-1,1,0|0),\\ \mathcal{F}^{\leftrightsquigarrow}_{\mathcal{M}}(-1,-1), & \big(\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\big)=(-1,-1,0|0).\end{cases} \tag{D.40}$$

Via the definitions (D.38)–(D.40), define the following compositions (isomorphisms on $\mathcal{M}$): for $\ell=0$,[58] set

$$\begin{aligned}
&\mathcal{F}^{\{0\}}_{-1,-1,-1} := \mathcal{F}^{\{0\}}_{0,-1,-1}\circ\mathcal{F}^{\{0\}}_{-1,0,0}, \qquad (\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)=(-1,-1,-1|0),\\
&\mathcal{F}^{\{0\}}_{1,-1,-1} := \mathcal{F}^{\{0\}}_{0,-1,-1}\circ\mathcal{F}^{\{0\}}_{1,0,0}, \qquad (\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)=(1,-1,-1|0),\\
&\mathcal{F}^{\{0\}}_{-1,-1,1} := \mathcal{F}^{\{0\}}_{0,-1,1}\circ\mathcal{F}^{\{0\}}_{-1,0,0}, \qquad (\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)=(-1,-1,1|0),\\
&\mathcal{F}^{\{0\}}_{1,-1,1} := \mathcal{F}^{\{0\}}_{0,-1,1}\circ\mathcal{F}^{\{0\}}_{1,0,0}, \qquad (\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)=(1,-1,1|0),\\
&\mathcal{F}^{\{0\}}_{-1,1,-1} := \mathcal{F}^{\{0\}}_{0,1,-1}\circ\mathcal{F}^{\{0\}}_{-1,0,0}, \qquad (\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)=(-1,1,-1|0),\\
&\mathcal{F}^{\{0\}}_{1,1,-1} := \mathcal{F}^{\{0\}}_{0,1,-1}\circ\mathcal{F}^{\{0\}}_{1,0,0}, \qquad (\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)=(1,1,-1|0),\\
&\mathcal{F}^{\{0\}}_{-1,1,1} := \mathcal{F}^{\{0\}}_{0,1,1}\circ\mathcal{F}^{\{0\}}_{-1,0,0}, \qquad (\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)=(-1,1,1|0),\\
&\mathcal{F}^{\{0\}}_{1,1,1} := \mathcal{F}^{\{0\}}_{0,1,1}\circ\mathcal{F}^{\{0\}}_{1,0,0}, \qquad (\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)=(1,1,1|0),
\end{aligned} \tag{D.41}$$

and, for $\hat{\ell}=0$, set

$$\begin{aligned}
&\hat{\mathcal{F}}^{\{0\}}_{1,0,-1} := \mathcal{F}^{\{0\}}_{0,-1,-1}\circ\hat{\mathcal{F}}^{\{0\}}_{1,1,0}, \qquad \big(\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\big)=(1,0,-1|0),\\
&\hat{\mathcal{F}}^{\{0\}}_{-1,0,-1} := \mathcal{F}^{\{0\}}_{0,-1,-1}\circ\hat{\mathcal{F}}^{\{0\}}_{-1,1,0}, \qquad \big(\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\big)=(-1,0,-1|0),\\
&\hat{\mathcal{F}}^{\{0\}}_{1,0,1} := \mathcal{F}^{\{0\}}_{0,1,1}\circ\hat{\mathcal{F}}^{\{0\}}_{1,-1,0}, \qquad \big(\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\big)=(1,0,1|0),\\
&\hat{\mathcal{F}}^{\{0\}}_{-1,0,1} := \mathcal{F}^{\{0\}}_{0,1,1}\circ\hat{\mathcal{F}}^{\{0\}}_{-1,-1,0}, \qquad \big(\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\big)=(-1,0,1|0).
\end{aligned} \tag{D.42}$$

The cases $\ell,\hat{\ell}=1$ are a bit more subtle, because there is no analogue, *per se*, of the (standard) identity map (D.38); rather, the rôle of the identity map for $\ell,\hat{\ell}=1$ is mimicked by the endomorphism $\mathcal{F}^{\lhd}_{\mathcal{M}}(\tilde{l})$, $\tilde{l}\in\{\pm1\}$, given in Appendix D.2 (cf. equations (D.15)); with conspicuous changes in notation (which are in line with the notations introduced in this subsection), it reads (for $\ell=1$):

$$\begin{aligned}
\mathcal{F}^{\{1\}}_{0,0,0}\colon\ \mathcal{M}\to\mathcal{M},\ &\big(a,s^0_0,s^\infty_0,s^\infty_1,g_{11},g_{12},g_{21},g_{22}\big)\\
&\mapsto \big(a,s^0_0(0,0,0|1),s^\infty_0(0,0,0|1),s^\infty_1(0,0,0|1),g_{11}(0,0,0|1),g_{12}(0,0,0|1),\\
&\quad g_{21}(0,0,0|1),g_{22}(0,0,0|1)\big),
\end{aligned} \tag{D.43}$$

where, for $\tilde{l}\in\{\pm1\}$,

$$\begin{aligned}
&s^0_0(0,0,0|1) := s^0_0\big(\tilde{l}\big), \qquad s^\infty_0(0,0,0|1) := s^\infty_0\big(\tilde{l}\big), \qquad s^\infty_1(0,0,0|1) := s^\infty_1\big(\tilde{l}\big),\\
&g_{ij}(0,0,0|1) := g_{ij}\big(\tilde{l}\big), \qquad i,j\in\{1,2\}.
\end{aligned} \tag{D.44}$$

To complete the list of the remaining $\ell,\hat{\ell}=1$ mappings, define, in analogy with the definitions (D.39)–(D.42), the following compositions (isomorphisms) on $\mathcal{M}$: for $\ell=1$,

$$\mathcal{F}^{\{1\}}_{-1,0,0} := \mathcal{F}^{\{0\}}_{-1,0,0}\circ\mathcal{F}^{\{1\}}_{0,0,0}, \qquad (\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)=(-1,0,0|1),$$

[58] Recall from Remarks 1.16 and 1.17 that $G_1\equiv G_2 \Leftrightarrow (G_1)_{ij}=-(G_2)_{ij}$, $i,j\in\{1,2\}$.

$$\begin{aligned}
&\mathcal{F}^{\{1\}}_{1,0,0} := \mathcal{F}^{\{0\}}_{1,0,0} \circ \mathcal{F}^{\{1\}}_{0,0,0}, \qquad (\varepsilon_1, \varepsilon_2, m(\varepsilon_2)|\ell) = (1,0,0|1),\\
&\mathcal{F}^{\{1\}}_{0,-1,-1} := \mathcal{F}^{\{0\}}_{0,-1,-1} \circ \mathcal{F}^{\{1\}}_{0,0,0}, \qquad (\varepsilon_1, \varepsilon_2, m(\varepsilon_2)|\ell) = (0,-1,-1|1),\\
&\mathcal{F}^{\{1\}}_{0,-1,1} := \mathcal{F}^{\{0\}}_{0,-1,1} \circ \mathcal{F}^{\{1\}}_{0,0,0}, \qquad (\varepsilon_1, \varepsilon_2, m(\varepsilon_2)|\ell) = (0,-1,1|1),\\
&\mathcal{F}^{\{1\}}_{0,1,-1} := \mathcal{F}^{\{0\}}_{0,1,-1} \circ \mathcal{F}^{\{1\}}_{0,0,0}, \qquad (\varepsilon_1, \varepsilon_2, m(\varepsilon_2)|\ell) = (0,1,-1|1),\\
&\mathcal{F}^{\{1\}}_{0,1,1} := \mathcal{F}^{\{0\}}_{0,1,1} \circ \mathcal{F}^{\{1\}}_{0,0,0}, \qquad (\varepsilon_1, \varepsilon_2, m(\varepsilon_2)|\ell) = (0,1,1|1),\\
&\mathcal{F}^{\{1\}}_{-1,-1,-1} := \mathcal{F}^{\{1\}}_{0,-1,-1} \circ \mathcal{F}^{\{0\}}_{-1,0,0}, \qquad (\varepsilon_1, \varepsilon_2, m(\varepsilon_2)|\ell) = (-1,-1,-1|1),\\
&\mathcal{F}^{\{1\}}_{1,-1,-1} := \mathcal{F}^{\{1\}}_{0,-1,-1} \circ \mathcal{F}^{\{0\}}_{1,0,0}, \qquad (\varepsilon_1, \varepsilon_2, m(\varepsilon_2)|\ell) = (1,-1,-1|1),\\
&\mathcal{F}^{\{1\}}_{-1,-1,1} := \mathcal{F}^{\{1\}}_{0,-1,1} \circ \mathcal{F}^{\{0\}}_{-1,0,0}, \qquad (\varepsilon_1, \varepsilon_2, m(\varepsilon_2)|\ell) = (-1,-1,1|1),\\
&\mathcal{F}^{\{1\}}_{1,-1,1} := \mathcal{F}^{\{1\}}_{0,-1,1} \circ \mathcal{F}^{\{0\}}_{1,0,0}, \qquad (\varepsilon_1, \varepsilon_2, m(\varepsilon_2)|\ell) = (1,-1,1|1),\\
&\mathcal{F}^{\{1\}}_{-1,1,-1} := \mathcal{F}^{\{1\}}_{0,1,-1} \circ \mathcal{F}^{\{0\}}_{-1,0,0}, \qquad (\varepsilon_1, \varepsilon_2, m(\varepsilon_2)|\ell) = (-1,1,-1|1),\\
&\mathcal{F}^{\{1\}}_{1,1,-1} := \mathcal{F}^{\{1\}}_{0,1,-1} \circ \mathcal{F}^{\{0\}}_{1,0,0}, \qquad (\varepsilon_1, \varepsilon_2, m(\varepsilon_2)|\ell) = (1,1,-1|1),\\
&\mathcal{F}^{\{1\}}_{-1,1,1} := \mathcal{F}^{\{1\}}_{0,1,1} \circ \mathcal{F}^{\{0\}}_{-1,0,0}, \qquad (\varepsilon_1, \varepsilon_2, m(\varepsilon_2)|\ell) = (-1,1,1|1),\\
&\mathcal{F}^{\{1\}}_{1,1,1} := \mathcal{F}^{\{1\}}_{0,1,1} \circ \mathcal{F}^{\{0\}}_{1,0,0}, \qquad (\varepsilon_1, \varepsilon_2, m(\varepsilon_2)|\ell) = (1,1,1|1);
\end{aligned} \tag{D.45}$$

and, for $\hat{\ell} = 1$,

$$\begin{aligned}
&\hat{\mathcal{F}}^{\{1\}}_{1,1,0} := \hat{\mathcal{F}}^{\{0\}}_{1,1,0} \circ \mathcal{F}^{\{1\}}_{0,0,0}, \qquad (\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{m}(\hat{\varepsilon}_2)|\hat{\ell}) = (1,1,0|1),\\
&\hat{\mathcal{F}}^{\{1\}}_{1,-1,0} := \hat{\mathcal{F}}^{\{0\}}_{1,-1,0} \circ \mathcal{F}^{\{1\}}_{0,0,0}, \qquad (\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{m}(\hat{\varepsilon}_2)|\hat{\ell}) = (1,-1,0|1),\\
&\hat{\mathcal{F}}^{\{1\}}_{-1,1,0} := \hat{\mathcal{F}}^{\{0\}}_{-1,1,0} \circ \mathcal{F}^{\{1\}}_{0,0,0}, \qquad (\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{m}(\hat{\varepsilon}_2)|\hat{\ell}) = (-1,1,0|1),\\
&\hat{\mathcal{F}}^{\{1\}}_{-1,-1,0} := \hat{\mathcal{F}}^{\{0\}}_{-1,-1,0} \circ \mathcal{F}^{\{1\}}_{0,0,0}, \qquad (\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{m}(\hat{\varepsilon}_2)|\hat{\ell}) = (-1,-1,0|1),\\
&\hat{\mathcal{F}}^{\{1\}}_{1,0,-1} := \mathcal{F}^{\{0\}}_{0,1,-1} \circ \hat{\mathcal{F}}^{\{1\}}_{1,-1,0}, \qquad (\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{m}(\hat{\varepsilon}_2)|\hat{\ell}) = (1,0,-1|1),\\
&\hat{\mathcal{F}}^{\{1\}}_{1,0,1} := \mathcal{F}^{\{0\}}_{0,1,1} \circ \hat{\mathcal{F}}^{\{1\}}_{1,-1,0}, \qquad (\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{m}(\hat{\varepsilon}_2)|\hat{\ell}) = (1,0,1|1),\\
&\hat{\mathcal{F}}^{\{1\}}_{-1,0,-1} := \mathcal{F}^{\{0\}}_{0,1,-1} \circ \hat{\mathcal{F}}^{\{1\}}_{-1,-1,0}, \qquad (\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{m}(\hat{\varepsilon}_2)|\hat{\ell}) = (-1,0,-1|1),\\
&\hat{\mathcal{F}}^{\{1\}}_{-1,0,1} := \mathcal{F}^{\{0\}}_{0,1,1} \circ \hat{\mathcal{F}}^{\{1\}}_{-1,-1,0}, \qquad (\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{m}(\hat{\varepsilon}_2)|\hat{\ell}) = (-1,0,1|1).
\end{aligned} \tag{D.46}$$

Via the elementary symmetries (D.7), (D.8), (D.15), (D.22)–(D.25), and (D.32)–(D.35), and the definitions (D.38)–(D.46), one arrives at the following explicit list of actions on $\mathcal{M}$ of the isomorphisms (cf. definition (D.36)) $\mathcal{F}^{\{\ell\}}_{\varepsilon_1,\varepsilon_2,m(\varepsilon_2)}$, relevant for real τ, and (cf. definition (D.37)) $\hat{\mathcal{F}}^{\{\hat{\ell}\}}_{\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)}$, relevant for pure-imaginary τ: for $\tilde{l}, l' \in \{\pm 1\}$,

(1) $\mathcal{F}^{\{0\}}_{0,0,0} \Rightarrow$

$$\begin{aligned}
&s_0^0(0,0,0|0) = s_0^0, \qquad s_0^{\infty}(0,0,0|0) = s_0^{\infty}, \qquad s_1^{\infty}(0,0,0|0) = s_1^{\infty},\\
&g_{ij}(0,0,0|0) = g_{ij}, \qquad i,j \in \{1,2\};
\end{aligned} \tag{D.47}$$

(2) $\mathcal{F}^{\{0\}}_{-1,0,0} \Rightarrow$

$$\begin{aligned}
&s_0^0(-1,0,0|0) = s_0^0, \qquad s_0^{\infty}(-1,0,0|0) = s_0^{\infty}\mathrm{e}^{\pi a}, \qquad s_1^{\infty}(-1,0,0|0) = s_1^{\infty}\mathrm{e}^{-\pi a},\\
&g_{11}(-1,0,0|0) = -\mathrm{i}\big(g_{21} + s_0^0 g_{11}\big)\mathrm{e}^{\pi a/2}, \qquad g_{12}(-1,0,0|0) = -\mathrm{i}\big(g_{22} + s_0^0 g_{12}\big)\mathrm{e}^{-\pi a/2},\\
&g_{21}(-1,0,0|0) = -\mathrm{i}g_{11}\mathrm{e}^{\pi a/2}, \qquad g_{22}(-1,0,0|0) = -\mathrm{i}g_{12}\mathrm{e}^{-\pi a/2};
\end{aligned} \tag{D.48}$$

(3) $\mathcal{F}^{\{0\}}_{1,0,0} \Rightarrow$

$$\begin{aligned}
&s^0_0(1,0,0|0)=s^0_0, \qquad s^{\infty}_0(1,0,0|0)=s^{\infty}_0\mathrm{e}^{-\pi a}, \qquad s^{\infty}_1(1,0,0|0)=s^{\infty}_1\mathrm{e}^{\pi a},\\
&g_{11}(1,0,0|0)=\mathrm{i}g_{21}\mathrm{e}^{-\pi a/2}, \qquad g_{12}(1,0,0|0)=\mathrm{i}g_{22}\mathrm{e}^{\pi a/2},\\
&g_{21}(1,0,0|0)=\mathrm{i}\big(g_{11}-s^0_0g_{21}\big)\mathrm{e}^{-\pi a/2}, \qquad g_{22}(1,0,0|0)=\mathrm{i}\big(g_{12}-s^0_0g_{22}\big)\mathrm{e}^{\pi a/2};
\end{aligned} \tag{D.49}$$

(4) $\mathcal{F}^{\{0\}}_{0,-1,-1} \Rightarrow$

$$\begin{aligned}
&s^0_0(0,-1,-1|0)=s^0_0, \qquad s^{\infty}_0(0,-1,-1|0)=-s^{\infty}_1\mathrm{e}^{\pi a},\\
&s^{\infty}_1(0,-1,-1|0)=-s^{\infty}_0\mathrm{e}^{\pi a},\\
&g_{11}(0,-1,-1|0)=\mathrm{i}l'\big(g_{22}-g_{21}s^{\infty}_1\mathrm{e}^{2\pi a}+s^0_0\big(g_{12}-g_{11}s^{\infty}_1\mathrm{e}^{2\pi a}\big)\big)\mathrm{e}^{-\pi a/2},\\
&g_{12}(0,-1,-1|0)=-\mathrm{i}l'\big(g_{21}+s^0_0g_{11}\big)\mathrm{e}^{\pi a/2},\\
&g_{21}(0,-1,-1|0)=\mathrm{i}l'\big(g_{12}-g_{11}s^{\infty}_1\mathrm{e}^{2\pi a}\big)\mathrm{e}^{-\pi a/2},\\
&g_{22}(0,-1,-1|0)=-\mathrm{i}l'g_{11}\mathrm{e}^{\pi a/2};
\end{aligned} \tag{D.50}$$

(5) $\mathcal{F}^{\{0\}}_{0,-1,1} \Rightarrow$

$$\begin{aligned}
&s^0_0(0,-1,1|0)=s^0_0, \qquad s^{\infty}_0(0,-1,1|0)=-s^{\infty}_1\mathrm{e}^{\pi a}, \qquad s^{\infty}_1(0,-1,1|0)=-s^{\infty}_0\mathrm{e}^{\pi a},\\
&g_{11}(0,-1,1|0)=l'g_{12}\mathrm{e}^{\pi a/2}, \qquad g_{12}(0,-1,1|0)=-l'(g_{11}+s^{\infty}_0g_{12})\mathrm{e}^{-\pi a/2},\\
&g_{21}(0,-1,1|0)=l'g_{22}\mathrm{e}^{\pi a/2}, \qquad g_{22}(0,-1,1|0)=-l'(g_{21}+s^{\infty}_0g_{22})\mathrm{e}^{-\pi a/2};
\end{aligned} \tag{D.51}$$

(6) $\mathcal{F}^{\{0\}}_{0,1,-1} \Rightarrow$

$$\begin{aligned}
&s^0_0(0,1,-1|0)=s^0_0, \qquad s^{\infty}_0(0,1,-1|0)=-s^{\infty}_1\mathrm{e}^{\pi a}, \qquad s^{\infty}_1(0,1,-1|0)=-s^{\infty}_0\mathrm{e}^{\pi a},\\
&g_{11}(0,1,-1|0)=l'(g_{12}-g_{11}s^{\infty}_1\mathrm{e}^{2\pi a})\mathrm{e}^{-\pi a/2}, \qquad g_{12}(0,1,-1|0)=-l'g_{11}\mathrm{e}^{\pi a/2},\\
&g_{21}(0,1,-1|0)=l'(g_{22}-g_{21}s^{\infty}_1\mathrm{e}^{2\pi a})\mathrm{e}^{-\pi a/2}, \qquad g_{22}(0,1,-1|0)=-l'g_{21}\mathrm{e}^{\pi a/2};
\end{aligned} \tag{D.52}$$

(7) $\mathcal{F}^{\{0\}}_{0,1,1} \Rightarrow$

$$\begin{aligned}
&s^0_0(0,1,1|0)=s^0_0, \qquad s^{\infty}_0(0,1,1|0)=-s^{\infty}_1\mathrm{e}^{\pi a}, \qquad s^{\infty}_1(0,1,1|0)=-s^{\infty}_0\mathrm{e}^{\pi a},\\
&g_{11}(0,1,1|0)=\mathrm{i}l'g_{22}\mathrm{e}^{\pi a/2}, \qquad g_{12}(0,1,1|0)=-\mathrm{i}l'(g_{21}+s^{\infty}_0g_{22})\mathrm{e}^{-\pi a/2},\\
&g_{21}(0,1,1|0)=\mathrm{i}l'\big(g_{12}-s^0_0g_{22}\big)\mathrm{e}^{\pi a/2},\\
&g_{22}(0,1,1|0)=\mathrm{i}l'\big(-g_{11}-g_{12}s^{\infty}_0+s^0_0(g_{21}+s^{\infty}_0g_{22})\big)\mathrm{e}^{-\pi a/2};
\end{aligned} \tag{D.53}$$

(8) $\mathcal{F}^{\{0\}}_{-1,-1,-1} \Rightarrow$

$$\begin{aligned}
&s^0_0(-1,-1,-1|0)=s^0_0, \qquad s^{\infty}_0(-1,-1,-1|0)=-s^{\infty}_1,\\
&s^{\infty}_1(-1,-1,-1|0)=-s^{\infty}_0\mathrm{e}^{2\pi a},\\
&g_{11}(-1,-1,-1|0)=l'\big(\big(g_{12}-g_{11}s^{\infty}_1\mathrm{e}^{2\pi a}\big)\big(1+\big(s^0_0\big)^2\big)+s^0_0\big(g_{22}-g_{21}s^{\infty}_1\mathrm{e}^{2\pi a}\big)\big)\mathrm{e}^{-\pi a},\\
&g_{12}(-1,-1,-1|0)=-l'\big(g_{11}\big(1+\big(s^0_0\big)^2\big)+s^0_0g_{21}\big)\mathrm{e}^{\pi a},\\
&g_{21}(-1,-1,-1|0)=l'\big(g_{22}-g_{21}s^{\infty}_1\mathrm{e}^{2\pi a}+s^0_0\big(g_{12}-g_{11}s^{\infty}_1\mathrm{e}^{2\pi a}\big)\big)\mathrm{e}^{-\pi a},\\
&g_{22}(-1,-1,-1|0)=-l'\big(g_{21}+s^0_0g_{11}\big)\mathrm{e}^{\pi a};
\end{aligned} \tag{D.54}$$

(9) $\mathcal{F}^{\{0\}}_{1,-1,-1} \Rightarrow$

$$\begin{aligned}
&s^0_0(1,-1,-1|0) = s^0_0, \qquad s^\infty_0(1,-1,-1|0) = -s^\infty_1 \mathrm{e}^{2\pi a}, \qquad s^\infty_1(1,-1,-1|0) = -s^\infty_0,\\
&g_{11}(1,-1,-1|0) = -l'\big(g_{12} - g_{11}s^\infty_1 \mathrm{e}^{2\pi a}\big), \qquad g_{12}(1,-1,-1|0) = l' g_{11},\\
&g_{21}(1,-1,-1|0) = -l'\big(g_{22} - g_{21}s^\infty_1 \mathrm{e}^{2\pi a}\big), \qquad g_{22}(1,-1,-1|0) = l' g_{21};
\end{aligned} \tag{D.55}$$

(10) $\mathcal{F}^{\{0\}}_{-1,-1,1} \Rightarrow$

$$\begin{aligned}
&s^0_0(-1,-1,1|0) = s^0_0, \qquad s^\infty_0(-1,-1,1|0) = -s^\infty_1, \qquad s^\infty_1(-1,-1,1|0) = -s^\infty_0 \mathrm{e}^{2\pi a},\\
&g_{11}(-1,-1,1|0) = -\mathrm{i}l'\big(g_{22} + s^0_0 g_{12}\big),\\
&g_{12}(-1,-1,1|0) = \mathrm{i}l'\big(g_{21} + s^\infty_0 g_{22} + s^0_0(g_{11} + s^\infty_0 g_{12})\big),\\
&g_{21}(-1,-1,1|0) = -\mathrm{i}l' g_{12}, \qquad g_{22}(-1,-1,1|0) = \mathrm{i}l'(g_{11} + s^\infty_0 g_{12});
\end{aligned} \tag{D.56}$$

(11) $\mathcal{F}^{\{0\}}_{1,-1,1} \Rightarrow$

$$\begin{aligned}
&s^0_0(1,-1,1|0) = s^0_0, \qquad s^\infty_0(1,-1,1|0) = -s^\infty_1 \mathrm{e}^{2\pi a}, \qquad s^\infty_1(1,-1,1|0) = -s^\infty_0,\\
&g_{11}(1,-1,1|0) = \mathrm{i}l' g_{22} \mathrm{e}^{\pi a}, \qquad g_{12}(1,-1,1|0) = -\mathrm{i}l'(g_{21} + s^\infty_0 g_{22})\mathrm{e}^{-\pi a},\\
&g_{21}(1,-1,1|0) = \mathrm{i}l'\big(g_{12} - s^0_0 g_{22}\big)\mathrm{e}^{\pi a},\\
&g_{22}(1,-1,1|0) = -\mathrm{i}l'\big(g_{11} + s^\infty_0 g_{12} - s^0_0(g_{21} + s^\infty_0 g_{22})\big)\mathrm{e}^{-\pi a};
\end{aligned} \tag{D.57}$$

(12) $\mathcal{F}^{\{0\}}_{-1,1,-1} \Rightarrow$

$$\begin{aligned}
&s^0_0(-1,1,-1|0) = s^0_0, \qquad s^\infty_0(-1,1,-1|0) = -s^\infty_1, \qquad s^\infty_1(-1,1,-1|0) = -s^\infty_0 \mathrm{e}^{2\pi a},\\
&g_{11}(-1,1,-1|0) = -\mathrm{i}l'\big(g_{22} - g_{21}s^\infty_1 \mathrm{e}^{2\pi a} + s^0_0\big(g_{12} - g_{11}s^\infty_1 \mathrm{e}^{2\pi a}\big)\big)\mathrm{e}^{-\pi a},\\
&g_{12}(-1,1,-1|0) = \mathrm{i}l'\big(g_{21} + s^0_0 g_{11}\big)\mathrm{e}^{\pi a},\\
&g_{21}(-1,1,-1|0) = -\mathrm{i}l'\big(g_{12} - g_{11}s^\infty_1 \mathrm{e}^{2\pi a}\big)\mathrm{e}^{-\pi a},\\
&g_{22}(-1,1,-1|0) = \mathrm{i}l' g_{11} \mathrm{e}^{\pi a};
\end{aligned} \tag{D.58}$$

(13) $\mathcal{F}^{\{0\}}_{1,1,-1} \Rightarrow$

$$\begin{aligned}
&s^0_0(1,1,-1|0) = s^0_0, \qquad s^\infty_0(1,1,-1|0) = -s^\infty_1 \mathrm{e}^{2\pi a}, \qquad s^\infty_1(1,1,-1|0) = -s^\infty_0,\\
&g_{11}(1,1,-1|0) = \mathrm{i}l'\big(g_{22} - g_{21}s^\infty_1 \mathrm{e}^{2\pi a}\big), \qquad g_{12}(1,1,-1|0) = -\mathrm{i}l' g_{21},\\
&g_{21}(1,1,-1|0) = \mathrm{i}l'\big(g_{12} - g_{11}s^\infty_1 \mathrm{e}^{2\pi a} - s^0_0\big(g_{22} - g_{21}s^\infty_1 \mathrm{e}^{2\pi a}\big)\big),\\
&g_{22}(1,1,-1|0) = -\mathrm{i}l'\big(g_{11} - s^0_0 g_{21}\big);
\end{aligned} \tag{D.59}$$

(14) $\mathcal{F}^{\{0\}}_{-1,1,1} \Rightarrow$

$$\begin{aligned}
&s^0_0(-1,1,1|0) = s^0_0, \qquad s^\infty_0(-1,1,1|0) = -s^\infty_1, \qquad s^\infty_1(-1,1,1|0) = -s^\infty_0 \mathrm{e}^{2\pi a},\\
&g_{11}(-1,1,1|0) = l' g_{12}, \qquad g_{12}(-1,1,1|0) = -l'(g_{11} + s^\infty_0 g_{12}),\\
&g_{21}(-1,1,1|0) = l' g_{22}, \qquad g_{22}(-1,1,1|0) = -l'(g_{21} + s^\infty_0 g_{22});
\end{aligned} \tag{D.60}$$

(15) $\mathcal{F}^{\{0\}}_{1,1,1} \Rightarrow$

$$s^0_0(1,1,1|0) = s^0_0, \qquad s^\infty_0(1,1,1|0) = -s^\infty_1 \mathrm{e}^{2\pi a}, \qquad s^\infty_1(1,1,1|0) = -s^\infty_0,$$

$$\begin{aligned}
&g_{11}(1,1,1|0) = -l'\big(g_{12} - s_0^0 g_{22}\big)\mathrm{e}^{\pi a},\\
&g_{12}(1,1,1|0) = -l'\big(-g_{11} - g_{12}s_0^\infty + s_0^0(g_{21} + g_{22}s_0^\infty)\big)\mathrm{e}^{-\pi a},\\
&g_{21}(1,1,1|0) = -l'\big(g_{22} - s_0^0\big(g_{12} - s_0^0 g_{22}\big)\big)\mathrm{e}^{\pi a},\\
&g_{22}(1,1,1|0) = l'\big((g_{21} + g_{22}s_0^\infty)\big(1 + \big(s_0^0\big)^2\big) - s_0^0(g_{11} + s_0^\infty g_{12})\big)\mathrm{e}^{-\pi a};
\end{aligned} \tag{D.61}$$

(16) $\hat{\mathfrak{F}}^{\{0\}}_{1,1,0} \Rightarrow$

$$\begin{aligned}
&\hat{s}_0^0(1,1,0|0) = s_0^0, \qquad \hat{s}_0^\infty(1,1,0|0) = s_0^\infty \mathrm{e}^{-\pi a/2}, \qquad \hat{s}_1^\infty(1,1,0|0) = s_1^\infty \mathrm{e}^{\pi a/2},\\
&\hat{g}_{11}(1,1,0|0) = -\mathrm{i}g_{21}\mathrm{e}^{-\pi a/4}, \qquad \hat{g}_{12}(1,1,0|0) = -\mathrm{i}g_{22}\mathrm{e}^{\pi a/4},\\
&\hat{g}_{21}(1,1,0|0) = -\mathrm{i}\big(g_{11} - s_0^0 g_{21}\big)\mathrm{e}^{-\pi a/4},\\
&\hat{g}_{22}(1,1,0|0) = -\mathrm{i}\big(g_{12} - s_0^0 g_{22}\big)\mathrm{e}^{\pi a/4};
\end{aligned} \tag{D.62}$$

(17) $\hat{\mathfrak{F}}^{\{0\}}_{1,-1,0} \Rightarrow$

$$\begin{aligned}
&\hat{s}_0^0(1,-1,0|0) = s_0^0, \qquad \hat{s}_0^\infty(1,-1,0|0) = s_0^\infty \mathrm{e}^{-\pi a/2}, \qquad \hat{s}_1^\infty(1,-1,0|0) = s_1^\infty \mathrm{e}^{\pi a/2},\\
&\hat{g}_{11}(1,-1,0|0) = g_{11}\mathrm{e}^{-\pi a/4}, \qquad \hat{g}_{12}(1,-1,0|0) = g_{12}\mathrm{e}^{\pi a/4},\\
&\hat{g}_{21}(1,-1,0|0) = g_{21}\mathrm{e}^{-\pi a/4}, \qquad \hat{g}_{22}(1,-1,0|0) = g_{22}\mathrm{e}^{\pi a/4};
\end{aligned} \tag{D.63}$$

(18) $\hat{\mathfrak{F}}^{\{0\}}_{-1,1,0} \Rightarrow$

$$\begin{aligned}
&\hat{s}_0^0(-1,1,0|0) = s_0^0, \qquad \hat{s}_0^\infty(-1,1,0|0) = s_0^\infty \mathrm{e}^{\pi a/2}, \qquad \hat{s}_1^\infty(-1,1,0|0) = s_1^\infty \mathrm{e}^{-\pi a/2},\\
&\hat{g}_{11}(-1,1,0|0) = g_{11}\mathrm{e}^{\pi a/4}, \qquad \hat{g}_{12}(-1,1,0|0) = g_{12}\mathrm{e}^{-\pi a/4},\\
&\hat{g}_{21}(-1,1,0|0) = g_{21}\mathrm{e}^{\pi a/4}, \qquad \hat{g}_{22}(-1,1,0|0) = g_{22}\mathrm{e}^{-\pi a/4};
\end{aligned} \tag{D.64}$$

(19) $\hat{\mathfrak{F}}^{\{0\}}_{-1,-1,0} \Rightarrow$

$$\hat{s}_0^0(-1,-1,0|0) = s_0^0, \qquad \hat{s}_0^\infty(-1,-1,0|0) = s_0^\infty \mathrm{e}^{\pi a/2}, \tag{D.65}$$

$$\begin{aligned}
&\hat{s}_1^\infty(-1,-1,0|0) = s_1^\infty \mathrm{e}^{-\pi a/2},\\
&\hat{g}_{11}(-1,-1,0|0) = \mathrm{i}\big(g_{21} + s_0^0 g_{11}\big)\mathrm{e}^{\pi a/4}, \qquad \hat{g}_{12}(-1,-1,0|0) = \mathrm{i}\big(g_{22} + s_0^0 g_{12}\big)\mathrm{e}^{-\pi a/4},\\
&\hat{g}_{21}(-1,-1,0|0) = \mathrm{i}g_{11}\mathrm{e}^{\pi a/4}, \qquad \hat{g}_{22}(-1,-1,0|0) = \mathrm{i}g_{12}\mathrm{e}^{-\pi a/4};
\end{aligned} \tag{D.66}$$

(20) $\hat{\mathfrak{F}}^{\{0\}}_{1,0,-1} \Rightarrow$

$$\begin{aligned}
&\hat{s}_0^0(1,0,-1|0) = s_0^0, \qquad \hat{s}_0^\infty(1,0,-1|0) = -s_1^\infty \mathrm{e}^{3\pi a/2}, \qquad \hat{s}_1^\infty(1,0,-1|0) = -s_0^\infty \mathrm{e}^{\pi a/2},\\
&\hat{g}_{11}(1,0,-1|0) = l'\big(g_{12} - g_{11}s_1^\infty \mathrm{e}^{2\pi a}\big)\mathrm{e}^{-\pi a/4}, \qquad \hat{g}_{12}(1,0,-1|0) = -l' g_{11}\mathrm{e}^{\pi a/4},\\
&\hat{g}_{21}(1,0,-1|0) = l'\big(g_{22} - g_{21}s_1^\infty \mathrm{e}^{2\pi a}\big)\mathrm{e}^{-\pi a/4}, \qquad \hat{g}_{22}(1,0,-1|0) = -l' g_{21}\mathrm{e}^{\pi a/4};
\end{aligned} \tag{D.67}$$

(21) $\hat{\mathfrak{F}}^{\{0\}}_{-1,0,-1} \Rightarrow$

$$\begin{aligned}
&\hat{s}_0^0(-1,0,-1|0) = s_0^0, \qquad \hat{s}_0^\infty(-1,0,-1|0) = -s_1^\infty \mathrm{e}^{\pi a/2},\\
&\hat{s}_1^\infty(-1,0,-1|0) = -s_0^\infty \mathrm{e}^{3\pi a/2},\\
&\hat{g}_{11}(-1,0,-1|0) = \mathrm{i}l'\big(g_{22} - g_{21}s_1^\infty \mathrm{e}^{2\pi a} + s_0^0\big(g_{12} - g_{11}s_1^\infty \mathrm{e}^{2\pi a}\big)\big)\mathrm{e}^{-3\pi a/4},\\
&\hat{g}_{12}(-1,0,-1|0) = -\mathrm{i}l'\big(g_{21} + s_0^0 g_{11}\big)\mathrm{e}^{3\pi a/4},\\
&\hat{g}_{21}(-1,0,-1|0) = \mathrm{i}l'\big(g_{12} - g_{11}s_1^\infty \mathrm{e}^{2\pi a}\big)\mathrm{e}^{-3\pi a/4},\\
&\hat{g}_{22}(-1,0,-1|0) = -\mathrm{i}l' g_{11}\mathrm{e}^{3\pi a/4};
\end{aligned} \tag{D.68}$$

(22) $\hat{\mathcal{F}}^{\{0\}}_{1,0,1} \Rightarrow$

$$\begin{aligned} &\hat{s}^{0}_{0}(1,0,1|0) = s^{0}_{0}, \qquad \hat{s}^{\infty}_{0}(1,0,1|0) = -s^{\infty}_{1}\mathrm{e}^{3\pi a/2}, \qquad \hat{s}^{\infty}_{1}(1,0,1|0) = -s^{\infty}_{0}\mathrm{e}^{\pi a/2},\\ &\hat{g}_{11}(1,0,1|0) = \mathrm{i}l' g_{22}\mathrm{e}^{3\pi a/4}, \qquad \hat{g}_{12}(1,0,1|0) = -\mathrm{i}l'(g_{21} + s^{\infty}_{0} g_{22})\mathrm{e}^{-3\pi a/4},\\ &\hat{g}_{21}(1,0,1|0) = \mathrm{i}l'\big(g_{12} - s^{0}_{0} g_{22}\big)\mathrm{e}^{3\pi a/4},\\ &\hat{g}_{22}(1,0,1|0) = \mathrm{i}l'\big({-g_{11}} - s^{\infty}_{0} g_{12} + s^{0}_{0}(g_{21} + s^{\infty}_{0} g_{22})\big)\mathrm{e}^{-3\pi a/4}; \end{aligned} \tag{D.69}$$

(23) $\hat{\mathcal{F}}^{\{0\}}_{-1,0,1} \Rightarrow$

$$\begin{aligned} &\hat{s}^{0}_{0}(-1,0,1|0) = s^{0}_{0}, \qquad \hat{s}^{\infty}_{0}(-1,0,1|0) = -s^{\infty}_{1}\mathrm{e}^{\pi a/2}, \qquad \hat{s}^{\infty}_{1}(-1,0,1|0) = -s^{\infty}_{0}\mathrm{e}^{3\pi a/2},\\ &\hat{g}_{11}(-1,0,1|0) = -l' g_{12}\mathrm{e}^{\pi a/4}, \qquad \hat{g}_{12}(-1,0,1|0) = l'(g_{11} + s^{\infty}_{0} g_{12})\mathrm{e}^{-\pi a/4},\\ &\hat{g}_{21}(-1,0,1|0) = -l' g_{22}\mathrm{e}^{\pi a/4}, \qquad \hat{g}_{22}(-1,0,1|0) = l'(g_{21} + s^{\infty}_{0} g_{22})\mathrm{e}^{-\pi a/4}; \end{aligned} \tag{D.70}$$

(24) $\mathcal{F}^{\{1\}}_{0,0,0} \Rightarrow$

$$\begin{aligned} &s^{0}_{0}(0,0,0|1) = s^{0}_{0}, \qquad s^{\infty}_{0}(0,0,0|1) = -s^{\infty}_{0}, \qquad s^{\infty}_{1}(0,0,0|1) = -s^{\infty}_{1},\\ &g_{11}(0,0,0|1) = \mathrm{i}\tilde{l} g_{11}, \qquad g_{12}(0,0,0|1) = -\mathrm{i}\tilde{l} g_{12}, \qquad g_{21}(0,0,0|1) = \mathrm{i}\tilde{l} g_{21},\\ &g_{22}(0,0,0|1) = -\mathrm{i}\tilde{l} g_{22}; \end{aligned} \tag{D.71}$$

(25) $\mathcal{F}^{\{1\}}_{-1,0,0} \Rightarrow$

$$\begin{aligned} &s^{0}_{0}(-1,0,0|1) = s^{0}_{0}, \qquad s^{\infty}_{0}(-1,0,0|1) = -s^{\infty}_{0}\mathrm{e}^{\pi a}, \qquad s^{\infty}_{1}(-1,0,0|1) = -s^{\infty}_{1}\mathrm{e}^{-\pi a},\\ &g_{11}(-1,0,0|1) = \tilde{l}\big(g_{21} + s^{0}_{0} g_{11}\big)\mathrm{e}^{\pi a/2}, \qquad g_{12}(-1,0,0|1) = -\tilde{l}\big(g_{22} + s^{0}_{0} g_{12}\big)\mathrm{e}^{-\pi a/2},\\ &g_{21}(-1,0,0|1) = \tilde{l} g_{11}\mathrm{e}^{\pi a/2}, \qquad g_{22}(-1,0,0|1) = -\tilde{l} g_{12}\mathrm{e}^{-\pi a/2}; \end{aligned} \tag{D.72}$$

(26) $\mathcal{F}^{\{1\}}_{1,0,0} \Rightarrow$

$$\begin{aligned} &s^{0}_{0}(1,0,0|1) = s^{0}_{0}, \qquad s^{\infty}_{0}(1,0,0|1) = -s^{\infty}_{0}\mathrm{e}^{-\pi a}, \qquad s^{\infty}_{1}(1,0,0|1) = -s^{\infty}_{1}\mathrm{e}^{\pi a},\\ &g_{11}(1,0,0|1) = -\tilde{l} g_{21}\mathrm{e}^{-\pi a/2}, \qquad g_{12}(1,0,0|1) = \tilde{l} g_{22}\mathrm{e}^{\pi a/2},\\ &g_{21}(1,0,0|1) = -\tilde{l}\big(g_{11} - s^{0}_{0} g_{21}\big)\mathrm{e}^{-\pi a/2}, \qquad g_{22}(1,0,0|1) = \tilde{l}\big(g_{12} - s^{0}_{0} g_{22}\big)\mathrm{e}^{\pi a/2}; \end{aligned} \tag{D.73}$$

(27) $\mathcal{F}^{\{1\}}_{0,-1,-1} \Rightarrow$

$$\begin{aligned} &s^{0}_{0}(0,-1,-1|1) = s^{0}_{0}, \qquad s^{\infty}_{0}(0,-1,-1|1) = s^{\infty}_{1}\mathrm{e}^{\pi a}, \qquad s^{\infty}_{1}(0,-1,-1|1) = s^{\infty}_{0}\mathrm{e}^{\pi a},\\ &g_{11}(0,-1,-1|1) = -\tilde{l}l'\big(g_{22} - g_{21}s^{\infty}_{1}\mathrm{e}^{2\pi a} + s^{0}_{0}\big(g_{12} - g_{11}s^{\infty}_{1}\mathrm{e}^{2\pi a}\big)\big)\mathrm{e}^{-\pi a/2},\\ &g_{12}(0,-1,-1|1) = -\tilde{l}l'\big(g_{21} + s^{0}_{0} g_{11}\big)\mathrm{e}^{\pi a/2},\\ &g_{21}(0,-1,-1|1) = -\tilde{l}l'\big(g_{12} - g_{11}s^{\infty}_{1}\mathrm{e}^{2\pi a}\big)\mathrm{e}^{-\pi a/2},\\ &g_{22}(0,-1,-1|1) = -\tilde{l}l' g_{11}\mathrm{e}^{\pi a/2}; \end{aligned} \tag{D.74}$$

(28) $\mathcal{F}^{\{1\}}_{0,-1,1} \Rightarrow$

$$\begin{aligned} &s^{0}_{0}(0,-1,1|1) = s^{0}_{0}, \qquad s^{\infty}_{0}(0,-1,1|1) = s^{\infty}_{1}\mathrm{e}^{\pi a}, \qquad s^{\infty}_{1}(0,-1,1|1) = s^{\infty}_{0}\mathrm{e}^{\pi a},\\ &g_{11}(0,-1,1|1) = \mathrm{i}\tilde{l}l' g_{12}\mathrm{e}^{\pi a/2}, \qquad g_{12}(0,-1,1|1) = \mathrm{i}\tilde{l}l'(g_{11} + s^{\infty}_{0} g_{12})\mathrm{e}^{-\pi a/2},\\ &g_{21}(0,-1,1|1) = \mathrm{i}\tilde{l}l' g_{22}\mathrm{e}^{\pi a/2}, \qquad g_{22}(0,-1,1|1) = \mathrm{i}\tilde{l}l'(g_{21} + s^{\infty}_{0} g_{22})\mathrm{e}^{-\pi a/2}; \end{aligned} \tag{D.75}$$

(29) $\mathcal{F}^{\{1\}}_{0,1,-1} \Rightarrow$

$$s_0^0(0,1,-1|1) = s_0^0, \qquad s_0^\infty(0,1,-1|1) = s_1^\infty e^{\pi a}, \qquad s_1^\infty(0,1,-1|1) = s_0^\infty e^{\pi a},$$
$$g_{11}(0,1,-1|1) = i\tilde{l}l'(g_{12} - g_{11}s_1^\infty e^{2\pi a})e^{-\pi a/2}, \qquad g_{12}(0,1,-1|1) = i\tilde{l}l' g_{11} e^{\pi a/2},$$
$$g_{21}(0,1,-1|1) = i\tilde{l}l'(g_{22} - g_{21}s_1^\infty e^{2\pi a})e^{-\pi a/2}, \qquad g_{22}(0,1,-1|1) = i\tilde{l}l' g_{21} e^{\pi a/2}; \tag{D.76}$$

(30) $\mathcal{F}^{\{1\}}_{0,1,1} \Rightarrow$

$$s_0^0(0,1,1|1) = s_0^0, \qquad s_0^\infty(0,1,1|1) = s_1^\infty e^{\pi a}, \qquad s_1^\infty(0,1,1|1) = s_0^\infty e^{\pi a},$$
$$g_{11}(0,1,1|1) = -\tilde{l}l' g_{22} e^{\pi a/2}, \qquad g_{12}(0,1,1|1) = -\tilde{l}l'(g_{21} + s_0^\infty g_{22})e^{-\pi a/2},$$
$$g_{21}(0,1,1|1) = -\tilde{l}l'\big(g_{12} - s_0^0 g_{22}\big)e^{\pi a/2},$$
$$g_{22}(0,1,1|1) = \tilde{l}l'\big(-g_{11} - s_0^\infty g_{12} + s_0^0(g_{21} + s_0^\infty g_{22})\big)e^{-\pi a/2}; \tag{D.77}$$

(31) $\mathcal{F}^{\{1\}}_{-1,-1,-1} \Rightarrow$

$$s_0^0(-1,-1,-1|1) = s_0^0, \qquad s_0^\infty(-1,-1,-1|1) = s_1^\infty,$$
$$s_1^\infty(-1,-1,-1|1) = s_0^\infty e^{2\pi a},$$
$$g_{11}(-1,-1,-1|1) = i\tilde{l}l'\Big(\big(g_{12} - g_{11}s_1^\infty e^{2\pi a}\big)\big(1 + \big(s_0^0\big)^2\big) + s_0^0\big(g_{22} - g_{21}s_1^\infty e^{2\pi a}\big)\Big)e^{-\pi a},$$
$$g_{12}(-1,-1,-1|1) = i\tilde{l}l'\big(g_{11}\big(1 + \big(s_0^0\big)^2\big) + s_0^0 g_{21}\big)e^{\pi a},$$
$$g_{21}(-1,-1,-1|1) = i\tilde{l}l'\big(g_{22} - g_{21}s_1^\infty e^{2\pi a} + s_0^0\big(g_{12} - g_{11}s_1^\infty e^{2\pi a}\big)\big)e^{-\pi a},$$
$$g_{22}(-1,-1,-1|1) = i\tilde{l}l'(g_{21} + s_0^0 g_{11})e^{\pi a}; \tag{D.78}$$

(32) $\mathcal{F}^{\{1\}}_{1,-1,-1} \Rightarrow$

$$s_0^0(1,-1,-1|1) = s_0^0, \qquad s_0^\infty(1,-1,-1|1) = s_1^\infty e^{2\pi a}, \qquad s_1^\infty(1,-1,-1|1) = s_0^\infty,$$
$$g_{11}(1,-1,-1|1) = -i\tilde{l}l'\big(g_{12} - g_{11}s_1^\infty e^{2\pi a}\big), \qquad g_{12}(1,-1,-1|1) = -i\tilde{l}l' g_{11},$$
$$g_{21}(1,-1,-1|1) = -i\tilde{l}l'\big(g_{22} - g_{21}s_1^\infty e^{2\pi a}\big), \qquad g_{22}(1,-1,-1|1) = -i\tilde{l}l' g_{21}; \tag{D.79}$$

(33) $\mathcal{F}^{\{1\}}_{-1,-1,1} \Rightarrow$

$$s_0^0(-1,-1,1|1) = s_0^0, \qquad s_0^\infty(-1,-1,1|1) = s_1^\infty, \qquad s_1^\infty(-1,-1,1|1) = s_0^\infty e^{2\pi a},$$
$$g_{11}(-1,-1,1|1) = \tilde{l}l'\big(g_{22} + s_0^0 g_{12}\big),$$
$$g_{12}(-1,-1,1|1) = \tilde{l}l'\big(g_{21} + s_0^\infty g_{22} + s_0^0(g_{11} + s_0^\infty g_{12})\big),$$
$$g_{21}(-1,-1,1|1) = \tilde{l}l' g_{12}, \qquad g_{22}(-1,-1,1|1) = \tilde{l}l'(g_{11} + s_0^\infty g_{12}); \tag{D.80}$$

(34) $\mathcal{F}^{\{1\}}_{1,-1,1} \Rightarrow$

$$s_0^0(1,-1,1|1) = s_0^0, \qquad s_0^\infty(1,-1,1|1) = s_1^\infty e^{2\pi a}, \qquad s_1^\infty(1,-1,1|1) = s_0^\infty,$$
$$g_{11}(1,-1,1|1) = -\tilde{l}l' g_{22} e^{\pi a}, \qquad g_{12}(1,-1,1|1) = -\tilde{l}l'(g_{21} + s_0^\infty g_{22})e^{-\pi a},$$
$$g_{21}(1,-1,1|1) = -\tilde{l}l'\big(g_{12} - s_0^0 g_{22}\big)e^{\pi a},$$
$$g_{22}(1,-1,1|1) = -\tilde{l}l'\big(g_{11} + s_0^\infty g_{12} - s_0^0(g_{21} + s_0^\infty g_{22})\big)e^{-\pi a}; \tag{D.81}$$

(35) $\mathcal{F}^{\{1\}}_{-1,1,-1} \Rightarrow$

$$\begin{aligned}
&s_0^0(-1,1,-1|1)=s_0^0, \qquad s_0^\infty(-1,1,-1|1)=s_1^\infty, \qquad s_1^\infty(-1,1,-1|1)=s_0^\infty \mathrm{e}^{2\pi a},\\
&g_{11}(-1,1,-1|1)=\tilde{l}l'\big(g_{22}-g_{21}s_1^\infty \mathrm{e}^{2\pi a}+s_0^0\big(g_{12}-g_{11}s_1^\infty \mathrm{e}^{2\pi a}\big)\big)\mathrm{e}^{-\pi a},\\
&g_{12}(-1,1,-1|1)=\tilde{l}l'\big(g_{21}+s_0^0 g_{11}\big)\mathrm{e}^{\pi a},\\
&g_{21}(-1,1,-1|1)=\tilde{l}l'\big(g_{12}-g_{11}s_1^\infty \mathrm{e}^{2\pi a}\big)\mathrm{e}^{-\pi a},\\
&g_{22}(-1,1,-1|1)=\tilde{l}l' g_{11}\mathrm{e}^{\pi a};
\end{aligned} \tag{D.82}$$

(36) $\mathcal{F}^{\{1\}}_{1,1,-1} \Rightarrow$

$$\begin{aligned}
&s_0^0(1,1,-1|1)=s_0^0, \qquad s_0^\infty(1,1,-1|1)=s_1^\infty \mathrm{e}^{2\pi a}, \qquad s_1^\infty(1,1,-1|1)=s_0^\infty,\\
&g_{11}(1,1,-1|1)=-\tilde{l}l'\big(g_{22}-g_{21}s_1^\infty \mathrm{e}^{2\pi a}\big), \qquad g_{12}(1,1,-1|1)=-\tilde{l}l' g_{21},\\
&g_{21}(1,1,-1|1)=-\tilde{l}l'\big(g_{12}-g_{11}s_1^\infty \mathrm{e}^{2\pi a}-s_0^0\big(g_{22}-g_{21}s_1^\infty \mathrm{e}^{2\pi a}\big)\big),\\
&g_{22}(1,1,-1|1)=-\tilde{l}l'\big(g_{11}-s_0^0 g_{21}\big);
\end{aligned} \tag{D.83}$$

(37) $\mathcal{F}^{\{1\}}_{-1,1,1} \Rightarrow$

$$\begin{aligned}
&s_0^0(-1,1,1|1)=s_0^0, \qquad s_0^\infty(-1,1,1|1)=s_1^\infty, \qquad s_1^\infty(-1,1,1|1)=s_0^\infty \mathrm{e}^{2\pi a},\\
&g_{11}(-1,1,1|1)=\mathrm{i}\tilde{l}l' g_{12}, \qquad g_{12}(-1,1,1|1)=\mathrm{i}\tilde{l}l'(g_{11}+s_0^\infty g_{12}),\\
&g_{21}(-1,1,1|1)=\mathrm{i}\tilde{l}l' g_{22}, \qquad g_{22}(-1,1,1|1)=\mathrm{i}\tilde{l}l'(g_{21}+s_0^\infty g_{22});
\end{aligned} \tag{D.84}$$

(38) $\mathcal{F}^{\{1\}}_{1,1,1} \Rightarrow$

$$\begin{aligned}
&s_0^0(1,1,1|1)=s_0^0, \qquad s_0^\infty(1,1,1|1)=s_1^\infty \mathrm{e}^{2\pi a}, \qquad s_1^\infty(1,1,1|1)=s_0^\infty,\\
&g_{11}(1,1,1|1)=-\mathrm{i}\tilde{l}l'\big(g_{12}-s_0^0 g_{22}\big)\mathrm{e}^{\pi a},\\
&g_{12}(1,1,1|1)=\mathrm{i}\tilde{l}l'\big(-g_{11}-s_0^\infty g_{12}+s_0^0(g_{21}+s_0^\infty g_{22})\big)\mathrm{e}^{-\pi a},\\
&g_{21}(1,1,1|1)=-\mathrm{i}\tilde{l}l'\big(g_{22}-s_0^0\big(g_{12}-s_0^0 g_{22}\big)\big)\mathrm{e}^{\pi a},\\
&g_{22}(1,1,1|1)=-\mathrm{i}\tilde{l}l'\big((g_{21}+s_0^\infty g_{22})\big(1+\big(s_0^0\big)^2\big)-s_0^0(g_{11}+s_0^\infty g_{12})\big)\mathrm{e}^{-\pi a};
\end{aligned} \tag{D.85}$$

(39) $\hat{\mathcal{F}}^{\{1\}}_{1,1,0} \Rightarrow$

$$\begin{aligned}
&\hat{s}_0^0(1,1,0|1)=s_0^0, \qquad \hat{s}_0^\infty(1,1,0|1)=-s_0^\infty \mathrm{e}^{-\pi a/2}, \qquad \hat{s}_1^\infty(1,1,0|1)=-s_1^\infty \mathrm{e}^{\pi a/2},\\
&\hat{g}_{11}(1,1,0|1)=\tilde{l}g_{21}\mathrm{e}^{-\pi a/4}, \qquad \hat{g}_{12}(1,1,0|1)=-\tilde{l}g_{22}\mathrm{e}^{\pi a/4},\\
&\hat{g}_{21}(1,1,0|1)=\tilde{l}\big(g_{11}-s_0^0 g_{21}\big)\mathrm{e}^{-\pi a/4},\\
&\hat{g}_{22}(1,1,0|1)=-\tilde{l}\big(g_{12}-s_0^0 g_{22}\big)\mathrm{e}^{\pi a/4};
\end{aligned} \tag{D.86}$$

(40) $\hat{\mathcal{F}}^{\{1\}}_{1,-1,0} \Rightarrow$

$$\begin{aligned}
&\hat{s}_0^0(1,-1,0|1)=s_0^0, \qquad \hat{s}_0^\infty(1,-1,0|1)=-s_0^\infty \mathrm{e}^{-\pi a/2},\\
&\hat{s}_1^\infty(1,-1,0|1)=-s_1^\infty \mathrm{e}^{\pi a/2},\\
&\hat{g}_{11}(1,-1,0|1)=\mathrm{i}\tilde{l}g_{11}\mathrm{e}^{-\pi a/4}, \qquad \hat{g}_{12}(1,-1,0|1)=-\mathrm{i}\tilde{l}g_{12}\mathrm{e}^{\pi a/4},\\
&\hat{g}_{21}(1,-1,0|1)=\mathrm{i}\tilde{l}g_{21}\mathrm{e}^{-\pi a/4}, \qquad \hat{g}_{22}(1,-1,0|1)=-\mathrm{i}\tilde{l}g_{22}\mathrm{e}^{\pi a/4};
\end{aligned} \tag{D.87}$$

(41) $\hat{\mathfrak{F}}^{\{1\}}_{-1,1,0} \Rightarrow$

$$\begin{aligned}
&\hat{s}_0^0(-1,1,0|1) = s_0^0, \qquad \hat{s}_0^\infty(-1,1,0|1) = -s_0^\infty \mathrm{e}^{\pi a/2},\\
&\hat{s}_1^\infty(-1,1,0|1) = -s_1^\infty \mathrm{e}^{-\pi a/2},\\
&\hat{g}_{11}(-1,1,0|1) = \mathrm{i}\tilde{l} g_{11} \mathrm{e}^{\pi a/4}, \qquad \hat{g}_{12}(-1,1,0|1) = -\mathrm{i}\tilde{l} g_{12} \mathrm{e}^{-\pi a/4},\\
&\hat{g}_{21}(-1,1,0|1) = \mathrm{i}\tilde{l} g_{21} \mathrm{e}^{\pi a/4}, \qquad \hat{g}_{22}(-1,1,0|1) = -\mathrm{i}\tilde{l} g_{22} \mathrm{e}^{-\pi a/4};
\end{aligned} \tag{D.88}$$

(42) $\hat{\mathfrak{F}}^{\{1\}}_{-1,-1,0} \Rightarrow$

$$\begin{aligned}
&\hat{s}_0^0(-1,-1,0|1) = s_0^0, \qquad \hat{s}_0^\infty(-1,-1,0|1) = -s_0^\infty \mathrm{e}^{\pi a/2},\\
&\hat{s}_1^\infty(-1,-1,0|1) = -s_1^\infty \mathrm{e}^{-\pi a/2}, \qquad \hat{g}_{11}(-1,-1,0|1) = -\tilde{l}\big(g_{21} + s_0^0 g_{11}\big) \mathrm{e}^{\pi a/4},\\
&\hat{g}_{12}(-1,-1,0|1) = \tilde{l}\big(g_{22} + s_0^0 g_{12}\big) \mathrm{e}^{-\pi a/4},\\
&\hat{g}_{21}(-1,-1,0|1) = -\tilde{l} g_{11} \mathrm{e}^{\pi a/4}, \qquad \hat{g}_{22}(-1,-1,0|1) = \tilde{l} g_{12} \mathrm{e}^{-\pi a/4};
\end{aligned} \tag{D.89}$$

(43) $\hat{\mathfrak{F}}^{\{1\}}_{1,0,-1} \Rightarrow$

$$\begin{aligned}
&\hat{s}_0^0(1,0,-1|1) = s_0^0, \qquad \hat{s}_0^\infty(1,0,-1|1) = s_1^\infty \mathrm{e}^{3\pi a/2}, \qquad \hat{s}_1^\infty(1,0,-1|1) = s_0^\infty \mathrm{e}^{\pi a/2},\\
&\hat{g}_{11}(1,0,-1|1) = \mathrm{i}\tilde{l}l'\big(g_{12} - g_{11} s_1^\infty \mathrm{e}^{2\pi a}\big) \mathrm{e}^{-\pi a/4}, \qquad \hat{g}_{12}(1,0,-1|1) = \mathrm{i}\tilde{l}l' g_{11} \mathrm{e}^{\pi a/4},\\
&\hat{g}_{21}(1,0,-1|1) = \mathrm{i}\tilde{l}l'\big(g_{22} - g_{21} s_1^\infty \mathrm{e}^{2\pi a}\big) \mathrm{e}^{-\pi a/4}, \qquad \hat{g}_{22}(1,0,-1|1) = \mathrm{i}\tilde{l}l' g_{21} \mathrm{e}^{\pi a/4};
\end{aligned} \tag{D.90}$$

(44) $\hat{\mathfrak{F}}^{\{1\}}_{1,0,1} \Rightarrow$

$$\begin{aligned}
&\hat{s}_0^0(1,0,1|1) = s_0^0, \qquad \hat{s}_0^\infty(1,0,1|1) = s_1^\infty \mathrm{e}^{3\pi a/2}, \qquad \hat{s}_1^\infty(1,0,1|1) = s_0^\infty \mathrm{e}^{\pi a/2},\\
&\hat{g}_{11}(1,0,1|1) = -\tilde{l}l' g_{22} \mathrm{e}^{3\pi a/4}, \qquad \hat{g}_{12}(1,0,1|1) = -\tilde{l}l'(g_{21} + g_{22} s_0^\infty) \mathrm{e}^{-3\pi a/4},\\
&\hat{g}_{21}(1,0,1|1) = -\tilde{l}l'\big(g_{12} - s_0^0 g_{22}\big) \mathrm{e}^{3\pi a/4},\\
&\hat{g}_{22}(1,0,1|1) = \tilde{l}l'\big({-g_{11}} - s_0^\infty g_{12} + s_0^0(g_{21} + s_0^\infty g_{22})\big) \mathrm{e}^{-3\pi a/4};
\end{aligned} \tag{D.91}$$

(45) $\hat{\mathfrak{F}}^{\{1\}}_{-1,0,-1} \Rightarrow$

$$\begin{aligned}
&\hat{s}_0^0(-1,0,-1|1) = s_0^0, \qquad \hat{s}_0^\infty(-1,0,-1|1) = s_1^\infty \mathrm{e}^{\pi a/2},\\
&\hat{s}_1^\infty(-1,0,-1|1) = s_0^\infty \mathrm{e}^{3\pi a/2},\\
&\hat{g}_{11}(-1,0,-1|1) = -\tilde{l}l'\big(g_{22} - g_{21} s_1^\infty \mathrm{e}^{2\pi a} + s_0^0\big(g_{12} - g_{11} s_1^\infty \mathrm{e}^{2\pi a}\big)\big) \mathrm{e}^{-3\pi a/4},\\
&\hat{g}_{12}(-1,0,-1|1) = -\tilde{l}l'\big(g_{21} + s_0^0 g_{11}\big) \mathrm{e}^{3\pi a/4},\\
&\hat{g}_{21}(-1,0,-1|1) = -\tilde{l}l'\big(g_{12} - g_{11} s_1^\infty \mathrm{e}^{2\pi a}\big) \mathrm{e}^{-3\pi a/4},\\
&\hat{g}_{22}(-1,0,-1|1) = -\tilde{l}l' g_{11} \mathrm{e}^{3\pi a/4};
\end{aligned} \tag{D.92}$$

(46) $\hat{\mathfrak{F}}^{\{1\}}_{-1,0,1} \Rightarrow$

$$\begin{aligned}
&\hat{s}_0^0(-1,0,1|1) = s_0^0, \qquad \hat{s}_0^\infty(-1,0,1|1) = s_1^\infty \mathrm{e}^{\pi a/2}, \qquad \hat{s}_1^\infty(-1,0,1|1) = s_0^\infty \mathrm{e}^{3\pi a/2},\\
&\hat{g}_{11}(-1,0,1|1) = -\mathrm{i}\tilde{l}l' g_{12} \mathrm{e}^{\pi a/4}, \qquad \hat{g}_{12}(-1,0,1|1) = -\mathrm{i}\tilde{l}l'(g_{11} + s_0^\infty g_{12}) \mathrm{e}^{-\pi a/4},\\
&\hat{g}_{21}(-1,0,1|1) = -\mathrm{i}\tilde{l}l' g_{22} \mathrm{e}^{\pi a/4},\\
&\hat{g}_{22}(-1,0,1|1) = -\mathrm{i}\tilde{l}l'(g_{21} + s_0^\infty g_{22}) \mathrm{e}^{-\pi a/4}.
\end{aligned} \tag{D.93}$$

Finally, applying the isomorphism $\mathcal{F}^{\{\ell\}}_{\varepsilon_1,\varepsilon_2,m(\varepsilon_2)}$ (resp., $\hat{\mathcal{F}}^{\{\hat{\ell}\}}_{\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)}$), whose action on $\mathcal{M}$ is given by equations (D.47)–(D.61) and (D.71)–(D.85) (resp., equations (D.62)–(D.70) and (D.86)–(D.93)), to the corresponding $(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)=(0,0,0|0)$ (resp., $\big(\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\big)=(0,0,0|0)$) asymptotics (as $\tau\to+\infty$ with $\varepsilon b>0$) for $u(\tau)$, $f_\pm(\tau)$, $\mathcal{H}(\tau)$, and $\sigma(\tau)$ derived in Section 4, one arrives at the asymptotics as $\tau\to\pm\infty$ (resp., $\tau\to\pm\mathrm{i}\infty$) for $u(\tau)$, $f_\pm(\tau)$, $\mathcal{H}(\tau)$, and $\sigma(\tau)$ stated in Theorem 2.4 (resp., Theorem 2.8).[59]

E Asymptotics of $\hat{\varphi}(\tau)$ as $\tau\to\pm\infty$ and $\tau\to\pm\mathrm{i}\infty$

In this appendix, asymptotics as $\tau\to\pm\infty$ (resp., $\tau\to\pm\mathrm{i}\infty$) for $\pm\varepsilon b>0$ of the function $\hat{\varphi}(\tau)$ (cf. Proposition 1.5) are presented in Theorem E.3 (resp., Theorem E.6). The results of this appendix are seminal for an upcoming series of works on asymptotics of integrals of solutions to the DP3E (1.1) and related functions.

Remark E.1. Since the function $\hat{\varphi}(\tau)$ is defined $\mathrm{mod}(2\pi)$, the reader should be cognizant of the fact that the asymptotics for $\hat{\varphi}(\tau)$ stated in Theorems E.3 and E.6 are defined $\mathrm{mod}(2\pi)$; this $\mathrm{mod}(2\pi)$ arbitrariness, however, is not important, because the requisite functions are $u(\tau)$ and $\exp(\mathrm{i}\hat{\varphi}(\tau))$.

Remark E.2. If one is only interested in the asymptotics as $\tau\to+\infty$ for $\varepsilon b>0$ of the function $\hat{\varphi}(\tau)$, then, in Theorem E.3, one sets $(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)=(0,0,0|0)$ and uses the fact that (cf. Appendix D.5, the identity map (D.47)) $s_0^0(0,0,0|0)=s_0^0$, $s_0^\infty(0,0,0|0)=s_0^\infty$, $s_1^\infty(0,0,0|0)=s_1^\infty$, and $g_{ij}(0,0,0|0)=g_{ij}$, $i,j\in\{1,2\}$.

Theorem E.3. *Let $u(\tau)$ be a solution of the* DP3E (1.1) *and $\hat{\varphi}(\tau)$ be the general solution of the* ODE $\hat{\varphi}'(\tau)=2a\tau^{-1}+b(u(\tau))^{-1}$ *for $\varepsilon b>0$ corresponding to the monodromy data $\big(a,s_0^0,s_0^\infty,s_1^\infty,g_{11},g_{12},g_{21},g_{22}\big)$. Let*

$$\varepsilon_1,\varepsilon_2\in\{0,\pm1\},\qquad m(\varepsilon_2)=\begin{cases}0, & \varepsilon_2=0,\\ \pm\varepsilon_2, & \varepsilon_2\in\{\pm1\},\end{cases}\qquad \ell\in\{0,1\},$$

and $\varepsilon b=|\varepsilon b|\mathrm{e}^{\mathrm{i}\pi\varepsilon_2}$. For $k=+1$, let

$$g_{11}(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)g_{12}(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)g_{21}(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)\neq0,\qquad g_{22}(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)=0,$$

and, for $k=-1$, let

$$g_{11}(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)=0,\qquad g_{12}(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)g_{21}(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)g_{22}(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)\neq0,$$

where explicit expressions for $g_{ij}(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)$, $i,j\in\{1,2\}$, are given in Appendix D, equations (D.47)–(D.61) *and* (D.71)–(D.85)*. Then, for $s_0^0(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)\neq\mathrm{i}\mathrm{e}^{(-1)^{1+\varepsilon_2}\pi a}$,*[60]

$$(-1)^{\varepsilon_2}\hat{\varphi}(\tau)\underset{\tau\to+\infty\mathrm{e}^{\mathrm{i}\pi\varepsilon_1}}{=}3\mathrm{e}^{\frac{\mathrm{i}2\pi k}{3}}(-1)^{\varepsilon_2}(\varepsilon b)^{1/3}\tau^{2/3}+2(-1)^{\varepsilon_2}a\ln\left(\frac{2\mathrm{e}^{-\frac{\mathrm{i}\pi k}{3}}\tau^{2/3}}{\big(\varepsilon b\mathrm{e}^{-\mathrm{i}\pi\varepsilon_2}\big)^{1/6}}\right)$$
$$+\mathrm{i}\mathcal{L}^{(m(\varepsilon_2)|\ell)}_{(\varepsilon_1,\varepsilon_2)}(k)-k\pi-\mathrm{i}\sum_{m=2}^{\infty}\big(2\widetilde{\nu}_m(k)$$

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[59] In Section 3 (resp., Section 2), p. 1174 (resp., p. 7) of [61] (resp., [57]), for item (9) in the definition of the mapping $\mathcal{F}_{1,1}$, the formula for $g_{21}(1,1)$ is missing: it reads $g_{21}(1,1)=\mathrm{i}g_{12}\mathrm{e}^{\pi a}$.

[60] Recall that (cf. Remark 2.1) $s_0^0(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)=s_0^0$. For $s_0^0(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)=\mathrm{i}\mathrm{e}^{(-1)^{1+\varepsilon_2}\pi a}$, the exponentially small correction term in the asymptotics (E.1) is absent.

$$+\sum_{\substack{n,\mathfrak{l}\in\mathbb{N}\\ \mathfrak{l}\geqslant n\\ n+\mathfrak{l}=m}}\sum_{\substack{\mathfrak{i}_1+2\mathfrak{i}_2+\cdots+\mathfrak{l}\mathfrak{i}_{\mathfrak{l}}=\mathfrak{l}\\ \mathfrak{i}_1+\mathfrak{i}_2+\cdots+\mathfrak{i}_{\mathfrak{l}}=n}}(-1)^{n-1}(n-1)!\prod_{\mathfrak{j}=1}^{\mathfrak{l}}\frac{(\mathfrak{u}_{\mathfrak{j}-1}(k))^{\mathfrak{i}_{\mathfrak{j}}}}{\mathfrak{i}_{\mathfrak{j}}!}\Bigg)$$
$$\times\left((-1)^{\varepsilon_1}\tau^{-1/3}\right)^m-\frac{k(-1)^{\varepsilon_1}\mathrm{e}^{-\frac{\mathrm{i}\pi k}{3}}\mathrm{e}^{\frac{\mathrm{i}\pi k}{4}}\left(2+\sqrt{3}\right)^{\mathrm{i}k(-1)^{\varepsilon_2}a}}{\sqrt{2\pi}3^{3/4}\left(\varepsilon b\mathrm{e}^{-\mathrm{i}\pi\varepsilon_2}\right)^{1/6}\tau^{1/3}}$$
$$\times\left(s_0^0(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)-\mathrm{i}\mathrm{e}^{(-1)^{1+\varepsilon_2}\pi a}\right)\mathrm{e}^{-\frac{3\sqrt{3}}{2}(\sqrt{3}+\mathrm{i}k)(-1)^{\varepsilon_2}(\varepsilon b)^{1/3}\tau^{2/3}}$$
$$\times\left(1+\mathcal{O}\left(\tau^{-1/3}\right)\right),\qquad k\in\{\pm1\},\tag{E.1}$$

where

$$\mathcal{L}_{(\varepsilon_1,\varepsilon_2)}^{(m(\varepsilon_2)|\ell)}(k)=\begin{cases}\ln\left(g_{11}(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)\mathrm{e}^{(-1)^{\varepsilon_2}\pi a}\right)^2, & k=+1,\\ \ln\left(g_{22}(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)\mathrm{e}^{(-1)^{\varepsilon_2}\pi a}\right)^{-2}, & k=-1,\end{cases}\tag{E.2}$$
$$\widetilde{\nu}_1(k)=0,\qquad\widetilde{\nu}_2(k)=\frac{a(1+\mathrm{i}(-1)^{\varepsilon_2}a)\mathrm{e}^{\mathrm{i}\pi k/3}}{6(\varepsilon b)^{1/3}},\qquad\widetilde{\nu}_3(k)=0,\tag{E.3}$$
$$\widetilde{\nu}_4(k)=-\frac{\mathrm{i}(-1)^{\varepsilon_2}a\mathrm{e}^{\mathrm{i}2\pi k/3}}{36(\varepsilon b)^{2/3}}\left(\frac{1-2a^2}{3}+\mathrm{i}(-1)^{\varepsilon_2}a\right),\tag{E.4}$$

and

$$(m+5)\widetilde{\nu}_{m+5}(k)=\frac{3\mathrm{i}}{2}\mathrm{e}^{-\mathrm{i}\pi k/3}(-1)^{\varepsilon_2}(\varepsilon b)^{1/3}\mathfrak{u}_{m+5}(k)+\frac{\mathrm{i}(-1)^{\varepsilon_2}\mathrm{e}^{\mathrm{i}\pi k/3}(1+2\mathrm{i}(-1)^{\varepsilon_2}a)}{12(\varepsilon b)^{1/3}}\mu_{m+1}^*(k)$$
$$+\frac{1}{4}\mu_{m+3}^*(k)-\frac{\mathrm{i}(-1)^{\varepsilon_2}\mathrm{e}^{\mathrm{i}\pi k/3}}{12(\varepsilon b)^{1/3}}\Bigg((m+3)(m+5+2\mathrm{i}(-1)^{\varepsilon_2}a)\widetilde{\nu}_{m+3}(k)$$
$$+\sum_{j=0}^{m-1}(j+1)\widetilde{\nu}_{j+1}(k)(\mu_{m-j}^*(k)-2(m+2-j)\widetilde{\nu}_{m+2-j}(k))$$
$$-\frac{\mathrm{i}(-1)^{\varepsilon_2}2a^2\mathrm{e}^{\mathrm{i}\pi k/3}}{3(\varepsilon b)^{1/3}}(m+1)\widetilde{\nu}_{m+1}(k)\Bigg),\qquad m\in\mathbb{Z}_+,\tag{E.5}$$

with

$$\mu_0^*(k)=\frac{2a\mathrm{e}^{\mathrm{i}\pi k/3}}{3(\varepsilon b)^{1/3}},\qquad\mu_1^*(k)=0,\tag{E.6}$$
$$\mu_{m_1+2}^*(k)=-2\left(\mathrm{P}_{m_1+2}^*(k)+\mathfrak{w}_{m_1+2}(k)+\sum_{j=0}^{m_1}\mathrm{P}_j^*(k)\mathfrak{w}_{m_1-j}(k)\right),\qquad m_1\in\mathbb{Z}_+,\tag{E.7}$$

and

$$\mathrm{P}_0^*(k)=-\frac{2a\mathrm{e}^{\mathrm{i}\pi k/3}}{3(\varepsilon b)^{1/3}},\qquad\mathrm{P}_1^*(k)=0,\tag{E.8}$$
$$\mathrm{P}_j^*(k)=\frac{3}{2}\left(\mathfrak{u}_j(k)-\mathrm{i}(-1)^{\varepsilon_2}\mathrm{e}^{\frac{\mathrm{i}2\pi k}{3}}(\varepsilon b)^{1/3}\left(\mathfrak{r}_{j+2}(k)-2\mathfrak{u}_{j+2}(k)+\sum_{m_2=0}^{j}\mathfrak{u}_{m_2}(k)\mathfrak{r}_{j-m_2}(k)\right)\right),$$
$$\mathbb{N}\ni j\geqslant2,\tag{E.9}$$

where the expansion coefficients $\mathfrak{u}_m(k)$ *and* $\mathfrak{w}_m(k)$ *(resp.,* $\mathfrak{r}_m(k)$*),* $m\in\mathbb{Z}_+$*,* $k\in\{\pm1\}$*, are given in equations (2.2)–(2.9) (resp., (2.13) and (2.14)).*[61]

[61]Note: $\sum_{j=0}^{-1}*:=0$.

Proof. The proof is presented for the case $\tau \to +\infty$ with $\varepsilon b > 0$, that is, $(\varepsilon_1, \varepsilon_2, m(\varepsilon_2)|\ell) = (0,0,0|0)$ (cf. Appendix D). Recall from Proposition 1.5 that, given any solution $u(\tau)$ of the DP3E (1.1), the function $\hat{\varphi}(\tau)$ is defined as the general solution of the ODE $\hat{\varphi}'(\tau) = 2a\tau^{-1} + b(u(\tau))^{-1}$. From [61, Propositions 1.2 and 4.1.1] (see also [58, Section 1]), it can be shown that, for $\varepsilon \in \{\pm 1\}$,

$$\hat{\varphi}(\tau) = -\mathrm{i}\ln\left(\frac{\varepsilon\tau^{\mathrm{i}a}u(\tau)}{\tau^{1/3}b(\tau)}\right), \tag{E.10}$$

the trans-series asymptotics (as $\tau \to +\infty$ with $\varepsilon b > 0$) for $u(\tau)$ is given in Theorem 2.4, whilst only the leading-order asymptotics for the function $b(\tau)$ is derived in Lemma 4.1 (cf. equations (4.3)–(4.5)); therefore, in order to proceed with the proof, trans-series asymptotics for $b(\tau)$ must be derived.

Commencing with the asymptotics (4.1) and (4.2), and repeating, *verbatim*, the asymptotic analysis of Section 4, one shows that the asymptotic representation (as $\tau \to +\infty$ with $\varepsilon b > 0$) for the function $b(\tau)$ reads

$$b(\tau) \underset{\tau\to+\infty}{=} \mathfrak{b}_0^*(k)\exp(-2\mathcal{B}_k(\tau)), \qquad k \in \{\pm 1\}, \tag{E.11}$$

where

$$\mathfrak{b}_0^*(k) := (\mathfrak{b}(k))^2(\varepsilon b)^{1/2}\exp\bigl(2\mathrm{i}(a-\mathrm{i}/2)\ln\bigl((\varepsilon b)^{1/6}\mathrm{e}^{\mathrm{i}\pi k/3}/2\bigr)\bigr), \tag{E.12}$$

with $\mathfrak{b}(k)$ given in equation (4.5), and

$$\begin{aligned}\mathcal{B}_k(\tau) := \frac{\mathrm{i}a}{6}\ln\tau - \frac{3k}{4}\bigl(\sqrt{3}+\mathrm{i}k\bigr)(\varepsilon b)^{1/3}\tau^{2/3} + \sum_{m=1}^{\infty}\widetilde{\nu}_m(k)\bigl(\tau^{-1/3}\bigr)^m \\ + \left(\sum_{m=0}^{\infty}\frac{\upsilon_m(k)}{\bigl(\tau^{1/3}\bigr)^m} + \mathcal{O}\bigl(\mathrm{e}^{-\frac{3\sqrt{3}}{2}(\sqrt{3}+\mathrm{i}k)(\varepsilon b)^{1/3}\tau^{2/3}}\bigr)\right)\mathrm{e}^{-\frac{3\sqrt{3}}{2}(\sqrt{3}+\mathrm{i}k)(\varepsilon b)^{1/3}\tau^{2/3}};\end{aligned} \tag{E.13}$$

it remains to determine the expansion coefficients $\{\widetilde{\nu}_m(k)\}_{m=1}^{\infty}$ and the first non-zero coefficient $\upsilon_m(k)$. Via the definitions (1.31), the isomonodromy deformations (1.36), the definitions (1.38), (1.39), and (3.2), and equation (E.11), one shows that the function $\mathcal{B}_k(\tau)$ solves the following inhomogeneous second-order nonlinear ODE

$$\begin{aligned}\mathcal{B}_k''(\tau) - 2(\mathcal{B}_k'(\tau))^2 - \left(\frac{\mathrm{d}}{\mathrm{d}\tau}\ln\left(\frac{u(\tau)}{\tau^{2/3}}\right)\right)\mathcal{B}_k'(\tau) \\ = \frac{1}{2\tau}\left(\frac{2}{3}\frac{\mathrm{d}}{\mathrm{d}\tau}\ln\left(\frac{u(\tau)}{\tau^{1/3}}\right) + \mathrm{i}a\frac{\mathrm{d}}{\mathrm{d}\tau}\ln\left(\frac{u(\tau)}{\tau^{1+\mathrm{i}a}}\right) + 8\varepsilon u(\tau)\right),\end{aligned} \tag{E.14}$$

where (cf. equation (3.20)) $u(\tau) = \frac{1}{2}\varepsilon(\varepsilon b)^{2/3}\mathrm{e}^{-\mathrm{i}2\pi k/3}\bigl(\tau^{1/3} + v_{0,k}(\tau)\bigr)$, with $v_{0,k}(\tau)$ given in the asymptotics (4.1). From the expression for $u'(\tau)$ given in the proof of [61, Proposition 5.7] and the definitions (1.31) and (3.2), it follows that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\ln(u(\tau)) = \frac{u'(\tau)}{u(\tau)} = \frac{1}{\tau} + 2\varepsilon\left(\frac{a(\tau)d(\tau) - b(\tau)c(\tau)}{u(\tau)}\right); \tag{E.15}$$

via equation (3.20), the asymptotics (4.1), (4.2), (4.99), and (4.100), and equation (E.15), one shows that, for $k \in \{\pm 1\}$,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\ln(u(\tau)) \underset{\tau\to+\infty}{=} \frac{1}{3\tau}\left(1 + \sum_{m=0}^{\infty}\frac{\mu_m^*(k)}{\bigl(\tau^{1/3}\bigr)^{m+2}}\right)$$

$$-\mathbb{V}_0(k)\tau^{-2/3}\mathrm{e}^{-\frac{3\sqrt{3}}{2}(\sqrt{3}+\mathrm{i}k)(\varepsilon b)^{1/3}\tau^{2/3}}\big(1+\mathcal{O}\big(\tau^{-1/3}\big)\big), \tag{E.16}$$

where the expansion coefficients $\{\mu_m^*(k)\}_{m=0}^{\infty}$ are given in equations (E.6)–(E.9), and

$$\mathbb{V}_0(k) := \frac{k2^{1/2}3^{1/4}\mathrm{e}^{\mathrm{i}\pi k/3}\mathrm{e}^{\mathrm{i}\pi k/4}(\varepsilon b)^{1/6}\big(s_0^0-\mathrm{i}\mathrm{e}^{-\pi a}\big)}{\sqrt{\pi}\big(2+\sqrt{3}\big)^{-\mathrm{i}ka}}.$$

Substituting the asymptotic expansions (2.1), (E.13), and (E.16) into the second-order nonlinear ODE (E.14), and equating coefficients of terms of orders

$$\mathcal{O}\big(\big(\tau^{-1/3}\big)^{m_1}\exp\big(-\tfrac{3\sqrt{3}}{2}\big(\sqrt{3}+\mathrm{i}k\big)(\varepsilon b)^{1/3}\tau^{2/3}\big)\big),$$

$m_1=2,3$, and $\mathcal{O}\big(\big(\tau^{-1/3}\big)^{m_2}\big)$, $\mathbb{N}\ni m_2\geqslant 2$, one arrives at, after simplification, for $k\in\{\pm1\}$, in the indicated order:

(i) $\mathcal{O}\big(\tau^{-2/3}\exp\big(-\tfrac{3\sqrt{3}}{2}\big(\sqrt{3}+\mathrm{i}k\big)(\varepsilon b)^{1/3}\tau^{2/3}\big)\big)\Rightarrow$

$$\sqrt{3}\big(\sqrt{3}+\mathrm{i}k\big)^2\big(\sqrt{3}-2k\big)(\varepsilon b)^{2/3}\upsilon_0(k)=0; \tag{E.17}$$

(ii) $\mathcal{O}\big(\tau^{-1}\exp\big(-\tfrac{3\sqrt{3}}{2}\big(\sqrt{3}+\mathrm{i}k\big)(\varepsilon b)^{1/3}\tau^{2/3}\big)\big)\Rightarrow$

$$\begin{aligned}&\sqrt{3}\big(\sqrt{3}+\mathrm{i}k\big)^2\big(\sqrt{3}-2k\big)(\varepsilon b)^{2/3}\upsilon_1(k)\\&\quad=\frac{\big(-\mathrm{i}2+\sqrt{3}\big(\sqrt{3}+\mathrm{i}k\big)\mathrm{e}^{\mathrm{i}\pi k/3}\big)\,\mathrm{e}^{\mathrm{i}\pi k/4}(\varepsilon b)^{1/2}\big(s_0^0-\mathrm{i}\mathrm{e}^{-\pi a}\big)}{\sqrt{2\pi}3^{1/4}\big(2+\sqrt{3}\big)^{-\mathrm{i}ka}};\end{aligned} \tag{E.18}$$

(iii) $\mathcal{O}\big(\tau^{-2/3}\big)\Rightarrow$

$$-4\mathrm{e}^{-\mathrm{i}2\pi k/3}=\big(k\sqrt{3}+\mathrm{i}\big)^2; \tag{E.19}$$

(iv) $\mathcal{O}\big(\tau^{-4/3}\big)\Rightarrow$

$$2\mathrm{i}\mathrm{e}^{-\mathrm{i}\pi k/3}=k\sqrt{3}+\mathrm{i}; \tag{E.20}$$

(v) $\mathcal{O}\big(\tau^{-5/3}\big)\Rightarrow$

$$\widetilde{\nu}_1(k)=0; \tag{E.21}$$

(vi) $\mathcal{O}\big(\tau^{-2}\big)\Rightarrow$

$$4\widetilde{\nu}_2(k)-\frac{a\mathrm{e}^{\mathrm{i}\pi k/3}}{3(\varepsilon b)^{1/3}}=\frac{2\mathrm{i}a(a-\mathrm{i}/2)\mathrm{e}^{\mathrm{i}\pi k/3}}{3(\varepsilon b)^{1/3}}; \tag{E.22}$$

(vii) $\mathcal{O}\big(\tau^{-7/3}\big)\Rightarrow$

$$\widetilde{\nu}_3(k)=0; \tag{E.23}$$

(viii) $\mathcal{O}\big(\tau^{-8/3}\big)\Rightarrow$

$$4\mathrm{i}\mathrm{e}^{-\mathrm{i}\pi k/3}(\varepsilon b)^{1/3}\widetilde{\nu}_4(k)=\frac{a\mathrm{e}^{\mathrm{i}\pi k/3}}{9(\varepsilon b)^{1/3}}\left(\frac{1-2a^2}{3}+\mathrm{i}a\right); \tag{E.24}$$

(ix) $\mathcal{O}\big(\tau^{-(m+9)/3}\big)$, $m\in\mathbb{Z}_{+}$, $\Rightarrow$

$$\begin{aligned}&4\mathrm{i}\mathrm{e}^{-\mathrm{i}\pi k/3}(\varepsilon b)^{1/3}(m+5)\widetilde{\nu}_{m+5}(k)\\&\quad=-6\mathrm{e}^{-\mathrm{i}2\pi k/3}(\varepsilon b)^{2/3}\mathfrak{u}_{m+5}(k)-\frac{(1+\mathrm{i}2a)}{3}\mu^{*}_{m+1}(k)+\mathrm{i}\mathrm{e}^{-\mathrm{i}\pi k/3}(\varepsilon b)^{1/3}\mu^{*}_{m+3}(k)\\&\qquad+\frac{1}{3}\Bigg((m+3)(m+5+\mathrm{i}2a)\widetilde{\nu}_{m+3}(k)+\sum_{j=0}^{m-1}(j+1)\widetilde{\nu}_{j+1}(k)(\mu^{*}_{m-j}(k)\\&\qquad-2(m+2-j)\widetilde{\nu}_{m+2-j}(k))-\frac{2\mathrm{i}a^{2}\mathrm{e}^{\mathrm{i}\pi k/3}}{3(\varepsilon b)^{1/3}}(m+1)\widetilde{\nu}_{m+1}(k)\Bigg),\end{aligned}\tag{E.25}$$

with the convention $\sum_{j=0}^{-1}*:=0$. Solving equations (E.17) and (E.18) for $\upsilon_0(k)$ and $\upsilon_1(k)$, $k\in\{\pm1\}$, respectively, one shows that

$$\upsilon_0(k)=0\qquad\text{and}\qquad\upsilon_1(k)=-\frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}\pi k/3}\mathrm{e}^{\mathrm{i}\pi k/4}\big(2+\sqrt{3}\big)^{\mathrm{i}ka}\big(s^{0}_{0}-\mathrm{i}\mathrm{e}^{-\pi a}\big)}{\sqrt{2\pi}3^{3/4}\big(\sqrt{3}-k\big)(\varepsilon b)^{1/6}}.$$

Equations (E.19) and (E.20) are identities. Solving equations (E.21)–(E.25) for the coefficients $\widetilde{\nu}_1(k)$, $\widetilde{\nu}_2(k)$, $\widetilde{\nu}_3(k)$, $\widetilde{\nu}_4(k)$, and $\widetilde{\nu}_{m+5}(k)$, $k\in\{\pm1\}$, $m\in\mathbb{Z}_{+}$, respectively, one arrives at equations (E.3)–(E.9); therefore, the trans-series asymptotics for the function $b(\tau)$ is now established via equations (E.11)–(E.13); in particular, for $k\in\{\pm1\}$,

$$\begin{aligned}\mathcal{B}_k(\tau)\underset{\tau\to+\infty}{=}&\frac{\mathrm{i}a}{6}\ln\tau-\frac{3k}{4}\big(\sqrt{3}+\mathrm{i}k\big)(\varepsilon b)^{1/3}\tau^{2/3}+\sum_{m=1}^{\infty}\widetilde{\nu}_m(k)\big(\tau^{-1/3}\big)^{m}\\&-\frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}\pi k/3}\mathrm{e}^{\mathrm{i}\pi k/4}\big(2+\sqrt{3}\big)^{\mathrm{i}ka}\big(s^{0}_{0}-\mathrm{i}\mathrm{e}^{-\pi a}\big)}{\sqrt{2\pi}3^{3/4}\big(\sqrt{3}-k\big)(\varepsilon b)^{1/6}\tau^{1/3}}\mathrm{e}^{-\frac{3\sqrt{3}}{2}(\sqrt{3}+\mathrm{i}k)(\varepsilon b)^{1/3}\tau^{2/3}}\\&\times\big(1+\mathcal{O}\big(\tau^{-1/3}\big)\big).\end{aligned}\tag{E.26}$$

Via equation (3.20), the asymptotics (4.1) and (4.2), equation (E.10), the definition (E.12) (cf. equation (4.5)), the asymptotics (E.26), and the expansion

$$\begin{aligned}&\ln\left(1+\sum_{m=0}^{\infty}\frac{\mathfrak{u}_m(k)}{\big(\tau^{1/3}\big)^{m+2}}\right)\\&\underset{\tau\to+\infty}{=}\sum_{m=2}^{\infty}\sum_{\substack{n,\mathfrak{l}\in\mathbb{N}\\ \mathfrak{l}\geqslant n\\ n+\mathfrak{l}=m}}\sum_{\substack{\mathfrak{i}_1+2\mathfrak{i}_2+\cdots+\mathfrak{l}\mathfrak{i}_{\mathfrak{l}}=\mathfrak{l}\\ \mathfrak{i}_1+\mathfrak{i}_2+\cdots+\mathfrak{i}_{\mathfrak{l}}=n}}\frac{\mathrm{S}^{(1)}_n(\mathfrak{u}_0(k))^{\mathfrak{i}_1}(\mathfrak{u}_1(k))^{\mathfrak{i}_2}\cdots(\mathfrak{u}_{\mathfrak{l}-1}(k))^{\mathfrak{i}_{\mathfrak{l}}}}{\mathfrak{i}_1!\mathfrak{i}_2!\cdots\mathfrak{i}_{\mathfrak{l}}!\big(\tau^{1/3}\big)^{m}},\end{aligned}\tag{E.27}$$

where $\mathrm{S}^{(1)}_n=(-1)^{n-1}(n-1)!$ is a special value of the Stirling number of the first kind [32], one arrives at, for $k\in\{\pm1\}$, the $(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)=(0,0,0|0)$ trans-series asymptotics (as $\tau\to+\infty$ with $\varepsilon b>0$) for the function $\hat{\varphi}(\tau)$

$$\begin{aligned}\hat{\varphi}(\tau)\underset{\tau\to+\infty}{=}&\mathrm{i}\mathcal{L}^{(0|0)}_{(0,0)}(k)-k\pi+\frac{\mathrm{i}3k}{2}\big(\sqrt{3}+\mathrm{i}k\big)(\varepsilon b)^{1/3}\tau^{2/3}+2a\ln\left(\frac{2\mathrm{e}^{-\mathrm{i}\pi k/3}\tau^{2/3}}{(\varepsilon b)^{1/6}}\right)\\&-\mathrm{i}\sum_{m=2}^{\infty}\left(2\widetilde{\nu}_m(k)+\sum_{\substack{n,\mathfrak{l}\in\mathbb{N}\\ \mathfrak{l}\geqslant n\\ n+\mathfrak{l}=m}}\sum_{\substack{\mathfrak{i}_1+2\mathfrak{i}_2+\cdots+\mathfrak{l}\mathfrak{i}_{\mathfrak{l}}=\mathfrak{l}\\ \mathfrak{i}_1+\mathfrak{i}_2+\cdots+\mathfrak{i}_{\mathfrak{l}}=n}}(-1)^{n-1}(n-1)!\prod_{\mathfrak{j}=1}^{\mathfrak{l}}\frac{(\mathfrak{u}_{\mathfrak{j}-1}(k))^{\mathfrak{i}_{\mathfrak{j}}}}{\mathfrak{i}_{\mathfrak{j}}!}\right)\end{aligned}$$

$$\times \left(\tau^{-1/3}\right)^{m} - \frac{k\mathrm{e}^{-\mathrm{i}\pi k/3}\mathrm{e}^{\mathrm{i}\pi k/4}\left(2+\sqrt{3}\right)^{\mathrm{i}ka}\left(s_0^0-\mathrm{i}\mathrm{e}^{-\pi a}\right)}{\sqrt{2\pi}3^{3/4}(\varepsilon b)^{1/6}\tau^{1/3}}\mathrm{e}^{-\frac{3\sqrt{3}}{2}(\sqrt{3}+\mathrm{i}k)(\varepsilon b)^{1/3}\tau^{2/3}}$$
$$\times \left(1+\mathcal{O}\left(\tau^{-1/3}\right)\right), \tag{E.28}$$

where

$$\mathfrak{L}_{(0,0)}^{(0|0)}(k) = \begin{cases} \ln\left(g_{11}\mathrm{e}^{\pi a}\right)^{2}, & k=+1, \\ \ln\left(g_{22}\mathrm{e}^{\pi a}\right)^{-2}, & k=-1. \end{cases}$$

Finally, applying the (map) isomorphism (cf. Appendix D) $\mathcal{F}^{\{\ell\}}_{\varepsilon_1,\varepsilon_2,m(\varepsilon_2)}$, whose action on $\mathcal{M}$ is given by equations (D.47)–(D.61) and (D.71)–(D.85), to the corresponding $(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell) = (0,0,0|0)$ asymptotics (E.28) for $\hat{\varphi}(\tau)$, one arrives at the trans-series asymptotics (E.1) (and equations (E.2)–(E.9)) stated in the theorem. ■

Remark E.4. Via equation (E.11), the definition (E.12) (cf. equation (4.5)), and the asymptotics (E.26), one arrives at, from the asymptotics (4.99), (4.100), and (4.101), respectively, the trans-series asymptotics (as $\tau \to +\infty$ with $\varepsilon b > 0$) for the functions $c(\tau)$, $d(\tau)$, and $a(\tau)$.

Remark E.5. It is instructive to illustrate the first few contributions of the multi-indexed double summation in equation (E.27) to the asymptotics of $\hat{\varphi}(\tau)$ for various values of the index m: (i) for $m=2$ (that is, $\mathcal{O}\left(\tau^{-2/3}\right)$), $(n,\mathfrak{l})=(1,1) \Rightarrow \mathfrak{i}_1=1$, thus, for $k\in\{\pm1\}$,[62]

$$\sum_{\substack{n,\mathfrak{l}\in\mathbb{N}\\ \mathfrak{l}\geqslant n\\ n+\mathfrak{l}=2}} \sum_{\substack{\mathfrak{i}_1+2\mathfrak{i}_2+\cdots+\mathfrak{l}\mathfrak{i}_{\mathfrak{l}}=\mathfrak{l}\\ \mathfrak{i}_1+\mathfrak{i}_2+\cdots+\mathfrak{i}_{\mathfrak{l}}=n}} (-1)^{n-1}(n-1)!\prod_{\mathfrak{j}=1}^{\mathfrak{l}}\frac{(\mathfrak{u}_{\mathfrak{j}-1}(k))^{\mathfrak{i}_{\mathfrak{j}}}}{\mathfrak{i}_{\mathfrak{j}}!} = \mathfrak{u}_0(k) = \frac{a\mathrm{e}^{-\mathrm{i}2\pi k/3}}{3(\varepsilon b)^{1/3}};$$

(ii) for $m=3$ (that is, $\mathcal{O}\left(\tau^{-1}\right)$), $(n,\mathfrak{l})=(1,2) \Rightarrow (\mathfrak{i}_1,\mathfrak{i}_2)=(0,1)$, thus, for $k\in\{\pm1\}$,

$$\sum_{\substack{n,\mathfrak{l}\in\mathbb{N}\\ \mathfrak{l}\geqslant n\\ n+\mathfrak{l}=3}} \sum_{\substack{\mathfrak{i}_1+2\mathfrak{i}_2+\cdots+\mathfrak{l}\mathfrak{i}_{\mathfrak{l}}=\mathfrak{l}\\ \mathfrak{i}_1+\mathfrak{i}_2+\cdots+\mathfrak{i}_{\mathfrak{l}}=n}} (-1)^{n-1}(n-1)!\prod_{\mathfrak{j}=1}^{\mathfrak{l}}\frac{(\mathfrak{u}_{\mathfrak{j}-1}(k))^{\mathfrak{i}_{\mathfrak{j}}}}{\mathfrak{i}_{\mathfrak{j}}!} = \mathfrak{u}_1(k) = 0;$$

(iii) for $m=4$ (that is, $\mathcal{O}\left(\tau^{-4/3}\right)$), $(n,\mathfrak{l})=(2,2) \Rightarrow (\mathfrak{i}_1,\mathfrak{i}_2)=(2,0)$, and $(n,\mathfrak{l})=(1,3) \Rightarrow (\mathfrak{i}_1,\mathfrak{i}_2,\mathfrak{i}_3)=(0,0,1)$, thus, for $k\in\{\pm1\}$,

$$\sum_{\substack{n,\mathfrak{l}\in\mathbb{N}\\ \mathfrak{l}\geqslant n\\ n+\mathfrak{l}=4}} \sum_{\substack{\mathfrak{i}_1+2\mathfrak{i}_2+\cdots+\mathfrak{l}\mathfrak{i}_{\mathfrak{l}}=\mathfrak{l}\\ \mathfrak{i}_1+\mathfrak{i}_2+\cdots+\mathfrak{i}_{\mathfrak{l}}=n}} (-1)^{n-1}(n-1)!\prod_{\mathfrak{j}=1}^{\mathfrak{l}}\frac{(\mathfrak{u}_{\mathfrak{j}-1}(k))^{\mathfrak{i}_{\mathfrak{j}}}}{\mathfrak{i}_{\mathfrak{j}}!} = \mathfrak{u}_2(k) - \frac{(\mathfrak{u}_0(k))^2}{2} = \frac{a^2\mathrm{e}^{-\mathrm{i}\pi k/3}}{18(\varepsilon b)^{2/3}};$$

(iv) for $m=5$ (that is, $\mathcal{O}\left(\tau^{-5/3}\right)$), $(n,\mathfrak{l})=(2,3) \Rightarrow (\mathfrak{i}_1,\mathfrak{i}_2,\mathfrak{i}_3)=(1,1,0)$, and $(n,\mathfrak{l})=(1,4) \Rightarrow (\mathfrak{i}_1,\mathfrak{i}_2,\mathfrak{i}_3,\mathfrak{i}_4)=(0,0,0,1)$, thus, for $k\in\{\pm1\}$,

$$\sum_{\substack{n,\mathfrak{l}\in\mathbb{N}\\ \mathfrak{l}\geqslant n\\ n+\mathfrak{l}=5}} \sum_{\substack{\mathfrak{i}_1+2\mathfrak{i}_2+\cdots+\mathfrak{l}\mathfrak{i}_{\mathfrak{l}}=\mathfrak{l}\\ \mathfrak{i}_1+\mathfrak{i}_2+\cdots+\mathfrak{i}_{\mathfrak{l}}=n}} (-1)^{n-1}(n-1)!\prod_{\mathfrak{j}=1}^{\mathfrak{l}}\frac{(\mathfrak{u}_{\mathfrak{j}-1}(k))^{\mathfrak{i}_{\mathfrak{j}}}}{\mathfrak{i}_{\mathfrak{j}}!} = \mathfrak{u}_3(k) - \mathfrak{u}_0(k)\mathfrak{u}_1(k) = 0;$$

[62]Recall that the expansion coefficients $\{\mathfrak{u}_j(k)\}_{j=0}^{\infty}$, $k\in\{\pm1\}$, are given in equations (2.2)–(2.9).

and (v) for $m=6$ (that is, $\mathcal{O}(\tau^{-2})$), $(n,\mathfrak{l})=(3,3) \Rightarrow (\mathfrak{i}_1,\mathfrak{i}_2,\mathfrak{i}_3)=(3,0,0)$, $(n,\mathfrak{l})=(2,4) \Rightarrow (\mathfrak{i}_1,\mathfrak{i}_2,\mathfrak{i}_3,\mathfrak{i}_4) \in \{(1,0,1,0),(0,2,0,0)\}$, and $(n,\mathfrak{l})=(1,5) \Rightarrow (\mathfrak{i}_1,\mathfrak{i}_2,\mathfrak{i}_3,\mathfrak{i}_4,\mathfrak{i}_5)=(0,0,0,0,1)$, thus, for $k\in\{\pm1\}$,

$$\sum_{\substack{n,\mathfrak{l}\in\mathbb{N}\\ \mathfrak{l}\geqslant n\\ n+\mathfrak{l}=6}} \sum_{\substack{\mathfrak{i}_1+2\mathfrak{i}_2+\cdots+\mathfrak{l}\mathfrak{i}_{\mathfrak{l}}=\mathfrak{l}\\ \mathfrak{i}_1+\mathfrak{i}_2+\cdots+\mathfrak{i}_{\mathfrak{l}}=n}} (-1)^{n-1}(n-1)! \prod_{\mathfrak{j}=1}^{\mathfrak{l}} \frac{(\mathfrak{u}_{\mathfrak{j}-1}(k))^{\mathfrak{i}_{\mathfrak{j}}}}{\mathfrak{i}_{\mathfrak{j}}!}$$
$$= \mathfrak{u}_4(k)-\mathfrak{u}_0(k)\mathfrak{u}_2(k)+\frac{(\mathfrak{u}_0(k))^3}{3}-\frac{(\mathfrak{u}_1(k))^2}{2} = -\frac{a}{3^4(\varepsilon b)}.$$

Theorem E.6. *Let $u(\tau)$ be a solution of the* DP3E (1.1) *and $\hat{\varphi}(\tau)$ be the general solution of the* ODE *$\hat{\varphi}'(\tau)=2a\tau^{-1}+b(u(\tau))^{-1}$ for $\varepsilon b>0$ corresponding to the monodromy data $(a,s_0^0,s_0^\infty,s_1^\infty,g_{11},g_{12},g_{21},g_{22})$. Let*

$$\hat{\varepsilon}_1\in\{\pm1\}, \qquad \hat{\varepsilon}_2\in\{0,\pm1\}, \qquad \hat{m}(\hat{\varepsilon}_2)=\begin{cases}0, & \hat{\varepsilon}_2\in\{\pm1\},\\ \pm\hat{\varepsilon}_1, & \hat{\varepsilon}_2=0,\end{cases} \qquad \hat{\ell}\in\{0,1\},$$

and $\varepsilon b=|\varepsilon b|\mathrm{e}^{\mathrm{i}\pi\hat{\varepsilon}_2}$. For $k=+1$, let

$$\hat{g}_{11}\big(\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\big)\hat{g}_{12}\big(\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\big)\hat{g}_{21}\big(\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\big)\neq0, \qquad \hat{g}_{22}\big(\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\big)=0,$$

and, for $k=-1$, let

$$\hat{g}_{11}\big(\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\big)=0, \qquad \hat{g}_{12}\big(\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\big)\hat{g}_{21}\big(\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\big)\hat{g}_{22}\big(\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\big)\neq0,$$

where explicit expressions for $\hat{g}_{ij}\big(\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\big)$, $i,j\in\{1,2\}$, are given in Appendix D, equations (D.62)–(D.70) and (D.86)–(D.93). Then, for $\hat{s}_0^0\big(\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\big)\neq\mathrm{i}\mathrm{e}^{(-1)^{\hat{\varepsilon}_2}\pi a}$,[63]

$$\begin{aligned}(-1)^{1+\hat{\varepsilon}_2}\hat{\varphi}(\tau) \underset{\tau\to+\infty\mathrm{e}^{\mathrm{i}\pi\hat{\varepsilon}_1/2}}{=} {}& 3\mathrm{e}^{\frac{\mathrm{i}2\pi k}{3}}(-1)^{\hat{\varepsilon}_2}(\varepsilon b)^{1/3}\tau_*^{2/3}+2(-1)^{1+\hat{\varepsilon}_2}a\ln\left(\frac{2\mathrm{e}^{-\frac{\mathrm{i}\pi k}{3}}\tau_*^{2/3}}{\big(\varepsilon b\mathrm{e}^{-\mathrm{i}\pi\hat{\varepsilon}_2}\big)^{1/6}}\right)\\
&+\mathrm{i}\hat{\mathcal{L}}^{(\hat{m}(\hat{\varepsilon}_2)|\hat{\ell})}_{(\hat{\varepsilon}_1,\hat{\varepsilon}_2)}(k)-k\pi-\mathrm{i}\sum_{m=2}^{\infty}\bigg(2\widehat{\nu}_m(k)\\
&+\sum_{\substack{n,\mathfrak{l}\in\mathbb{N}\\ \mathfrak{l}\geqslant n\\ n+\mathfrak{l}=m}} \sum_{\substack{\mathfrak{i}_1+2\mathfrak{i}_2+\cdots+\mathfrak{l}\mathfrak{i}_{\mathfrak{l}}=\mathfrak{l}\\ \mathfrak{i}_1+\mathfrak{i}_2+\cdots+\mathfrak{i}_{\mathfrak{l}}=n}} (-1)^{n-1}(n-1)! \prod_{\mathfrak{j}=1}^{\mathfrak{l}} \frac{(\hat{\mathfrak{u}}_{\mathfrak{j}-1}(k))^{\mathfrak{i}_{\mathfrak{j}}}}{\mathfrak{i}_{\mathfrak{j}}!}\bigg)\\
&\times\big(\tau_*^{-1/3}\big)^m-\frac{k\mathrm{e}^{-\frac{\mathrm{i}\pi k}{3}}\mathrm{e}^{\frac{\mathrm{i}\pi k}{4}}\big(2+\sqrt{3}\big)^{\mathrm{i}k(-1)^{1+\hat{\varepsilon}_2}a}}{\sqrt{2\pi}3^{3/4}\big(\varepsilon b\mathrm{e}^{-\mathrm{i}\pi\hat{\varepsilon}_2}\big)^{1/6}\tau_*^{1/3}}\\
&\times\big(\hat{s}_0^0\big(\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)|\hat{\ell}\big)-\mathrm{i}\mathrm{e}^{(-1)^{\hat{\varepsilon}_2}\pi a}\big)\mathrm{e}^{-\frac{3\sqrt{3}}{2}(\sqrt{3}+\mathrm{i}k)(-1)^{\hat{\varepsilon}_2}(\varepsilon b)^{1/3}\tau_*^{2/3}}\\
&\times\big(1+\mathcal{O}\big(\tau^{-1/3}\big)\big), \qquad k\in\{\pm1\},\end{aligned} \tag{E.29}$$

where τ_* is defined by equation (2.20),

$$\hat{\mathcal{L}}^{(\hat{m}(\hat{\varepsilon}_2)|\hat{\ell})}_{(\hat{\varepsilon}_1,\hat{\varepsilon}_2)}(k)=\begin{cases}\ln\big(\hat{g}_{11}(\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)|\hat{\ell})\mathrm{e}^{(-1)^{1+\hat{\varepsilon}_2}\pi a}\big)^2, & k=+1,\\ \ln\big(\hat{g}_{22}(\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)|\hat{\ell})\mathrm{e}^{(-1)^{1+\hat{\varepsilon}_2}\pi a}\big)^{-2}, & k=-1,\end{cases} \tag{E.30}$$

[63]Recall that (cf. Remark 2.1) $\hat{s}_0^0(\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)|\hat{\ell})=s_0^0$. For $\hat{s}_0^0(\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)|\hat{\ell})=\mathrm{i}\mathrm{e}^{(-1)^{\hat{\varepsilon}_2}\pi a}$, the exponentially small correction term in the asymptotics (E.29) is absent.

$$\widehat{\nu}_1(k)=0, \qquad \widehat{\nu}_2(k)=-\frac{a\big(1+\mathrm{i}(-1)^{1+\hat{\varepsilon}_2}a\big)\mathrm{e}^{\mathrm{i}\pi k/3}}{6(\varepsilon b)^{1/3}}, \qquad \widehat{\nu}_3(k)=0, \tag{E.31}$$

$$\widehat{\nu}_4(k)=\frac{\mathrm{i}(-1)^{\hat{\varepsilon}_2}a\mathrm{e}^{\mathrm{i}2\pi k/3}}{36(\varepsilon b)^{2/3}}\left(\frac{1-2a^2}{3}+\mathrm{i}(-1)^{1+\hat{\varepsilon}_2}a\right), \tag{E.32}$$

and

$$\begin{aligned}(m+5)\widehat{\nu}_{m+5}(k) &= \frac{3\mathrm{i}}{2}\mathrm{e}^{-\mathrm{i}\pi k/3}(-1)^{\hat{\varepsilon}_2}(\varepsilon b)^{1/3}\hat{\mathfrak{u}}_{m+5}(k)+\frac{\mathrm{i}(-1)^{\hat{\varepsilon}_2}\mathrm{e}^{\mathrm{i}\pi k/3}\big(1+2\mathrm{i}(-1)^{1+\hat{\varepsilon}_2}a\big)}{12(\varepsilon b)^{1/3}}\hat{\mu}^{*}_{m+1}(k)\\ &\quad+\frac{1}{4}\hat{\mu}^{*}_{m+3}(k)-\frac{\mathrm{i}(-1)^{\hat{\varepsilon}_2}\mathrm{e}^{\mathrm{i}\pi k/3}}{12(\varepsilon b)^{1/3}}\Bigg((m+3)\big(m+5+2\mathrm{i}(-1)^{1+\hat{\varepsilon}_2}a\big)\widehat{\nu}_{m+3}(k)\\ &\quad+\sum_{j=0}^{m-1}(j+1)\widehat{\nu}_{j+1}(k)(\hat{\mu}^{*}_{m-j}(k)-2(m+2-j)\widehat{\nu}_{m+2-j}(k))\\ &\quad-\frac{\mathrm{i}(-1)^{\hat{\varepsilon}_2}2a^2\mathrm{e}^{\mathrm{i}\pi k/3}}{3(\varepsilon b)^{1/3}}(m+1)\widehat{\nu}_{m+1}(k)\Bigg), \qquad m\in\mathbb{Z}_+,\end{aligned} \tag{E.33}$$

with

$$\hat{\mu}^{*}_0(k)=-\frac{2a\mathrm{e}^{\mathrm{i}\pi k/3}}{3(\varepsilon b)^{1/3}}, \qquad \hat{\mu}^{*}_1(k)=0, \tag{E.34}$$

$$\hat{\mu}^{*}_{m_1+2}(k)=-2\left(\widehat{\mathrm{P}}^{*}_{m_1+2}(k)+\hat{\mathfrak{w}}_{m_1+2}(k)+\sum_{j=0}^{m_1}\widehat{\mathrm{P}}^{*}_j(k)\hat{\mathfrak{w}}_{m_1-j}(k)\right), \qquad m_1\in\mathbb{Z}_+, \tag{E.35}$$

and

$$\widehat{\mathrm{P}}^{*}_0(k)=\frac{2a\mathrm{e}^{\mathrm{i}\pi k/3}}{3(\varepsilon b)^{1/3}}, \qquad \widehat{\mathrm{P}}^{*}_1(k)=0, \tag{E.36}$$

$$\widehat{\mathrm{P}}^{*}_j(k)=\frac{3}{2}\left(\hat{\mathfrak{u}}_j(k)-\mathrm{i}(-1)^{\hat{\varepsilon}_2}\mathrm{e}^{\frac{\mathrm{i}2\pi k}{3}}(\varepsilon b)^{1/3}\left(\hat{\mathfrak{r}}_{j+2}(k)-2\hat{\mathfrak{u}}_{j+2}(k)+\sum_{m_2=0}^{j}\hat{\mathfrak{u}}_{m_2}(k)\hat{\mathfrak{r}}_{j-m_2}(k)\right)\right),$$
$$\mathbb{N}\ni j\geqslant 2, \tag{E.37}$$

where the expansion coefficients $\hat{\mathfrak{u}}_m(k)$ *and* $\hat{\mathfrak{w}}_m(k)$ *(resp.,* $\hat{\mathfrak{r}}_m(k)$*),* $m\in\mathbb{Z}_+$*,* $k\in\{\pm 1\}$*, are given in equations* (2.22)–(2.27) *(resp.,* (2.30) *and* (2.31)*).*

Proof. Applying the (map) isomorphism (cf. Appendix D) $\hat{\mathcal{F}}^{\{\hat{\ell}\}}_{\hat{\varepsilon}_1,\hat{\varepsilon}_2,\hat{m}(\hat{\varepsilon}_2)}$, whose action on $\mathcal{M}$ is given by equations (D.62)–(D.70) and (D.86)–(D.93), to the $(\varepsilon_1,\varepsilon_2,m(\varepsilon_2)|\ell)=(0,0,0|0)$ asymptotics (E.28) (as $\tau\to+\infty$ with $\varepsilon b>0$) for $\hat{\varphi}(\tau)$, one arrives at the trans-series asymptotics (E.29) (and equations (E.30)–(E.37)) stated in the theorem. ■

F Literature survey of the DP3E

The interested reader will find representative samples of the ubiquitous manifestations of the DP3E (1.1) in this appendix.

(i) It was shown in [75] that a variant of the DP3E (1.1) appears in the characterisation of the effect of the small dispersion on the self-focusing of solutions of the fundamental equations of nonlinear optics in the one-dimensional case, where the main order of the influence of this effect is described via a universal special monodromic solution of the nonlinear Schrödinger equation (NLSE); in particular, the author studies the asymptotics of a function that can be identified as a solution (the so-called 'Suleimanov solution') of a slightly modified, yet equivalent, version of the DP3E (1.1) for the parameter values $a=\mathrm{i}/2$ and $b=64k^{-3}$, where $k>0$ is a physical variable.

(ii) In [56], an extensive number-theoretic and asymptotic analysis of the universal special monodromic solution considered in [75] is presented: the author studies a particular meromorphic solution of the DP3E (1.1) that vanishes at the origin; more specifically, it is proved that, for $-2\mathrm{i}a\in\mathbb{Z}$, the aforementioned solution exists and is unique, and, for the case $a-\mathrm{i}/2\in\mathbb{Z}$, this solution exists and is unique provided that $u(\tau)=-u(-\tau)$. The bulk of the analysis presented in [56] focuses on the study of the Taylor-series expansion coefficients of the solution to the DP3E (1.1) that is holomorphic at $\tau=0$; in particular, upon invoking the 'normalisation condition' $b=a$ and taking $\varepsilon=+1$, it is shown that, for general values of the parameter a, these coefficients are rational functions of a^2 that possess remarkable number-theoretic properties: en route, novel notions such as super-generating functions and quasi-periodic fences are introduced. The author also studies the connection problem for the "Suleimanov solution" [60] of the DP3E (1.1).

(iii) Unlike the physical optics context adopted in [75], the authors of [8] provide a colossal Riemann–Hilbert problem (RHP) asymptotic analysis of the solution of the focusing NLSE, $\mathrm{i}\partial_{\mathrm{T}}\Psi+\frac{1}{2}\partial_{\mathrm{X}}^2\Psi+|\Psi|^2\Psi=0$, by considering the rogue wave solution $\Psi(\mathrm{X},\mathrm{T})$ of infinite order, that is, a scaling limit of a sequence of particular solutions of the focusing NLSE modelling so-called rogue waves of ever-increasing amplitude, and show that, in the regime of large variables $\mathbb{R}^2\ni(\mathrm{X},\mathrm{T})$ when $|\mathrm{X}|\to+\infty$ in such a way that $\mathrm{T}|\mathrm{X}|^{-3/2}-54^{-1/2}=\mathcal{O}\big(|\mathrm{X}|^{-1/3}\big)$, the rogue wave of infinite order $\Psi(\mathrm{X},\mathrm{T})$ can be expressed explicitly in terms of a function $\mathcal{V}(y)$ extracted from the solution of the Jimbo–Miwa Painlevé II (PII) RHP for parameters $p=\ln(2)/2\pi$ and $\tau=1$;[64] in particular, [8, Corollary 6] presents the leading term of the $\mathrm{T}\to+\infty$ asymptotics of the rogue wave of infinite order $\Psi(0,\mathrm{T})$ (see also [7, Theorem 2 and Section 4]),[65] which, in the context of the DP3E (1.1), coincides, up to a scalar, τ-independent factor, with $\exp(\mathrm{i}\hat{\varphi}(\tau))$, $\mathrm{T}=\tau^2$, where, given the solution, denoted by $\hat{u}(\tau)$, say, of the DP3E (1.1) studied in [56] for the monodromy data corresponding to $a=\mathrm{i}/2$ (and a suitable choice for the parameter b), $\hat{\varphi}(\tau)$ is the general solution of the ODE $\hat{\varphi}'(\tau)=2a\tau^{-1}+b(\hat{u}(\tau))^{-1}$.

(iv) The authors of [13] present an expansive study of algebraic solutions (rational functions of $\tau^{1/3}$) of the DP3E (1.1) for the parameter values $\varepsilon=-1$, $b=\mathrm{i}$, and $a=-\mathrm{i}n$, $n\in\mathbb{Z}$. By

[64] Not to be confused with the independent variable τ that appears in the DP3E (1.1) and throughout this work.

[65] For the rogue wave of infinite order [8], one needs to consider asymptotics of tronquée/tritronquée solutions of the inhomogeneous PII equation,

$$\frac{\mathrm{d}^2u(x;\alpha)}{\mathrm{d}x^2}=2(u(x;\alpha))^3+xu(x;\alpha)-\alpha,$$

for the special complex value of $\alpha=\frac{1}{2}+\mathrm{i}\frac{\ln(2)}{2\pi}$ (asymptotics for tronquée/tritronquée solutions of the PII equation with $\alpha=0$ are given in the monograph [29]), and to know that the increasing tritronquée solution, denoted $u_{\mathrm{TT}}^{-}(x;\alpha)$ in [64], is void of poles on $\mathbb{R}$; furthermore, for the function $\mathcal{V}(y)$ to have sense as a meaningful asymptotic representation of the rogue wave of infinite order $\Psi(\mathrm{X},\mathrm{T})$, it is, additionally, necessary that $u_{\mathrm{TT}}^{-}(x;\alpha)$ be a global solution (analytic $\forall\ x\in\mathbb{R}$) of the PII equation for $\alpha=\frac{1}{2}+\mathrm{i}\frac{\ln(2)}{2\pi}$. In [64], the author provides a complete RHP asymptotic analysis of the global nature of tritronquée solutions of the PII equation for various complex values of α, including the particular value $\alpha=\frac{1}{2}+\mathrm{i}\frac{\ln(2)}{2\pi}$, and relates the function $\mathcal{V}(y)$ to the PII equation, subsequently identifying the particular solution that is requisite in order to construct $\mathcal{V}(y)$ as the increasing tritronquée solution $u_{\mathrm{TT}}^{-}(x;\alpha)$ for the special parameter value $\alpha=\frac{1}{2}+\mathrm{i}\frac{\ln(2)}{2\pi}$; moreover, the value of the total, regularised integral over $\mathbb{R}$ for the increasing tritronquée solution is evaluated.

considering the Lax-pair equations associated with the DP3E (1.1), the authors construct their simultaneous solutions, called the 'seed' lax-pair solutions, corresponding to the simplest algebraic solution of the DP3E (1.1), $u(\tau) := u_0(\tau) = \frac{1}{2}\tau^{1/3}$, for $\varepsilon = -1$, $b = \mathrm{i}$, and $a = 0$ in terms of Airy functions, and then formulate, as Riemann–Hilbert Problem 1 (RHP1), the inverse monodromy problem for the rational solution $u(\tau) := u_n(\tau)$ for $a = -\mathrm{i}n$, $n \in \mathbb{Z} \setminus \{0\}$ (the case $a = -\mathrm{i}n$ for $n = 0$ is solved via the 'seed' Lax-pair solutions); in particular, the authors show that, if RHP1 is solvable for $\tau > 0$ and $n \in \mathbb{Z}$, then the function $u_n(\tau)$ defined by [13, equation (101)] is the unique solution of the DP3E (1.1) with $\varepsilon = -1$, $b = \mathrm{i}$, and $a = -\mathrm{i}n$, $n \in \mathbb{Z}$, that is a rational function of $\tau^{1/3}$ (see [13, Theorem 1]). The authors then use the RHP1 representation for the algebraic solution $u_n(\tau)$ of the DP3E (1.1) to consider the large-positive-n asymptotic behaviour of the solution (as a consequence of an inherent symmetry of the DP3E (1.1) that is discussed at the beginning of [13, Section 4.1], it is sufficient to consider large $n \in \mathbb{N}$); in particular, after a rescaling argument for both the independent variable and the spectral parameter, the authors present a rigorous asymptotic analysis of RHP1 and derive $\mathbb{N} \ni n \to \infty$ (for sufficiently large rescaled $\tau > 0$) asymptotics of the function $u_n(\tau)$ (see, in particular, [13, Theorems 2 and 3]). (In this context, see also [14].)

(v) Introducing the substitution $\varepsilon\tau u = (x/3)^2 y$, $\varepsilon b\tau^2 = 2(x/3)^3$, the author of [72] transforms the DP3E (1.1) into the second-order nonlinear ODE

$$y''(x) = \frac{(y'(x))^2}{y(x)} - \frac{y'(x)}{x} - 2(y(x))^2 + \frac{3a}{x} + \frac{1}{y(x)},$$

where the prime denotes differentiation with respect to x, and then, via additional auxiliary changes of variables, shows that, with $x = t\mathrm{e}^{\mathrm{i}\phi}$, the latter ODE for y governs the isomonodromy deformation of a 2×2 linear system $\partial_\lambda \Psi(\lambda, t) = \frac{t}{3}\mathcal{B}(\lambda, t)\Psi(\lambda, t)$, where $\mathrm{M}_2(\mathbb{C}) \ni \mathcal{B}(\lambda, t)$ is given in equation (1.4), or, equivalently, equation (3.2), of [72]. By applying the isomonodromy deformation method [45], the author demonstrates the Boutroux ansatz (near the point at infinity) by deriving an elliptic asymptotic representation of the general solution $y(x)$ in terms of the Weierstrass $\wp$-function as $x = t\mathrm{e}^{\mathrm{i}\phi} \to \infty$ in cheese-like strip domains along generic directions; see, in particular, the leading-order asymptotics of $y(x)$ stated in [72, Theorems 2.1–2.3].

(vi) In [82], the authors study the eigenvalue correlation kernel, denoted by $K_n(x, y, t)$, for the singularly perturbed Laguerre unitary ensemble (pLUE)[66] on the space $\mathcal{H}_n^+$ of $n \times n$ positive-definite Hermitian matrices $M = (M)_{i,j=1}^n$ defined by the probability measure $Z_n^{-1}(\det M)^\alpha \exp(-\operatorname{tr} V_t(M))\mathrm{d}M$, $n \in \mathbb{N}$, $\alpha > 0$, $t > 0$, where

$$Z_n := \int_{\mathcal{H}_n^+} (\det M)^\alpha \mathrm{e}^{-\operatorname{tr} V_t(M)} \mathrm{d}M$$

is the normalisation constant, $\mathrm{d}M := \prod_{i=1}^n \mathrm{d}M_{ii} \prod_{j=1}^{n-1} \prod_{k=j+1}^n \mathrm{d}\,\mathrm{Re}(M_{jk})\mathrm{d}\,\mathrm{Im}(M_{jk})$, and $V_t(x) := x + t/x$, $x \in (0, +\infty)$. By considering, for example, a variety of double-scaling limits such as $n \to \infty$ and $(0, d] \ni t \to 0^+$, $d > 0$, such that $s := 2nt$ belongs to compact subsets of $(0, +\infty)$, or $n \to \infty$ and $t \to 0^+$ such that $s \to 0^+$, or $n \to \infty$ and $(0, d] \ni t$ such that $s \to +\infty$, the authors derive the corresponding limiting behaviours of the eigenvalue correlation kernel by studying the large-n asymptotics of the orthogonal polynomials associated with the singularly perturbed Laguerre weight $w(x; t, \alpha) = x^\alpha \mathrm{e}^{-V_t(x)}$, and, en route, demonstrate that some of the limiting kernels involve certain functions related to a special solution of $(P_{\mathrm{III}'})_{D_7}$ (1.2); moreover, in the follow-up work [83] on the pLUE, the authors

[66] The pLUE and its relation to the Painlevé III (PIII) equation was introduced and studied in [16].

derive the large-n asymptotic formula (uniformly valid for $(0,d]\ni t$, $d>0$ and fixed) for the Hankel determinant

$$D_n[w;t]:=\det\left(\int_0^{+\infty}x^{j+k}w(x;t,\alpha)\mathrm{d}x\right)_{j,k=0}^{n-1}$$

associated with the singularly perturbed Laguerre weight $w(x;t,\alpha)$, and show that the asymptotic representation for $D_n[w;t]$ involves a function related to a particular solution of $(P_{\mathrm{III}'})_{D_7}$ (1.2). In the study of the Hankel determinant

$$D_n(t,\alpha,\beta):=\det\left(\int_0^{1}\xi^{j+k}w(\xi;t,\alpha,\beta)\mathrm{d}\xi\right)_{j,k=0}^{n-1}$$

generated by the Pollaczek–Jacobi-type weight $w(x;t,\alpha,\beta)=x^{\alpha}(1-x)^{\beta}\mathrm{e}^{-t/x}$, $x\in[0,1]$, $t\geqslant 0$, $\alpha,\beta>0$, which is a fundamental object in unitary random matrix theory, under a double-scaling limit where n, the dimension of the Hankel matrix, tends to ∞ and $t\to 0^+$ in such a way that $s:=2n^2t$ remains bounded, the authors of [15] show that the double-scaled Hankel determinant has an integral representation in terms of particular asymptotic solutions of a scaled version of the DP3E (1.1) (or, equivalently, $(P_{\mathrm{III}'})_{D_7}$ (1.2)). In [4], the authors study singularly perturbed unitary invariant random matrix ensembles on $\mathcal{H}_n^+$ defined by the probability measure

$$C_n^{-1}(\det M)^{\alpha}\exp(-n\operatorname{tr}V_k(M))\mathrm{d}M,\qquad n,k\in\mathbb{N},\quad \alpha>-1,$$

where $C_n:=\int_{\mathcal{H}_n^+}(\det M)^{\alpha}\mathrm{e}^{-n\operatorname{tr}V_k(M)}\mathrm{d}M$, and the (perturbed) potential $V_k(x)$ has a pole of order k at the origin, $V_k(x):=V(x)+(t/x)^k$, $t>0$, with the regular part V of the potential being real analytic on $[0,+\infty)$ and satisfying certain constraints; in particular, for the pLUE, the authors obtain, in various double-scaling limits when the size of the matrix $n\to\infty$ (at an appropriately adjusted rate) and the "strength" of the perturbation $t\to 0$, asymptotics of the associated eigenvalue correlation kernel and partition function, which are characterised in terms of special, pole-free solutions of a hierarchy (indexed by k) of higher-order analogues of the PIII equation: the first ($k=1$) member of this PIII hierarchy, denoted by $\ell_1(s)$, $s>0$, solves a rescaled version of the DP3E (1.1). (Analogous results for the singularly perturbed Gaussian unitary ensemble (pGUE) on the set $\mathcal{H}_n$ of $n\times n$ Hermitian matrices are also obtained in [4].) For the pLUE with perturbed potential $V_k(x):=V(x)+(t/x)^k$, $k\in\mathbb{N}$, $x\in(0,+\infty)$, $t>0$, studied in [4], the authors of [19] consider a related Fredholm determinant of an integral operator, denoted by $\mathcal{K}_{\mathrm{PIII}}$, acting on the space $L^2((0,+\infty))$, whose kernel is constructed from a certain $\mathrm{M}_2(\mathbb{C})$-valued function associated with a hierarchy (indexed by k) of higher-order analogues of the PIII equation; more precisely, for the Fredholm determinant $F(s;\lambda):=\ln\det(\mathrm{I}-\mathcal{K}_{\mathrm{PIII}})$, $s,\lambda>0$, the authors of [19] obtain $s\to+\infty$ asymptotics of $F(s;\lambda)$ characterised in terms of an explicit integral representation of a special, pole-free solution for the first ($k=1$) member of the corresponding PIII hierarchy: this solution is denoted by $\ell_1(\lambda)$, and it solves a rescaled version of the DP3E (1.1).

(vii) In [77], the authors compute small-t asymptotics of a class of solutions to the two-dimensional cylindrical Toda equations (2DCTE),[67]

$$q_k''(t)+t^{-1}q_k'(t)=4\left(\mathrm{e}^{q_k(t)-q_{k-1}(t)}-\mathrm{e}^{q_{k+1}(t)-q_k(t)}\right),$$

[67] See, also, its generalisations [35, 36, 37, 38].

$k \in \mathbb{Z}$, satisfying the periodicity conditions $q_{k+n}(t) = q_k(t)$, where the integer n is arbitrary but fixed. Solutions that are valid for all $t > 0$ have the representation $q_k(t) = \log\det(\mathrm{I} - \lambda\mathcal{K}_k) - \log\det(\mathrm{I} - \lambda\mathcal{K}_{k-1})$, where $\mathcal{K}_k$ is the integral operator on $\mathbb{R}_+$ with kernel

$$\sum_{\{\omega^n=1\}\setminus\{1\}} \omega^k c_\omega \frac{\mathrm{e}^{-t((1-\omega)u+(1-\omega^{-1})u^{-1})}}{-\omega u + v},$$

for some coefficients c_ω, and λ is a free parameter. For $n = 3$ and the imposition of an additional constraint, which implies $q_1(t) = 0$ and $q_2(t) = -q_3(t)$, the 2DCTE gives rise to the radial Bullough–Dodd equation (for $q_3(t)$), $q_3''(t) + t^{-1}q_3'(t) = 4\big(\mathrm{e}^{2q_3(t)} - \mathrm{e}^{-q_3(t)}\big)$, which, via the dependent-variable transformation $w(t) = \mathrm{e}^{-q_3(t)}$, reduces to the nonlinear ODE

$$w''(t) = \frac{(w'(t))^2}{w(t)} - \frac{w'(t)}{t} + 4(w(t))^2 - \frac{4}{w(t)};$$

by making one more change of variables, namely, $t = \lambda^{2/3}$ and $w(t) = \lambda^{-1/3}\mathcal{W}(\lambda)$, this ODE can, in turn, be transformed to the PIII equation with parameter values $(16/9, 0, 0, -16/9)$,

$$\mathcal{W}''(\lambda) = \frac{(\mathcal{W}'(\lambda))^2}{\mathcal{W}(\lambda)} - \frac{\mathcal{W}'(\lambda)}{\lambda} + \frac{16}{9}\frac{(\mathcal{W}(\lambda))^2}{\lambda} - \frac{16}{9}\frac{1}{\mathcal{W}(\lambda)},$$

where the prime denotes differentiation with respect to λ, which can be identified as a special reduction of the DP3E (1.1) for $a = 0$. The small-t asymptotics of $q_k(t)$ are derived by computing the asymptotics $\det(\mathrm{I} - \lambda\mathcal{K}_k) \underset{t\to 0^+}{\sim} b_k(t/n)^{a_k}$, $n = 2, 3$, where explicit expressions for the coefficients a_k and b_k are presented in [77].

(viii) The DP3E (1.1) also plays a prominent rôle in the description of surfaces with constant negative Gaussian curvature (K-surfaces) and two straight asymptotic lines (*Amsler surfaces*) [9]. A non-degenerate surface in $\mathbb{R}^3$ is called an *affine sphere* if all affine normal directions intersect at a point: this class of surfaces is described by an integrable equation first derived by Tzitzéica. As discussed in [9], for affine spheres characterised by the property that they possess two intersecting straight affine lines, the corresponding Tzitzéica equation reduces to the PIII equation with parameter values $(1, 0, 0, -1)$,

$$y''(t) = \frac{(y'(t))^2}{y(t)} - \frac{y'(t)}{t} + \frac{(y(t))^2}{t} - \frac{1}{y(t)},$$

where the prime denotes differentiation with respect to t, with $y(t) = t^{1/3}H(r)$ and $t = \frac{8}{3^{3/2}}r^{3/4}$, and where $H(r)$, with $r := xy$, is a Lorentz invariant solution of the Tzitzéica equation that satisfies the second-order nonlinear ODE

$$H''(r) = \frac{(H'(r))^2}{H(r)} - \frac{H'(r)}{r} + \frac{1}{r}\left((H(r))^2 - \frac{1}{H(r)}\right),$$

where the prime denotes differentiation with respect to r; in fact, the ODE for the function $y(t)$ can be identified as a special reduction of the DP3E (1.1) for $a = 0$: letting $\tau = 2^{-3/2}\mathrm{e}^{\mathrm{i}(2m+1)\pi/4}t$ and $u(\tau) = -2^{-3/2}\mathrm{e}^{-\mathrm{i}(2m+1)\pi/4}y(t)$, $m = 0, 1, 2, 3$, and choosing the (external) parameter values $\varepsilon = b = +1$ and $a = 0$, it follows that the DP3E (1.1) reduces to the ODE for $y(t)$. The algebroid theory for solutions of the ODE for $H(r)$ is presented in [59].

(ix) Let $\mathcal{X}$ be a six-dimensional Calabi–Yau (CY) manifold (a complex Kähler three-fold with covariantly constant holomorphic three-form Ω). The Strominger–Yau–Zaslow (SYZ) conjecture (see [24] for details) states that, near the large complex structure limit, both $\mathcal{X}$ and its mirror should be the fibrations over the moduli space of special Lagrangian tori (submanifolds admitting a unitary flat connection). As an examination of the SYZ conjecture, Loftin–Yau–Zaslow (LYZ) set out to prove the existence of the metric of Hessian form $g_B = \frac{\partial^2\phi}{\partial x^j \partial x^k} \mathrm{d}x^j \otimes \mathrm{d}x^k$, where x^j, $j = 1, 2, 3$, are local coordinates on a real three-dimensional manifold, and ϕ (a Kähler potential) is homogeneous of degree two in x^j and satisfies the real Monge–Ampére equation $\det\big(\frac{\partial^2\phi}{\partial x^j \partial x^k}\big) = 1$: LYZ showed that the construction of the metric is tantamount to searching for solutions of the definite affine sphere equation (DASE) $\psi_{z\overline{z}} + \frac{1}{2}\mathrm{e}^{\psi} + |U|^2\mathrm{e}^{-2\psi} = 0$, $U_{\overline{z}} = 0$, where ψ and U are real- and complex-valued functions, respectively, on an open subset of $\mathbb{C}$. For $U = z^{-2}$, LYZ proved the existence of the radially symmetric solution ψ of the DASE with a prescribed behaviour near the singularity $z = 0$, and established the existence of the global solution to the coordinate-independent version of the DASE on $\mathbb{S}^2$ with three points excised. In [24], the authors show that the DASE, and a closely related equation called the Tzitzéica equation, arise as reductions of anti-self-dual Yang–Mills (ASDYM) system by two translations; moreover, they show that the ODE characterising its radial solutions give rise to an isomonodromy problem described by the PIII equation for special values of its parameters. In particular (see [24, Proposition 1.3]), the authors show that, for $U = z^{-2}$, solutions of the DASE that are invariant under the group of rotations (rotational symmetry) $z \to \mathrm{e}^{\mathrm{i}\mathfrak{c}}z$, $\mathfrak{c} \in \mathbb{R}$, are of the form $\psi(z, \overline{z}) = \ln(\mathcal{H}(s)) - 3\ln(s)$, with $s := |z|^{1/2}$, where $\mathcal{H}(s)$ solves the PIII equation with parameter values $(-8, 0, 0, -16)$,

$$\mathcal{H}''(s) = \frac{(\mathcal{H}'(s))^2}{\mathcal{H}(s)} - \frac{\mathcal{H}'(s)}{s} - \frac{8(\mathcal{H}(s))^2}{s} - \frac{16}{\mathcal{H}(s)},$$

where the prime denotes differentiation with respect to s, which can be identified as a special reduction of the DP3E (1.1) for $a = 0$. The authors of [24] demonstrate that the existence theorem for Hessian metrics with prescribed monodromy reduces to the study of the PIII equation with parameters $(-8, 0, 0, -16)$, that is, a class of semi-flat CY metrics is obtained in terms of real solutions of the DP3E (1.1) for $a = 0$ (see also [17, 18, 22, 23]).

(x) In [39], the author introduces affine spheres as immersions of a manifold $\mathcal{M}$ as a hypersurface in $\mathbb{R}^n$ with certain properties and defines the affine metric h and the cubic form C on $\mathcal{M}$. By identifying, for 3-dimensional cones and, correspondingly, affine 2-spheres, the manifold $\mathcal{M}$ with a non-compact, simply-connected domain in $\mathbb{C}$, one can introduce complex isothermal co-ordinates z on $\mathcal{M}$, in terms of which the affine metric h may equivalently be described by a real conformal factor $u(z)$ and the cubic form C by a holomorphic function $U(z)$ on $\mathcal{M}$, the relations being $h = \mathrm{e}^u|\mathrm{d}z|^2$ and $C = 2\,\mathrm{Re}(U(z))\mathrm{d}z^3$: the compatibility condition of the pair (u, U) is referred to as *Wang's equation*, $\mathrm{e}^u = \frac{1}{2}\Delta u + 2|U|^2\mathrm{e}^{-2u}$, where $\Delta u = u_{xx} + u_{yy} = 4u_{z\overline{z}}$ is the Laplacian of u, $\partial_z := \frac{1}{2}(\partial_x - \mathrm{i}\partial_y)$, and $\partial_{\overline{z}} := \frac{1}{2}(\partial_x + \mathrm{i}\partial_y)$. By classifying pairs (ψ, U), where ψ is a vector field on $\mathcal{M}$ generating a one-parameter group of conformal automorphisms on $\mathcal{M}$ which multiply U by unimodular complex constants, the author finds, for every pair (ψ, U), a unique solution u of Wang's equation such that the corresponding affine metric h is complete on $\mathcal{M}$ and ψ is a Killing vector field for h: this latter property permits Wang's equation to be reduced to a second-order nonlinear ODE that is equivalent to the DP3E (1.1), a detailed qualitative study for which is presented in [39]. The author presents a complete classification of self-associated cones (one calls a cone self-associated if it is linearly isomorphic to all its associated cones, with two cones said to be associated with each other if the Blaschke metrics on the corresponding affine spheres are related by an orientation-preserving isometry) and computes isothermal parametrisations of the corresponding affine spheres, the solution(s) of which can be expressed in terms of degenerate PIII transcendents (solutions of the DP3E (1.1)).

Whilst not directly relevant to the DP3E (1.1), the following facts are worth mentioning: (1) elliptic asymptotic representations in terms of the Jacobi sn-function in cheese-like strip domains along generic directions are obtained for the general solution of the 'complete' PIII equation in [73]; (2) a detailed study of the PIII monodromy maps under the $D_6 \to D_8$ confluence has recently been presented in [6]; (3) parametric Stokes phenomena for the D_6 and D_7 cases of the PIII equation are studied in [46]; (4) application of the PIII equation to the study of transformation phenomena for parametric Painlevé equations for the D_6 and D_7 cases is considered in [47], whilst the D_8 case is studied in [76, 79]; (5) the monograph [34] studies the relation of the PIII equation of type $(P_{\mathrm{III}})_{D_6}$ to isomonodromic families of vector bundles on $\mathbb{P}^1$ with meromorphic connections; (6) in [31], the $\boldsymbol{\tau}$-function associated with the degenerate PIII equation of type D_8 is shown to admit a Fredholm determinant representation in terms of a generalised Bessel kernel; and (7) by using the universal example of the Gross–Witten–Wadia (GWW) third-order phase transition in the unitary matrix model, concomitant with the explicit Tracy–Widom mapping of the GWW partition function to a solution of a PIII equation, the transmutation (change in the resurgent asymptotic properties) of a trans-series in two parameters (a coupling g^2 and a gauge index N) at all coupling and all finite N is studied in [1] (see also [25]).

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References

[1] Ahmed A., Dunne G.V., Transmutation of a trans-series: the Gross–Witten–Wadia phase transition, *J. High Energy Phys.* **2017** (2017), no. 11, 054, 51 pages, arXiv:1710.01812.

[2] Andreev F.V., Kitaev A.V., Exponentially small corrections to divergent asymptotic expansions of solutions of the fifth Painlevé equation, *Math. Res. Lett.* **4** (1997), 741–759.

[3] Aniceto I., Başar G., Schiappa R., A primer on resurgent transseries and their asymptotics, *Phys. Rep.* **809** (2019), 1–135, arXiv:1802.10441.

[4] Atkin M.R., Claeys T., Mezzadri F., Random matrix ensembles with singularities and a hierarchy of Painlevé III equations, *Int. Math. Res. Not.* **2016** (2016), 2320–2375, arXiv:1501.04475.

[5] Baldino S., Schiappa R., Schwick M., Vega R., Resurgent stokes data for Painlevé equations and two-dimensional quantum (super) gravity, *Commun. Number Theory Phys.* **17** (2023), 385–552, arXiv:2203.13726.

[6] Barhoumi A., Lisovyy O., Miller P.D., Prokhorov A., Painlevé-III monodromy maps under the $D_6 \to D_8$ confluence and applications to the large-parameter asymptotics of rational solutions, *SIGMA* **20** (2024), 019, 77 pages, arXiv:2307.11217.

[7] Bilman D., Buckingham R., Large-order asymptotics for multiple-pole solitons of the focusing nonlinear Schrödinger equation, *J. Nonlinear Sci.* **29** (2019), 2185–2229, arXiv:1807.09058.

[8] Bilman D., Ling L., Miller P.D., Extreme superposition: rogue waves of infinite order and the Painlevé-III hierarchy, *Duke Math. J.* **169** (2020), 671–760, arXiv:1806.00545.

[9] Bobenko A.I., Eitner U., Painlevé equations in the differential geometry of surfaces, *Lecture Notes in Math.*, Vol. 1753, Springer, Berlin, 2000.

[10] Bolibruch A., Its A., Kapaev A., On the Riemann–Hilbert–Birkhoff inverse monodromy problem and the Painlevé equations, *St. Petersburg Math. J.* **16** (2005), 105–142.

[11] Bothner T., Miller P.D., Rational solutions of the Painlevé-III equation: large parameter asymptotics, *Constr. Approx.* **51** (2020), 123–224, arXiv:1808.01421.

[12] Brzeziński J., Galois theory through exercises, Springer Undergrad. Math. Ser., Springer, Cham, 2018.

[13] Buckingham R.J., Miller P.D., On the algebraic solutions of the Painlevé-III (D_7) equation, *Phys. D* **441** (2022), 133493, 22 pages, arXiv:2202.04217.

[14] Buckingham R.J., Miller P.D., Differential equations for approximate solutions of Painlevé equations: application to the algebraic solutions of the Painlevé-III (D_7) equation, *SIGMA* **20** (2024), 008, 27 pages, arXiv:2308.16051.

[15] Chen M., Chen Y., Fan E., Perturbed Hankel determinant, correlation functions and Painlevé equations, *J. Math. Phys.* **57** (2016), 023501, 31 pages, arXiv:1507.05261.

[16] Chen Y., Its A., Painlevé III and a singular linear statistics in Hermitian random matrix ensembles. I, *J. Approx. Theory* **162** (2010), 270–297, arXiv:0808.3590.

[17] Contatto F., Integrable Abelian vortex-like solitons, *Phys. Lett. B* **768** (2017), 23–29, arXiv:1612.01879.

[18] Contatto F., Dorigoni D., Instanton solutions from Abelian sinh-Gordon and Tzitzeica vortices, *J. Geom. Phys.* **98** (2015), 429–445, arXiv:1412.8312.

[19] Dai D., Xu S.X., Zhang L., Gap probability at the hard edge for random matrix ensembles with pole singularities in the potential, *SIAM J. Math. Anal.* **50** (2018), 2233–2279, arXiv:1710.08132.

[20] Deift P.A., Zhou X., Asymptotics for the Painlevé II equation, *Comm. Pure Appl. Math.* **48** (1995), 277–337.

[21] Delabaere E., Divergent series, summability and resurgence III. Resurgent methods and the first Painlevé equation, *Lecture Notes in Math.*, Vol. 2155, Springer, Cham, 2016.

[22] Dunajski M., Abelian vortices from sinh-Gordon and Tzitzeica equations, *Phys. Lett. B* **710** (2012), 236–239, arXiv:1201.0105.

[23] Dunajski M., Gavrea N., Elizabethan vortices, *Nonlinearity* **36** (2023), 4169–4186, arXiv:2301.06191.

[24] Dunajski M., Plansangkate P., Strominger–Yau–Zaslow geometry, affine spheres and Painlevé III, *Comm. Math. Phys.* **290** (2009), 997–1024, arXiv:0809.3015.

[25] Dunne G.V., Resurgence, Painlevé equations and conformal blocks, *J. Phys. A* **52** (2019), 463001, 31 pages, arXiv:1901.02076.

[26] Edgar G.A., Transseries for beginners, *Real Anal. Exchange* **35** (2010), 253–309, arXiv:0801.4877.

[27] Erdélyi A., Magnus W., Oberhettinger F., Tricomi F.G., Higher transcendental functions, Vol. II, Robert E. Krieger Publishing, Melbourne, Fla., 1981.

[28] Fedoryuk M.V., Asymptotic analysis. Linear ordinary differential equations, Springer, Berlin, 1993.

[29] Fokas A.S., Its A.R., Kapaev A.A., Novokshenov V.Y., Painlevé transcendents: The Riemann–Hilbert approach, *Math. Surveys Monogr.*, Vol. 128, American Mathematical Society, Providence, RI, 2006.

[30] Garoufalidis S., Its A., Kapaev A., Mariño M., Asymptotics of the instantons of Painlevé I, *Int. Math. Res. Not.* **2012** (2012), 561–606, arXiv:1002.3634.

[31] Gavrylenko P., Lisovyy O., Pure SU(2) gauge theory partition function and generalized Bessel kernel, in String-Math 2016, *Proc. Sympos. Pure Math.*, Vol. 98, American Mathematical Society, Providence, RI, 2018, 181–205, arXiv:1705.01869.

[32] Gradshteyn I.S., Ryzhik I.M., Table of integrals, series, and products, 5th ed., Academic Press, Boston, MA, 1994.

[33] Gromak V.I., Laine I., Shimomura S., Painlevé differential equations in the complex plane, *De Gruyter Stud. Math.*, Vol. 28, Walter de Gruyter & Co., Berlin, 2002.

[34] Guest M.A., Hertling C., Painlevé III: a case study in the geometry of meromorphic connections, *Lecture Notes in Math.*, Vol. 2198, Springer, Cham, 2017.

[35] Guest M.A., Its A.R., Lin C.S., Isomonodromy aspects of the tt^* equations of Cecotti and Vafa I. Stokes data, *Int. Math. Res. Not.* **2015** (2015), 11745–11784, arXiv:1209.2045.

[36] Guest M.A., Its A.R., Lin C.S., Isomonodromy aspects of the tt* equations of Cecotti and Vafa II: Riemann–Hilbert problem, *Comm. Math. Phys.* **336** (2015), 337–380, arXiv:1312.4825.

[37] Guest M.A., Its A.R., Lin C.S., Isomonodromy aspects of the tt* equations of Cecotti and Vafa III: Iwasawa factorization and asymptotics, *Comm. Math. Phys.* **374** (2020), 923–973, arXiv:1707.00259.

[38] Guest M.A., Its A.R., Lin C.S., The tt*-Toda equations of A_n type, arXiv:2302.04597.

[39] Hildebrand R., Self-associated three-dimensional cones, *Beitr. Algebra Geom.* **63** (2022), 867–906, arXiv:1806.06588.

[40] Hille E., Ordinary differential equations in the complex domain, Dover Publications, Mineola, NY, 1997.

[41] Its A.R., "Isomonodromic" solutions of equations of zero curvature, *Math. USSR Izv.* **26** (1986), 497–529.

[42] Its A.R., Fokas A.S., Kapaev A.A., On the asymptotic analysis of the Painlevé equations via the isomonodromy method, *Nonlinearity* **7** (1994), 1291–1325.

[43] Its A.R., Kapaev A.A., Connection formulae for the fourth Painlevé transcendent; Clarkson–McLeod solution, *J. Phys. A* **31** (1998), 4073–4113.

[44] Its A.R., Kapaev A.A., Quasi-linear Stokes phenomenon for the second Painlevé transcendent, *Nonlinearity* **16** (2003), 363–386, arXiv:nlin/0108010.

[45] Its A.R., Novokshenov V.Yu., The isomonodromic deformation method in the theory of Painlevé equations, *Lecture Notes in Math.*, Vol. 1191, Springer, Berlin, 1986.

[46] Iwaki K., Voros coefficients of the third Painlevé equation and parametric Stokes phenomena, arXiv:1303.3603.

[47] Iwaki K., On WKB theoretic transformations for Painlevé transcendents on degenerate Stokes segments, *Publ. Res. Inst. Math. Sci.* **51** (2015), 1–57, arXiv:1312.1874.

[48] Janson S., Roots of polynomials of degrees 3 and 4, arXiv:1009.2373.

[49] Joshi N., Lustri C.J., Luu S., Stokes phenomena in discrete Painlevé II, *Proc. A.* **473** (2017), 20160539, 20 pages, arXiv:1607.04494.

[50] Kapaev A.A., Quasi-linear Stokes phenomenon for the Painlevé first equation, *J. Phys. A* **37** (2004), 11149–11167, arXiv:nlin/0404026.

[51] Kapaev A.A., Quasi-linear Stokes phenomenon for the Hastings–McLeod solution of the second Painlevé equation, arXiv:nlin.SI/0411009.

[52] Kitaev A.V., The method of isomonodromic deformations and the asymptotics of the solutions of the "complete" third Painlevé equation, *Math. USSR Izv.* **31** (1988), 193–207.

[53] Kitaev A.V., Method of isomonodromic deformations for the "degenerate" third Painlevé equation, *J. Sov. Math.* **46** (1989), 2077–2083.

[54] Kitaev A.V., The justification of asymptotic formulas that can be obtained by the method of isomonodromic deformations, *J. Sov. Math.* **57** (1991), 3131–3135.

[55] Kitaev A.V., Elliptic asymptotics of the first and second Painlevé transcendents, *Russian Math. Surveys* **49** (1994), 81–150.

[56] Kitaev A.V., Meromorphic solution of the degenerate third Painlevé equation vanishing at the origin, *SIGMA* **15** (2019), 046, 53 pages, arXiv:1809.00122.

[57] Kitaev A.V., Vartanian A., Connection formulae for asymptotics of solutions of the degenerate third Painlevé equation: II, *Inverse Problems* **26** (2010), 105010, 58 pages, arXiv:1005.2677.

[58] Kitaev A.V., Vartanian A., Asymptotics of integrals of some functions related to the degenerate third Painlevé equation, *J. Math. Sci.* **242** (2019), 715–721.

[59] Kitaev A.V., Vartanian A., Algebroid solutions of the degenerate third Painlevé equation for vanishing formal monodromy parameter, *J. Math. Anal. Appl.* **532** (2024), 127917, 86 pages, arXiv:2304.05671.

[60] Kitaev A.V., Vartanian A., One-parameter meromorphic solution of the degenerate third Painlevé equation with formal monodromy parameter $a = \pm \mathrm{i}/2$ vanishing at the origin, *J. Math. Sci.* **284** (2024), 700–725, arXiv:2305.17278.

[61] Kitaev A.V., Vartanian A.H., Connection formulae for asymptotics of solutions of the degenerate third Painlevé equation. I, *Inverse Problems* **20** (2004), 1165–1206, arXiv:math.CA/0312075.

[62] Lin Y., Dai D., Tibboel P., Existence and uniqueness of tronquée solutions of the third and fourth Painlevé equations, *Nonlinearity* **27** (2014), 171–186, arXiv:1306.1317.

[63] Loday-Richaud M., Divergent series, summability and resurgence II. Simple and multiple summability, *Lecture Notes in Math.*, Vol. 2154, Springer, Cham, 2016.

[64] Miller P.D., On the increasing tritronquée solutions of the Painlevé-II equation, *SIGMA* **14** (2018), 125, 38 pages, arXiv:1804.03173.

[65] Murata Y., Classical solutions of the third Painlevé equation, *Nagoya Math. J.* **139** (1995), 37–65.

[66] Ohyama Y., Kawamuko H., Sakai H., Okamoto K., Studies on the Painlevé equations. V. Third Painlevé equations of special type $P_{\mathrm{III}}(D_7)$ and $P_{\mathrm{III}}(D_8)$, *J. Math. Sci. Univ. Tokyo* **13** (2006), 145–204.

[67] Olde Daalhuis A.B., Exponentially-improved asymptotics and numerics for the (un)perturbed first Painlevé equation, *J. Phys. A* **55** (2022), 304004, 16 pages, arXiv:2205.12800.

[68] Shimomura S., Truncated solutions of the fifth Painlevé equation, *Funkcial. Ekvac.* **54** (2011), 451–471.

[69] Shimomura S., Series expansions of Painlevé transcendents near the point at infinity, *Funkcial. Ekvac.* **58** (2015), 277–319.

[70] Shimomura S., Three-parameter solutions of the PV Schlesinger-type equation near the point at infinity and the monodromy data, *SIGMA* **14** (2018), 113, 50 pages, arXiv:1804.10369.

[71] Shimomura S., Elliptic asymptotic representation of the fifth Painlevé transcendents, *Kyushu J. Math.* **76** (2022), 43–99, arXiv:2012.07321.

[72] Shimomura S., Boutroux ansatz for the degenerate third Painlevé transcendents, *Publ. Res. Inst. Math. Sci.* **60** (2024), 651–698, arXiv:2207.11495.

[73] Shimomura S., Elliptic asymptotics for the complete third Painlevé transcendents, *Funkcial. Ekvac.* **68** (2025), 69–117, arXiv:2211.00886.

[74] Steinmetz N., Nevanlinna theory, normal families, and algebraic differential equations, Universitext, Springer, Cham, 2017.

[75] Suleimanov B.I., Effect of a small dispersion on self-focusing in a spatially one-dimensional case, *JETP Lett.* **106** (2017), 400–405.

[76] Takei Y., On the role of the degenerate third Painlevé equation of type $(D8)$ in the exact WKB analysis, in Exact WKB Analysis and Microlocal Analysis, *RIMS Kôkyûroku Bessatsu*, Vol. B37, Research Institute for Mathematical Sciences, Kyoto, 2013, 211–222.

[77] Tracy C.A., Widom H., Asymptotics of a class of solutions to the cylindrical Toda equations, *Comm. Math. Phys.* **190** (1998), 697–721, arXiv:solv-int/9701003.

[78] van Spaendonck A., Vonk M., Painlevé I and exact WKB: Stokes phenomenon for two-parameter transseries, *J. Phys. A* **55** (2022), 454003, 64 pages, arXiv:2204.09062.

[79] Wakako H., Takei Y., Exact WKB analysis for the degenerate third Painlevé equation of type (D_8), *Proc. Japan Acad. Ser. A Math. Sci.* **83** (2007), 63–68.

[80] Wasow W., Linear turning point theory, *Appl. Math. Sci.*, Vol. 54, Springer, New York, 1985.

[81] Xia X., Tronquée solutions of the third and fourth Painlevé equations, *SIGMA* **14** (2018), 095, 28 pages, arXiv:1803.11230.

[82] Xu S.X., Dai D., Zhao Y.Q., Critical edge behavior and the Bessel to Airy transition in the singularly perturbed Laguerre unitary ensemble, *Comm. Math. Phys.* **332** (2014), 1257–1296, arXiv:1309.4354.

[83] Xu S.X., Dai D., Zhao Y.Q., Painlevé III asymptotics of Hankel determinants for a singularly perturbed Laguerre weight, *J. Approx. Theory* **192** (2015), 1–18, arXiv:1407.7334.