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Mode Stability of Hermitian Instantons

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Abstract. In this note, we prove the Riemannian analog of black hole mode stability for Hermitian, non-self-dual gravitational instantons which are either asymptotically locally flat (ALF) and Ricci-flat, or compact and Einstein with positive cosmological constant. We show that the Teukolsky equation on any such manifold is a positive definite operator. We also discuss the compatibility of the results with the existence of negative modes associated to variational instabilities.

Key words: gravitational instantons; stability; spinor methods

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1 Introduction

A gravitational instanton is a complete and orientable 4-dimensional, Ricci-flat and asymptotically flat Riemannian manifold. In this note, we are interested in Hermitian asymptotically locally flat (ALF) instantons, as well as compact Einstein–Hermitian 4-spaces. Recall that a Riemannian manifold is Hermitian if there is an integrable almost-complex structure which is compatible with the metric. In both the Ricci-flat and Einstein cases, the Goldberg–Sachs theorem adapted to Riemannian signature implies that the above is equivalent to the curvature being algebraically special, see [22, 26]. For the definition of a Riemannian manifold being ALF, see [12, Definition 1.1]. A classification of Hermitian, toric, ALF gravitational instantons has been given by Biquard and Gauduchon [12]. The toric assumption was recently removed in [33]. In fact an ALF Hermitian non-Kähler instanton belongs to the Euclidean Kerr, Taub-bolt, or Chen–Teo families, or is Taub-NUT with the opposite orientation. We shall ignore the last case, since the Taub-NUT manifold is half-flat, see Remark 1.6 below. The Euclidean Kerr and Taub-bolt instantons are both Petrov type D, that is both the self-dual and anti-self dual parts of the Weyl tensor are algebraically special and the manifold is non-Kähler with respect to both orientations. The Chen–Teo instanton [19], on the other hand is Hermitian but has algebraically general anti-self dual Weyl tensor [1]. The classification of Hermitian non-Kähler ALF instantons complements the classification of hyperkähler instantons [15, 16, 17, 30, 34, 42], and furthermore bears a close similarity to the classification of compact Einstein–Hermitian non-Kähler manifolds by LeBrun [32].

Local rigidity of Hermitian ALF instantons was proved by Biquard, Gauduchon, and LeBrun, see [13]. Given a Hermitian gravitational instanton (M, g_{ab}), there is an open neighborhood $\mathcal{O}$ of g_{ab} in the space of metrics on M such that any gravitational instanton $g'_{ab} \in \mathcal{O}$ is also Hermitian.

Remark 1.1. The above result leaves open the question of whether a Hermitian ALF gravitational instanton is integrable in the sense of [11, Section 12.E]. The notion of mode stability for instantons that we study in this paper is a step towards addressing this problem, see below.

Let g_{ab} be a Riemannian metric on a 4-dimensional manifold M with Ricci tensor $R_{ab}(g)$ and scalar curvature $S(g)$. We define the Einstein tensor $E_{ab}(g)$ by

$$E_{ab}(g) = R_{ab}(g) - \frac{S(g)}{4} g_{ab}, \tag{1.1}$$

see [11, Section 12.26]. The metric g_{ab} is Einstein if $E_{ab}(g) = 0$. We shall be interested in two classes of Einstein metrics: ALF Ricci-flat manifolds, and compact Einstein manifolds with $S(g) \neq 0$. Let $g(s)_{ab}$ be a 1-parameter family of metrics, with

$$\frac{\mathrm{d}}{\mathrm{d}s} g(s)_{ab}|_{s=0} = h_{ab}, \tag{1.2}$$

and let $E(s)_{ab}$ be the Einstein operator of $g(s)_{ab}$. Assume that $g(0)_{ab} = g_{ab}$ is Einstein and that h_{ab} is a linearized Einstein perturbation, that is h_{ab} satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}s} E(s)_{ab}|_{s=0} = 0. \tag{1.3}$$

If (M, g_{ab}) is ALF Ricci-flat, then $E(s)_{ab}$ in (1.3) can be replaced by $R(s)_{ab}$, and we say that h_{ab} is an ALF vacuum perturbation if for any integer $k \geq 0$, $\nabla^k h_{ab} = O\big(r^{-1-k}\big)$. See Definition 2.2 below for notation. The ALF Ricci-flat instanton (M, g_{ab}) is said to be integrable if for any ALF vacuum perturbation h_{ab}, there is a 1-parameter family $g(s)_{ab}$ of ALF Ricci flat metrics such that $\mathrm{d}/\mathrm{d}s\, g(s)_{ab}|_{s=0} = h_{ab}$. Similarly, if (M, g_{ab}) is compact Einstein, then we say that it is integrable if for any Einstein perturbation h_{ab}, there is a 1-parameter family $g(s)_{ab}$ of Einstein metrics such that $\mathrm{d}/\mathrm{d}s\, g(s)_{ab}|_{s=0} = h_{ab}$.

The above notion of integrability (which follows [11, Section 12.E]) intends to describe the space of solutions to the Einstein equations around the given solution g_{ab}. This space is not necessarily a manifold, so the required curve of metrics may not exist. Integrability in this sense is also known as *linearization stability* as defined by Fischer and Marsden [24]. For Lorentzian metrics satisfying the Einstein equations, this concept was introduced by Choquet-Bruhat and Deser [20], and then thoroughly studied by Fischer, Marsden, Moncrief, and Arms [9, 25, 23, 35, 36]. In particular, for vacuum spacetimes with compact Cauchy hypersurfaces, Moncrief showed [35, 36] that linearization stability *fails* if the solution has Killing vector fields.

A step towards addressing integrability of Hermitian instantons is the problem of mode stability, which is a concept that originates in the study of dynamical stability of Lorentzian black holes, but can be adapted to Riemannian metrics. Let (M, g_{ab}) be Hermitian, with complex structure $J^a{}_b$ and unprimed Weyl spinor Ψ_{ABCD} (see Section 2 for notation). Then $J^a{}_b$ can be represented by a spinor o^A as in (2.1) and (2.2) below, with respect to which $\Psi_0 = \Psi_{ABCD} o^A o^B o^C o^D = 0$. This follows from (2.10) below and its integrability condition, see, for example, the discussion around [29, equation (12.3)]. Let $g(s)_{ab}$ be a 1-parameter family of metrics on M, with $g(0)_{ab} = g_{ab}$. The linearized Ψ_0 is given by

$$\vartheta \Psi_0 = \frac{\mathrm{d}}{\mathrm{d}s} \Psi_0[g(s)_{ab}]|_{s=0}, \tag{1.4}$$

where ϑ denotes the variation operator introduced in [10] (adapted to Riemannian signature) which we shall use in this paper to treat perturbations, see Section 2.3. Since the background Ψ_0 vanishes, $\vartheta\Psi_0$ is a gauge invariant quantity. If the Hermitian manifold (M, g_{ab}) is Ricci-flat or Einstein, then for a perturbation satisfying (1.3), we have that $\vartheta\Psi_0$ satisfies the Teukolsky equation, see (2.18) below. The Teukolsky equation in the Riemannian case is the analog of the Teukolsky equation which governs perturbations of the Kerr black hole and other Lorentzian Petrov type D Einstein metrics. In the Lorentzian case, mode stability means that there are no solutions of the Teukolsky equation with frequency in the upper half plane and satisfying a condition of no incoming radiation, see [5, 45]. In the Riemannian case, we have the following analog of the notion of mode stability.

Definition 1.2. Let (M, g_{ab}) be Hermitian. If (M, g_{ab}) is ALF Ricci-flat, then we say that mode stability holds for (M, g_{ab}) if for any ALF vacuum perturbation we have $\vartheta\Psi_0 = 0$. If (M, g_{ab}) is Einstein and compact, then we say that mode stability holds for (M, g_{ab}) if for any Einstein perturbation we have $\vartheta\Psi_0 = 0$.

Remark 1.3. In Lorentzian signature, mode stability for non-vacuum spacetimes such as the Kerr–Newman solution to the Einstein–Maxwell system, or the Kerr–de Sitter black hole in the Einstein case, remains open. In Riemannian signature, Einstein–Maxwell instantons are also of interest, and the problem of their mode stability is also worth studying. The instanton version of the Kerr–de Sitter black hole is a compact Einstein space found by Page [38], whose mode stability is proven in Theorem 1.5 below.

It was recently shown by Nilsson that mode stability holds for the Petrov type D Euclidean Kerr and Taub-bolt families of instantons, see [37]. In this paper, we give a new proof of this result which is valid for all Hermitian ALF instantons, that is including the Chen–Teo case. Our argument also applies for compact Einstein–Hermitian 4-manifolds.

Theorem 1.4. *Let (M, g_{ab}) be a Hermitian non-Kähler ALF instanton. Then mode stability holds for (M, g_{ab}).*

Theorem 1.5. *Let (M, g_{ab}) be a compact Einstein–Hermitian 4-manifold with positive cosmological constant. Then mode stability holds for (M, g_{ab}).*

Remark 1.6. Recall that a hyperkähler instanton is half-flat. Due to the Hitchin–Thorpe inequality for closed four-manifolds M, a Ricci-flat manifold (M, g_{ab}) is half-flat if and only if $\tau(M) = \frac{2}{3}\chi(M)$, see, for example, [11, Section 13.8]. Since $\tau(M)$ and $\chi(M)$ are topological invariants, any Ricci-flat metric on a closed manifold admitting a half-flat metric will satisfy the same equality and will thus be half-flat.

For ALF manifolds (M, g_{ab}), the Hitchin–Thorpe inequality has an additional term giving the contribution from the ALF end, see [18, 21]. Again, equality holds if and only if (M, g_{ab}) is hyperkähler. Since neither $\tau(M)$, $\chi(M)$ nor the contribution at infinity depends on the particular ALF metric, any other Ricci-flat metric will again satisfy the same equality and will thus be half-flat.

This gives the analog of mode stability for hyperkähler instantons. Therefore, we shall consider mode stability only for Hermitian, non-Kähler manifolds.

Mode stability is a step towards proving the following conjecture.

Conjecture 1.7. *Let (M, g_{ab}) be a Hermitian non-Kähler ALF instanton. Then (M, g_{ab}) is integrable.*

Remark 1.8. It follows from the classification of Hermitian non-Kähler ALF instantons that the corresponding moduli spaces are smooth and 2-dimensional. Therefore, Conjecture 1.7 can be restated in terms of infinitesimal rigidity, that is the statement that an ALF vacuum perturbation is, modulo gauge, a perturbation with respect to the moduli parameters, see [37]. The conjecture can be addressed by analyzing the compatibility complex along the lines in [2]. In the Lorentzian case, the corresponding result for Kerr has been shown to hold in [4].

2 Preliminaries and notation

In this section, (M, g_{ab}) denotes a four-dimensional orientable Riemannian manifold with Levi-Civita connection ∇_a.

2.1 Spinors and complex structures

We shall use the formalism of 2-spinors as developed by Penrose and Rindler [40, 41], with the exception that we adapt the framework to Riemannian signature following Woodhouse [47] (see also [27]). This can be done by first using that the constructions in [40, 41] formally apply in a complex space as explained in [41, Section 6.9], and then noticing that one can specialize to Riemannian signature by equipping the spin spaces with the Riemannian spinor conjugation † defined in [47, equation (2.5)].

We shall in fact only use spinors up to scale, so the existence of a global spin structure is not assumed. Note that the Taub-bolt and Chen–Teo instantons do not admit spin structures [19, 39].

Abstract spinor indices are denoted by $A, B, \dots$ and $A', B', \dots$, and can be raised and lowered with the symplectic forms ϵ_{AB}, $\epsilon_{A'B'}$ and their inverses. Tensor indices correspond to pairs of primed and unprimed spinor indices, $a = AA'$, $b = BB'$, etc. For example, the metric is $g_{ab} = \epsilon_{AB}\epsilon_{A'B'}$, and the Weyl tensor is

$$W_{abcd} = \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \tilde{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD},$$

where $\tilde{\Psi}_{A'B'C'D'}$ and Ψ_{ABCD} are the (totally symmetric) self-dual and anti-self-dual Weyl curvature spinors, respectively.

We shall assume that (M, g_{ab}) has an almost-complex structure $J^a{}_b$ compatible with g_{ab}. From in [31, Chapter IV, Proposition 9.8], $J^a{}_b$ can be represented by a projective spinor, say o^A. The explicit representation is

$$J^{AA'}{}_{BB'} = \mathrm{i}(o^A\iota_B + \iota^A o_B)\delta^{A'}_{B'}, \tag{2.1}$$

see [47, equation (4.7)], where ι^A is the complex conjugate of o^A:

$$\iota^A = o^{\dagger A}. \tag{2.2}$$

In (2.1), we chose the normalization $o_A\iota^A = 1$. The pair $\left(o^A, \iota^A\right)$ is called spin dyad. The components of Ψ_{ABCD} in this dyad are

$$\Psi_0 = \Psi_{ABCD}o^Ao^Bo^Co^D, \qquad \Psi_1 = \Psi_{ABCD}o^Ao^Bo^C\iota^D, \qquad \Psi_2 = \Psi_{ABCD}o^Ao^B\iota^C\iota^D, \tag{2.3}$$

together with $\Psi_3 = -\overline{\Psi_1}$ and $\Psi_4 = \overline{\Psi_0}$. Note that Ψ_2 is real.

2.2 Conformally invariant GHP connections

An almost-complex structure is invariant under two kinds of transformations: a rescaling of the spinors in (2.1) of the form

$$o^A \to \lambda o^A, \qquad \iota^A \to \lambda^{-1}\iota^A, \tag{2.4}$$

where $\lambda\colon M \to U(1)$, and conformal transformations

$$g_{ab} \to \hat{g}_{ab} = \Omega^2 g_{ab}, \tag{2.5}$$

where Ω is a positive function.

We shall use a framework that is covariant under both of the above transformations. This is closely related to the conformally invariant Geroch–Held–Penrose (GHP) formalism given in [40, Section 5.6], but with three main differences arising from the following requirements: it should apply to arbitrary spinor/tensor fields (the framework in [40, Section 5.6] applies only to scalar fields), it should be independent of a choice of primed spin dyad, it should be adapted to Riemann signature. A framework satisfying these requirements can be found in [8].

Under conformal transformations (2.5), the spin dyad in (2.1) transforms as

$$\hat{o}^A = \Omega^{-1/2} o^A, \qquad \hat{\iota}^A = \Omega^{-1/2}\iota^A. \tag{2.6}$$

A metric-dependent tensor/spinor field $\varphi^{\mathcal{A}}$ which transforms with respect to (2.4) and (2.5) according to

$$\varphi^{\mathcal{A}} \to \lambda^p \Omega^w \varphi^{\mathcal{A}} \tag{2.7}$$

is said to have conformal weight w and GHP weight p. Here, $\mathcal{A}$ represents an arbitrary collection of tensor/spinor indices. We shall refer to fields satisfying (2.7) as properly weighted fields with weights (w, p). For example, o^A and ι^A have weights $\left(-\frac{1}{2}, 1\right)$ and $\left(-\frac{1}{2}, -1\right)$, respectively, while o_A and ι_A have weights $\left(\frac{1}{2}, 1\right)$ and $\left(\frac{1}{2}, -1\right)$. The components Ψ_0, Ψ_1, Ψ_2 of the Weyl spinor (equation (2.3)) have conformal weight $w = -2$ and GHP weights $4, 2, 0$, respectively.

Let χ be a scalar field, φ_A a spinor field, and v_a a covector field, all of them with conformal weight w and GHP weight p. We define the covariant derivative $\mathcal{C}_a$ by

$$\begin{aligned}
&\mathcal{C}_a\chi = \nabla_a\chi + w f_a\chi + pP_a\chi, \\
&\mathcal{C}_{AA'}\varphi_B = \nabla_{AA'}\varphi_B - f_{A'B}\varphi_A + w f_{AA'}\varphi_B + pP_{AA'}\varphi_B, \\
&\mathcal{C}_a v_b = \nabla_a v_b + w f_a v_b + pP_a v_b - Q_{ab}{}^c v_c,
\end{aligned}$$

where

$$f_a = -\frac{1}{2}J^c{}_b\nabla_c J^b{}_a, \tag{2.8}$$

$$P_a = \omega_a - \frac{1}{2}\mathrm{i}J^b{}_a f_b, \tag{2.9}$$

$$Q_{ab}{}^c = f_a\delta^c_b + f_b\delta^c_a - f^c g_{ab}.$$

Here $\omega_a = \iota_B\nabla_a o^B$ is the GHP connection 1-form, and f_a is the Lee form. The action of $\mathcal{C}_a$ on fields with an arbitrary index structure is defined in the standard way.

Remark 2.1. We have the following facts, which generalize similar results for the standard GHP formalism and its corresponding covariant derivative Θ_a.

1. If χ has weights (w, p), then $\bar{\chi}$ has weights $(w, -p)$. This follows from (2.2), (2.4) and (2.6). This is different from Lorentzian GHP, since in that case there is also a "q-weight" associated to a primed spin dyad, and complex conjugation interchanges p and q.
2. $\mathcal{C}_a$ is real (it commutes with complex conjugation) and metric ($\mathcal{C}_a g_{bc} = 0$). Reality can be seen by making use of the previous item together with the fact that the 1-form P_a in (2.9) is purely imaginary, $\bar{P}_a = -P_a$.
3. $\mathcal{C}_a$ is covariant under GHP and conformal transformations [8, Section 2.3]: if (2.7) holds, then

$$\mathcal{C}_a\varphi^{\mathcal{A}} \to \lambda^p\Omega^w\mathcal{C}_a\varphi^{\mathcal{A}}.$$

This generalizes the transformation rule of the standard GHP derivative, that is $\Theta_a\varphi^{\mathcal{A}} \to \lambda^p\Theta_a\varphi^{\mathcal{A}}$ under GHP scalings (2.4).

4. (M, g_{ab}) is Hermitian if and only if $\mathcal{C}_a o^B = 0$. This follows since

$$\mathcal{C}_{AA'} o^B = \sigma_{A'} \iota_A \iota^B, \qquad \sigma_{A'} = o^A o^B \nabla_{AA'} o_B,$$

and, using GHP notation [40],

$$o^A o^B \nabla_{AA'} o_B = 0 \qquad \Leftrightarrow \qquad \kappa = \sigma = 0, \tag{2.10}$$

which is the condition for the existence of a shear-free null geodesic congruence [41, equation (7.3.1)], or, equivalently, an integrable complex structure [8, Section 2.4]. This generalizes the characterization of Kähler manifolds in terms of the GHP connection: a Riemannian 4-manifold is Kähler if and only if $\Theta_a o^B = 0$, see [31, Chapter IV, Proposition 9.8].

5. (M, g_{ab}) is conformally Kähler iff it is Hermitian and $f_a = \nabla_a \log \phi$ for some scalar field ϕ (with $w = -1$, $p = 0$). This field satisfies

$$\mathcal{C}_a \phi = 0.$$

For example, in the Einstein–Hermitian case we have that [8, Remark 5.1]

$$\phi \propto \Psi_2^{1/3}. \tag{2.11}$$

6. If (M, g_{ab}) is conformally Kähler, and u_a has weights w, $p = 0$, then

$$\mathcal{C}_a u^a = \phi^{-(w+2)} \nabla_a \big(\phi^{w+2} u^a\big). \tag{2.12}$$

2.3 Perturbations and the Teukolsky equation

Here we introduce some notation for gravitational perturbations and prove an identity that will be needed in Section 3.

The gravitational perturbations we consider are of two types: either compactly supported in the compact Einstein case, or they satisfy certain fall-off conditions in the ALF Ricci-flat case. For the latter, we use the following notation, which is taken from [3].

Definition 2.2 ([3, Definition 2.6])**.** Let (M, g_{ab}) be an ALF manifold as defined in [3, Definition 2.1]. Let t and s be any two tensor fields on (M, g_{ab}). We write

$$t = O(r^\alpha), \qquad s = O^*(r^\alpha)$$

if there is a constant C such that $|t| \le C r^\alpha$ for $r \ge A$, and $\left|\nabla^k s\right| = O\big(r^{\alpha - k}\big)$ for all non-negative integers k. Here, $|t|^2 = t_{a\ldots d} \bar{t}^{a\ldots d}$.

We treat variations of spinor and tensor fields following the approach introduced in [10], which can be adapted to Riemannian signature. In particular, given a symmetric 2-tensor h_{ab} on (M, g_{ab}), viewed as a linear perturbation of the metric, the corresponding perturbations of spinor and tensor fields are given by the variation operator ϑ. For example, the variation of the unprimed Weyl spinor is $\vartheta \Psi_{ABCD}$, and the variations of the scalars Ψ_i, $i = 0, 1, 2$ are given by $\vartheta \Psi_i$. The formula for $\vartheta \Psi_{ABCD}$ is

$$\vartheta \Psi_{ABCD} = \frac{1}{2} \nabla^{A'}_{(A} \nabla^{B'}_{B} h_{CD)A'B'} + \frac{1}{4} g^{ef} h_{ef} \Psi_{ABCD}, \tag{2.13}$$

which coincides with [40, equation (5.7.15)].

Let h_{ab} be an arbitrary metric perturbation. Recall that we defined the Einstein operator in (1.1). We denote its linearization by ϑE_{ab}, and the linearized Ricci tensor and scalar curvature by ϑR_{ab} and ϑS. From [11, Theorem 1.174], we have

$$\begin{aligned}\vartheta E_{ab} &= \vartheta R_{ab} - \frac{1}{4}\vartheta S g_{ab} - \frac{1}{4}S(g)h_{ab} \\ &= \frac{1}{2}\Delta h_{ab} - \frac{1}{2}\nabla_a\nabla_b\big(g^{cd}h_{cd}\big) + \frac{1}{2}\nabla^c\nabla_a h_{bc} + \frac{1}{2}\nabla^c\nabla_b h_{ac} \\ &\quad - \frac{1}{4}g_{ab}\big[\nabla^c\nabla^d h_{cd} + \Delta\big(g^{cd}h_{cd}\big) - h^{cd}R_{cd}\big] - \frac{1}{4}S(g)h_{ab},\end{aligned} \tag{2.14}$$

where $\Delta = -g^{ab}\nabla_a\nabla_b$. Define the operator

$$\mathrm{L} = g^{ab}\mathcal{C}_a\mathcal{C}_b - 18\Psi_2 \tag{2.15}$$

acting on scalar fields of weight (w,p). We have the following lemma.

Lemma 2.3. *Let (M, g_{ab}) be Einstein–Hermitian, with (possibly vanishing) cosmological constant λ. Let h_{ab} be an arbitrary metric perturbation (1.2), ϑE_{ab} the linearized Einstein operator (2.14), and $\vartheta\Psi_0$ the linearized Weyl scalar (1.4). Furthermore, let f_a be the Lee form (2.8), and Q^{abcd} the tensor field*

$$Q^{abcd} = o^A o^B o^C o^D \epsilon^{A'B'}\epsilon^{C'D'}. \tag{2.16}$$

Then

$$-\mathring{\Omega}^{-1}Q^{acbd}(\nabla_a - 4f_a)\nabla_d\vartheta E_{bc} = \mathrm{L}\big[\mathring{\Omega}^{-1}\vartheta\Psi_0\big], \tag{2.17}$$

where L is the operator (2.15) and $\mathring{\Omega}$ is an auxiliary constant conformal factor, that is a scalar field with weights $w = 1$, $p = 0$ and $\nabla_a\mathring{\Omega} = 0$.

Corollary 2.4. *Let h_{ab} be a linearized Einstein perturbation $\vartheta E_{ab} = 0$, and let $\chi = \mathring{\Omega}^{-1}\vartheta\Psi_0$. Then χ solves the Teukolsky equation*

$$\mathrm{L}[\chi] = 0. \tag{2.18}$$

Remark 2.5.

1. The auxiliary constant conformal factor $\mathring{\Omega}$ is necessary for conformal invariance (note that $\nabla_a\mathring{\Omega} = 0$ but $\mathcal{C}_a\mathring{\Omega} \neq 0$), see the proof below. Once the operator $\mathcal{C}_a$ is written in terms of the ordinary Levi-Civita connection (see (2.19) below), one can set $\mathring{\Omega} = 1$.
2. If (M, g_{ab}) is Hermitian, and χ has $w(\chi) = -3$ and $p(\chi) = 4$, then in Newman–Penrose notation, we have

$$\begin{aligned}\mathrm{L}[\chi] = 2\big[&(D + \tilde{\varepsilon} - 3\varepsilon - 4\rho - \tilde{\rho})\big(D' + 4\varepsilon' - \rho'\big) \\ &- \big(\delta + \tilde{\beta}' - 3\beta - 4\tau - \tau'\big)\big(\delta' + 2\beta' - \tau'\big) - 3\Psi_2\big]\chi,\end{aligned} \tag{2.19}$$

so we see that L coincides with the Teukolsky operator [43, equation (2.12)].

Proof of Lemma 2.3. The strategy is to consider identities for an arbitrary Riemannian manifold (M, g_{ab}), and then to linearize around an Einstein–Hermitian metric. Consider then an arbitrary (M, g_{ab}), with Levi-Civita connection ∇_a and unprimed Weyl curvature spinor Ψ_{ABCD}. Let $J^a{}_b$ be a (locally defined) compatible almost-complex structure, and let o^A be the associated spinor field as in Section 2.1. We can then define conformally and GHP weighted fields as in Section 2.2, together with the connection $\mathcal{C}_a$ on the corresponding bundles.

Let $\mathring{\Omega}$ be an arbitrary constant conformal factor, that is a scalar field with weights $w = 1$, $p = 0$ and $\nabla_a \mathring{\Omega} = 0$, and define

$$\varphi_{ABCD} := \mathring{\Omega}^{-1} \Psi_{ABCD}. \tag{2.20}$$

This object has weights $w = -1$, $p = 0$, and it is essentially the "gravitational spin 2 field" of Penrose and Rindler [41, equation (9.6.40)]. We have

$$\begin{aligned} \mathcal{C}_{AA'} \mathcal{C}^{A'E} \varphi_{BCDE} &= (\nabla_{AA'} - 4f_{AA'}) \nabla^{A'E} \varphi_{BCDE} \\ &= \mathring{\Omega}^{-1} (\nabla_{AA'} - 4f_{AA'}) \nabla^{A'E} \Psi_{BCDE} \\ &= \mathring{\Omega}^{-1} (\nabla_{AA'} - 4f_{AA'}) \nabla^{B'}_{(B} \Phi_{CD)B'}{}^{A'}, \end{aligned}$$

where in the second line we used the definition (2.20) and the fact that $\mathring{\Omega}$ is constant, and in the third line we used the spinor form of the Bianchi identities [40, equation (4.10.7)] adapted to Riemann signature (recall Section 2.1). Here $\Phi_{ABA'B'}$ is the trace-free Ricci spinor. Contracting with $o^A o^B o^C o^D$:

$$\begin{aligned} o^A o^B o^C o^D \mathcal{C}_{AA'} \mathcal{C}^{A'E} \varphi_{BCDE} &= \mathring{\Omega}^{-1} o^A o^B o^C o^D (\nabla_{AA'} - 4f_{AA'}) \nabla^{B'}_{B} \Phi_{CDB'}{}^{A'} \\ &= \mathring{\Omega}^{-1} o^A o^C o^B o^D \epsilon^{A'C'} \epsilon^{B'D'} (\nabla_{AA'} - 4f_{AA'}) \nabla_{DD'} \Phi_{BCB'C'} \\ &= -\frac{1}{2} \mathring{\Omega}^{-1} Q^{acbd} (\nabla_a - 4f_a) \nabla_d E_{bc}, \end{aligned} \tag{2.21}$$

where in the last line we used the definition (2.16) and the identity $\Phi_{bc} = -\frac{1}{2} E_{bc}$ (see [40, equation (4.6.25)] and recall (1.1)).

It remains to find a convenient expression for the first line in (2.21). This can be done using [7, equations (3.6) and (3.15)]. We have

$$o^A o^B o^C o^D \mathcal{C}_{AA'} \mathcal{C}^{A'E} \varphi_{BCDE} = \frac{1}{2} \big(g^{ab} \mathcal{C}_a \mathcal{C}_b - 18\Psi_2 \big) \varphi_0 + B, \tag{2.22}$$

where $\varphi_0 = o^A o^B o^C o^D \varphi_{ABCD} = \mathring{\Omega}^{-1} \Psi_0$ and B is a term which couples the GHP quantities κ, σ, Ψ_0, Ψ_1 quadratically. Equating the last line of (2.21) to the right-hand side of (2.22), and taking a linearization around a metric which satisfies $E_{ab}|_{s=0} = 0$, $\Psi_0|_{s=0} = \Psi_1|_{s=0} = \kappa|_{s=0} = \sigma|_{s=0} = 0$, that is, an Einstein–Hermitian metric, see (2.10), the result (2.17) follows. ■

Lemma 2.6. *Let (M, g_{ab}) be Einstein–Hermitian, with volume form $\mathrm{d}\mu$. Let V be a four-dimensional region in M with boundary ∂V, whose unit normal and induced volume form are n^a, $\mathrm{d}\Sigma$, respectively. For any scalar field χ with conformal weight $w = -3$ and GHP weight $p = 4$ satisfying (2.18), we have*

$$0 = \int_{\partial V} \Psi_2^{-4/3} \bar{\chi} (n^a \mathcal{C}_a \chi) \, \mathrm{d}\Sigma - \int_V \Psi_2^{-4/3} \big(|\mathcal{C}\chi|^2 + 18\Psi_2 |\chi|^2 \big) \mathrm{d}\mu. \tag{2.23}$$

Proof. We have that (M, g_{ab}) is Einstein and conformally Kähler, see [22], so identities (2.11) and (2.12) hold. Let χ be a solution to (2.18), with $w = -3$, $p = 4$. First notice that the covector field $\bar{\chi} \mathcal{C}_a \chi$ has weights $w = -6$, $p = 0$, so using (2.11) and (2.12), we have

$$\mathcal{C}_a (\bar{\chi} \mathcal{C}^a \chi) = \Psi_2^{4/3} \nabla_a \big[\Psi_2^{-4/3} \bar{\chi} \mathcal{C}^a \chi \big]. \tag{2.24}$$

Now we multiply (2.18) by $\Psi_2^{-4/3} \bar{\chi}$ and use the Leibniz property of $\mathcal{C}_a$ together with (2.24),

$$\begin{aligned} 0 = \Psi_2^{-4/3} \bar{\chi} \mathrm{L}[\chi] &= \Psi_2^{-4/3} \bar{\chi} g^{ab} \mathcal{C}_a \mathcal{C}_b \chi - 18 \Psi_2^{-1/3} |\chi|^2 \\ &= \Psi_2^{-4/3} g^{ab} \mathcal{C}_a (\bar{\chi} \mathcal{C}_b \chi) - \Psi_2^{-4/3} g^{ab} (\mathcal{C}_a \bar{\chi})(\mathcal{C}_b \chi) - 18 \Psi_2^{-1/3} |\chi|^2 \\ &= \nabla_a \big(\Psi_2^{-4/3} \bar{\chi} \mathcal{C}^a \chi \big) - \Psi_2^{-4/3} \big(|\mathcal{C}\chi|^2 + 18\Psi_2 |\chi|^2 \big). \end{aligned}$$

Integrating this equation over a four-dimensional region V and using the divergence theorem, we get (2.23). ■

3 Mode stability

3.1 ALF instantons

The proof of the following lemma is similar to the proof of [13, Theorem A].

Lemma 3.1. *Let (M, g_{ab}) be a Hermitian non-Kähler ALF instanton. Then $\Psi_2 > 0$ in M.*

Remark 3.2.

1. In view of the classification of Hermitian non-Kähler ALF instantons [33], one could prove Lemma 3.1 by an explicit calculation for the relevant families of instantons. For the Chen–Teo case, the calculation needed is lengthy but can be done along the lines in [1].
2. For a conformally Kähler manifold (M, g_{ab}), where the Kähler metric and its scalar curvature are $\hat{g}_{ab} = \varphi^2 g_{ab}$ and $\hat{S}$, it holds

$$\Psi_2 = \varphi^2 \frac{\hat{S}}{12}, \tag{3.1}$$

see [6]. Thus $\operatorname{sign}\Psi_2 = \operatorname{sign}\hat{S}$, so it is sufficient to show that $\hat{S} > 0$.

Proof. Let $\mathcal{W}^+$ be the self-dual part of the Weyl tensor. By [22, Proposition 5, p. 420], we have that $\mathcal{W}^+$ does not have zeros in M, so Ψ_2 does not have zeros either. Hence, by (3.1), $\hat{S}$ does not change sign. With the conformal factor

$$\varphi = 24^{1/6} \left|\mathcal{W}^+\right|_g^{1/3} \tag{3.2}$$

the metric $\hat{g}_{ab} = \varphi^2 g_{ab}$ is extremal Kähler with scalar curvature $\hat{S}$ satisfying

$$\hat{S}\varphi^3 = 6\Delta\varphi,$$

where $\Delta = -g^{ab}\nabla_a\nabla_b$. By construction, $\varphi > 0$. Recalling Definition 2.2, we have $\left|\mathcal{W}^+\right| = O\big(r^{-3}\big)$ so $\varphi \to 0$ at ∞. Hence φ must have a local maximum at some $x \in M$, and $\Delta\varphi|_x \geq 0$. This implies $\hat{S}\varphi^3\big|_x > 0$, which since $\varphi > 0$ implies $\hat{S}(x) > 0$. By point (2) of Remark 3.2, we find that $\Psi_2 > 0$. ∎

We are now ready to prove our main theorem.

Proof of Theorem 1.4. Consider an ALF vacuum perturbation h_{ab}, that is, h_{ab} satisfies (1.3) and $\nabla^k h_{ab} = O\big(r^{-1-k}\big)$ for any integer $k \geq 0$. We recall that the symbols O, O^* used here and below were introduced in Definition 2.2. Let $\chi = \vartheta\Psi_0$ be the linearized extreme Weyl scalar. Since this involves two derivatives of h_{ab} (see (2.13)), the ALF assumption for the perturbation implies $\chi = O^*\big(r^{-3}\big)$. In particular, we have $\mathcal{C}_a\chi = O\big(r^{-4}\big)$. In addition, the ALF condition for the background instanton implies $\Psi_2 = O\big(r^{-3}\big)$, so $\Psi_2^{-4/3} = O\big(r^4\big)$. Therefore, letting $V = \{r < R\}$, from the above we deduce that on ∂V we have

$$\Psi_2^{-4/3}\bar{\chi}(\mathcal{C}_a\chi) = O\big(r^{-3}\big)$$

while $A(\partial V) = O\big(r^2\big)$. This shows that the boundary term in (2.23) is $O\big(r^{-1}\big)$ and hence letting $r \to \infty$, we have

$$0 = \int_M \Psi_2^{-4/3}\big(|\mathcal{C}\chi|^2 + 18\Psi_2|\chi|^2\big)\mathrm{d}\mu.$$

Since by Lemma 3.1 $\Psi_2 > 0$, we get $\chi = 0$ and the result follows. ∎

3.2 The compact case

In this section, we extend our mode stability result (Theorem 1.4) to the compact case. The only known Ricci-flat compact 4-manifolds are the flat 4-torus and K3 surfaces (which are half-flat), so we need to include a cosmological constant $\lambda \neq 0$ (see Remark 1.6).

A classification of compact Einstein–Hermitian (non-Kähler) 4-manifolds with $\lambda > 0$ is known from LeBrun [32], the only possibilities are the Fubini–Study metric on $\mathbb{CP}^2$ (with orientation opposite to the Kähler one), the Page metric on $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$, or the Chen–LeBrun–Weber metric on $\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2$. We note that the Page metric corresponds to a special limit of the Riemannian Kerr–de Sitter solution [38].

Proof of Theorem 1.5. Let (M, g_{ab}) be a compact Einstein–Hermitian 4-manifold with $\lambda > 0$, and consider a metric perturbation h_{ab}. Note that Lemmas 2.3 and 2.6 apply also to the compact case. By item (2) in Remark 2.5, if h_{ab} solves $\vartheta E_{ab} = 0$, then we have a solution χ to equation (2.18), so identity (2.23) applies. From the above list of compact Einstein–Hermitian instantons, we see that all of them are closed, so the boundary term in (2.23) vanishes:

$$0 = \int_M \Psi_2^{-4/3}\big(|\mathcal{C}\chi|^2 + 18\Psi_2|\chi|^2\big)\mathrm{d}\mu.$$

The sign of Ψ_2 can be determined by an analog of Lemma 3.1: from item (2) in Remark 3.2, we need only focus on the sign of the scalar curvature $\hat{S}$ of the conformally related Kähler metric $\hat{g}_{ab} = \varphi^2 g_{ab}$, where φ is still given by (3.2). Using [44, equation (D.9)] with $\Omega = \varphi$, the conformal behaviour of scalar curvature is

$$\varphi^2\hat{S} = S + 6\varphi^{-1}\Delta\varphi.$$

Since $S = 4\lambda > 0$, and since the proof of Lemma 3.1 applies to show that $\varphi^{-1}\Delta\varphi > 0$, we have $\hat{S} > 0$, and thus $\Psi_2 > 0$. So $\chi = 0$, and the result follows. ■

3.3 Negative modes

Here we comment on the compatibility of our mode stability results with other notions of stability in the literature, both in the ALF and compact cases.

A frequently used definition of Riemannian linear stability for Einstein metrics, see, for example, [11, Definition 4.63], is in terms of a variational problem: given the Einstein–Hilbert functional S, an Einstein metric g_{ab} is said to be stable if the second variation of S at g_{ab} is negative for all compactly supported, trace-free metric perturbations. If, on the other hand, one can find a perturbation such that the second variation of S is positive, then g_{ab} is said to be unstable.

The above definition is often formulated as an eigenvalue problem: if h_{ab} satisfies the TT conditions $\nabla^a h_{ab} = 0$ and $g^{ab}h_{ab} = 0$, one considers the problem $L(h)_{ab} = \mu h_{ab}$, where $L(h)_{ab} = -g^{cd}\nabla_c\nabla_d h_{ab} - 2R_a{}^c{}_b{}^d h_{cd}$, and the solution is unstable if there is a negative mode $\mu < 0$. See, for example, [28, Section V], where a negative mode is found for the Schwarzschild instanton, and used to argue about the *semi-classical* instability of the solution; see also Witten's work [46].

It was recently shown in [14] that if (M, g_{ab}) is a conformally Kähler 4-manifold which is either compact and Einstein, or ALF and Ricci-flat, then it is unstable in the above variational sense. Here we point out the following.

Proposition 3.3. *The unstable metric perturbations found in [14] are conformally half-flat: the unprimed linearized Weyl curvature spinor identically vanishes.*

Remark 3.4. The above result means that $\vartheta\Psi_{ABCD} = 0$, thus in particular $\vartheta\Psi_0 = 0$, so we see that the variational instability is still compatible with mode stability in the sense of Definition 1.2.

Proof. In both the compact and ALF cases, the unstable metric perturbations in [14] are given by the composition of a closed anti-self-dual 2-form ω^- and the conformal Killing–Yano tensor τ associated to the conformal Kähler structure; see [14]. In spinor notation, this can be expressed as follows: $\omega^-_{ab} = \phi_{A'B'}\epsilon_{AB}$, $\tau_{ab} = K_{AB}\epsilon_{A'B'}$, and the unstable perturbation is

$$h_{ab} = (\omega^- \circ \tau)_{ab} = \phi_{A'B'}K_{AB}, \tag{3.3}$$

where $\phi_{A'B'}$ and K_{AB} satisfy the Maxwell and Killing spinor equations, respectively,

$$\nabla^{AA'}\phi_{A'B'} = 0, \qquad \nabla_{A'(A}K_{BC)} = 0. \tag{3.4}$$

In the compact case, ω^-_{ab} can be any closed anti-self-dual 2-form, that is any Maxwell field $\phi_{A'B'}$. In the ALF case, $\phi_{A'B'} = \nabla_{A(A'}X^A_{B')}$, where X^a is the Killing field associated to the Killing spinor K_{AB} [14].

We can compute the linearized Weyl spinor using formula (2.13). Since the trace of (3.3) vanishes, we have

$$\vartheta\Psi_{ABCD} = \frac{1}{2}\nabla^{A'}_{(A}\nabla^{B'}_{B}\left[K_{CD)}\phi_{A'B'}\right] = 0$$

where the second equality follows from (3.4). ■

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