Symmetry, Integrability and Geometry: Methods and Applications SIGMA **21** (2025), 020, 24 pages

Non-Integrability of the Sasano System of Type $D_5^{(1)}$ and Stokes Phenomena

Tsvetana STOYANOVA

Faculty of Mathematics and Informatics, Sofia University "St. Kliment Ohridski", 5 J. Bourchier Blvd., Sofia 1164, Bulgaria
E-mail: *cveti@fmi.uni-sofia.bg*

Received November 28, 2023, in final form March 10, 2025; Published online March 27, 2025
https://doi.org/10.3842/SIGMA.2025.020

Abstract. In 2006, Y. Sasano proposed higher-order Painlevé systems, which admit affine Weyl group symmetry of type $D_l^{(1)}$, $l = 4, 5, 6, \dots$. In this paper, we study the integrability of a four-dimensional Painlevé system, which has symmetry under the extended affine Weyl group $\widetilde{W}\big(D_5^{(1)}\big)$ and which we call the Sasano system of type $D_5^{(1)}$. We prove that one family of the Sasano system of type $D_5^{(1)}$ is not integrable by rational first integrals. We describe Stokes phenomena relative to a subsystem of the second normal variational equations. This approach allows us to compute in an explicit way the corresponding differential Galois group and therefore to determine whether the connected component of its unit element is not Abelian. Applying the Morales–Ramis–Simó theory, we establish a non-integrable result.

Key words: Sasano systems; non-integrability of Hamiltonian systems; differential Galois theory; Stokes phenomenon

2020 Mathematics Subject Classification: 34M55; 37J30; 34M40; 37J65

1 Introduction

In 2006, using the methods from algebraic geometry Y. Sasano [27] introduced higher-order Painlevé systems, which have symmetry under the affine Weyl group of type $D_l^{(1)}$, $l = 4, 5, 6, \dots$. Subsequently Fuji and Suzuki [13] derived the higher-order Painlevé systems of type $D_{2n+2}^{(1)}$ from the Drinfeld–Sokolov hierarchy by similarity reduction. These higher-order Painlevé systems have four essential properties:

1. They are Hamiltonian systems.
2. They admit an affine Weyl group symmetry of type $D_l^{(1)}$ as Bäcklund transformations.
3. They can be considered as higher-order analogues of the Painlevé V and Painlevé VI systems.
4. They have several symplectic coordinate systems, on which the Hamiltonians are polynomial.

In this paper, we study the integrability of the following fourth-order Painlevé system

$$\begin{aligned}
\dot{x} &= \frac{2x^2y}{t} + x^2 - \frac{2xy}{t} - \left(1 + \frac{\beta}{t}\right)x + \frac{\alpha_2+\alpha_5}{t} + \frac{2z((z-1)w+\alpha_3)}{t},\\
\dot{y} &= -\frac{2xy^2}{t} + \frac{y^2}{t} - 2xy + \left(1 + \frac{\beta}{t}\right)y - \alpha_1,\\
\dot{z} &= \frac{2z^2w}{t} + z^2 - \frac{2zw}{t} - \left(1 + \frac{\alpha_5+\alpha_4}{t}\right)z + \frac{\alpha_5}{t} + \frac{2yz(z-1)}{t},
\end{aligned}$$

$$\dot{w} = -\frac{2zw^2}{t} + \frac{w^2}{t} - 2zw + \left(1 + \frac{\alpha_5 + \alpha_4}{t}\right)w - \alpha_3 - \frac{2y(-w + 2zw + \alpha_3)}{t} \tag{1.1}$$

with $\beta = 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5$, where $\alpha_0, \alpha_1, \ldots, \alpha_5$ are complex parameters, which satisfy the relation

$$\alpha_0 + \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 = 1.$$

The system (1.1) is one of the systems introduced by Sasano in [27]. It admits the extended affine Weyl group $\widetilde{W}\big(D_5^{(1)}\big)$ as a group of Bäcklund transformations. For these reasons, throughout this paper we call the system (1.1) the Sasano system of type $D_5^{(1)}$ or in short the Sasano system. The system (1.1) is a two-degree-of-freedom non-autonomous Hamiltonian system with the Hamiltonian

$$\begin{aligned} H = {} & H_V(x, y, t; \alpha_2 + \alpha_5, \alpha_1, \alpha_2 + 2\alpha_3 + \alpha_4) + H_V(z, w, t; \alpha_5, \alpha_3, \alpha_4) \\ & + \frac{2yz((z-1)w + \alpha_3)}{t}, \end{aligned} \tag{1.2}$$

where $H_V(q, p, t; \gamma_1, \gamma_2, \gamma_3)$ is the Hamiltonian associated with the fifth Painlevé equation, i.e.,

$$H_V(q, p, t; \gamma_1, \gamma_2, \gamma_3) = \frac{q(q-1)p(p+t) - (\gamma_1 + \gamma_3)qp + \gamma_1 p + \gamma_2 tq}{t}.$$

Hence the system (1.1) is considered as coupled Painlevé V systems in dimension 4. The non-autonomous Hamiltonian system (1.1) can be turned into an autonomous one with three degrees of freedom by introducing two new dynamical variables: t and its conjugate variable $-F$. The new Hamiltonian becomes

$$\widetilde{H} = H + F,$$

where H is given by (1.2). Then the extended Hamiltonian system (1.1) becomes

$$\begin{aligned} &\frac{\mathrm{d}x}{\mathrm{d}s} = \frac{\partial \widetilde{H}}{\partial y}, \qquad \frac{\mathrm{d}y}{\mathrm{d}s} = -\frac{\partial \widetilde{H}}{\partial x}, \\ &\frac{\mathrm{d}z}{\mathrm{d}s} = \frac{\partial \widetilde{H}}{\partial w}, \qquad \frac{\mathrm{d}w}{\mathrm{d}s} = -\frac{\partial \widetilde{H}}{\partial z}, \\ &\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{\partial \widetilde{H}}{\partial F}, \qquad \frac{\mathrm{d}F}{\mathrm{d}s} = -\frac{\partial \widetilde{H}}{\partial t}. \end{aligned} \tag{1.3}$$

The symplectic structure ω is canonical in the variables (x, y, z, w, t, F), i.e., $\omega = \mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}z \wedge \mathrm{d}w + \mathrm{d}t \wedge \mathrm{d}F$.

In this paper, we are interested in the non-integrability of the Hamiltonian system (1.3). Recall that from the theorem of Liouville–Arnold [1] this means the non-existence of three first integrals $f_1 = \widetilde{H}, f_2, f_3$ functionally independent and in involution. We prove that the Sasano system (1.1) is non-integrable by rational first integrals. Our approach comes under the frame of the Morales–Ramis–Simó theory. This theory reduces the problem of integrability of a given analytic Hamiltonian system to the problem of integrability of the variational equation of this Hamiltonian system along one particular non-stationary solution. Since the variational equations are linear ordinary differential equations their integrability is well defined in the context of the differential Galois theory. The Morales–Ramis–Simó theory finds applications in the study of non-integrability of a huge range of dynamical systems like N-body problems [4, 9, 16, 23, 35, 36], problems with homogeneous potentials [6, 10, 11], the Painlevé equation and their q-analogues [7, 12, 18, 31, 32, 33], the higher-order analogues of Painlevé systems [34], as well as in the study of non-integrability of non-Hamiltonian systems [2, 5], in the study of integrability in the Jacobi sense [15], in the study of the irreducibility of the Painlevé equations [8], etc.

In this paper, we study the extended Sasano system (1.3) when $\alpha_1 = \alpha_2 = \alpha_3 = 0$, $\alpha_4 = 1$, $\alpha_0 = -\alpha_5$. It turns out that the differential Galois group of the first variational equations is a commutative group. To establish a non-integrable result, we find an obstruction to integrability studying the linearized second normal variational equations $(\mathrm{LNVE})_2$, which is a linear homogeneous system of thirteen order. To determine the Galois group of such a higher-order linear system, we find a subsystem of the $(\mathrm{LNVE})_2$, whose Galois group can be computed explicitly. It turns out that this subsystem is a system with non-trivial Stokes phenomena at the infinity point. Computing the corresponding Stokes matrices we deduce that the connected component of the unit element of the differential Galois group of this subsystem and hence the Galois group of the $(\mathrm{LNVE})_2$ is not an Abelian group. Then the key result of this paper (Theorem 3.11 in Section 3) states

Theorem 1.1. *Assume that $\alpha_1 = \alpha_2 = \alpha_3 = 0$, $\alpha_4 = 1$, $\alpha_0 = -\alpha_5$, where α_5 is arbitrary. Then the Sasano system (1.1) is not integrable in the Liouville–Arnold sense by rational first integrals.*

Using Bäcklund transformations of the Sasano system (1.1), which are rational canonical transformations [28], we can extend the result of the key Theorem 1.1 to the main results of this paper (Theorems 4.6 and 4.7 in Section 4).

Theorem 1.2. *Let α be an arbitrary complex parameter, which is not an integer. Assume that the parameters α_j are either of the kind $\pm\alpha + n_j$ or of the kind $l_j, n_j, l_j \in \mathbb{Z}$ in such a way that $1 - \alpha_0 - \alpha_1$ and $\alpha_4 + \alpha_5$ are together of the kind $\pm\alpha + m_i$, $m_i \in \mathbb{Z}$, $i = 1, 2$. Then the Sasano system (1.1) is not integrable in the Liouville–Arnold sense by rational first integrals.*

Theorem 1.3. *Assume that all of the parameters α_j, $0 \le j \le 5$, are integer in such a way that $1-\alpha_0-\alpha_1$ and $\alpha_4+\alpha_5$ are together either even or odd integer. Then the Sasano system (1.1) is not integrable in the Liouville–Arnold sense by rational first integrals.*

In fact, the result of Theorem 1.2 contains the result of Theorem 1.3. We present Theorem 1.3 as an independent result because of the additional specification of the quantities $1 - \alpha_0 - \alpha_1$ and $\alpha_4 + \alpha_5$ when $\alpha \in \mathbb{Z}$.

This paper is organized as follows. In the next section, we briefly review the basics of the Morales–Rams–Simó theory of the non-integrability of the Hamiltonian systems and the relation of the differential Galois theory to the linear systems of ordinary differential equations. In Section 3, we prove non-integrability of the Sasano system (1.1) when $\alpha_1 = \alpha_2 = \alpha_3 = 0$, $\alpha_4 = 1$ and $\alpha_0 = -\alpha_5$. In Section 4, using Bäcklund transformations of the Sasano system (1.1), we extend the result of the Section 3 to the entire orbits of the parameters α_j and establish the main theorems of this paper.

2 Preliminaries

2.1 Non-integrability of Hamiltonian systems and differential Galois theory

In this subsection, we briefly recall Morales-Ruiz–Ramis–Simó theory of non-integrability of Hamiltonian systems following [19, 20, 21, 22].

Let M be a symplectic analytical complex manifold of complex dimension $2n$. Consider on M a Hamiltonian system

$$\dot{x} = X_H(x) \tag{2.1}$$

with a Hamiltonian $H\colon M \to \mathbb{C}$. Let $x(t)$ be a particular solution of (2.1), which is not an equilibrium point. Denote by Γ the phase curve corresponding to this solution. The first variational equations $(\mathrm{VE})_1$ of (2.1) along Γ are written

$$\dot{\xi} = \frac{\partial X_H}{\partial x}(x(t))\xi, \qquad \xi \in T_\Gamma M. \tag{2.2}$$

Using the Hamiltonian H, we can always reduce the degrees of freedom of the variational equations (2.2) by one in the following sense. Consider the normal bundle of Γ on the level variety $M_h = \{x \mid H(x) = h\}$. The projection of the variational equations (2.2) on this bundle induces the so called first normal variational equations $(\mathrm{NVE})_1$ along Γ. The dimension of the $(\mathrm{NVE})_1$ is $2n-2$. Assume now that $x(t)$ is a rational non stationary particular solution of (2.1) and let as above Γ be the phase curve corresponding to it. Assume also that the field K of the coefficients of the $(\mathrm{NVE})_1$ is the field of rational functions in t, that is $K = \mathbb{C}(t)$. Assume also that $t = \infty$ is an irregular singularity for the $(\mathrm{NVE})_1$. The entries of a fundamental matrix solution of $(\mathrm{NVE})_1$ define a Picard–Vessiot extension L_1 of the field K. This in its turn defines a differential Galois group $G_1 = \mathrm{Gal}(L_1/K)$. Then the main theorem of the Morales-Ruiz–Ramis theory states [19, 20, 22].

Theorem 2.1 (Morales–Ramis)**.** *Assume that the Hamiltonian system* (2.1) *is completely integrable with rational first integrals in a neighbourhood of* Γ*, not necessarily independent on* Γ *itself. Then the identity component* $(G_1)^0$ *of the differential Galois group* $G_1 = \mathrm{Gal}(L_1/K)$ *is Abelian.*

The problem considered in this paper is one among many examples illustrating that the opposite is not true in general. That is if the connected component $(G_1)^0$ of the unit element of the differential Galois group $\mathrm{Gal}(L_1/K)$ is Abelian, one cannot deduce that the corresponding Hamiltonian system (2.1) is completely integrable. Beyond the first variational equations Morales-Ruiz, Ramis and Simó suggest in [22] to use higher-order variational equations to solve such integrability problems. Let as above $x(t)$ be a particular rational non stationary solution of the Hamiltonian system (2.1). We write the general solution as $x(t,z)$, where z parametrizes it near $x(t)$ as $x(t,z_0) = x(t)$. Then we can write the system (2.1) as

$$\dot{x}(t,z) = X_H(x(t,z)). \tag{2.3}$$

Denote by $x^{(k)}(t,z)$, $k \geq 1$ the derivatives of $x(t,z)$ with respect to z and by $X_H^{(k)}(x)$, $k \geq 1$ the derivatives of $X_H(x)$ with respect to x. By successive derivations of (2.3) with respect to z and evaluations at z_0, we obtain the so called k-th variational equations $(\mathrm{VE})_k$ along the solution $x(t)$

$$\dot{x}^{(k)}(t) = X_H^{(1)}(x(t))x^{(k)}(t) + P\big(x^{(1)}(t), x^{(2)}(t), \ldots, x^{(k-1)}(t)\big). \tag{2.4}$$

Here P denotes polynomial terms in the monomials of order $|k|$ of the components of its arguments. The coefficients of P depend on t through $X_H^{(j)}(x(t))$, $j < k$. For every $k > 1$, the linear non-homogeneous system (2.4) can be arranged as a linear homogeneous system of higher dimension by making the monomials of order $|k|$ in P new variables and adding to (2.4) their differential equations. If we restrict the system (2.4) to the variables that define the $(\mathrm{NVE})_1$, the corresponding linear homogeneous system is the so called k-th linearized normal variational equations $(\mathrm{LNVE})_k$. The solutions of the chain of $(\mathrm{LNVE})_k$ define a chain of Picard–Vessiot extensions of the main field $K = \mathbb{C}(t)$ of the coefficients of $(\mathrm{NVE})_1$, i.e., we have $K \subset L_1 \subset L_2 \subset \cdots \subset L_k$, where L_1 is above, L_2 is the Picard–Vessiot extension of K associated with $(\mathrm{LNVE})_2$, etc. Then we can define the differential Galois groups $G_1 = \mathrm{Gal}(L_1/K), G_2 = \mathrm{Gal}(L_2/K), \ldots, G_k = \mathrm{Gal}(L_k/K)$. Assume as above that $t = \infty$ is an irregular singularity for the $(\mathrm{NVE})_1$ and therefore for $(\mathrm{NVE})_k$ for all $k \geq 2$. Then the main theorem of the Morales-Ruiz–Ramis–Simó theory states the following.

Theorem 2.2 (Morales–Ramis–Simó)**.** *If the Hamiltonian system* (2.1) *is completely integrable with rational first integrals, then for every* $k \in \mathbb{N}$ *the connected component of the unit element* $(G_k)^0$ *of the differential Galois group* $G_k = \operatorname{Gal}(L_k/K)$ *is Abelian.*

From Theorem 2.2, it follows that if we find a group $(G_k)^0$, which is not Abelian, then the Hamiltonian system (2.1) will be non-integrable by means of rational first integrals. Note that this non-commutative group $(G_k)^0$ will be a solvable group. In this way, non-integrability in the sense of the Hamiltonian dynamics will correspond to integrability in the Picard–Vessiot sense.

2.2 Differential Galois group of a linear system of ordinary differential equations

In this subsection, we briefly recall some facts, notations and definitions from the differential Galois theory, needed to compute the differential Galois group of a linear system with one irregular and one regular singularity. We follow the works of van der Put, Mitschi, Singer and Ramis [17, 24, 30, 37]. Throughout this paper, all angular directions and sectors are defined on the Riemann surface of the natural logarithm.

Consider a linear system of ordinary differential equations of order n

$$\dot{\upsilon} = A(t)\upsilon, \tag{2.5}$$

where $A(t) \in \mathrm{GL}_n(\mathbb{C}(t))$.

Definition 2.3. The differential Galois group G of the system (2.5) over $\mathbb{C}(t)$ is the group of all differential $\mathbb{C}(t)$-automorphisms of a Picard–Vessiot extension of $\mathbb{C}(t)$ relative to (2.5). This group is isomorphic to an algebraic subgroup of $\mathrm{GL}_n(\mathbb{C})$ with respect to a fundamental matrix solution of (2.5).

Assume that the system (2.5) has two singular points over $\mathbb{CP}^1$ taken at $t = 0$ and $t = \infty$. Assume that the origin is a regular singularity while $t = \infty$ is a non-resonant irregular singularity of Poincaré rank 1. Denote by S the set of singular points of the system (2.5), that is, $S = \{0, \infty\}$. If we replace in Definition 2.3 the field $\mathbb{C}(t)$ with the field of germs of meromorphic functions at $a \in S$, we define the so called local differential Galois group G_a of (2.5). In what follows, we present effective theorems for computing the local differential Galois groups G_a, $a \in S$ of the system (2.5).

Let $\Phi(t)$ be a local fundamental matrix solution near the origin of the system (2.5). The following result of Schlesinger [29] describes the local differential Galois group at the origin of the system (2.5).

Theorem 2.4 (Schlesinger)**.** *Under the above assumptions the monodromy group around the origin with respect to the fundamental matrix solution* $\Phi(t)$ *is a Zariski dense subgroup of the differential Galois group* G *of the system* (2.5)*.*

Since we prefer to work with an irregular singularity at the origin to at $t = \infty$, we make the change $t = 1/\tau$ in the system (2.5). This transformation takes the system (2.5) into the system

$$\upsilon' = A(\tau)\upsilon, \qquad {}' = \frac{\mathrm{d}}{\mathrm{d}\tau}, \tag{2.6}$$

for which the origin is a non-resonant irregular singularity of Poincaré rank 1. Denote by $\mathbb{C}(\tau)$, $\mathbb{C}((\tau))$ and $\mathbb{C}\{\tau\}$ the differential fields of rational functions, formal power series and convergent power series, respectively. Note that

$$\mathbb{C}(\tau) \subset \mathbb{C}\{\tau\} \subset \mathbb{C}((\tau)).$$

In what follows, we determine the local differential Galois groups of the system (2.6) around the origin.

From the Hukuhara–Turrittin theorem [38], it follows that the system (2.6) admits a formal fundamental matrix solution at the origin of the form

$$\hat{\Psi}(\tau) = \hat{H}(\tau)\tau^{\Lambda}\exp\left(\frac{Q}{\tau}\right), \tag{2.7}$$

where

$$\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n), \qquad Q = \operatorname{diag}(q_1, q_2, \dots, q_n), \qquad \hat{H}(t) \in \mathrm{GL}_n(\mathbb{C}((t)))$$

with λ_j, $q_j \in \mathbb{C}$, $j = 1, \dots, n$. Consider the system (2.6) and its formal fundamental matrix solution $\hat{\Psi}(\tau)$ over the field $\mathbb{C}((\tau))$.

Definition 2.5. With respect to the formal fundamental matrix solution $\hat{\Psi}(\tau)$ from (2.7), we define the formal monodromy matrix $\hat{M}_0 \in \mathrm{GL}_n(\mathbb{C})$ around the origin as

$$\hat{\Psi}\big(\tau.\mathrm{e}^{2\pi\mathrm{i}}\big) = \hat{\Psi}(\tau)\hat{M}_0.$$

In particular,

$$\hat{M}_0 = \mathrm{e}^{2\pi\mathrm{i}\Lambda}.$$

Definition 2.6. With respect to the formal fundamental matrix solution $\hat{\Psi}(\tau)$ from (2.7) we define the exponential torus $\mathcal{T}$ as the differential Galois group $\mathrm{Gal}(E/F)$, where

$$F = \mathbb{C}((\tau))\big(\tau^{\lambda_1}, \tau^{\lambda_2}, \dots, \tau^{\lambda_n}\big) \qquad \text{and} \qquad E = F\big(\mathrm{e}^{q_1/\tau}, \mathrm{e}^{q_2/\tau}, \dots, \mathrm{e}^{q_n/\tau}\big).$$

We may consider $\mathcal{T}$ as a subgroup of $(\mathbb{C}^*)^n$. The Zariski closure of the group generated by the formal monodromy matrix and exponential torus yields the so called formal differential Galois group at the origin of the system (2.6) (see [30, 37]).

Consider now the system (2.6) over the field $\mathbb{C}\{\tau\}$. In general, the entries $\hat{h}_{ij}(\tau)$, $1 \le i, j \le n$ of the matrix $\hat{H}(\tau)$ in (2.7) are either divergent or convergent power series in τ. The existence of divergent power series entries in $\hat{H}(\tau)$ ensures an observation of a non-trivial Stokes phenomenon at the origin.

Definition 2.7. Under the above notations for every divergent power series $\hat{h}_{ij}(\tau)$, we define a set Θ_j of admissible singular directions θ_{ji}, $0 \le \theta_{ji} < 2\pi$, where θ_{ji} is the bisector of the maximal angular sector $\big\{\mathrm{Re}\big(\frac{q_i - q_j}{\tau}\big) < 0\big\}$. In particular,

$$\Theta_j = \{\theta_{ji},\, 0 \le \theta_{ji} < 2\pi,\, \theta_{ji} = \arg(q_j - q_i),\, 1 \le i, j \le n,\, i \ne j\}.$$

In order to compute the analytic invariants at the origin of the system (2.6), we have to lift the formal fundamental matrix solution $\hat{\Psi}(\tau)$ from (2.7) to such an actual. To solve this problem, in this paper we utilize the summability theory. The application of the summability theory to ordinary differential equations generalizes the theorem of Hukuhara–Turrittin to the following theorem of Ramis [25].

Theorem 2.8. *In the formal fundamental matrix solution at the origin $\hat{\Psi}(\tau)$ from (2.7) the entries of the matrix $\hat{H}(\tau)$ are 1-summable along any non-singular direction θ. If we denote by $H_\theta(\tau)$ the 1-sum of the matrix $\hat{H}(\tau)$ along θ, then $\Psi_\theta(\tau) = H_\theta(\tau)\tau^{\Lambda}\exp(Q/\tau)$ is an actual fundamental matrix solution at the origin of the system (2.6).*

Since this paper is not devoted to the summability theory, rather we only use it, we will not consider it in details. For the needed facts, notation and definitions, we refer to the works of Loday-Richaud [14], as well as the works of Ramis [24, 25].

Let $\varepsilon > 0$ be a small number. Let $\theta - \varepsilon$ and $\theta + \varepsilon$ be two non-singular neighboring directions to the singular direction $\theta \in \Theta_j$. Let $\Psi_{\theta-\varepsilon}(\tau)$ and $\Psi_{\theta+\varepsilon}(\tau)$ be the actual fundamental matrix solutions at the origin of the system (2.6) corresponding to the directions $\theta - \varepsilon$ and $\theta + \varepsilon$ in the sense of Theorem 2.8.

Definition 2.9. With respect to the actual fundamental matrix solutions $\Psi_{\theta-\varepsilon}(\tau)$ and $\Psi_{\theta+\varepsilon}(\tau)$ the Stokes matrix $St_\theta \in \mathrm{GL}_n(\mathbb{C})$ related to the singular direction θ is defined as

$$St_\theta = \left(\Psi_{\theta+\varepsilon}(\tau)\right)^{-1} \Psi_{\theta-\varepsilon}(\tau).$$

The next theorem of Ramis [24] determines the differential Galois group at the origin of the system (2.6) over the field $\mathbb{C}(\tau\}$.

Theorem 2.10 (Ramis)**.** *The differential Galois group at the origin of the system* (2.6) *over* $\mathbb{C}\{\tau\}$ *is the Zariski closure of the group generated by the formal differential Galois group at the origin and the collection of the Stokes matrices* $\{St_\theta\}$ *for all singular directions* θ*.*

For more details about the relation between the Stokes phenomenon and the differential Galois theory, we refer to the very recent work of Ramis [26]. We make note that one can introduce the differential Galois group at $t = \infty$ of the system (2.5) in the same way.

Let t_0 be a base point of $\mathbb{CP}^1 \backslash S$ and let Σ_{t_0} denote an analytic germ of a fundamental matrix solution of (2.5) at t_0. Let U_a, $a \in S$, be an open disc with center a, together with a local parameter t_a at a, and such that $U_a \cap S = \{a\}$. Let d_a be a fixed ray from a in U_a, together with a point $b_a \in d_a$ in U_a and a path γ_a from t_0 to b_a. Analytic continuation of Σ_{t_0} along γ_a and d_a provides an analytic germ Σ_a of fundamental matrix solution on a germ of open sector with vertex a, bisected by d_a. Let G_a be the local differential Galois group of the system (2.5) over the field of germs of meromorphic function at a with respect to Σ_a. If we conjugate elements of G_a by the analytic continuation described above, we get an injective morphism of algebraic groups $G_a \hookrightarrow G$ with respect to the representation of these groups in $\mathrm{GL}_n(\mathbb{C})$ given by Σ_a and Σ_{t_0}, respectively. In this way all G_a, $a \in S$, can be simultaneously identified with closed subgroups of G. Then we have the following important result of Mitschi [17, Proposition 1.3].

Theorem 2.11 (Mitschi)**.** *The differential Galois group G of the system* (2.5) *is topologically generated in* $\mathrm{GL}_n(\mathbb{C})$ *by the local differential Galois groups G_a, where a runs over S.*

3 Non-integrability for $\alpha_1 = \alpha_2 = \alpha_3 = 0$, $\alpha_4 = 1$, $\alpha_0 = -\alpha_5$

In this section, we deal with the non-integrability of the Hamiltonian system (1.3) when $\alpha_1 = \alpha_2 = \alpha_3 = 0$, $\alpha_4 = 1$, $\alpha_0 = -\alpha_5$. Denote $\alpha = \alpha_5$. For these values of the parameters the autonomous Hamiltonian system (1.3) becomes

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}s} &= \frac{2x^2y}{t} + x^2 - \frac{2xy}{t} - \left(1 + \frac{1+\alpha}{t}\right)x + \frac{\alpha}{t} + \frac{2z(z-1)w}{t},\\
\frac{\mathrm{d}y}{\mathrm{d}s} &= -\frac{2xy^2}{t} + \frac{y^2}{t} - 2xy + \left(1 + \frac{1+\alpha}{t}\right)y,\\
\frac{\mathrm{d}z}{\mathrm{d}s} &= \frac{2z^2w}{t} + z^2 - \frac{2zw}{t} - \left(1 + \frac{1+\alpha}{t}\right)z + \frac{\alpha}{t} + \frac{2yz(z-1)}{t},\\
\frac{\mathrm{d}w}{\mathrm{d}s} &= -\frac{2zw^2}{t} + \frac{w^2}{t} - 2zw + \left(1 + \frac{1+\alpha}{t}\right)w - \frac{2y(-w+2zw)}{t},
\end{aligned}$$

$$\frac{\mathrm{d}t}{\mathrm{d}s} = 1,$$
$$\frac{\mathrm{d}F}{\mathrm{d}s} = \frac{1}{t^2}(x(x-1)y(y+t) + z(z-1)w(w+t) - (1+\alpha)(xy+zw) + \alpha(y+w)) - \frac{x(x-1)y}{t} - \frac{z(z-1)w}{t}.$$

We choose

$$x = z = \frac{\alpha}{s}, \qquad y = w = 0, \qquad t = s, \qquad F = 0 \tag{3.1}$$

as a non-equilibrium particular solution, along which we will write the variational equations. Because of the equation $\frac{\mathrm{d}t}{\mathrm{d}s} = 1$, from here on we use t instead of s.

For the first normal variational equations $(\mathrm{NVE})_1$ along the solution (3.1), we obtain the system

$$\begin{aligned}
\dot{x}_1 &= \left(-1 + \frac{\alpha-1}{t}\right)x_1 + \left(\frac{2\alpha^2}{t^3} - \frac{2\alpha}{t^2}\right)y_1 + \left(\frac{2\alpha^2}{t^3} - \frac{2\alpha}{t^2}\right)w_1,\\
\dot{z}_1 &= \left(-1 + \frac{\alpha-1}{t}\right)z_1 + \left(\frac{2\alpha^2}{t^3} - \frac{2\alpha}{t^2}\right)w_1 + \left(\frac{2\alpha^2}{t^3} - \frac{2\alpha}{t^2}\right)y_1,\\
\dot{w}_1 &= \left(1 - \frac{\alpha-1}{t}\right)w_1,\\
\dot{y}_1 &= \left(1 - \frac{\alpha-1}{t}\right)y_1.
\end{aligned} \tag{3.2}$$

Note that the $(\mathrm{NVE})_k$, $k \in \mathbb{N}$ of the system (1.3) along the solution (3.1) are nothing but the $(\mathrm{VE})_k$, $k \in \mathbb{N}$, of the system (1.1) along the solution $x = z = \frac{\alpha}{t}$, $y = w = 0$ for $\alpha_1 = \alpha_2 = \alpha_3 = 0$, $\alpha_4 = 1$, $\alpha_0 = -\alpha_5 = -\alpha$.

The system (3.2) is solvable in quadratures and therefore its differential Galois group G_1 is a solvable subgroup in $\mathrm{GL}_4(\mathbb{C})$.

Theorem 3.1. *The connected component $(G_1)^0$ of the unit element of the differential Galois group G_1 of the $(\mathrm{NVE})_1$ is Abelian.*

Proof. The $(\mathrm{NVE})_1$ have two singular points over $\mathbb{CP}^1$: the points $t = 0$ and $t = \infty$. The origin is a regular singularity while $t = \infty$ is an irregular singularity. The system (3.2) admits a fundamental matrix solution $\Phi(t)$ in the form

$$\Phi(t) = \begin{pmatrix} \mathrm{e}^{-t}t^{\alpha-1} & 0 & -\alpha\mathrm{e}^{t}t^{-\alpha-1} & -\alpha\mathrm{e}^{t}t^{-\alpha-1} \\ 0 & \mathrm{e}^{-t}t^{\alpha-1} & -\alpha\mathrm{e}^{t}t^{-\alpha-1} & -\alpha\mathrm{e}^{t}t^{-\alpha-1} \\ 0 & 0 & \mathrm{e}^{t}t^{-\alpha+1} & 0 \\ 0 & 0 & 0 & \mathrm{e}^{t}t^{-\alpha+1} \end{pmatrix}.$$

We will compute the Galois group G_1 of the $(\mathrm{NVE})_1$ with respect to this fundamental matrix solution. In this case, from Theorem 2.11, it follows that the differential Galois group G_1 of the system (3.2) is generated topologically by the local Galois groups G_0 and G_∞, corresponding to the singularities $t = 0$ and $t = \infty$, respectively.

Let us first determine the group G_0. In a neighborhood of the origin, the above solution $\Phi(t)$ is written as

$$\Phi(t) = P(t)t^A,$$

where $P(t)$ is the holomorphic matrix

$$P(t) = \begin{pmatrix} \mathrm{e}^{-t} & 0 & -\alpha \mathrm{e}^{t} & -\alpha \mathrm{e}^{t} \\ 0 & \mathrm{e}^{-t} & -\alpha \mathrm{e}^{t} & -\alpha \mathrm{e}^{t} \\ 0 & 0 & t^2 \mathrm{e}^{t} & 0 \\ 0 & 0 & 0 & t^2 \mathrm{e}^{t} \end{pmatrix}.$$

For the constant matrix A we have that $A = \mathrm{diag}(\alpha-1, \alpha-1, -\alpha-1, -\alpha-1)$. From Theorem 2.4, it follows that the Galois group G_0 over $\mathbb{C}(t)$ is generated topologically by the monodromy matrix M_0 around the origin. With respect to the fundamental matrix solution $\Phi(t)$, we obtain

$$M_0 = \mathrm{e}^{2\pi \mathrm{i} A} = \begin{pmatrix} \mathrm{e}^{2\pi \mathrm{i}\alpha} & 0 & 0 & 0 \\ 0 & \mathrm{e}^{2\pi \mathrm{i}\alpha} & 0 & \\ 0 & 0 & \mathrm{e}^{-2\pi \mathrm{i}\alpha} & 0 \\ 0 & 0 & 0 & \mathrm{e}^{-2\pi \mathrm{i}\alpha} \end{pmatrix}.$$

When $\alpha \in \mathbb{Q}$ but $\alpha \notin \mathbb{Z}$, the group generated by M_0 is not connected but it is a finite and cyclic group. In this case,

$$G_0 = \left\{ \begin{pmatrix} \nu^{-1} & 0 & 0 & 0 \\ 0 & \nu^{-1} & 0 & 0 \\ 0 & 0 & \nu & 0 \\ 0 & 0 & 0 & \nu \end{pmatrix}, \ \nu \text{ is a root of unity} \right\}, \qquad (G_0)^0 = \{I_4\},$$

where I_4 is the identity matrix. When $\alpha \in \mathbb{Z}$, we have that $G_0 = (G_0)^0 = \{I_4\}$. When $\alpha \notin \mathbb{Q}$, the group generated by M_0 is a connected group and

$$G_0 = (G_0)^0 = \left\{ \begin{pmatrix} \nu^{-1} & 0 & 0 & 0 \\ 0 & \nu^{-1} & 0 & 0 \\ 0 & 0 & \nu & 0 \\ 0 & 0 & 0 & \nu \end{pmatrix}, \ \nu \in \mathbb{C} \right\}.$$

Consider now the $(\mathrm{NVE})_1$ and the fundamental matrix solution $\Phi(t)$ at $t = \infty$. In a neighborhood of the irregular singularity $t = \infty$, the matrix $\Phi(t)$ is written as

$$\Phi(t) = H_1(t) t^{\Lambda_1} \exp(Q_1 t),$$

where $H_1(t)$ is the holomorphic matrix

$$H_1(t) = \begin{pmatrix} 1 & 0 & -\alpha t^{-2} & -\alpha t^{-2} \\ 0 & 1 & -\alpha t^{-2} & -\alpha t^{-2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The matrices Λ_1 and Q_1 are given by

$$\Lambda_1 = \mathrm{diag}(\alpha - 1, \alpha - 1, -\alpha, -\alpha), \qquad Q_1 = \mathrm{diag}(-1, -1, 1, 1).$$

Since we do not observe non-trivial Stokes phenomena, the local Galois group G_∞ is generated topologically by the formal monodromy $\hat{M}_\infty$ and the exponential torus $\mathcal{T}_\infty$. The formal monodromy corresponding to a loop around $t = \infty$ is nothing but $(M_0)^{-1}$, that is, $\hat{M}_\infty = (M_0)^{-1}$. Therefore, we can consider the local differential Galois group G_0 at the origin as a subgroup of the local differential Galois group G_∞ at $t = \infty$. Thus the Galois group G_1 of the $(\mathrm{NVE})_1$

coincides with the local differential Galois group G_∞ at $t=\infty$. For the exponential torus, we have

$$\mathcal{T}_\infty = \begin{pmatrix} \lambda^{-1} & 0 & 0 & 0 \\ 0 & \lambda^{-1} & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix},$$

where $\lambda \in \mathbb{C}^*$.

As a result, we find that when $\alpha \in \mathbb{Q}$ the connected component $(G_1)^0$ of the differential Galois group G_1 coincides with $\mathcal{T}_\infty$. When $\alpha \notin \mathbb{Q}$ the group $(G_1)^0$ is generated by $\hat{M}_\infty$ and $\mathcal{T}_\infty$. Summarily, the group $(G_1)^0$ is defined as

$$(G_1)^0 = \left\{ \begin{pmatrix} \mu^{-1} & 0 & 0 & 0 \\ 0 & \mu^{-1} & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix},\ \mu \in \mathbb{C}^* \right\},$$

which is an Abelian group. ■

For the second normal variational equations $(\mathrm{NVE})_2$ along the solution (3.1), we obtain the non-homogeneous system

$$\begin{aligned}
\dot{x}_2 &= \left(-1 + \frac{\alpha-1}{t}\right)x_2 + \left(\frac{2\alpha^2}{t^3} - \frac{2\alpha}{t^2}\right)y_2 + \left(\frac{2\alpha^2}{t^3} - \frac{2\alpha}{t^2}\right)w_2 + x_1^2 \\
&\quad + \left(\frac{4\alpha}{t^2} - \frac{2}{t}\right)x_1y_1 + \left(\frac{4\alpha}{t^2} - \frac{2}{t}\right)w_1z_1, \\
\dot{y}_2 &= \left(1 - \frac{\alpha-1}{t}\right)y_2 - 2x_1y_1 + \left(\frac{1}{t} - \frac{2\alpha}{t^2}\right)y_1^2, \\
\dot{z}_2 &= \left(-1 + \frac{\alpha-1}{t}\right)z_2 + \left(\frac{2\alpha^2}{t^3} - \frac{2\alpha}{t^2}\right)w_2 + \left(\frac{2\alpha^2}{t^3} - \frac{2\alpha}{t^2}\right)y_2 + z_1^2 \\
&\quad + \left(\frac{4\alpha}{t^2} - \frac{2}{t}\right)y_1z_1 + \left(\frac{4\alpha}{t^2} - \frac{2}{t}\right)w_1z_1, \\
\dot{w}_2 &= \left(1 - \frac{\alpha-1}{t}\right)w_2 - \left(\frac{4\alpha}{t^2} - \frac{2}{t}\right)y_1w_1 - 2w_1z_1 + \left(\frac{1}{t} - \frac{2\alpha}{t^2}\right)w_1^2.
\end{aligned}$$

Introducing more 9 new variables x_1^2, x_1y_1, x_1w_1, w_1z_1, w_1^2, w_1y_1, y_1^2, z_1^2, z_1y_1 and their differential equations, we extend the $(\mathrm{NVE})_2$ to the $(\mathrm{LNVE})_2$ [22]. The $(\mathrm{LNVE})_2$ is a system of thirteenth order. The very high order of the $(\mathrm{LNVE})_2$ make the problem of the description of its differential Galois group too complicated. Fortunately, it is not necessary to study the whole $(\mathrm{LNVE})_2$. If we find a subsystem of $(\mathrm{LNVE})_2$, for which the connected component G^0 of the unit element of the corresponding differential Galois group is not Abelian and so is $(G_2)^0$.

For this reason, from here on we study the differential Galois group of the following fourth-order linear homogeneous system:

$$\begin{aligned}
\dot{w}_2 &= \left(1 - \frac{\alpha-1}{t}\right)w_2 - 2p - \left(\frac{4\alpha}{t^2} - \frac{2}{t}\right)q + \left(\frac{1}{t} - \frac{2\alpha}{t^2}\right)v, \\
\dot{p} &= \left(\frac{2\alpha^2}{t^3} - \frac{2\alpha}{t^2}\right)q + \left(\frac{2\alpha^2}{t^3} - \frac{2\alpha}{t^2}\right)v, \\
\dot{q} &= 2\left(1 - \frac{\alpha-1}{t}\right)q,
\end{aligned}$$

$$\dot{v} = 2\left(1 - \frac{\alpha-1}{t}\right)v, \tag{3.3}$$

where we have denoted $p := w_1 z_1$, $q := y_1 w_1$, $v := w_1^2$. The system (3.3) as the $(\mathrm{NVE})_1$ has two singular points over $\mathbb{CP}^1$: $t = 0$ and $t = \infty$. The origin is a regular singularity, while $t = \infty$ is an irregular singularity of Poincaré rank 1.

In order to determine the local differential Galois group at $t = \infty$ of the system (3.3), we make the change $t = 1/\tau$. This transformation takes the system (3.3) into the system

$$\begin{aligned}
w_2' &= \left(\frac{\alpha-1}{\tau} - \frac{1}{\tau^2}\right)w_2 + \frac{2}{\tau^2}p + \left(4\alpha - \frac{2}{\tau}\right)q + \left(2\alpha - \frac{1}{\tau}\right)v,\\
p' &= \left(2\alpha - 2\alpha^2\tau\right)q + \left(2\alpha - 2\alpha^2\tau\right)v,\\
q' &= 2\left(\frac{\alpha-1}{\tau} - \frac{1}{\tau^2}\right)q,\\
v' &= 2\left(\frac{\alpha-1}{\tau} - \frac{1}{\tau^2}\right)v,
\end{aligned} \tag{3.4}$$

where $' = \frac{\mathrm{d}}{\mathrm{d}\tau}$. Now the origin is an irregular singularity of Poincaré rank 1 for the system (3.4). We note that from here on we use the standard notations $(a)_n$ and $(a)^{(n)}$ for the falling and the rising factorials

$$\begin{aligned}
&(a)_n = a(a-1)(a-2)\cdots(a-n+1), \qquad (a)_0 = 1,\\
&(a)^{(n)} = a(a+1)(a+2)\cdots(a+n-1), \qquad a^{(0)} = 1,
\end{aligned}$$

respectively.

Proposition 3.2. *The system (3.4) possesses an unique formal fundamental matrix solution at the origin in the form*

$$\hat{\Psi}(\tau) = \hat{H}(\tau)\tau^{\Lambda}\exp\left(\frac{Q}{\tau}\right),$$

where the matrices Λ and Q are given by

$$\Lambda = \operatorname{diag}(\alpha-1, 0, 2\alpha-2, 2\alpha-2), \qquad Q = \operatorname{diag}(1, 0, 2, 2).$$

The matrix $\hat{H}(\tau)$ is defined as

$$\hat{H}(\tau) = \begin{pmatrix} 1 & 2+\hat{\varphi}(\tau) & 2\tau & \hat{\phi}(\tau)\\ 0 & 1 & -\alpha\tau^2 & -\alpha\tau^2\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The elements $\hat{\varphi}(\tau)$ and $\hat{\phi}(\tau)$ are defined as follows:

1. *If $\alpha \in \mathbb{N}$, the function $\hat{\varphi}(\tau)$ is the polynomial*

$$\hat{\varphi}(\tau) = 2(\alpha-1)\tau + 2(\alpha-1)(\alpha-2)\tau^2 + \cdots + 2(\alpha-1)!\tau^{\alpha-1}. \tag{3.5}$$

Otherwise, $\hat{\varphi}(\tau)$ is given by the following divergent power series:

$$\hat{\varphi}(\tau) = 2\sum_{n=1}^{\infty}(\alpha-1)_n\tau^n.$$

2. *If $\alpha \in \mathbb{Z}_{\leq 0}$, the function $\hat{\phi}(\tau)$ is the polynomial*

$$\hat{\phi}(\tau) = \tau + \alpha\tau^2 + \alpha(\alpha+1)\tau^3 + \dots + (-1)^{-\alpha}(-\alpha)!\tau^{-\alpha+1}. \tag{3.6}$$

Otherwise, $\hat{\phi}(\tau)$ is given by the following divergent power series:

$$\hat{\phi}(\tau) = \sum_{n=0}^{\infty} \alpha^{(n)}\tau^{n+1}.$$

Proof. The formulas

$$p(\tau) = C_3 - C_2\alpha e^{\frac{2}{\tau}}\tau^{2\alpha} - C_1\alpha e^{\frac{2}{\tau}}\tau^{2\alpha}, \qquad q(\tau) = C_2 e^{\frac{2}{\tau}}\tau^{2(\alpha-1)}, \qquad v(\tau) = C_1 e^{\frac{2}{\tau}}\tau^{2(\alpha-1)},$$

where C_1, C_2, C_3 are constant of integration, give the general solutions of the last three equations of the system (3.4). To build a local fundamental matrix solution $\hat{\Psi}(\tau)$ at the origin, we use that each column of such a matrix is a solution of the system (3.4). Denote by $\hat{\Psi}_j(\tau)$, $j = 1, 2, 3, 4$, the columns of the matrix $\hat{\Psi}(\tau)$. Then

$$\hat{\Psi}_1(\tau) = \begin{pmatrix} \hat{w}_2^{(1)}(\tau) \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where $\hat{w}_2^{(1)}(\tau)$ is a solution of the equation

$$w_2' = \left(\frac{\alpha-1}{\tau} - \frac{1}{\tau^2}\right)w_2.$$

We choose $\hat{w}_2^{(1)}(\tau) = e^{\frac{1}{\tau}}\tau^{\alpha-1}$. For the second column $\hat{\Psi}_2(\tau)$, we have

$$\hat{\Psi}_2(\tau) = \begin{pmatrix} \hat{w}_2^{(2)}(\tau) \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

where $\hat{w}_2^{(2)}(\tau)$ is a formal solution of the equation

$$w_2' = \left(\frac{\alpha-1}{\tau} - \frac{1}{\tau^2}\right)w_2 + \frac{2}{\tau^2}. \tag{3.7}$$

The equation (3.7) admits a formal solution near the origin in the form

$$\hat{w}_2^{(2)}(\tau) = 2 + 2\sum_{n=1}^{\infty}(\alpha-1)_n\tau^n,$$

which is a polynomial when $\alpha \in \mathbb{N}$. When $\alpha \notin \mathbb{N}$, the solution $\hat{w}_2^{(2)}(\tau)$ is a divergent power series.

For the next column, we have

$$\hat{\Psi}_3(\tau) = \begin{pmatrix} \hat{w}_2^{(3)}(\tau) \\ -\alpha e^{\frac{2}{\tau}}\tau^{2\alpha} \\ e^{\frac{2}{\tau}}\tau^{2(\alpha-1)} \\ 0 \end{pmatrix},$$

where $\hat{w}_2^{(3)}(\tau)$ is a solution of the equation

$$w_2' = \left(\frac{\alpha-1}{\tau} - \frac{1}{\tau^2}\right) w_2 + 2\alpha e^{\frac{2}{\tau}}\tau^{2\alpha-2} - 2e^{\frac{2}{\tau}}\tau^{2\alpha-3}.$$

We choose $\hat{w}_2^{(3)}(\tau) = 2e^{\frac{2}{\tau}}\tau^{2\alpha-1}$.

For the last column $\hat{\Psi}_4(\tau)$, we have

$$\hat{\Psi}_4(\tau) = \begin{pmatrix} \hat{w}_2^{(4)}(\tau) \\ -\alpha e^{\frac{2}{\alpha}}\tau^{2\alpha} \\ 0 \\ e^{\frac{2}{\tau}}\tau^{2(\alpha-1)} \end{pmatrix},$$

where $\hat{w}_2^{(4)}(\tau)$ is a solution of the equation

$$w_2' = \left(\frac{\alpha-1}{\tau} - \frac{1}{\tau^2}\right) w_2 - e^{\frac{2}{\tau}}\tau^{2\alpha-3}. \tag{3.8}$$

Looking for a solution of the equation (3.8) in the form $\hat{w}_2^{(4)}(\tau) = e^{\frac{2}{\tau}}\tau^{2\alpha-3}\hat{g}(\tau)$, we find that $\hat{g}(\tau)$ must satisfy the equation

$$\tau^2\hat{g}' = (1-(\alpha-2))\hat{g} - \tau^2. \tag{3.9}$$

The equation (3.9) admits a formal solution near the origin in the form

$$\hat{g}(\tau) = \tau\sum_{n=0}^{\infty}\alpha^{(n)}\tau^{n+1},$$

which is a polynomial when $\alpha \in \mathbb{Z}_{\leq 0}$. Otherwise, $\hat{g}(\tau)$ is a divergent power series.

Fitting together the so building columns $\hat{\Psi}_j(\tau)$, $j = 1,2,3,4$, we complete the proof. ■

Now we can determine explicitly the formal invariants at the origin of the system (3.4).

Proposition 3.3. *With respect to the formal fundamental matrix solution $\hat{\Psi}(\tau)$ introduced by Proposition 3.2, the exponential torus $\mathcal{T}$ and the formal monodromy $\hat{M}_0$ at the origin of the system (3.4) are given by*

$$\mathcal{T} = \left\{\begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda^2 & 0 \\ 0 & 0 & 0 & \lambda^2 \end{pmatrix},\ \lambda \in \mathbb{C}^*\right\}, \qquad \hat{M}_0 = e^{2\pi i\Lambda} = \begin{pmatrix} e^{2\pi i\alpha} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{4\pi i\alpha} & 0 \\ 0 & 0 & 0 & e^{4\pi i\alpha} \end{pmatrix}.$$

The application of Definition 2.7 to the divergent power series $\hat{\varphi}(\tau)$ and $\hat{\phi}(\tau)$ gives the following sets of admissible singular directions:

$$\Theta_2 = \{\theta = \arg(0-1) = \arg(0-2) = \pi\}$$

for the series $\hat{\varphi}(\tau)$ and

$$\Theta_3 = \{\theta = \arg(2-1) = \arg(2-0) = 0\}$$

for the series $\hat{\phi}(\tau)$.

In the next lemma, we compute the 1-sums of the divergent power series $\hat{\varphi}(\tau)$ and $\hat{\phi}(\tau)$. These 1-sums illustrate explicitly the dependence of the power series $\hat{\varphi}(\tau)$ and $\hat{\phi}(\tau)$ on the suggested admissible singular direction. Denote $a = \mathrm{Re}(\alpha)$ and by $[|a|]$, $[a]$ the integer parts of the real numbers $|a|$ and a, respectively.

Lemma 3.4. *Under the above notations, we have*

1. *Assume that* $\alpha \notin \mathbb{N}$. *Then for every direction* $\theta \neq \pi$ *the function*

$$\varphi_\theta(\tau) = 2(\alpha-1)\int_0^{+\infty e^{i\theta}} (1+\zeta)^{\alpha-2} e^{-\frac{\zeta}{\tau}} d\zeta$$

defines the 1*-sum of the power series* $\hat{\varphi}(\tau)$ *in such a direction.*

2. *Assume that* $\alpha \notin \mathbb{Z}_{\leq 0}$. *Then for every direction* $\theta \neq 0$ *the function*

$$\phi_\theta(\tau) = \int_0^{+\infty e^{i\theta}} (1-\zeta)^{-\alpha} e^{-\frac{\zeta}{\tau}} d\zeta$$

defines the 1*-sum of the power series* $\hat{\phi}(\tau)$ *in such a direction.*

When $\mathrm{Re}(\alpha) \leq 2$ *(resp.* $\mathrm{Re}(\alpha) \geq 0$*) the function* $\varphi_\theta(\tau)$ *(resp.* $\phi_\theta(\theta)$*) is a holomorphic function in the open unlimited disc*

$$\mathcal{D}_\theta = \left\{\tau \in \mathbb{C} \mid \mathrm{Re}\left(\frac{e^{i\theta}}{\tau}\right) > 0\right\}$$

for any direction $\theta \neq \pi$ *(resp.* $\theta \neq 0$*). Otherwise, they are holomorphic functions in the open bounded disc*

$$\mathcal{D}_\theta(1) = \left\{\tau \in \mathbb{C} \mid \mathrm{Re}\left(\frac{e^{i\theta}}{\tau}\right) > 1\right\}.$$

Proof. Consider the divergent power series $\hat{\varphi}(\tau)$ and $\hat{\phi}(\tau)$ defined by Proposition 3.2. For the corresponding formal Borel transforms, we obtain the power series

$$\hat{\mathcal{B}}_1\hat{\varphi}(\zeta) = 2\sum_{n=0}^{\infty}(\alpha-1)_{n+1}\frac{\zeta^n}{n!} = 2(\alpha-1)\sum_{n=0}^{\infty}(\alpha-2)_n\frac{\zeta^n}{n!}, \qquad \hat{\mathcal{B}}_1\hat{\phi}(\zeta) = \sum_{n=0}^{\infty}\alpha^{(n)}\frac{\zeta^n}{n!}.$$

Both of the series $\hat{\mathcal{B}}_1\hat{\varphi}(\zeta)$ and $\hat{\mathcal{B}}_1\hat{\phi}(\zeta)$ are convergent near the origin of the Borel ζ-plane with finite radiuses of convergence. Therefore, both of the divergent series $\hat{\varphi}(\tau)$ and $\hat{\phi}(\tau)$ are Gevrey series of order 1. The functions

$$\varphi(\zeta) = 2(\alpha-1)(1+\zeta)^{\alpha-2}, \qquad \phi(\zeta) = (1-\zeta)^{-\alpha}$$

present the sums of the power series $\hat{\mathcal{B}}_1\hat{\varphi}(\zeta)$ and $\hat{\mathcal{B}}_1\hat{\phi}(\zeta)$, respectively.

Consider the function $(1+\zeta)^{\alpha-2}$. We have that $\left|(1+\zeta)^{\alpha-2}\right| = A|1+\zeta|^{\mathrm{Re}(\alpha)-2}$, where $A = e^{-\mathrm{Im}(\alpha)\arg(1+\zeta)}$. Let $\theta = \arg(\zeta)$. If $\mathrm{Re}(\alpha)-2 \leq 0$, then

$$\frac{A}{|1+\zeta|^{2-\mathrm{Re}(\alpha)}} \leq A$$

when $\cos\theta \geq 0$, while

$$\frac{A}{|1+\zeta|^{2-\mathrm{Re}(\alpha)}} \leq \frac{A}{|\sin\theta|^{2-\mathrm{Re}(\alpha)}}$$

when $\cos\alpha < 0$. If $\mathrm{Re}(\alpha)-2 > 0$, then

$$A|1+\zeta|^{\mathrm{Re}(\alpha)-2} \leq Be^{|\zeta|}$$

for an appropriate constant $B > 0$. Therefore, the function $\varphi(\zeta)$ is of exponential size at most 1 at ∞ along any direction $\theta \neq \pi$ from 0 to $+\infty e^{i\theta}$. Moreover, the function $\varphi(\zeta)$ is continued analytically along any such a direction. Then the corresponding Laplace transform $(\mathcal{L}_\theta \varphi)(\tau)$ is well defined and gives the 1-sum of the divergent power series $\hat{\varphi}(\tau)$ in such a direction. If we denote by $\varphi_\theta(\tau)$ this 1-sum, we have that

$$\varphi_\theta(\tau) = 2(\alpha - 1) \int_0^{+\infty e^{i\theta}} (1+\zeta)^{\alpha-2} e^{-\frac{\zeta}{\tau}} d\zeta$$

for every $\theta \neq \pi$. From the above estimates, it follows that when $\mathrm{Re}(\alpha) - 2 \leq 0$, the 1-sum $\varphi_\theta(\tau)$ is a holomorphic function in the open unlimited disc

$$\mathcal{D}_\theta = \left\{ \tau \in \mathbb{C} \mid \mathrm{Re}\left(\frac{e^{i\theta}}{\tau}\right) > 0 \right\}, \tag{3.10}$$

whose opening is π. When $\mathrm{Re}(\alpha) - 2 > 0$, the 1-sum is a holomorphic function in the open bounded disc

$$\mathcal{D}_\theta(1) = \left\{ \tau \in \mathbb{C} \mid \mathrm{Re}\left(\frac{e^{i\theta}}{\tau}\right) > 1 \right\} \tag{3.11}$$

with opening $< \pi$.

Using similar arguments, we find that for any direction $\theta \neq 0$ the Laplace transform

$$\phi_\theta(\tau) = \int_0^{+\infty e^{i\theta}} (1-\zeta)^{-\alpha} e^{-\frac{\zeta}{\tau}} d\zeta$$

defines the 1-sum of the divergent power series $\hat{\phi}(\tau)$ in such a direction. When $\mathrm{Re}(\alpha) \geq 0$, the 1-sum $\phi_\theta(\tau)$ is a holomorphic function in the disc $\mathcal{D}_\theta$ from (3.10). Otherwise, the 1-sum is a holomorphic function in the disc $\mathcal{D}_\theta(1)$ from (3.11). ∎

Remark 3.5. Let $I = (-\pi, \pi) \subset \mathbb{R}$ and $J = (0, 2\pi) \subset \mathbb{R}$. When we move the direction $\theta \in I$, the holomorphic functions $\varphi_\theta(\tau)$ glue together analytically and define a holomorphic function $\tilde{\varphi}(\tau)$ on a sector $\widetilde{\mathcal{D}}_1$ with opening 3π, $-\frac{3\pi}{2} < \arg(\tau) < \frac{3\pi}{2}$ when $\mathrm{Re}(\alpha) \leq 2$ or on a sector

$$\widetilde{\mathcal{D}}_1(1) = \bigcup_{\theta \in I} \mathcal{D}_\theta(1)$$

with opening $> \pi$ when $\mathrm{Re}(\alpha) > 2$. Similarly, when we move the direction $\theta \in J$, the holomorphic functions $\phi_\theta(\tau)$ glue together analytically and define a holomorphic function $\tilde{\phi}(\tau)$ on a sector $\widetilde{\mathcal{D}}_2$ with opening 3π, $-\frac{\pi}{2} < \arg(\tau) < \frac{5\pi}{2}$ when $\mathrm{Re}(\alpha) \geq 0$ or on a sector

$$\widetilde{\mathcal{D}}_2(1) = \bigcup_{\theta \in J} \mathcal{D}_\theta(1)$$

with opening $> \pi$ when $\mathrm{Re}(\alpha) < 0$. On these sectors, the functions $\tilde{\varphi}(\tau)$ and $\tilde{\phi}(\tau)$ are asymptotic to the power series $\hat{\varphi}(\tau)$ and $\hat{\phi}(\tau)$, respectively, in Gevrey 1-sense and define the 1-sums of these power series there. Their restrictions on $\mathbb{C}^*$ are multivalued functions. In any direction $\theta \neq \pi$ (resp. $\theta \neq 0$) the function $\tilde{\varphi}(\tau)$ $\big($resp. $\tilde{\phi}(\tau)\big)$ has only one value which coincides with the function $\varphi_\theta(\tau)$ (resp. $\phi_\theta(\tau)$) defined by Lemma 3.4. Near the singular direction $\theta = \pi$ (resp. $\theta = 0$) the function $\tilde{\varphi}(\tau)$ $\big($resp. $\tilde{\phi}(\tau)\big)$ has two different values: $\varphi_\pi^+(\tau) = \varphi_{\pi+\varepsilon}(\tau)$ $\big($resp. $\phi_0^+(\tau) = \phi_{0+\varepsilon}(\tau)\big)$ and $\varphi_\pi^-(\tau) = \varphi_{\pi-\varepsilon}(\tau)$ (resp. $\phi_0^-(\tau) = \phi_{0-\varepsilon}(\tau)$) for a small number $\varepsilon > 0$.

Replacing the divergent power series entries of the matrix $\hat{H}(\tau)$ with their 1-sums, we get an actual fundamental matrix solution at the origin of the system (3.4). More precisely, denote $F(\tau) = \tau^{\Lambda}\exp\left(\frac{Q}{\tau}\right)$.

Proposition 3.6. *For every non-singular direction θ, the system (3.4) possesses an unique actual fundamental matrix solution at the origin in the form*

$$\Psi_\theta(\tau) = H_\theta(\tau)F_\theta(\tau), \tag{3.12}$$

where $F_\theta(\tau)$ is the branch of the matrix $F(\tau)$ for $\theta = \arg(\tau)$. The matrix $H_\theta(\tau)$ is given by

$$H_\theta(\tau) = \begin{pmatrix} 1 & 2+h_{12}(\tau) & 2\tau & h_{14}(\tau) \\ 0 & 1 & -\alpha\tau^2 & -\alpha\tau^2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The entries $h_{12}(\tau)$ and $h_{14}(\tau)$ of the matrix $H_\theta(\tau)$ are defined as follows:

1. *If $\alpha \in \mathbb{N}$, then $h_{14}(\tau) = \phi_\theta(\tau)$, where $\phi_\theta(\tau)$ is defined by Lemma 3.4 and extended by Remark 3.5. The element $h_{12}(\tau)$ coincides with the function $\hat{\varphi}(\tau)$ from (3.5).*
2. *If $\alpha \in \mathbb{Z}_{\leq 0}$, then $h_{12}(\tau) = \varphi_\theta(\tau)$, where $\varphi_\theta(\tau)$ is defined by Lemma 3.4 and extended by Remark 3.5. The element $h_{14}(\tau)$ coincides with the function $\hat{\phi}(\tau)$ from (3.6).*
3. *If $\alpha \notin \mathbb{Z}$, then $h_{12}(\tau) = \varphi_\theta(\tau)$, $h_{14}(\tau) = \phi_\theta(\tau)$, where $\varphi_\theta(\tau)$ and $\phi_\theta(\tau)$ are defined by Lemma 3.4 and extended by Remark 3.5.*

Near the singular direction $\theta = 0$ or $\theta = \pi$, the system (3.4) possesses two different actual fundamental matrix solutions at the origin in the form

$$\Psi_0^+(\tau) = \Psi_{0+\varepsilon}(\tau) \qquad \text{and} \qquad \Psi_0^-(\tau) = \Psi_{0-\varepsilon}(\tau)$$

or

$$\Psi_\pi^+(\tau) = \Psi_{\pi+\varepsilon}(\tau) \qquad \text{and} \qquad \Psi_\pi^-(\tau) = \Psi_{\pi-\varepsilon}(\tau),$$

where $\Psi_{0\pm\varepsilon}(\tau)$ and $\Psi_{\pi\pm\varepsilon}(\tau)$ are defined by (3.12) for a small number $\varepsilon > 0$.

Now we can compute the analytic invariants at the origin of the system (3.4).

Theorem 3.7. *With respect to the actual fundamental matrix solution at the origin, defined by Proposition 3.6, the system (3.4) has a Stokes matrix St_π in the form*

$$St_\pi = \begin{pmatrix} 1 & \mu_1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The multiplier μ_1 is defined as follows:

1. *If* $\operatorname{Re}(\alpha-1) > 0$, *then*
$$\mu_1 = 4\mathrm{i}(\alpha-1)(-1)^{\alpha-1}\sin((2-\alpha)\pi)\Gamma(\alpha-1).$$
2. *If* $\operatorname{Re}(\alpha-1) \leq 0$ *but* $\alpha \notin \mathbb{Z}_{\leq 0}$, *then*
$$\mu_1 = \frac{4\pi\mathrm{i}(\alpha-1)(-1)^{\alpha-1}}{\Gamma(2-\alpha)}.$$

3. *If* $\alpha \in \mathbb{Z}_{\leq 0}$, *then*

$$\mu_1 = \frac{4\pi\mathrm{i}(-1)^{-\alpha}}{(-\alpha)!}.$$

Similarly, with respect to the actual fundamental matrix solution at the origin, defined by Proposition 3.6, the system (3.4) has a Stokes matrix St_0 *in the form*

$$St_0 = \begin{pmatrix} 1 & 0 & 0 & \mu_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The multiplier μ_2 *is defined as follows:*

1. *If* $\mathrm{Re}(1-\alpha) > 0$, *then*

$$\mu_2 = -2\mathrm{i}\sin((1-\alpha)\pi)\Gamma(1-\alpha).$$

2. *If* $\mathrm{Re}(1-\alpha) \leq 0$ *but* $\alpha \notin \mathbb{N}$, *then*

$$\mu_2 = -\frac{2\pi\mathrm{i}}{\Gamma(\alpha)}.$$

3. *If* $\alpha \in \mathbb{N}$, *then*

$$\mu_2 = -\frac{2\pi\mathrm{i}}{(\alpha-1)!}.$$

Proof. From the Definition 2.9, it follows that the multiplier μ_1 is computed by comparing the solutions $\varphi_\pi^-(\tau)$ and $\varphi_\pi^+(\tau)$. Denote

$$J_1 = \varphi_\pi^-(\tau) - \varphi_\pi^+(\tau).$$

Then

$$J_1 = 2(\alpha-1)\int_\gamma (1+\zeta)^{\alpha-2}\mathrm{e}^{-\zeta/\tau}\mathrm{d}\zeta \qquad \text{for } \frac{\pi}{2} < \arg(\tau) < \frac{3\pi}{2},$$

where $\gamma = (\pi-\varepsilon) - (\pi+\varepsilon)$ for a small number $\varepsilon > 0$. Assume that $\alpha \notin \mathbb{Z}_{\leq 0}$. Then without changing the integral, we can deform the path γ into a Hankel type contour γ_1 winding around the branch cut on $\mathbb{R}^-$ of the function $(1+\zeta)^{\alpha-2}$, starting on $-\infty$, encircling -1 in the positive sense and returning to $-\infty$. Then J_1 becomes

$$J_1 = 2(\alpha-1)\int_{\gamma_1} (1+\zeta)^{\alpha-2}\mathrm{e}^{-\zeta/\tau}\mathrm{d}\zeta.$$

The transformation $1+\zeta = u$ takes the contour γ_1 into a Hankel type contour γ_2 going along the branch cut on $\mathbb{R}^-$ of the function $u^{\alpha-2}$, starting on $-\infty$, encircling 0 in the positive sense and backing to $-\infty$. Then we have

$$J_1 = 2(\alpha-1)\mathrm{e}^{1/\tau}\int_{\gamma_2} u^{\alpha-2}\mathrm{e}^{-u/\tau}\mathrm{d}u.$$

Now the change $u/\tau = -\eta$ takes the contour γ_2 into itself and we find that

$$J_1 = 2(\alpha-1)(-1)^{\alpha-1}\tau^{\alpha-1}\mathrm{e}^{1/\tau}\int_{\gamma_2}\eta^{\alpha-2}\mathrm{e}^{\eta}\mathrm{d}\eta.$$

To obtain the formula for μ_1, we use the well-known Hankel's representation of the Gamma function $\Gamma(1-\alpha)$ and reciprocal Gamma function $1/\Gamma(\alpha)$ when $\alpha \neq 0, -1, -2, \dots$ as a contour integral (see [3, 1.6 (1) and 1.6 (2)])

$$\int_{\gamma_2}\eta^{-\alpha}\mathrm{e}^{\eta}\mathrm{d}\eta = 2\mathrm{i}\sin(\alpha\pi)\Gamma(1-\alpha), \qquad \frac{1}{2\pi\mathrm{i}}\int_{\gamma_2}\eta^{-\alpha}\mathrm{e}^{\eta}\mathrm{d}\eta = \frac{1}{\Gamma(\alpha)}.$$

Hence

$$\varphi_\pi^-(\tau) - \varphi_\pi^+(\tau) = 4\mathrm{i}(\alpha-1)(-1)^{\alpha-1}\sin((2-\alpha)\pi)\Gamma(\alpha-1)\tau^{\alpha-1}\mathrm{e}^{1/\tau}$$

when $\operatorname{Re}(\alpha-1) > 0$, and

$$\varphi_\pi^-(\tau) - \varphi_\pi^+(\tau) = \frac{4\pi\mathrm{i}(\alpha-1)(-1)^{\alpha-1}}{\Gamma(2-\alpha)}\tau^{\alpha-1}\mathrm{e}^{1/\tau}$$

when $\operatorname{Re}(\alpha-1) \leq 0$ but $\alpha \notin \mathbb{Z}_{\leq 0}$.

Assume now that $\alpha \in \mathbb{Z}_{\leq 0}$. Then from the Cauchy's differential formula it follows that

$$\varphi_\pi^-(\tau) - \varphi_\pi^+(\tau) = \frac{4(\alpha-1)\pi\mathrm{i}}{(1-\alpha)!}\left[D_\zeta^{1-\alpha}\left(\mathrm{e}^{-\frac{\zeta}{\tau}}\right)_{|\zeta=-1}\right] = \frac{4\pi\mathrm{i}(-1)^{-\alpha}}{(-\alpha)!}\tau^{\alpha-1}\mathrm{e}^{\frac{1}{\tau}},$$

where $D_\zeta^n = \frac{d^n}{\mathrm{d}\zeta^n}$.

In a similar manner comparing the solutions $\tau^{2\alpha-2}\mathrm{e}^{2/\tau}\phi_0^-(\tau)$ and $\tau^{2\alpha-2}\mathrm{e}^{2/\tau}\phi_0^+(\tau)$, one can derive the multiplier μ_2. ∎

Now we can describe the local differential Galois group G at the origin of the system (3.4).

Theorem 3.8. *With respect to the formal and actual fundamental matrix solutions, given by Propositions 3.2 and 3.6, the connected component G^0 of the unit element of the local differential Galois group G at the origin of the system (3.4) is defined as follows:*

1. *If $\alpha \in \mathbb{N}$, then*

$$G^0 = \left\{\begin{pmatrix}\lambda & 0 & 0 & \mu\\ 0 & 1 & 0 & 0\\ 0 & 0 & \lambda^2 & 0\\ 0 & 0 & 0 & \lambda^2\end{pmatrix},\ \lambda \in \mathbb{C}^*,\ \mu \in \mathbb{C}\right\}.$$

2. *If $\alpha \in \mathbb{Z}_{\leq 0}$, then*

$$G^0 = \left\{\begin{pmatrix}\lambda & \mu & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \lambda^2 & 0\\ 0 & 0 & 0 & \lambda^2\end{pmatrix},\ \lambda \in \mathbb{C}^*,\ \mu \in \mathbb{C}\right\}.$$

3. *If $\alpha \notin \mathbb{Z}$, then*

$$G^0 = \left\{\begin{pmatrix}\lambda & \mu & 0 & \nu\\ 0 & 1 & 0 & 0\\ 0 & 0 & \lambda^2 & 0\\ 0 & 0 & 0 & \lambda^2\end{pmatrix},\ \lambda \in \mathbb{C}^*,\ \mu, \nu \in \mathbb{C}\right\}.$$

Proof. From Theorem 2.10, it follows that the local differential Galois group G at $\tau = 0$ of the system (3.4) is the Zariski closure of the group generated by the formal differential Galois group and the Stokes matrices St_π and St_0. Since when $\alpha \in \mathbb{Z}$ the formal monodromy $\hat{M}_0$ is equal to the identity matrix I_4 then the formal differential Galois group coincides with the exponential torus $\mathcal{T}$. Therefore, in this case G is generated by the exponential torus and the Stokes matrices St_π and St_0. Since when $\alpha \in \mathbb{N}$ the Stokes matrix St_π is equal to I_4 the Galois group G is generated by the exponential torus $\mathcal{T}$ and the Stokes matrix St_0. Hence for $\alpha \in \mathbb{N}$ the differential Galois group coincides with its connected component G^0 of the unit element and

$$G = G^0 = \left\{ \begin{pmatrix} \lambda & 0 & 0 & \mu \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda^2 & 0 \\ 0 & 0 & 0 & \lambda^2 \end{pmatrix}, \ \lambda \in \mathbb{C}^*, \ \mu \in \mathbb{C} \right\}.$$

Similarly, when $\alpha \in \mathbb{Z}_{\leq 0}$ we have that G is generated topologically by $\mathcal{T}$ and St_π and

$$G = G^0 = \left\{ \begin{pmatrix} \lambda & \mu & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda^2 & 0 \\ 0 & 0 & 0 & \lambda^2 \end{pmatrix}, \ \lambda \in \mathbb{C}^*, \ \mu \in \mathbb{C} \right\}.$$

When $\alpha \in \mathbb{Q}$ but $\alpha \notin \mathbb{Z}$, the formal differential Galois group is not connected since the group generated by $\hat{M}_0$ is not connected. However, in this case the connected component of the unit element of the formal differential Galois group coincides with $\mathcal{T}$. Therefore, in this case G does not coincides with G^0 but G^0 is generated by $\mathcal{T}$ and the Stokes matrices St_π and St_0. Hence

$$G^0 = \left\{ \begin{pmatrix} \lambda & \mu & 0 & \nu \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda^2 & 0 \\ 0 & 0 & 0 & \lambda^2 \end{pmatrix}, \ \lambda \in \mathbb{C}^*, \ \mu, \nu \in \mathbb{C} \right\}.$$

In the last case when $\alpha \notin \mathbb{Q}$, we have that

$$G = G^0 = \left\{ \begin{pmatrix} \lambda & \mu & 0 & \nu \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda^2 & 0 \\ 0 & 0 & 0 & \lambda^2 \end{pmatrix}, \ \lambda \in \mathbb{C}^*, \ \mu, \nu \in \mathbb{C} \right\}.$$

This ends the proof. ■

Directly from Theorem 3.8, we obtain the following important result

Theorem 3.9. *The connected component of the unit element of the differential Galois group of the system* (3.3) *is not Abelian.*

Proof. If we prove that the connected component G^0 of the unit element of the local differential Galois group at the origin of the system (3.4) is not Abelian group, then we will have that the connected component of the unit element of the differential Galois group of the system (3.3) will be not Abelian too. To prove that G^0, defined by Theorem 3.8, is not an Abelian group it is enough to show that the matrices

$$T_\lambda = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda^2 & 0 \\ 0 & 0 & 0 & \lambda^2 \end{pmatrix} \qquad \text{and} \qquad S_{\mu,\nu} = \begin{pmatrix} 1 & \mu & 0 & \nu \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

do not commute.

When μ and ν do not vanish together, the commutator between $S_{\mu,\nu}$ and T_λ

$$S_{\mu,\nu}T_\lambda S_{\mu,\nu}^{-1}T_\lambda^{-1} = \begin{pmatrix} 1 & \mu(1-\lambda) & 0 & \nu\big(1-\lambda^{-1}\big) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is not identically equal to the identity matrix. The condition $\lambda = 1$ implies than for every $\sigma \in G$ we have that $\sigma\big(e^{\frac{1}{\tau}}\big) = e^{\frac{1}{\tau}}$, i.e., $e^{\frac{1}{\tau}} \in \mathbb{C}(\tau)$, which is a contradiction. Thus the connected component of the unit element of the differential Galois group of the system (3.3) is not Abelian. ∎

As a consequence of Theorem 3.9, we have the following.

Corollary 3.10. *The connected component* $(G_2)^0$ *of the unit element of the differential Galois group of the* $(\mathrm{LNVE})_2$ *is not Abelian.*

Proof. We always can put the independent variables in $(\mathrm{LNVE})_2$ in such an order that the variables w_2, w_1z_1, y_1w_1, w_1^2 stay in a block. For example, let the system (3.3) forms the tail of the $(\mathrm{LNVE})_2$. Then after the transformation $t = 1/\tau$ the $(\mathrm{LNVE})_2$ admit such formal $\widehat{\Psi}_{(\mathrm{LNVE}_2)}(\tau)$ and actual $\Psi^\theta_{(\mathrm{LNVE})_2}(\tau)$ fundamental matrix solutions at $\tau = 0$ which contain the matrices $\hat{\Psi}(\tau)$ and $\Psi_\theta(\tau)$ from Propositions 3.2 and 3.6, respectively, as right-hand lower corner blocks. The differential Galois group G_2 of the $(\mathrm{LNVE})_2$ is a subgroup of $\mathrm{GL}_{13}(\mathbb{C})$, so is $(G_2)^0$. With respect to the fundamental matrix solutions $\widehat{\Psi}_{(\mathrm{LNVE}_2)}(\tau)$ and $\Psi^\theta_{(\mathrm{LNVE})_2}(\tau)$ the connected component of the unit element $(G_2)^0$ of G_2 is not Abelian since it has a proper subgroup, which is not Abelian. ∎

Combining Corollary 3.10 with the Morales–Ramis–Simó theory, we establish the main result of this section.

Theorem 3.11. *Assume that* $\alpha_1 = \alpha_2 = \alpha_3 = 0$, $\alpha_4 = 1$, $\alpha_0 = -\alpha_5$, *where* α_5 *is arbitrary. Then the Sasano system* (1.1) *is not integrable in the Liouville–Arnold sense by rational first integrals.*

4 Bäcklund transformations and generalization

In this section with the aid of the Bäcklund transformations of the Sasano system (1.1), we extend the result of Theorem 3.11 to the entire orbit of the parameters α_j, $j = 0, \ldots, 5$, and establish the main results of this paper.

Denote $(*) := (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$. The action of the generators of the extended affine Weyl group $\widetilde{W}\big(D_5^{(1)}\big)$ on $(*)$ is defined as follows (see [27]):

$$s_0(*) = \left(x + \frac{\alpha_0}{y+t}, y, z, w, t; -\alpha_0, \alpha_1, \alpha_2 + \alpha_0, \alpha_3, \alpha_4, \alpha_5\right),$$
$$s_1(*) = \left(x + \frac{\alpha_1}{y}, y, z, w, t; \alpha_0, -\alpha_1, \alpha_2 + \alpha_1, \alpha_3, \alpha_4, \alpha_5\right),$$
$$s_2(*) = \left(x, y - \frac{\alpha_2}{x-z}, z, w + \frac{\alpha_2}{x-z}, t; \alpha_0 + \alpha_2, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3 + \alpha_2, \alpha_4, \alpha_5\right),$$
$$s_3(*) = \left(x, y, z + \frac{\alpha_3}{w}, w, t; \alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3, \alpha_4 + \alpha_3, \alpha_5 + \alpha_3\right),$$
$$s_4(*) = \left(x, y, z, w - \frac{\alpha_4}{z-1}, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3 + \alpha_4, -\alpha_4, \alpha_5\right),$$

$$
\begin{aligned}
&s_5(*) = \left(x, y, z, w - \frac{\alpha_5}{z}, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3 + \alpha_5, \alpha_4, -\alpha_5\right),\\
&\pi_1(*) = (1 - x, -y - t, 1 - z, -w, t; \alpha_1, \alpha_0, \alpha_2, \alpha_3, \alpha_5, \alpha_4),\\
&\pi_2(*) = \left(\frac{y + w + t}{t}, -t(z - 1), \frac{y + t}{t}, -t(x - z), -t; \alpha_5, \alpha_4, \alpha_3, \alpha_2, \alpha_1, \alpha_0\right),\\
&\pi_3(*) = (1 - x, -y, 1 - z, -w, -t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_4),\\
&\pi_4(*) = (x, y + t, z, w, -t; \alpha_1, \alpha_0, \alpha_2, \alpha_3, \alpha_4, \alpha_5). \qquad (4.1)
\end{aligned}
$$

In fact, the actions s_j, $0 \le j \le 5$ define a representation of the affine Weyl group $W\big(D_5^{(1)}\big)$.

Remark 4.1. When $y = -t$ (resp. $y = 0$) is a particular solution of the Sasano system (1.1), then the parameter α_0 (resp. α_1) must be equal to zero. So, when $y = -t$ (resp. $y = 0$), we consider the transformation s_0 (resp. s_1) as an identity transformation. Note that $\alpha_1 = 0$ does not imply $y = 0$ of necessity. For example, if $\alpha_1 = 0$, the function $y = -t$ is a particular solution of the system (1.1), provided that $\alpha_0 = 0$. Next, when $z = 0$ (resp. $z = 1$) is a particular solution of (1.1), we find that $\alpha_5 = 0$ (resp. $\alpha_4 = 0$). This time we consider the transformation s_5 (resp. s_4) as an identity transformation. Note that $\alpha_5 = 0$ does not imply $z = 0$ of necessity. For example, when $\alpha_5 = 0$, the function $z = t$ is a particular solution of the system (1.1), provided that $\alpha_4 = -1$ and $y = -\frac{t}{2}$, $w = 0$ solve the same system. When $w = 0$, $\alpha_3 = 0$ we consider the transformation s_3 as an identity transformation. Finally, when $x = z$, $\alpha_2 = 0$, the transformation s_2 is considered as an identity transformation.

Denote by $V := (\alpha_0, \dots, \alpha_5) = (-\alpha, 0, 0, 0, 1, \alpha)$ the vector of parameters corresponding to the particular solution $\mathrm{Sol} := (x, y, z, w) = \left(\frac{\alpha}{t}, 0, \frac{\alpha}{t}, 0\right)$. In order to describe the orbit of the vector V under the action of the group $\widetilde{W}\big(D_5^{(1)}\big)$, we define using the ideas of Sasano the following translation operators:

$$
\begin{aligned}
&T_1 = \pi_1 s_5 s_3 s_2 s_1 s_0 s_2 s_3 s_5, && T_2 = \pi_2 T_1 \pi_2, && T_3 = s_1 s_4 T_1 s_4 s_1,\\
&T_4 = s_2 s_3 T_3 s_3 s_2, && T_5 = s_1 T_4 s_1, && T_6 = s_3 T_3 s_3.
\end{aligned}
$$

These operators act on the parameters as follows:

$$
\begin{aligned}
&T_1(\alpha_0, \alpha_1, \dots, \alpha_5) = (\alpha_0, \alpha_1, \dots, \alpha_5) + (0, 0, 0, 0, 1, -1),\\
&T_2(\alpha_0, \alpha_1, \dots, \alpha_5) = (\alpha_0, \alpha_1, \dots, \alpha_5) + (-1, 1, 0, 0, 0, 0),\\
&T_3(\alpha_0, \alpha_1, \dots, \alpha_5) = (\alpha_0, \alpha_1, \dots, \alpha_5) + (0, 0, 0, 1, -1, -1),\\
&T_4(\alpha_0, \alpha_1, \dots, \alpha_5) = (\alpha_0, \alpha_1, \dots, \alpha_5) + (1, 1, -1, 0, 0, 0),\\
&T_5(\alpha_0, \alpha_1, \dots, \alpha_5) = (\alpha_0, \alpha_1, \dots, \alpha_5) + (1, -1, 0, 0, 0, 0),\\
&T_6(\alpha_0, \alpha_1, \dots, \alpha_5) = (\alpha_0, \alpha_1, \dots, \alpha_5) + (0, 0, 1, -1, 0, 0).
\end{aligned}
$$

Note that the particular solution Sol is transformed under the transformations T_j as follows:

$$
\begin{aligned}
&T_1(\mathrm{Sol}) = \left(\frac{\alpha}{t} - \frac{1}{t - \alpha + 1}, 0, \frac{\alpha}{t} - \frac{1}{t - \alpha + 1}, 0\right),\\
&T_2(\mathrm{Sol}) = \left(1 - \frac{1}{\alpha - t + 1}, -1 + \frac{\alpha}{\alpha - t}, 1 - \frac{1}{\alpha - t + 1}, 0\right),\\
&T_3(\mathrm{Sol}) = (1, 0, 1, -t + \alpha - 1),\\
&T_4(\mathrm{Sol}) = \left(1, -t + \alpha - 1, 1 - \frac{1}{t - \alpha + 1}, 0\right),\\
&T_5(\mathrm{Sol}) = \left(1 - \frac{1}{t - \alpha + 1}, -t + \alpha - 1, 1 - \frac{1}{t - \alpha + 1}, 0\right),
\end{aligned}
$$

$$T_6(\mathrm{Sol}) = \left(1, 0, 1 - \frac{1}{t - \alpha + 1}, -t + \alpha - 1\right).$$

Note also that from Remark 4.1 it follows that all of the Bäcklund transformations (4.1) make sense for the particular solution Sol.

Denote $M_1 = 1 - \alpha_0 - \alpha_1$, $M_2 = \alpha_4 + \alpha_5$. The next two lemmas describe the orbit of the vector V under the group $\widetilde{W}\big(D_5^{(1)}\big)$ with generators (4.1).

Lemma 4.2. *Assume that $\alpha \notin \mathbb{Z}$. Let $(x, y, z, t) = \left(\frac{\alpha}{t}, 0, \frac{\alpha}{t}, 0\right)$ be a particular rational solution of the Sasano system (1.1) with vector of parameters V. Then applying Bäcklund transformations (4.1) to this solution, we obtain rational solutions of (1.1) with parameters α_j, $0 \le j \le 5$, which are either of the kind $\pm\alpha + n_j$, $n_j \in \mathbb{Z}$ or of the kind l_j, $l_j \in \mathbb{Z}$ in such a way that M_1 and M_2 are together of the kind $\pm\alpha + m_j$, $m_j \in \mathbb{Z}$, $j = 1, 2$.*

Proof. This lemma is proved by an induction on the numbers of the applied transformations (4.1) on the vector V and with the aid of above translation operators T_j, $1 \le j \le 6$. ■

Remark 4.3. The conditions imposed on the parameters α_j by Lemma 4.2 ensure that when $\alpha \notin \mathbb{Z}$, at least two of the new-obtained parameters α_j are integer numbers and at least two of them are not integer numbers.

With the next lemma, we specify the orbit of the vector V when $\alpha \in \mathbb{Z}$.

Lemma 4.4. *Assume that $\alpha \in \mathbb{Z}$. Let $(x, y, z, t) = \left(\frac{\alpha}{t}, 0, \frac{\alpha}{t}, 0\right)$ be a particular rational solution of the Sasano system (1.1) with vector of parameters V. Then applying Bäcklund transformations (4.1) to this solution we obtain rational solutions of (1.1), for which all of the parameters α_j, $0 \le j \le 5$, are integer numbers in such a way that M_1 and M_2 are together either even or odd integer.*

Proof. This lemma is proved inductively. ■

Following [28], we define the symplectic transformations r_i, $0 \le i \le 5$, which correspond to the symmetries s_i, $0 \le i \le 5$, from (4.1)

$$\begin{aligned}
r_0(x, y, z, w) &= \left(\frac{1}{x}, -t - x(x(y + t) + \alpha_0), z, w\right),\\
r_1(x, y, z, w) &= \left(\frac{1}{x}, -x(xy + \alpha_1), z, w\right),\\
r_2(x, y, z, w) &= \left(z - y(y(x - z) - \alpha_2), \frac{1}{y}, z, w + y - \frac{1}{y}\right),\\
r_3(x, y, z, w) &= \left(x, y, \frac{1}{z}, -z(zw + \alpha_3)\right),\\
r_4(x, y, z, w) &= \left(x, y, 1 - w(w(z - 1) - \alpha_4), \frac{1}{w}\right),\\
r_5(x, y, z, w) &= \left(x, y, -w(wz - \alpha_5), \frac{1}{w}\right).
\end{aligned}$$

The transformations π_j, $1 \le j \le 5$, from (4.1) are also canonical symplectic transformations since

$$\mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}z \wedge \mathrm{d}w = \mathrm{d}\pi_j(x) \wedge \mathrm{d}\pi_j(y) + \mathrm{d}\pi_j(z) \wedge \mathrm{d}\pi_j(w).$$

Denote, in short, by r_i, $0 \le i \le 5$, and π_j, $1 \le j \le 4$, the image of the canonical coordinates x, y, z, w under the action of r_i and π_j, respectively. Then the following important result is an extension of [28, Theorem 4.1].

Theorem 4.5. *There exists an unique polynomial Hamiltonian system of degree* 4*, which is holomorphic in each coordinates* r_i*,* $0 \le i \le 5$*, and* π_j*,* $1 \le j \le 5$*. This system is invariant under the extended Weyl group* $\widetilde{W}\big(D_5^{(1)}\big)$ *and coincides with the system* (1.1)*.*

Theorem 4.5 says that the transformations s_i, $0 \le i \le 5$, and π_j, $1 \le j \le 4$, from (4.1) are canonical transformations, which are rational on the functions x, y, z, w. Using this fact, we establish the main results of this paper.

Theorem 4.6. *Let* α *be an arbitrary complex parameter, which is not an integer. Assume that the parameters* α_j *are either of the kind* $\pm\alpha+n_j$ *or of the kind* $l_j, n_j, l_j \in \mathbb{Z}$ *in such a way that* M_1 *and* M_2 *are together of the kind* $\pm\alpha + m_i$*,* $m_i \in \mathbb{Z}$*,* $i = 1, 2$*. Then the Sasano system* (1.1) *is not integrable in the Liouville–Arnold sense by rational first integrals.*

Theorem 4.7. *Assume that all of the parameters* α_j*,* $0 \le j \le 5$*, are integer in such a way that* M_1 *and* M_2 *are together either even or odd integer. Then the Sasano system* (1.1) *is not integrable in the Liouville–Arnold sense by rational first integrals.*

Acknowledgments

The author is indebted to the referees for critical remarks and advices towards the improvement of the paper. The author was partially supported by Grant KP-06-N 62/5 of the Bulgarian Fund “Scientific research”.

References

[1] Arnol'd V.I., Mathematical methods of classical mechanics, *Grad. Texts in Math.*, Vol. 60, Springer, New York, 1989.

[2] Ayoul M., Zung N.T., Galoisian obstructions to non-Hamiltonian integrability, *C. R. Math. Acad. Sci. Paris* **348** (2010), 1323–1326, arXiv:0901.4586.

[3] Bateman H., Erdélyi A., Higher transcendental functions. Vol. I, McGraw-Hill Book Co., Inc., New York, 1953.

[4] Boucher D., Weil J.-A., Application of J.-J. Morales and J.-P. Ramis' theorem to test the non-complete intagrability of the planar three-body problem, in From Combinatorics to Dynamical Systems, De Gruyter, Berlin, 2003, 163–178.

[5] Casale G., Morales–Ramis theorems via Malgrange pseudogroup, *Ann. Inst. Fourier (Grenoble)* **59** (2009), 2593–2610.

[6] Casale G., Duval G., Maciejewski A.J., Przybylska M., Integrability of Hamiltonian systems with homogeneous potentials of degree zero, *Phys. Lett. A* **374** (2010), 448–452, arXiv:0903.5199.

[7] Casale G., Roques J., Non-integrability by discrete quadratures, *J. Reine Angew. Math.* **687** (2014), 87–112.

[8] Casale G., Weil J.A., Galoisian methods for testing irreducibility of order two nonlinear differential equations, *Pacific J. Math.* **297** (2018), 299–337, arXiv:1504.08134.

[9] Combot T., Non-integrability of the equal mass n-body problem with non-zero angular momentum, *Celestial Mech. Dynam. Astronom.* **114** (2012), 319–340, arXiv:1112.1889.

[10] Combot T., Integrable planar homogeneous potentials of degree -1 with small eigenvalues, *Ann. Inst. Fourier (Grenoble)* **66** (2016), 2253–2298.

[11] Combot T., Maciejewski A.J., Przybylska M., Bi-homogeneity and integrability of rational potentials, *J. Differential Equations* **268** (2020), 7012–7028.

[12] Filipuk G., A remark about quasi-Painlevé equations of P_{II} type, *C. R. Acad. Bulgare Sci.* **68** (2015), 427–430.

[13] Fuji K., Suzuki T., Higher order Painlevé system of type $D_{2n+2}^{(1)}$ arising from integrable hierarchy, *Int. Math. Res. Not.* **2008** (2008), 129, 21 pages, arXiv:0704.2574.

[14] Loday-Richaud M., Divergent series, summability and resurgence. II. Simple and multiple summability, *Lecture Notes in Math.*, Vol. 2154, Springer, Cham, 2016.

[15] Maciejewski A.J., Przybylska M., Gyrostatic Suslov problem, *Russ. J. Nonlinear Dyn.* **18** (2022), 609–627.

[16] Maciejewski A.J., Przybylska M., Simpson L., Szumiński W., Non-integrability of the dumbbell and point mass problem, *Celestial Mech. Dynam. Astronom.* **117** (2013), 315–330, arXiv:1304.6369.

[17] Mitschi C., Differential Galois groups of confluent generalized hypergeometric equations: an approach using Stokes multipliers, *Pacific J. Math.* **176** (1996), 365–405.

[18] Morales-Ruiz J.J., A remark about the Painlevé transcendents, in Théories Asymptotiques et Équations de Painlevé, *Sémin. Congr.*, Vol. 14, Soc. Math. France, Paris, 2006, 229–235.

[19] Morales-Ruiz J.J., Ramis J.-P., Galoisian obstructions to integrability of Hamiltonian systems. I, *Methods Appl. Anal.* **8** (2001), 33–96.

[20] Morales-Ruiz J.J., Ramis J.-P., Galoisian obstructions to integrability of Hamiltonian systems. II, *Methods Appl. Anal.* **8** (2001), 97–112.

[21] Morales-Ruiz J.J., Ramis J.-P., Integrability of dynamical systems through differential Galois theory: a practical guide, in Differential Algebra, Complex Analysis and Orthogonal Polynomials, *Contemp. Math.*, Vol. 509, American Mathematical Society, Providence, RI, 2010, 143–220.

[22] Morales-Ruiz J.J., Ramis J.-P., Simo C., Integrability of Hamiltonian systems and differential Galois groups of higher variational equations, *Ann. Sci. École Norm. Sup.* **40** (2007), 845–884.

[23] Przybylska M., Maciejewski A.J., Non-integrability of the planar elliptic restricted three-body problem, *Celestial Mech. Dynam. Astronom.* **135** (2023), 13, 22 pages.

[24] Ramis J.-P., Phénomène de Stokes et filtration Gevrey sur le groupe de Picard–Vessiot, *C. R. Acad. Sci. Paris Sér. I Math.* **301** (1985), 165–167.

[25] Ramis J.-P., Gevrey asymptotics and applications to holomorphic ordinary differential equations, in Differential Equations & Asymptotic Theory in Mathematical Physics, *Ser. Anal.*, Vol. 2, World Scientific Publishing, Hackensack, NJ, 2004, 44–99.

[26] Ramis J.-P., Epilogue: Stokes phenomena. Dynamics, classification problems and avatars, in Handbook of Geometry and Topology of Singularities VI: Foliations, Springer, Cham, 2024, 383–482.

[27] Sasano Y., Higher order Painlevé equations of type $D_l^{(1)}$, in From Soliton Theory to a Mathematics of Integrable Systems: "New Perspectives", *RIMS Kôkyûroku Bessatsu*, Vol. 1473, Res. Inst. Math. Sci. (RIMS), Kyoto, 2006, 143–163.

[28] Sasano Y., Yamada Y., Symmetry and holomorphy of Painlevé type systems, in Algebraic, Analytic and Geometric Aspects of Complex Differential Equations and Their Deformations. Painlevé Hierarchies, *RIMS Kôkyûroku Bessatsu*, Vol. B2, Res. Inst. Math. Sci. (RIMS), Kyoto, 2007, 215–225.

[29] Schlesinger L., Handbuch der Theorie der linearen Differentialgleichungen, Teubner, Leipzig, 1897.

[30] Singer M.F., Introduction to the Galois theory of linear differential equations, in Algebraic Theory of Differential Equations, *London Math. Soc. Lecture Note Ser.*, Vol. 357, Cambridge University Press, Cambridge, 2009, 1–82, arXiv:0712.4124.

[31] Stoyanova Ts., Non-integrability of Painlevé VI equations in the Liouville sense, *Nonlinearity* **22** (2009), 2201–2230.

[32] Stoyanova Ts., Non-integrability of the fourth Painlevé equation in the Liouville–Arnold sense, *Nonlinearity* **27** (2014), 1029–1044.

[33] Stoyanova Ts., Nonintegrability of the Painlevé IV equation in the Liouville–Arnold sense and Stokes phenomena, *Stud. Appl. Math.* **151** (2023), 1380–1405, arXiv:2302.13732.

[34] Stoyanova Ts., Nonintegrability of coupled Painlevé systems with affine Weyl group symmetry of type $A_4^{(2)}$, *Eur. J. Math.* **10** (2024), 62, 15 pages.

[35] Tsygvintsev A., The meromorphic non-integrability of the three-body problem, *J. Reine Angew. Math.* **537** (2001), 127–149, arXiv:math.DS/0009218.

[36] Tsygvintsev A., On some exceptional cases in the integrability of the three-body problem, *Celestial Mech. Dynam. Astronom.* **99** (2007), 23–29, arXiv:math.DS/0610951.

[37] van der Put M., Singer M.F., Galois theory of linear differential equations, *Grundlehren Math. Wiss.*, Vol. 328, Springer, Berlin, 2003.

[38] Wasow W., Asymptotic expansions for ordinary differential equations, *Pure Appl. Math.*, Vol. 14, Interscience Publishers John Wiley & Sons, Inc., New York, 1965.