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The Algebraic and Geometric Classification of Compatible Pre-Lie Algebras

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Abstract. In this paper, we develop a method to obtain the algebraic classification of compatible pre-Lie algebras from the classification of pre-Lie algebras of the same dimension. We use this method to obtain the algebraic classification of complex 2-dimensional compatible pre-Lie algebras. As a byproduct, we obtain the classification of complex 2-dimensional compatible commutative associative, compatible associative and compatible Novikov algebras. In addition, we consider the geometric classification of varieties of cited algebras, that is the description of its irreducible components.

Key words: compatible algebra; compatible associative algebra; compatible pre-Lie algebra; algebraic classification; geometric classification

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1 Introduction

The algebraic classification (up to isomorphism) of algebras of small dimensions from a certain variety defined by a family of polynomial identities is a classic problem in the theory of non-associative algebras. Another interesting approach to studying algebras of a fixed dimension is to study them from a geometric point of view (that is, to study the degenerations and deformations of these algebras). The results in which the complete information about degenerations of a certain variety is obtained are generally referred to as the geometric classification of the algebras of these varieties. There are many results related to the algebraic and geometric classification of Jordan, Lie, Leibniz, Zinbiel, and other algebras (see [1, 2, 3, 5, 6, 15, 16, 20, 23, 25, 27] and references in [22, 24, 31]). The geometric classification of algebras from a certain variety is based on the notion of degeneration, that is a "dual" notion to deformation [4, 11, 15, 21, 29, 32].

Pre-Lie algebras, also known as right symmetric algebras, appeared in some papers by Gerstenhaber, Koszul, and Vinberg in the 1960s. It is a generalization of associative algebras and the most popular non-associative subvariety of Lie-admissible algebras. They have various applications in geometry and physics [7]; recently some connections between them and trees, braces and F-manifold algebras were established in [8, 29, 37, 38]. Novikov algebras are pre-Lie algebras with an additional identity. They were introduced in papers by Gel'fand and Dorfman, Balinskii and Novikov (about Novikov algebras, see [14] and references therein). Let Ω be a variety of algebras. We say that an algebra $(\mathbf{A}, \cdot, *)$ is a compatible Ω-algebra, if and only if $(\mathbf{A}, \cdot)$, $(\mathbf{A}, *)$ and $(\mathbf{A}, \cdot + *)$ are Ω-algebras. Compatible Lie algebras are considered in the study of the classical Yang–Baxter equation [18], integrable equations of the principal chiral model type [17], elliptic theta functions [34], and other areas. The study of non-Lie compatible algebras is also very popular. So, cohomology and deformations were studied for compatible Lie algebras [30], compatible associative algebras [9], compatible Hom-Lie algebras [11], compatible L_∞-algebras [10], compatible 3-Lie algebras [21], compatible dendriform algebras [12], and so on. Free compatible algebras were studied in the associative algebra case in [13] and in the Lie algebra case in [19, 28]. Compatible associative structures on matrix algebras were studied in [33, 35, 36]. General compatible structures have been studied from an operadic point of view in [39, 40]. A generalization of compatible algebras was considered in [26]. At this moment there is only one paper about the classification of small-dimensional compatible algebras. Namely, all 4-dimensional nilpotent compatible Lie algebras were classified in [27].

The main goal of the present paper is to obtain the algebraic and geometric description of the variety of complex 2-dimensional compatible pre-Lie algebras. To do so, we first determine all such 2-dimensional algebra structures, up to isomorphism (what we call the algebraic classification), and then proceed to determine the geometric properties of the corresponding variety, namely its dimension and description of the irreducible components (the geometric classification). As some corollaries, we have the algebraic and geometric classification of complex 2-dimensional compatible commutative associative, compatible associative and compatible Novikov algebras.

Our main results regarding the algebraic classification are summarized below.

Theorem A1. *There are infinitely many isomorphism classes of complex* 2*-dimensional compatible pre-Lie algebras, described explicitly in Theorem* 2.7 *in terms of* 6 *three-parameter families,* 14 *two-parameter families,* 13 *one-parameter families, and* 8 *additional isomorphism classes.*

Theorem A2. *There are infinitely many isomorphism classes of complex* 2*-dimensional compatible commutative associative algebras, described explicitly in Theorem* 2.8 *in terms of* 1 *three-parameter family,* 3 *two-parameter families,* 8 *one-parameter families, and* 6 *additional isomorphism classes.*

Theorem A3. *There are infinitely many isomorphism classes of complex* 2*-dimensional compatible associative algebras, described explicitly in Theorem* 2.10 *in terms of* 1 *three-parameter family,* 3 *two-parameter families,* 10 *one-parameter families, and* 10 *additional isomorphism classes.*

Theorem A4. *There are infinitely many isomorphism classes of complex* 2*-dimensional compatible Novikov algebras, described explicitly in Theorem* 2.12 *in terms of* 3 *three-parameter families,* 7 *two-parameter families,* 11 *one-parameter families, and* 7 *additional isomorphism classes.*

The geometric part of our work aims to generalize previously obtained results about the geometric classification of 2-dimensional Novikov [6] and pre-Lie [5] algebras. Our main results regarding the geometric classification are summarized below.

Theorem G1. *The variety of complex* 2*-dimensional compatible pre-Lie algebras has dimension* 7*. It is defined by* 2 *rigid algebras,* 1 *one-parametric family of algebras,* 7 *two-parametric families of algebras, and* 4 *three-parametric families of algebras and can be described as the closure of the union of* $\mathrm{GL}_2(\mathbb{C})$*-orbits of the algebras given in Theorem* 3.10*.*

Theorem G2. *The variety of complex* 2*-dimensional compatible commutative associative algebras has dimension* 7*. It is defined by* 1 *two-parametric family of algebras and* 1 *three-parametric family of algebras and can be described as the closure of the union of* $\mathrm{GL}_2(\mathbb{C})$*-orbits of the algebras given in Theorem* 3.7*.*

Theorem G3. *The variety of complex 2-dimensional compatible associative algebras has dimension 7. It is defined by 2 rigid algebras, 2 two-parametric families of algebras, and 1 three-parametric family of algebras and can be described as the closure of the union of* $\mathrm{GL}_2(\mathbb{C})$*-orbits of the algebras given in Theorem 3.8.*

Theorem G4. *The variety of complex 2-dimensional compatible Novikov algebras has dimension 7. It is defined by 1 rigid algebra, 3 two-parametric families of algebras, and 1 three-parametric family of algebras and can be described as the closure of the union of* $\mathrm{GL}_2(\mathbb{C})$*-orbits of the algebras given in Theorem 3.9.*

2 The algebraic classification of compatible algebras

All the algebras below will be over $\mathbb{C}$ and all the linear maps will be $\mathbb{C}$-linear. For simplicity, every time we write the multiplication table of an algebra the products of basic elements whose values are zero are omitted.

2.1 The algebraic classification of algebras

In this paper, we work with compatible pre-Lie algebras with two multiplications. Let us review the method we will use to obtain the algebraic classification for the variety of compatible pre-Lie algebras (the present method, in the case of Poisson algebras, is given with more details in [2, 3]).

Definition 2.1. An algebra is called a pre-Lie algebra if it satisfies the identity

$$(x \cdot y) \cdot z - x \cdot (y \cdot z) = (y \cdot x) \cdot z - y \cdot (x \cdot z).$$

Definition 2.2. A compatible pre-Lie algebra is a vector space $\mathbf{A}$ equipped with two multiplications: $\cdot$ and another multiplication $*$ such that $(\mathbf{A}, \cdot)$, $(\mathbf{A}, *)$ and $(\mathbf{A}, \cdot + *)$ are pre-Lie algebras. These two operations are required to satisfy the following identities:

$$\begin{aligned}
&(x \cdot y) \cdot z - x \cdot (y \cdot z) = (y \cdot x) \cdot z - y \cdot (x \cdot z),\\
&(x * y) * z - x * (y * z) = (y * x) * z - y * (x * z),\\
&(x * y) \cdot z - x * (y \cdot z) + (x \cdot y) * z - x \cdot (y * z)\\
&\qquad = (y * x) \cdot z - y * (x \cdot z) + (y \cdot x) * z - y \cdot (x * z).
\end{aligned}$$

The main examples of compatible pre-Lie algebras are the following: compatible commutative associative, compatible associative and compatible Novikov algebras.

Definition 2.3. Let $(\mathbf{A}, \cdot)$ be a pre-Lie algebra. Define $\mathrm{Z}^2(\mathbf{A}, \mathbf{A})$ to be the set of all bilinear maps $\theta\colon \mathbf{A} \times \mathbf{A} \longrightarrow \mathbf{A}$ such that

$$\begin{aligned}
&\theta(\theta(x, y), z) - \theta(x, \theta(y, z)) = \theta(\theta(y, x), z) - \theta(y, \theta(x, z)),\\
&\theta(x, y) \cdot z - \theta(x, y \cdot z) + \theta(x \cdot y, z) - x \cdot \theta(y, z)\\
&\qquad = \theta(y, x) \cdot z - \theta(y, x \cdot z) + \theta(y \cdot x, z) - y \cdot \theta(x, z).
\end{aligned}$$

Then $\mathrm{Z}^2(\mathbf{A}, \mathbf{A}) \neq \varnothing$ since $\theta = 0 \in \mathrm{Z}^2(\mathbf{A}, \mathbf{A})$.

Now, for $\theta \in \mathrm{Z}^2(\mathbf{A}, \mathbf{A})$, let us define a multiplication $\bullet_\theta$ on $\mathbf{A}$ by $x \bullet_\theta y = \theta(x, y)$ for all $x, y \in \mathbf{A}$. Then $(\mathbf{A}, \cdot, \bullet_\theta)$ is a compatible pre-Lie algebra. Conversely, if $(\mathbf{A}, \cdot, \bullet)$ is a compatible pre-Lie algebra, then there exists $\theta \in \mathrm{Z}^2(\mathbf{A}, \mathbf{A})$ such that $(\mathbf{A}, \cdot, \bullet_\theta) \cong (\mathbf{A}, \cdot, \bullet)$. To see this, consider the bilinear map $\theta\colon \mathbf{A} \times \mathbf{A} \longrightarrow \mathbf{A}$ defined by $\theta(x, y) = x \bullet y$ for all x, y in $\mathbf{A}$. Then $\theta \in \mathrm{Z}^2(\mathbf{A}, \mathbf{A})$ and $(\mathbf{A}, \cdot, \bullet_\theta) = (\mathbf{A}, \cdot, \bullet)$.

Let $(\mathbf{A},\cdot)$ be a pre-Lie algebra. The automorphism group $\mathrm{Aut}(\mathbf{A})$ of $\mathbf{A}$ acts on $\mathrm{Z}^2(\mathbf{A},\mathbf{A})$ by

$$(\theta*\phi)(x,y)=\phi^{-1}(\theta(\phi(x),\phi(y)))$$

for $\phi\in\mathrm{Aut}(\mathbf{A})$ and $\theta\in\mathrm{Z}^2(\mathbf{A},\mathbf{A})$.

Lemma 2.4. *Let $(\mathbf{A},\cdot)$ be a pre-Lie algebra and $\theta,\vartheta\in\mathrm{Z}^2(\mathbf{A},\mathbf{A})$. Then $(\mathbf{A},\cdot,\bullet_\theta)$ and $(\mathbf{A},\cdot,\bullet_\vartheta)$ are isomorphic if and only if there is a linear map $\phi\in\mathrm{Aut}(\mathbf{A})$ such that $\theta*\phi=\vartheta$.*

Proof. If $\theta*\phi=\vartheta$, then $\phi\colon(\mathbf{A},\cdot,\bullet_\vartheta)\longrightarrow(\mathbf{A},\cdot,\bullet_\theta)$ is an isomorphism since $\phi(\vartheta(x,y))=\theta(\phi(x),\phi(y))$. On the other hand, if $\phi\colon(\mathbf{A},\cdot,\bullet_\vartheta)\longrightarrow(\mathbf{A},\cdot,\bullet_\theta)$ is an isomorphism of compatible pre-Lie algebras, then $\phi\in\mathrm{Aut}(\mathbf{A})$ and $\phi(x\bullet_\vartheta y)=\phi(x)\bullet_\theta\phi(y)$. Hence

$$\vartheta(x,y)=\phi^{-1}(\theta(\phi(x),\phi(y)))=(\theta*\phi)(x,y),$$

and therefore $\theta*\phi=\vartheta$. ■

Consequently, we have a procedure to classify the compatible pre-Lie algebras with the given associated pre-Lie algebra $(\mathbf{A},\cdot)$. It consists of three steps:

(1) compute $\mathrm{Z}^2(\mathbf{A},\mathbf{A})$,

(2) find the orbits of $\mathrm{Aut}(\mathbf{A})$ on $\mathrm{Z}^2(\mathbf{A},\mathbf{A})$,

(3) choose a representative θ from each orbit and then construct the compatible pre-Lie algebra $(\mathbf{A},\cdot,\bullet_\theta)$.

2.2 2-dimensional pre-Lie algebras

Lemma 2.5. *Let $\mathcal{C}$ be a nonzero 2-dimensional pre-Lie algebra. Then $\mathcal{C}$ is isomorphic to one and only one of the following algebras:*

$$\begin{array}{ll}
\mathcal{C}_{01}: & e_1\cdot e_1=e_1+e_2,\quad e_2\cdot e_1=e_2,\\
\mathcal{C}_{02}: & e_1\cdot e_1=e_1+e_2,\quad e_1\cdot e_2=e_2,\\
\mathcal{C}_{03}: & e_1\cdot e_1=e_2,\\
\mathcal{C}_{04}: & e_2\cdot e_1=e_1,\\
\mathcal{C}_{05}^{\alpha}: & e_1\cdot e_1=e_1,\quad e_1\cdot e_2=\alpha e_2,\\
\mathcal{C}_{06}^{\alpha}: & e_1\cdot e_1=e_1,\quad e_1\cdot e_2=\alpha e_2,\quad e_2\cdot e_1=e_2,\\
\mathcal{C}_{07}: & e_1\cdot e_1=e_1,\quad e_2\cdot e_2=e_2,\\
\mathcal{C}_{08}: & e_1\cdot e_1=e_1,\quad e_1\cdot e_2=2e_2,\quad e_2\cdot e_1=\frac{1}{2}e_1+e_2,\quad e_2\cdot e_2=e_2.
\end{array}$$

Lemma 2.6. *The description of the group of automorphisms of every 2-dimensional pre-Lie algebra is given as follows:*

(1) *If $\phi\in\mathrm{Aut}(\mathcal{C}_{01})$, then $\phi(e_1)=e_1+\nu e_2$ and $\phi(e_2)=e_2$ for $\nu\in\mathbb{C}$.*

(2) *If $\phi\in\mathrm{Aut}(\mathcal{C}_{02})$, then $\phi(e_1)=e_1+\nu e_2$ and $\phi(e_2)=e_2$ for $\nu\in\mathbb{C}$.*

(3) *If $\phi\in\mathrm{Aut}(\mathcal{C}_{03})$, then $\phi(e_1)=\xi e_1+\nu e_2$ and $\phi(e_2)=\xi^2e_2$ for $\xi\in\mathbb{C}^*$ and $\nu\in\mathbb{C}$.*

(4) *If $\phi\in\mathrm{Aut}(\mathcal{C}_{04})$, then $\phi(e_1)=\xi e_1$ and $\phi(e_2)=e_2$ for $\xi\in\mathbb{C}^*$.*

(5) *If $\phi\in\mathrm{Aut}\big(\mathcal{C}_{05}^{\alpha\neq1}\big)$, then $\phi(e_1)=e_1$ and $\phi(e_2)=\xi e_2$ for $\xi\in\mathbb{C}^*$.*

(6) *If $\phi\in\mathrm{Aut}\big(\mathcal{C}_{05}^{1}\big)$, then $\phi(e_1)=e_1+\nu e_2$ and $\phi(e_2)=\xi e_2$ for $\xi\in\mathbb{C}^*$ and $\nu\in\mathbb{C}$.*

(7) *If $\phi\in\mathrm{Aut}\big(\mathcal{C}_{06}^{\alpha\neq0}\big)$, then $\phi(e_1)=e_1$ and $\phi(e_2)=\xi e_2$ for $\xi\in\mathbb{C}^*$.*

(8) *If $\phi\in\mathrm{Aut}\big(\mathcal{C}_{06}^{0}\big)$, then $\phi(e_1)=e_1+\nu e_2$ and $\phi(e_2)=\xi e_2$ for $\xi\in\mathbb{C}^*$ and $\nu\in\mathbb{C}$.*

(9) *If $\phi\in\mathrm{Aut}(\mathcal{C}_{07})$, then $\phi\in\mathbb{S}_2$, i.e., $\phi(e_1)=e_1$, $\phi(e_2)=e_2$ or $\phi(e_1)=e_2$, $\phi(e_2)=e_1$.*

(10) *If $\phi\in\mathrm{Aut}(\mathcal{C}_{08})$, then $\phi(e_1)=e_1$, $\phi(e_2)=e_2$ or $\phi(e_1)=-e_1+4e_2$, $\phi(e_2)=e_2$.*

2.3 The algebraic classification of compatible pre-Lie algebras

The main aim of the present section is to prove the following results.

Theorem 2.7. *Let $\mathcal{C}$ be a nonzero 2-dimensional compatible pre-Lie algebra. Then $\mathcal{C}$ is isomorphic to one and only one of the following algebras:*

$$\mathcal{C}_{01}:\ e_1*e_1=e_1+e_2,\quad e_2*e_1=e_2,$$
$$\mathcal{C}_{02}:\ e_1*e_1=e_1+e_2,\quad e_1*e_2=e_2,$$
$$\mathcal{C}_{03}:\ e_1*e_1=e_2,$$
$$\mathcal{C}_{04}:\ e_2*e_1=e_1,$$
$$\mathcal{C}_{05}^{\alpha}:\ e_1*e_1=e_1,\quad e_1*e_2=\alpha e_2,$$
$$\mathcal{C}_{06}^{\alpha}:\ e_1*e_1=e_1,\quad e_1*e_2=\alpha e_2,\quad e_2*e_1=e_2,$$
$$\mathcal{C}_{07}:\ e_1*e_1=e_1,\quad e_2*e_2=e_2,$$
$$\mathcal{C}_{08}:\ e_1*e_1=e_1,\quad e_1*e_2=2e_2,\quad e_2*e_1=\tfrac{1}{2}e_1+e_2,\quad e_2*e_2=e_2,$$
$$\mathcal{C}_{13}^{1}:\ e_1\cdot e_1=e_2,\quad e_1*e_1=e_1,\quad e_1*e_2=e_2,\quad e_2*e_1=e_2,$$
$$\mathcal{C}_{15}^{\alpha}:\ e_1\cdot e_1=e_2,\quad e_1*e_1=\alpha e_2,$$
$$\mathcal{C}_{16}^{0}:\ e_1\cdot e_1=e_2,\quad e_1*e_1=e_1,$$
$$\mathcal{C}_{18}^{\alpha}:\ e_1\cdot e_1=e_2,\quad e_1*e_1=\alpha e_2,\quad e_1*e_2=e_1,\quad e_2*e_1=e_1,\quad e_2*e_2=e_2,$$
$$\mathcal{C}_{24}^{0,\beta,0}:\ e_1\cdot e_1=e_1,\quad e_1*e_1=\beta e_1+e_2,$$
$$\mathcal{C}_{25}^{0,\beta,0}:\ e_1\cdot e_1=e_1,\quad e_1*e_1=\beta e_1,$$
$$\mathcal{C}_{29}^{\alpha}:\ e_1\cdot e_1=e_1,\quad e_1*e_1=\alpha e_1,\quad e_2*e_2=e_2,$$
$$\mathcal{C}_{30}^{\alpha,\beta}:\ e_1\cdot e_1=e_1,\quad e_1*e_1=\alpha e_1+\beta e_2,\quad e_1*e_2=e_1,\quad e_2*e_1=e_1,\quad e_2*e_2=e_2,$$
$$\mathcal{C}_{31}^{1,\beta,\beta}:\ e_1\cdot e_1=e_1,\quad e_1\cdot e_2=e_2,\quad e_2\cdot e_1=e_2,$$
$$e_1*e_1=\beta e_1+e_2,\quad e_1*e_2=\beta e_2,\quad e_2*e_1=\beta e_2,$$
$$\mathcal{C}_{32}^{1,\beta,\beta}:\ e_1\cdot e_1=e_1,\quad e_1\cdot e_2=e_2,\quad e_2\cdot e_1=e_2,$$
$$e_1*e_1=\beta e_1,\quad e_1*e_2=\beta e_2,\quad e_2*e_1=\beta e_2,$$
$$\mathcal{C}_{34}^{\alpha,\beta}:\ e_1\cdot e_1=e_1,\quad e_1\cdot e_2=e_2,\quad e_2\cdot e_1=e_2,$$
$$e_1*e_1=\alpha e_1,\quad e_1*e_2=\alpha e_2,\quad e_2*e_1=\alpha e_2,\quad e_2*e_2=e_1+\beta e_2,$$
$$\mathcal{C}_{35}^{\alpha}:\ e_1\cdot e_1=e_1,\quad e_1\cdot e_2=e_2,\quad e_2\cdot e_1=e_2,$$
$$e_1*e_1=\alpha e_1,\quad e_1*e_2=\alpha e_2,\quad e_2*e_1=\alpha e_2,\quad e_2*e_2=e_2,$$
$$\mathcal{C}_{38}^{\alpha,\beta,\gamma}:\ e_1\cdot e_1=e_1,\quad e_2\cdot e_2=e_2,\quad e_1*e_1=(\gamma+\beta-\alpha)e_1-\beta e_2,$$
$$e_1*e_2=\alpha e_1+\beta e_2,\quad e_2*e_1=\alpha e_1+\beta e_2,\quad e_2*e_2=-\alpha e_1+\gamma e_2,$$
$$\mathcal{C}_{39}^{\alpha,\beta\neq\alpha}:\ e_1\cdot e_1=e_1,\quad e_2\cdot e_2=e_2,\quad e_1*e_1=\alpha e_1,\quad e_2*e_2=\beta e_2.$$

All algebras are non-isomorphic, except

$$\mathcal{C}_{24}^{1,\beta,\gamma}\cong\mathcal{C}_{25}^{1,\beta,\gamma},\quad \mathcal{C}_{31}^{0,\beta,\gamma}\cong\mathcal{C}_{32}^{0,\beta,\gamma},\quad \mathcal{C}_{38}^{\alpha,\beta,\gamma}\cong\mathcal{C}_{38}^{\beta,\alpha,-\alpha+\beta+\gamma},$$
$$\mathcal{C}_{39}^{\alpha,\beta}\cong\mathcal{C}_{39}^{\beta,\alpha},\quad \mathcal{C}_{40}^{\alpha,\beta,\gamma}\cong\mathcal{C}_{40}^{4\gamma-\alpha,\beta-8\alpha+16\gamma,\gamma},\quad \mathcal{C}_{41}^{\alpha,\beta}\cong\mathcal{C}_{41}^{\alpha+4\beta,-\beta}.$$

2.3.1 Compatible pre-Lie algebras defined on $\mathcal{C}_{01}$

From the computation of $\mathrm{Z}^2_{\mathrm{CPL}}(\mathcal{C}_{01},\mathcal{C}_{01})$, the compatible pre-Lie algebra structures defined on $\mathcal{C}_{01}$ are of the form $e_1\cdot e_1=e_1+e_2$, $e_2\cdot e_1=e_2$, $e_1*e_1=\alpha_1e_1+\alpha_2e_2$, $e_1*e_2=\alpha_3e_2$, $e_2*e_1=\alpha_1e_2$. Then we have the following cases:

- If $\alpha_3 \neq 0$, then choose $\nu = -\frac{\alpha_2}{\alpha_3}$ and obtain the parametric family

$$\begin{aligned} \mathcal{C}_{09}^{\alpha,\beta\neq 0}: \ & e_1 \cdot e_1 = e_1 + e_2, \quad e_2 \cdot e_1 = e_2, \\ & e_1 * e_1 = \alpha e_1, \quad e_1 * e_2 = \beta e_2, \quad e_2 * e_1 = \alpha e_2. \end{aligned}$$

  The algebras $\mathcal{C}_{09}^{\alpha,\beta}$ and $\mathcal{C}_{09}^{\alpha',\beta'}$ are isomorphic if and only if $(\alpha,\beta) = (\alpha',\beta')$.

- If $\alpha_3 = 0$, we obtain the parametric family

$$\mathcal{C}_{10}^{\alpha,\beta}: \ e_1 \cdot e_1 = e_1 + e_2, \quad e_2 \cdot e_1 = e_2, \quad e_1 * e_1 = \alpha e_1 + \beta e_2, \quad e_2 * e_1 = \alpha e_2.$$

  The algebras $\mathcal{C}_{10}^{\alpha,\beta}$ and $\mathcal{C}_{10}^{\alpha',\beta'}$ are isomorphic if and only if $(\alpha,\beta) = (\alpha',\beta')$.

2.3.2 Compatible pre-Lie algebras defined on $\mathcal{C}_{02}$

From the computation of $\mathrm{Z}^2_{\mathrm{CPL}}(\mathcal{C}_{02},\mathcal{C}_{02})$, the compatible pre-Lie algebra structures defined on $\mathcal{C}_{02}$ are of the form $e_1 \cdot e_1 = e_1 + e_2$, $e_1 \cdot e_2 = e_2$, $e_1 * e_1 = \alpha_1 e_1 + \alpha_2 e_2$, $e_1 * e_2 = \alpha_3 e_2$. Then we have the following cases:

- If $\alpha_1 \neq \alpha_3$, then choose $\nu = \frac{\alpha_2}{\alpha_1 - \alpha_3}$ and obtain the parametric family

$$\mathcal{C}_{11}^{\alpha,\beta\neq\alpha}: \ e_1 \cdot e_1 = e_1 + e_2, \quad e_1 \cdot e_2 = e_2, \quad e_1 * e_1 = \alpha e_1, \quad e_1 * e_2 = \beta e_2.$$

  The algebras $\mathcal{C}_{11}^{\alpha,\beta}$and $\mathcal{C}_{11}^{\alpha',\beta'}$ are isomorphic if and only if $(\alpha,\beta) = (\alpha',\beta')$.

- If $\alpha_1 = \alpha_3$, we obtain the parametric family

$$\mathcal{C}_{12}^{\alpha,\beta}: \ e_1 \cdot e_1 = e_1 + e_2, \quad e_1 \cdot e_2 = e_2, \quad e_1 * e_1 = \alpha e_1 + \beta e_2, \quad e_1 * e_2 = \alpha e_2.$$

  The algebras $\mathcal{C}_{12}^{\alpha,\beta}$ and $\mathcal{C}_{12}^{\alpha',\beta'}$ are isomorphic if and only if $(\alpha,\beta) = (\alpha',\beta')$.

2.3.3 Compatible pre-Lie algebras defined on $\mathcal{C}_{03}$

From the computation of $\mathrm{Z}^2_{\mathrm{CPL}}(\mathcal{C}_{03},\mathcal{C}_{03})$, the compatible pre-Lie algebra structures defined on $\mathcal{C}_{03}$ are of the following forms:

(1) $e_1 \cdot e_1 = e_2$, $e_1 * e_1 = \alpha_1 e_1 + \alpha_2 e_2$, $e_1 * e_2 = \alpha_3 e_2$, $e_2 * e_1 = \alpha_1 e_2$,

(2) $e_1 \cdot e_1 = e_2$, $e_1 * e_1 = \alpha_4 e_1 + \alpha_5 e_2$, $e_1 * e_2 = \alpha_6 e_2$,

(3) $e_1 \cdot e_1 = e_2$, $e_1 * e_1 = \alpha_7 e_1 + \alpha_8 e_2$, $e_1 * e_2 = \alpha_9 e_1$, $e_2 * e_1 = \alpha_9 e_1$, $e_2 * e_2 = \alpha_9 e_2$,

(4) $e_1 \cdot e_1 = e_2$, $e_1 * e_1 = \alpha_{10} e_1 + \alpha_{11} e_2$, $e_1 * e_2 = 2\alpha_{10} e_2$, $e_2 * e_1 = \alpha_{12} e_1 + \alpha_{10} e_2$, $e_2 * e_2 = 2\alpha_{12} e_2$.

We may assume $\alpha_4 \alpha_9 \alpha_{12} \neq 0$. First, we consider the first form. Then we have the following cases:

- If $\alpha_3 \neq 0$, then choose $\xi = \alpha_3^{-1}$, $\nu = -\alpha_2 \alpha_3^{-2}$ and obtain the parametric family

$$\mathcal{C}_{13}^{\alpha}: \ e_1 \cdot e_1 = e_2, \quad e_1 * e_1 = \alpha e_1, \quad e_1 * e_2 = e_2, \quad e_2 * e_1 = \alpha e_2.$$

  The algebras $\mathcal{C}_{13}^{\alpha}$ and $\mathcal{C}_{13}^{\alpha'}$ are isomorphic if and only if $\alpha = \alpha'$.

- If $\alpha_3 = 0, \alpha_1 \neq 0$, then choose $\xi = \alpha_1^{-1}$ and obtain the parametric family

$$\mathcal{C}_{14}^{\alpha}: \ e_1 \cdot e_1 = e_2, \quad e_1 * e_1 = e_1 + \alpha e_2, \quad e_2 * e_1 = e_2.$$

  The algebras $\mathcal{C}_{14}^{\alpha}$ and $\mathcal{C}_{14}^{\alpha'}$ are isomorphic if and only if $\alpha = \alpha'$.

- If $\alpha_1 = \alpha_3 = 0$, then we obtain the parametric family

  $$\mathcal{C}_{15}^{\alpha}: \ e_1 \cdot e_1 = e_2, \quad e_1 * e_1 = \alpha e_2.$$

  The algebras $\mathcal{C}_{15}^{\alpha}$ and $\mathcal{C}_{15}^{\alpha'}$ are isomorphic if and only if $\alpha = \alpha'$.

Second, we consider the second form. Then we have the following cases:

- If $\alpha_4 \neq \alpha_6$, then choose $\xi = \alpha_4^{-1}$, $\nu = \frac{\alpha_5}{\alpha_4 - \alpha_6}$ and obtain the parametric family

  $$\mathcal{C}_{16}^{\alpha}: \ e_1 \cdot e_1 = e_2, \quad e_1 * e_1 = e_1, \quad e_1 * e_2 = \alpha e_2.$$

  The algebras $\mathcal{C}_{16}^{\alpha}$ and $\mathcal{C}_{16}^{\alpha'}$ are isomorphic if and only if $\alpha = \alpha'$.

- If $\alpha_4 = \alpha_6$, then choose $\xi = \alpha_4^{-1}$ and obtain the parametric family

  $$\mathcal{C}_{17}^{\alpha}: \ e_1 \cdot e_1 = e_2, \quad e_1 * e_1 = e_1 + \alpha e_2, \quad e_1 * e_2 = e_2.$$

  The algebras $\mathcal{C}_{17}^{\alpha}$ and $\mathcal{C}_{17}^{\alpha'}$ are isomorphic if and only if $\alpha = \alpha'$.

Third, we consider the third form. Then, we choose $\xi = \alpha_9^{-\frac{1}{2}}$, $\nu = -\frac{\alpha_7 \alpha_9^{-\frac{3}{2}}}{2}$ and obtain the parametric family

$$\mathcal{C}_{18}^{\alpha}: \ e_1 \cdot e_1 = e_2, \quad e_1 * e_1 = \alpha e_2, \quad e_1 * e_2 = e_1, \quad e_2 * e_1 = e_1, \quad e_2 * e_2 = e_2.$$

The algebras $\mathcal{C}_{18}^{\alpha}$ and $\mathcal{C}_{18}^{\alpha'}$ are isomorphic if and only if $\alpha = \alpha'$.

Finally, we consider the fourth form. Then, we choose $\xi = \alpha_{12}^{-\frac{1}{2}}$, $\nu = -\alpha_{10}\alpha_{12}^{-\frac{3}{2}}$ and obtain the parametric family

$$\mathcal{C}_{19}^{\alpha}: \ e_1 \cdot e_1 = e_2, \quad e_1 * e_1 = \alpha e_2, \quad e_2 * e_1 = e_1, \quad e_2 * e_2 = 2e_2.$$

The algebras $\mathcal{C}_{19}^{\alpha}$ and $\mathcal{C}_{19}^{\alpha'}$ are isomorphic if and only if $\alpha = \alpha'$.

2.3.4 Compatible pre-Lie algebras defined on $\mathcal{C}_{04}$

From the computation of $\mathrm{Z}^2_{\mathrm{CPL}}(\mathcal{C}_{04}, \mathcal{C}_{04})$, the compatible pre-Lie algebra structures defined on $\mathcal{C}_{04}$ are of the following forms:

(1) $e_2 \cdot e_1 = e_1$, $e_2 * e_1 = \alpha_1 e_1$, $e_2 * e_2 = \alpha_2 e_1 + \alpha_3 e_2$,

(2) $e_2 \cdot e_1 = e_1$, $e_1 * e_2 = \alpha_4 e_1$, $e_2 * e_1 = \alpha_5 e_1$, $e_2 * e_2 = \alpha_6 e_1 + \alpha_4 e_2$.

We may assume $\alpha_4 \neq 0$. First, we study the first form. Then we have the following cases:

- If $\alpha_2 \neq 0$, then choose $\xi = \alpha_2$ and obtain the parametric family

  $$\mathcal{C}_{20}^{\alpha,\beta}: \ e_2 \cdot e_1 = e_1, \quad e_2 * e_1 = \alpha e_1, \quad e_2 * e_2 = e_1 + \beta e_2.$$

  The algebras $\mathcal{C}_{20}^{\alpha,\beta}$ and $\mathcal{C}_{20}^{\alpha',\beta'}$ are isomorphic if and only if $(\alpha, \beta) = (\alpha', \beta')$.

- If $\alpha_2 = 0$, we obtain the parametric family

  $$\mathcal{C}_{21}^{\alpha,\beta}: \ e_2 \cdot e_1 = e_1, \quad e_2 * e_1 = \alpha e_1, \quad e_2 * e_2 = \beta e_2.$$

  The algebras $\mathcal{C}_{21}^{\alpha,\beta}$ and $\mathcal{C}_{21}^{\alpha',\beta'}$ are isomorphic if and only if $(\alpha, \beta) = (\alpha', \beta')$.

Second, we consider the second form. Then we have the following cases:

- If $\alpha_6 \neq 0$, then choose $\xi = \alpha_6$ and obtain the parametric family

  $$\mathcal{C}_{22}^{\alpha\neq 0,\beta}: \ e_2 \cdot e_1 = e_1, \quad e_1 * e_2 = \alpha e_1, \quad e_2 * e_1 = \beta e_1, \quad e_2 * e_2 = e_1 + \alpha e_2.$$

  The algebras $\mathcal{C}_{22}^{\alpha,\beta}$ and $\mathcal{C}_{22}^{\alpha',\beta'}$ are isomorphic if and only if $(\alpha, \beta) = (\alpha', \beta')$.

- If $\alpha_6 = 0$, we obtain the parametric family

  $$\mathcal{C}_{23}^{\alpha\neq 0,\beta}: \ e_2 \cdot e_1 = e_1, \quad e_1 * e_2 = \alpha e_1, \quad e_2 * e_1 = \beta e_1, \quad e_2 * e_2 = \alpha e_2.$$

  The algebras $\mathcal{C}_{23}^{\alpha,\beta}$ and $\mathcal{C}_{23}^{\alpha',\beta'}$ are isomorphic if and only if $(\alpha, \beta) = (\alpha', \beta')$.

2.3.5 Compatible pre-Lie algebras defined on $\mathcal{C}_{05}^{\alpha\neq 0,\frac{1}{2},1}$

From the computation of $\mathrm{Z}^2_{\mathrm{CPL}}(\mathcal{C}_{05}^{\alpha},\mathcal{C}_{05}^{\alpha})$, the compatible pre-Lie algebra structures defined on $\mathcal{C}_{05}^{\alpha}$ are of the form $e_1\cdot e_1=e_1$, $e_1\cdot e_2=\alpha e_2$, $e_1*e_1=\alpha_1e_1+\alpha_2e_2$, $e_1*e_2=\alpha_3e_2$. Then we have the following cases:

- If $\alpha_2\neq 0$, then choose $\xi=\alpha_2$ and obtain the parametric family

  $$\mathcal{C}_{24}^{\alpha,\beta,\gamma}:\ e_1\cdot e_1=e_1,\quad e_1\cdot e_2=\alpha e_2,\quad e_1*e_1=\beta e_1+e_2,\quad e_1*e_2=\gamma e_2.$$

  The algebras $\mathcal{C}_{24}^{\alpha,\beta,\gamma}$ and $\mathcal{C}_{24}^{\alpha',\beta',\gamma'}$ are isomorphic if and only if $(\alpha,\beta,\gamma)=(\alpha',\beta',\gamma')$.

- If $\alpha_2=0$, we obtain the parametric family

  $$\mathcal{C}_{25}^{\alpha,\beta,\gamma}:\ e_1\cdot e_1=e_1,\quad e_1\cdot e_2=\alpha e_2,\quad e_1*e_1=\beta e_1,\quad e_1*e_2=\gamma e_2.$$

  The algebras $\mathcal{C}_{25}^{\alpha,\beta,\gamma}$ and $\mathcal{C}_{25}^{\alpha',\beta',\gamma'}$ are isomorphic if and only if $(\alpha,\beta,\gamma)=(\alpha',\beta',\gamma')$.

2.3.6 Compatible pre-Lie algebras defined on $\mathcal{C}_{05}^{\frac{1}{2}}$

From the computation of $\mathrm{Z}^2_{\mathrm{CPL}}\big(\mathcal{C}_{05}^{\frac{1}{2}},\mathcal{C}_{05}^{\frac{1}{2}}\big)$, the compatible pre-Lie algebra structures defined on $\mathcal{C}_{05}^{\frac{1}{2}}$ are of the following forms:

(1) $e_1\cdot e_1=e_1$, $e_1\cdot e_2=\frac{1}{2}e_2$, $e_1*e_1=\alpha_1e_1+\alpha_2e_2$, $e_1*e_2=\alpha_3e_2$,

(2) $e_1\cdot e_1=e_1$, $e_1\cdot e_2=\frac{1}{2}e_2$, $e_1*e_1=2\alpha_4e_1$, $e_1*e_2=\alpha_4e_2$, $e_2*e_2=\alpha_5e_1$,

(3) $e_1\cdot e_1=e_1$, $e_1\cdot e_2=\frac{1}{2}e_2$, $e_1*e_1=2\alpha_6e_1$, $e_1*e_2=\alpha_7e_1+\alpha_6e_2$, $e_2*e_1=2\alpha_7e_1$, $e_2*e_2=\alpha_8e_1+\alpha_7e_2$.

We may assume $\alpha_5\alpha_7\neq 0$. If the compatible pre-Lie algebra structures defined on $\mathcal{C}_{05}^{\frac{1}{2}}$ is of the first form, then we obtain the algebras $\mathcal{C}_{24}^{\frac{1}{2},\beta,\gamma}$ and $\mathcal{C}_{25}^{\frac{1}{2},\beta,\gamma}$.

Assume now that the compatible pre-Lie algebra structures defined on $\mathcal{C}_{05}^{\frac{1}{2}}$ is of the second form. Then, choose $\xi=\alpha_5^{-\frac{1}{2}}$ and obtain the parametric family

$$\mathcal{C}_{26}^{\alpha}:\ e_1\cdot e_1=e_1,\quad e_1\cdot e_2=\frac{1}{2}e_2,\quad e_1*e_1=2\alpha e_1,\quad e_1*e_2=\alpha e_2,\quad e_2*e_2=e_1.$$

The algebras $\mathcal{C}_{26}^{\alpha}$ and $\mathcal{C}_{26}^{\alpha'}$ are isomorphic if and only if $\alpha=\alpha'$.

Finally, assume that the compatible pre-Lie algebra structures defined on $\mathcal{C}_{05}^{\frac{1}{2}}$ is of the third form. Then, choose $\xi=\alpha_7^{-1}$ and obtain the parametric family

$$\begin{aligned}\mathcal{C}_{27}^{\alpha,\beta}:\ & e_1\cdot e_1=e_1,\quad e_1\cdot e_2=\tfrac{1}{2}e_2,\quad e_1*e_1=2\alpha e_1,\\ & e_1*e_2=e_1+\alpha e_2,\quad e_2*e_1=2e_1,\quad e_2*e_2=\beta e_1+e_2.\end{aligned}$$

The algebras $\mathcal{C}_{27}^{\alpha,\beta}$ and $\mathcal{C}_{27}^{\alpha',\beta'}$ are isomorphic if and only if $(\alpha,\beta)=(\alpha',\beta')$.

2.3.7 Compatible pre-Lie algebras defined on $\mathcal{C}_{05}^{1}$

From the computation of $\mathrm{Z}^2_{\mathrm{CPL}}\big(\mathcal{C}_{05}^{1},\mathcal{C}_{05}^{1}\big)$, the compatible pre-Lie algebra structures defined on $\mathcal{C}_{05}^{1}$ are of the following forms:

(1) $e_1\cdot e_1=e_1$, $e_1\cdot e_2=e_2$, $e_1*e_1=\alpha_1e_1+\alpha_2e_2$, $e_1*e_2=\alpha_3e_2$,

(2) $e_1\cdot e_1=e_1$, $e_1\cdot e_2=e_2$, $e_1*e_1=\alpha_4e_1$, $e_1*e_2=\alpha_4e_2$, $e_2*e_1=\alpha_5e_1$, $e_2*e_2=\alpha_5e_2$.

We may assume $\alpha_5\neq 0$. First, suppose that the compatible pre-Lie algebra structures defined on $\mathcal{C}_{05}^{1}$ is of the first form. Then we have the following cases:

- If $\alpha_1 \neq \alpha_3$, then choose $\nu = \frac{\alpha_2}{\alpha_1 - \alpha_3}$ and obtain the parametric family $\mathcal{C}_{25}^{1,\beta,\gamma\neq\beta}$.
- If $\alpha_1 = \alpha_3$, then we will consider the following two cases:
  - If $\alpha_2 \neq 0$, then choose $\xi = \alpha_2$ and obtain the parametric family $\mathcal{C}_{24}^{1,\beta,\beta}$.
  - If $\alpha_2 = 0$, then we obtain the parametric family $\mathcal{C}_{25}^{1,\beta,\beta}$.

Assume now that the compatible pre-Lie algebra structures defined on $\mathcal{C}_{05}^{1}$ is of the second form. Then, choose $\xi = \alpha_5^{-1}$, $\nu = -\alpha_4\alpha_5^{-1}$ and obtain the algebra

$$\mathcal{C}_{28}\colon\ e_1 \cdot e_1 = e_1, \quad e_1 \cdot e_2 = e_2, \quad e_2 * e_1 = e_1, \quad e_2 * e_2 = e_2.$$

2.3.8 Compatible pre-Lie algebras defined on $\mathcal{C}_{05}^{0}$

From the computation of $\mathrm{Z}^2_{\mathrm{CPL}}\big(\mathcal{C}_{05}^{0}, \mathcal{C}_{05}^{0}\big)$, the compatible pre-Lie algebra structures defined on $\mathcal{C}_{05}^{0}$ are of the following forms:

(1) $e_1 \cdot e_1 = e_1$, $e_1 * e_1 = \alpha_1 e_1 + \alpha_2 e_2$, $e_1 * e_2 = \alpha_3 e_2$,

(2) $e_1 \cdot e_1 = e_1$, $e_1 * e_1 = \alpha_4 e_1$, $e_2 * e_2 = \alpha_5 e_2$,

(3) $e_1 \cdot e_1 = e_1$, $e_1 * e_1 = \alpha_6 e_1 + \alpha_7 e_2$, $e_1 * e_2 = \alpha_8 e_1$, $e_2 * e_1 = \alpha_8 e_1$, $e_2 * e_2 = \alpha_8 e_2$.

We may assume $\alpha_5\alpha_8 \neq 0$. If the compatible pre-Lie algebra structures defined on $\mathcal{C}_{05}^{0}$ is of the first form, then we obtain the algebras $\mathcal{C}_{24}^{0,\beta,\gamma}$ and $\mathcal{C}_{25}^{0,\beta,\gamma}$.

Assume now that the compatible pre-Lie algebra structures defined on $\mathcal{C}_{05}^{0}$ are of the second form. Then, choose $\xi = \alpha_5^{-1}$ and obtain the parametric family

$$\mathcal{C}_{29}^{\alpha}\colon\ e_1 \cdot e_1 = e_1, \quad e_1 * e_1 = \alpha e_1, \quad e_2 * e_2 = e_2.$$

The algebras $\mathcal{C}_{29}^{\alpha}$ are $\mathcal{C}_{29}^{\alpha'}$ are isomorphic if and only if $\alpha = \alpha'$.

Finally, assume that the compatible pre-Lie algebra structures defined on $\mathcal{C}_{05}^{0}$ is of the third form. Then, choose $\xi = \alpha_8^{-1}$ and obtain the parametric family

$$\mathcal{C}_{30}^{\alpha,\beta}\colon\ e_1 \cdot e_1 = e_1, \quad e_1 * e_1 = \alpha e_1 + \beta e_2, \quad e_1 * e_2 = e_1, \quad e_2 * e_1 = e_1, \quad e_2 * e_2 = e_2.$$

The algebras $\mathcal{C}_{30}^{\alpha,\beta}$ and $\mathcal{C}_{30}^{\alpha',\beta'}$ are isomorphic if and only if $(\alpha,\beta) = (\alpha',\beta')$.

2.3.9 Compatible pre-Lie algebras defined on $\mathcal{C}_{06}^{\alpha\neq 0,1,2}$

From the computation of $\mathrm{Z}^2_{\mathrm{CPL}}(\mathcal{C}_{06}^{\alpha}, \mathcal{C}_{06}^{\alpha})$, the compatible pre-Lie algebra structures defined on $\mathcal{C}_{06}^{\alpha}$ are of the form $e_1 \cdot e_1 = e_1$, $e_1 \cdot e_2 = \alpha e_2$, $e_2 \cdot e_1 = e_2$, $e_1 * e_1 = \alpha_1 e_1 + \alpha_2 e_2$, $e_1 * e_2 = \alpha_3 e_2$, $e_2 * e_1 = \alpha_1 e_2$. Then we have the following cases:

- If $\alpha_2 \neq 0$, then choose $\xi = \alpha_2$ and obtain the parametric family

$$\begin{aligned}\mathcal{C}_{31}^{\alpha,\beta,\gamma}\colon\ & e_1 \cdot e_1 = e_1, \quad e_1 \cdot e_2 = \alpha e_2, \quad e_2 \cdot e_1 = e_2,\\ & e_1 * e_1 = \beta e_1 + e_2, \quad e_1 * e_2 = \gamma e_2, \quad e_2 * e_1 = \beta e_2.\end{aligned}$$

  The algebras $\mathcal{C}_{31}^{\alpha,\beta,\gamma}$ and $\mathcal{C}_{31}^{\alpha',\beta',\gamma'}$ are isomorphic if and only if $(\alpha,\beta,\gamma) = (\alpha',\beta',\gamma')$.

- If $\alpha_2 = 0$, we obtain the parametric family

$$\begin{aligned}\mathcal{C}_{32}^{\alpha,\beta,\gamma}\colon\ & e_1 \cdot e_1 = e_1, \quad e_1 \cdot e_2 = \alpha e_2, \quad e_2 \cdot e_1 = e_2,\\ & e_1 * e_1 = \beta e_1, \quad e_1 * e_2 = \gamma e_2, \quad e_2 * e_1 = \beta e_2.\end{aligned}$$

  The algebras $\mathcal{C}_{32}^{\alpha,\beta,\gamma}$, $\mathcal{C}_{32}^{\alpha',\beta',\gamma'}$ are isomorphic if and only if $(\alpha,\beta,\gamma) = (\alpha',\beta',\gamma')$.

2.3.10 Compatible pre-Lie algebras defined on $\mathcal{C}_{06}^0$

From the computation of $\mathrm{Z}^2_{\mathrm{CPL}}(\mathcal{C}_{06}^0, \mathcal{C}_{06}^0)$, the compatible pre-Lie algebra structures defined on $\mathcal{C}_{06}^0$ are of the following forms:

(1) $e_1 \cdot e_1 = e_1$, $e_2 \cdot e_1 = e_2$, $e_1 * e_1 = \alpha_1 e_1 + \alpha_2 e_2$, $e_1 * e_2 = \alpha_3 e_2$, $e_2 * e_1 = \alpha_1 e_2$,

(2) $e_1 \cdot e_1 = e_1$, $e_2 \cdot e_1 = e_2$, $e_1 * e_1 = \alpha_4 e_1$, $e_1 * e_2 = \alpha_5 e_1$, $e_2 * e_1 = \alpha_4 e_2$, $e_2 * e_2 = \alpha_5 e_2$.

We may assume $\alpha_5 \neq 0$. Let us first consider the first form. Then we have the following cases:

- If $\alpha_3 \neq 0$, then $\nu = -\frac{\alpha_2}{\alpha_3}$ choose and obtain the parametric family $\mathcal{C}_{32}^{0,\beta,\gamma\neq 0}$.
- If $\alpha_3 = 0$, then we consider the following two cases:
  - If $\alpha_2 \neq 0$, then choose $\xi = \alpha_2$ and obtain the parametric family $\mathcal{C}_{31}^{0,\beta,0}$.
  - If $\alpha_2 = 0$, then we obtain the parametric family $\mathcal{C}_{32}^{0,\beta,0}$.

Now, we consider the second form. Then choose $\xi = \alpha_5^{-1}$, $\nu = -\alpha_4\alpha_5^{-1}$ and obtain the algebra

$$\mathcal{C}_{33}: \ e_1 \cdot e_1 = e_1, \quad e_2 \cdot e_1 = e_2, \quad e_1 * e_2 = e_1, \quad e_2 * e_2 = e_2.$$

2.3.11 Compatible pre-Lie algebras defined on $\mathcal{C}_{06}^1$

From the computation of $\mathrm{Z}^2_{\mathrm{CPL}}(\mathcal{C}_{06}^1, \mathcal{C}_{06}^1)$, the compatible pre-Lie algebra structures defined on $\mathcal{C}_{06}^1$ are of the following forms:

(1) $e_1 \cdot e_1 = e_1$, $e_1 \cdot e_2 = e_2$, $e_2 \cdot e_1 = e_2$, $e_1 * e_1 = \alpha_1 e_1 + \alpha_2 e_2$, $e_1 * e_2 = \alpha_3 e_2$, $e_2 * e_1 = \alpha_1 e_2$,

(2) $e_1 \cdot e_1 = e_1$, $e_1 \cdot e_2 = e_2$, $e_2 \cdot e_1 = e_2$, $e_1 * e_1 = \alpha_4 e_1$, $e_1 * e_2 = \alpha_4 e_2$, $e_2 * e_1 = \alpha_4 e_2$, $e_2 * e_2 = \alpha_5 e_1 + \alpha_6 e_2$.

We may assume $(\alpha_5, \alpha_6) \neq (0,0)$. If the compatible pre-Lie algebra structures defined on $\mathcal{C}_{06}^1$ is of the first form, then we obtain the algebras $\mathcal{C}_{31}^{1,\beta,\gamma}$ and $\mathcal{C}_{32}^{1,\beta,\gamma}$.

Assume now that the compatible pre-Lie algebra structures defined on $\mathcal{C}_{06}^1$ is of the second form. Then we have the following cases:

- If $\alpha_5 \neq 0$, then choose $\xi = \alpha_5^{-\frac{1}{2}}$ and obtain the parametric family

$$\begin{aligned} \mathcal{C}_{34}^{\alpha,\beta}: \ & e_1 \cdot e_1 = e_1, \quad e_1 \cdot e_2 = e_2, \quad e_2 \cdot e_1 = e_2, \\ & e_1 * e_1 = \alpha e_1, \quad e_1 * e_2 = \alpha e_2, \quad e_2 * e_1 = \alpha e_2, \quad e_2 * e_2 = e_1 + \beta e_2. \end{aligned}$$

  The algebras $\mathcal{C}_{34}^{\alpha,\beta}$ and $\mathcal{C}_{34}^{\alpha',\beta'}$ are isomorphic if and only if $(\alpha,\beta) = (\alpha',\beta')$.

- If $\alpha_5 = 0$, then choose $\xi = \alpha_6^{-1}$ and obtain the parametric family

$$\begin{aligned} \mathcal{C}_{35}^{\alpha}: \ & e_1 \cdot e_1 = e_1, \quad e_1 \cdot e_2 = e_2, \quad e_2 \cdot e_1 = e_2, \\ & e_1 * e_1 = \alpha e_1, \quad e_1 * e_2 = \alpha e_2, \quad e_2 * e_1 = \alpha e_2, \quad e_2 * e_2 = e_2. \end{aligned}$$

  The algebras $\mathcal{C}_{35}^{\alpha}$ and $\mathcal{C}_{35}^{\alpha'}$ are isomorphic if and only if $\alpha = \alpha'$.

2.3.12 Compatible pre-Lie algebras defined on $\mathcal{C}_{06}^2$

From the computation of $\mathrm{Z}^2_{\mathrm{CPL}}\big(\mathcal{C}_{06}^2,\mathcal{C}_{06}^2\big)$, the compatible pre-Lie algebra structures defined on $\mathcal{C}_{06}^2$ are of the following forms:

(1) $e_1\cdot e_1=e_1$, $e_1\cdot e_2=2e_2$, $e_2\cdot e_1=e_2$, $e_1*e_1=\alpha_1e_1+\alpha_2e_2$, $e_1*e_2=\alpha_3e_2$, $e_2*e_1=\alpha_1e_2$,

(2) $e_1\cdot e_1=e_1$, $e_1\cdot e_2=2e_2$, $e_2\cdot e_1=e_2$, $e_1*e_1=\alpha_4e_1+\alpha_5e_2$, $e_1*e_2=2\alpha_4e_2$, $e_2*e_1=\alpha_6e_1+\alpha_4e_2$, $e_2*e_2=2\alpha_6e_2$,

(3) $e_1\cdot e_1=e_1$, $e_1\cdot e_2=2e_2$, $e_2\cdot e_1=e_2$, $e_1*e_1=\alpha_7e_1$, $e_1*e_2=\alpha_8e_1+2\alpha_7e_2$, $e_2*e_1=2\alpha_8e_1+\alpha_7e_2$, $e_2*e_2=\alpha_8e_2$.

We may assume $\alpha_6\alpha_8\neq 0$. If the compatible pre-Lie algebra structures defined on $\mathcal{C}_{06}^2$ is of the first form, then we obtain the algebras $\mathcal{C}_{31}^{2,\beta,\gamma}$ and $\mathcal{C}_{32}^{2,\beta,\gamma}$.

Assume now that the compatible pre-Lie algebra structures defined on $\mathcal{C}_{06}^2$ are of the second form. Then, choose $\xi=\alpha_6^{-1}$ and obtain the parametric family

$$\begin{aligned}\mathcal{C}_{36}^{\alpha,\beta}\colon\ & e_1\cdot e_1=e_1,\quad e_1\cdot e_2=2e_2,\quad e_2\cdot e_1=e_2,\quad e_1*e_1=\alpha e_1+\beta e_2,\\ & e_1*e_2=2\alpha e_2,\quad e_2*e_1=e_1+\alpha e_2,\quad e_2*e_2=2e_2.\end{aligned}$$

The algebras $\mathcal{C}_{36}^{\alpha,\beta}$ and $\mathcal{C}_{36}^{\alpha',\beta'}$ are isomorphic if and only if $(\alpha,\beta)=(\alpha',\beta')$.

Finally, assume that the compatible pre-Lie algebra structures defined on $\mathcal{C}_{06}^2$ are of the third form. Then, choose $\xi=\alpha_8^{-1}$ and obtain the parametric family

$$\begin{aligned}\mathcal{C}_{37}^{\alpha}\colon\ & e_1\cdot e_1=e_1,\quad e_1\cdot e_2=2e_2,\quad e_2\cdot e_1=e_2,\\ & e_1*e_1=\alpha e_1,\quad e_1*e_2=e_1+2\alpha e_2,\quad e_2*e_1=2e_1+\alpha e_2,\quad e_2*e_2=e_2.\end{aligned}$$

The algebras $\mathcal{C}_{37}^{\alpha}$ and $\mathcal{C}_{37}^{\alpha'}$ are isomorphic if and only if $\alpha=\alpha'$.

2.3.13 Compatible pre-Lie algebras defined on $\mathcal{C}_{07}$

From the computation of $\mathrm{Z}^2_{\mathrm{CPL}}(\mathcal{C}_{07},\mathcal{C}_{07})$, the compatible pre-Lie algebra structures defined on $\mathcal{C}_{07}$ are of the following forms:

(1) $e_1\cdot e_1=e_1$, $e_2\cdot e_2=e_2$, $e_1*e_1=(\alpha_3+\alpha_2-\alpha_1)e_1-\alpha_2e_2$, $e_1*e_2=\alpha_1e_1+\alpha_2e_2$, $e_2*e_1=\alpha_1e_1+\alpha_2e_2$, $e_2*e_2=-\alpha_1e_1+\alpha_3e_2$,

(2) $e_1\cdot e_1=e_1$, $e_2\cdot e_2=e_2$, $e_1*e_1=\alpha_4e_1$, $e_2*e_2=\alpha_5e_2$.

We may assume that $\alpha_4\neq\alpha_5$. Suppose first that the compatible pre-Lie algebra structures defined on $\mathcal{C}_{07}$ has the first form. Then we obtain the algebras

$$\begin{aligned}\mathcal{C}_{38}^{\alpha,\beta,\gamma}\colon\ & e_1\cdot e_1=e_1,\quad e_2\cdot e_2=e_2,\quad e_1*e_1=(\gamma+\beta-\alpha)e_1-\beta e_2,\\ & e_1*e_2=\alpha e_1+\beta e_2,\quad e_2*e_1=\alpha e_1+\beta e_2,\quad e_2*e_2=-\alpha e_1+\gamma e_2.\end{aligned}$$

The algebras $\mathcal{C}_{38}^{\alpha,\beta,\gamma}$ and $\mathcal{C}_{38}^{\alpha',\beta',\gamma'}$ are isomorphic if and only if $(\alpha,\beta,\gamma)=\big(\alpha',\beta',\gamma'\big)$ or $(\alpha,\beta,\gamma)=\big(\beta',\alpha',-\alpha'+\beta'+\gamma'\big)$.

Now, assume that the compatible pre-Lie algebra structures defined on $\mathcal{C}_{07}$ has the second form. Then we obtain the algebras

$$\mathcal{C}_{39}^{\alpha,\beta\neq\alpha}\colon\ e_1\cdot e_1=e_1,\quad e_2\cdot e_2=e_2,\quad e_1*e_1=\alpha e_1,\quad e_2*e_2=\beta e_2.$$

The algebras $\mathcal{C}_{39}^{\alpha,\beta}$ and $\mathcal{C}_{39}^{\alpha',\beta'}$ are isomorphic if and only if $(\alpha,\beta)=\big(\alpha',\beta'\big)$ or $(\alpha,\beta)=\big(\beta',\alpha'\big)$.

2.3.14 Compatible pre-Lie algebras defined on $\mathcal{C}_{08}$

From the computation of $\mathrm{Z}^2_{\mathrm{CPL}}(\mathcal{C}_{08},\mathcal{C}_{08})$, the compatible pre-Lie algebra structures defined on $\mathcal{C}_{08}$ are of the following forms:

(1) $e_1\cdot e_1=e_1$, $e_1\cdot e_2=2e_2$, $e_2\cdot e_1=\frac{1}{2}e_1+e_2$, $e_2\cdot e_2=e_2$, $e_1*e_1=\alpha_1e_1+\alpha_2e_2$, $e_1*e_2=2\alpha_1e_2$, $e_2*e_1=\alpha_3e_1+\alpha_1e_2$, $e_2*e_2=2\alpha_3e_2$,

(2) $e_1\cdot e_1=e_1$, $e_1\cdot e_2=2e_2$, $e_2\cdot e_1=\frac{1}{2}e_1+e_2$, $e_2\cdot e_2=e_2$, $e_1*e_1=\alpha_4e_1$, $e_1*e_2=\alpha_5e_1+2\alpha_4e_2$, $e_2*e_1=\left(2\alpha_5+\frac{1}{2}\alpha_4\right)e_1+\alpha_4e_2$, $e_2*e_2=\frac{1}{2}\alpha_5e_1+(\alpha_4+\alpha_5)e_2$.

We may assume $\alpha_5\neq 0$. Then we obtain the following compatible pre-Lie algebras:

$$\begin{array}{ll}
\mathcal{C}_{40}^{\alpha,\beta,\gamma}: & e_1\cdot e_1=e_1, \quad e_1\cdot e_2=2e_2, \quad e_2\cdot e_1=\frac{1}{2}e_1+e_2, \quad e_2\cdot e_2=e_2,\\
& e_1*e_1=\alpha e_1+\beta e_2, \quad e_1*e_2=2\alpha e_2, \quad e_2*e_1=\gamma e_1+\alpha e_2, \quad e_2*e_2=2\gamma e_2,\\
\mathcal{C}_{41}^{\alpha,\beta\neq 0}: & e_1\cdot e_1=e_1, \quad e_1\cdot e_2=2e_2, \quad e_2\cdot e_1=\frac{1}{2}e_1+e_2, \quad e_2\cdot e_2=e_2,\\
& e_1*e_1=\alpha e_1, \quad e_1*e_2=\beta e_1+2\alpha e_2,\\
& e_2*e_1=\left(2\beta+\frac{1}{2}\alpha\right)e_1+\alpha e_2, \quad e_2*e_2=\frac{1}{2}\beta e_1+(\alpha+\beta)e_2.
\end{array}$$

Moreover, the algebras $\mathcal{C}_{40}^{\alpha,\beta,\gamma}$ and $\mathcal{C}_{40}^{\alpha',\beta',\gamma'}$ are isomorphic if and only if $\left(\alpha',\beta',\gamma'\right)=(\alpha,\beta,\gamma)$ or $\left(\alpha',\beta',\gamma'\right)=(4\gamma-\alpha,\beta-8\alpha+16\gamma,\gamma)$. Also, the algebras $\mathcal{C}_{41}^{\alpha,\beta}$ and $\mathcal{C}_{41}^{\alpha',\beta'}$ are isomorphic if and only if $\left(\alpha',\beta'\right)=(\alpha,\beta)$ or $\left(\alpha',\beta'\right)=(\alpha+4\beta,-\beta)$.

2.4 The algebraic classification of compatible commutative associative algebras

It is easy to see that each commutative pre-Lie algebra is associative. Hence, we can choose only commutative compatible pre-Lie algebras from Theorem 2.7.

Theorem 2.8. *Let $\mathcal{C}$ be a nonzero 2-dimensional compatible commutative associative algebra. Then $\mathcal{C}$ is isomorphic to one and only one and only one of the following algebras:*

$$\begin{array}{ll}
\mathcal{C}_{03}: & e_1*e_1=e_2,\\
\mathcal{C}_{05}^{0}: & e_1*e_1=e_1,\\
\mathcal{C}_{06}^{1}: & e_1*e_1=e_1, \quad e_1*e_2=e_2, \quad e_2*e_1=e_2,\\
\mathcal{C}_{07}: & e_1*e_1=e_1, \quad e_2*e_2=e_2,\\
\mathcal{C}_{13}^{1}: & e_1\cdot e_1=e_2, \quad e_1*e_1=e_1, \quad e_1*e_2=e_2, \quad e_2*e_1=e_2,\\
\mathcal{C}_{15}^{\alpha}: & e_1\cdot e_1=e_2, \quad e_1*e_1=\alpha e_2,\\
\mathcal{C}_{16}^{0}: & e_1\cdot e_1=e_2, \quad e_1*e_1=e_1,\\
\mathcal{C}_{18}^{\alpha}: & e_1\cdot e_1=e_2, \quad e_1*e_1=\alpha e_2, \quad e_1*e_2=e_1, \quad e_2*e_1=e_1, \quad e_2*e_2=e_2,\\
\mathcal{C}_{24}^{0,\beta,0}: & e_1\cdot e_1=e_1, \quad e_1*e_1=\beta e_1+e_2,\\
\mathcal{C}_{25}^{0,\beta,0}: & e_1\cdot e_1=e_1, \quad e_1*e_1=\beta e_1,\\
\mathcal{C}_{29}^{\alpha}: & e_1\cdot e_1=e_1, \quad e_1*e_1=\alpha e_1, \quad e_2*e_2=e_2,\\
\mathcal{C}_{30}^{\alpha,\beta}: & e_1\cdot e_1=e_1, \quad e_1*e_1=\alpha e_1+\beta e_2, \quad e_1*e_2=e_1, \quad e_2*e_1=e_1, \quad e_2*e_2=e_2,\\
\mathcal{C}_{31}^{1,\beta,\beta}: & e_1\cdot e_1=e_1, \quad e_1\cdot e_2=e_2, \quad e_2\cdot e_1=e_2,\\
& e_1*e_1=\beta e_1+e_2, \quad e_1*e_2=\beta e_2, \quad e_2*e_1=\beta e_2,\\
\mathcal{C}_{32}^{1,\beta,\beta}: & e_1\cdot e_1=e_1, \quad e_1\cdot e_2=e_2, \quad e_2\cdot e_1=e_2,\\
& e_1*e_1=\beta e_1, \quad e_1*e_2=\beta e_2, \quad e_2*e_1=\beta e_2,
\end{array}$$

$$\begin{aligned}
\mathcal{C}_{34}^{\alpha,\beta}\colon\ & e_1\cdot e_1=e_1,\quad e_1\cdot e_2=e_2,\quad e_2\cdot e_1=e_2,\\
& e_1*e_1=\alpha e_1,\quad e_1*e_2=\alpha e_2,\quad e_2*e_1=\alpha e_2,\quad e_2*e_2=e_1+\beta e_2,\\
\mathcal{C}_{35}^{\alpha}\colon\ & e_1\cdot e_1=e_1,\quad e_1\cdot e_2=e_2,\quad e_2\cdot e_1=e_2,\\
& e_1*e_1=\alpha e_1,\quad e_1*e_2=\alpha e_2,\quad e_2*e_1=\alpha e_2,\quad e_2*e_2=e_2,\\
\mathcal{C}_{38}^{\alpha,\beta,\gamma}\colon\ & e_1\cdot e_1=e_1,\quad e_2\cdot e_2=e_2,\quad e_1*e_1=(\gamma+\beta-\alpha)e_1-\beta e_2,\\
& e_1*e_2=\alpha e_1+\beta e_2,\quad e_2*e_1=\alpha e_1+\beta e_2,\quad e_2*e_2=-\alpha e_1+\gamma e_2,\\
\mathcal{C}_{39}^{\alpha,\beta\neq\alpha}\colon\ & e_1\cdot e_1=e_1,\quad e_2\cdot e_2=e_2,\quad e_1*e_1=\alpha e_1,\quad e_2*e_2=\beta e_2.
\end{aligned}$$

All algebras are non-isomorphic, except $\mathcal{C}_{38}^{\alpha,\beta,\gamma}\cong\mathcal{C}_{38}^{\beta,\alpha,-\alpha+\beta+\gamma}$, $\mathcal{C}_{39}^{\alpha,\beta}\cong\mathcal{C}_{39}^{\beta,\alpha}$.

2.5 The algebraic classification of compatible associative algebras

Definition 2.9. A compatible associative algebra is a vector space $\mathbf{A}$ equipped with two multiplications: $\cdot$ and another multiplication $*$, such that, $(\mathbf{A},\cdot)$, $(\mathbf{A},*)$ and $(\mathbf{A},\cdot+*)$ are associative algebras. These two operations are required to satisfy the following identities:

$$\begin{aligned}
&(x\cdot y)\cdot z=x\cdot(y\cdot z),\quad (x*y)*z=x*(y*z),\\
&(x*y)\cdot z+(x\cdot y)*z=x*(y\cdot z)+x\cdot(y*z).
\end{aligned}$$

Theorem 2.10. *Let $\mathcal{C}$ be a nonzero 2-dimensional compatible associative algebra. Then $\mathcal{C}$ is isomorphic to one and only one compatible commutative associative algebra listed in Theorem 2.8 or one of the following algebras:*

$$\begin{aligned}
\mathcal{C}_{05}^{1}\colon\ & e_1*e_1=e_1,\quad e_1*e_2=e_2,\\
\mathcal{C}_{06}^{0}\colon\ & e_1*e_1=e_1,\quad e_2*e_1=e_2,\\
\mathcal{C}_{25}^{1,\beta,\beta}\colon\ & e_1\cdot e_1=e_1,\quad e_1\cdot e_2=e_2,\quad e_1*e_1=\beta e_1,\quad e_1*e_2=\beta e_2,\\
\mathcal{C}_{28}\colon\ & e_1\cdot e_1=e_1,\quad e_1\cdot e_2=e_2,\quad e_2*e_1=e_1,\quad e_2*e_2=e_2,\\
\mathcal{C}_{32}^{0,\beta,0}\colon\ & e_1\cdot e_1=e_1,\quad e_2\cdot e_1=e_2,\quad e_1*e_1=\beta e_1,\quad e_2*e_1=\beta e_2,\\
\mathcal{C}_{33}\colon\ & e_1\cdot e_1=e_1,\quad e_2\cdot e_1=e_2,\quad e_1*e_2=e_1,\quad e_2*e_2=e_2.
\end{aligned}$$

2.6 The algebraic classification of compatible Novikov algebras

Definition 2.11. A compatible Novikov algebra is a vector space $\mathbf{A}$ equipped with two multiplications: $\cdot$ and another multiplication $*$, such that, $(\mathbf{A},\cdot)$, $(\mathbf{A},*)$ and $(\mathbf{A},\cdot+*)$ are Novikov algebras. These two operations are required to satisfy the following identities:

$$\begin{aligned}
&(x\cdot y)\cdot z-x\cdot(y\cdot z)=(y\cdot x)\cdot z-y\cdot(x\cdot z),\quad (x\cdot y)\cdot z=(x\cdot z)\cdot y,\\
&(x*y)*z-x*(y*z)=(y*x)*z-y*(x*z),\quad (x*y)*z=(x*z)*y,\\
&(x*y)\cdot z-x*(y\cdot z)+(x\cdot y)*z-x\cdot(y*z)\\
&\qquad=(y*x)\cdot z-y*(x\cdot z)+(y\cdot x)*z-y\cdot(x*z),\\
&(x*y)\cdot z+(x\cdot y)*z=(x*z)\cdot y+(x\cdot z)*y.
\end{aligned}$$

Theorem 2.12. *Let $\mathcal{C}$ be a nonzero 2-dimensional compatible Novikov algebra. Then $\mathcal{C}$ is isomorphic to one and only one compatible commutative associative algebra listed in Theorem 2.8 or one of the following algebras:*

$$\begin{aligned}
\mathcal{C}_{01}\colon\ & e_1*e_1=e_1+e_2,\quad e_2*e_1=e_2,\\
\mathcal{C}_{04}\colon\ & e_2*e_1=e_1,
\end{aligned}$$

$\mathcal{C}_{06}^{\alpha\neq 1}$: $e_1 * e_1 = e_1, \quad e_1 * e_2 = \alpha e_2, \quad e_2 * e_1 = e_2,$

$\mathcal{C}_{09}^{\alpha,\beta\neq 0}$: $e_1 \cdot e_1 = e_1 + e_2, \quad e_2 \cdot e_1 = e_2, \quad e_1 * e_1 = \alpha e_1, \quad e_1 * e_2 = \beta e_2, e_2 * e_1 = \alpha e_2,$

$\mathcal{C}_{10}^{\alpha,\beta}$: $e_1 \cdot e_1 = e_1 + e_2, \quad e_2 \cdot e_1 = e_2, \quad e_1 * e_1 = \alpha e_1 + \beta e_2, \quad e_2 * e_1 = \alpha e_2,$

$\mathcal{C}_{13}^{\alpha\neq 1}$: $e_1 \cdot e_1 = e_2, \quad e_1 * e_1 = \alpha e_1, \quad e_1 * e_2 = e_2, \quad e_2 * e_1 = \alpha e_2,$

$\mathcal{C}_{14}^{\alpha}$: $e_1 \cdot e_1 = e_2, \quad e_1 * e_1 = e_1 + \alpha e_2, \quad e_2 * e_1 = e_2,$

$\mathcal{C}_{20}^{\alpha,0}$: $e_2 \cdot e_1 = e_1, \quad e_2 * e_1 = \alpha e_1, \quad e_2 * e_2 = e_1,$

$\mathcal{C}_{21}^{\alpha,0}$: $e_2 \cdot e_1 = e_1, \quad e_2 * e_1 = \alpha e_1,$

$\mathcal{C}_{22}^{\alpha\neq 0,\beta}$: $e_2 \cdot e_1 = e_1, \quad e_1 * e_2 = \alpha e_1, \quad e_2 * e_1 = \beta e_1, \quad e_2 * e_2 = e_1 + \alpha e_2,$

$\mathcal{C}_{23}^{\alpha\neq 0,\beta}$: $e_2 \cdot e_1 = e_1, \quad e_1 * e_2 = \alpha e_1, \quad e_2 * e_1 = \beta e_1, \quad e_2 * e_2 = \alpha e_2,$

$\mathcal{C}_{31}^{(\alpha,\beta,\gamma)\neq(1,\beta,\beta)}$: $e_1 \cdot e_1 = e_1, \quad e_1 \cdot e_2 = \alpha e_2, \quad e_2 \cdot e_1 = e_2,$
$e_1 * e_1 = \beta e_1 + e_2, \quad e_1 * e_2 = \gamma e_2, \quad e_2 * e_1 = \beta e_2,$

$\mathcal{C}_{32}^{(\alpha,\beta,\gamma)\neq(1,\beta,\beta)}$: $e_1 \cdot e_1 = e_1, \quad e_1 \cdot e_2 = \alpha e_2, \quad e_2 \cdot e_1 = e_2,$
$e_1 * e_1 = \beta e_1, \quad e_1 * e_2 = \gamma e_2, \quad e_2 * e_1 = \beta e_2,$

$\mathcal{C}_{33}$: $e_1 \cdot e_1 = e_1, \quad e_2 \cdot e_1 = e_2, \quad e_1 * e_2 = e_1, \quad e_2 * e_2 = e_2.$

All algebras are non-isomorphic, except $\mathcal{C}_{31}^{0,\beta,\gamma} \cong \mathcal{C}_{32}^{0,\beta,\gamma}$.

3 The geometric classification of compatible algebras

3.1 Degenerations and the geometric classification of algebras

Let us introduce the techniques used to obtain the geometric classification for an arbitrary variety of compatible Ω-algebras. Given a complex vector space $\mathbb{V}$ of dimension n, the set of bilinear maps

$$\mathrm{Bil}(\mathbb{V}\times\mathbb{V},\mathbb{V}) \cong \mathrm{Hom}\big(\mathbb{V}^{\otimes 2},\mathbb{V}\big) \cong (\mathbb{V}^*)^{\otimes 2}\otimes\mathbb{V}$$

is a vector space of dimension n^3. The set of pairs of bilinear maps

$$\mathrm{Bil}(\mathbb{V}\times\mathbb{V},\mathbb{V}) \oplus \mathrm{Bil}(\mathbb{V}\times\mathbb{V},\mathbb{V}) \cong (\mathbb{V}^*)^{\otimes 2}\otimes\mathbb{V} \oplus (\mathbb{V}^*)^{\otimes 2}\otimes\mathbb{V},$$

which is a vector space of dimension $2n^3$. This vector space has the structure of the affine space $\mathbb{C}^{2n^3}$ in the following sense: fixed a basis $e_1,\ldots,e_n$ of $\mathbb{V}$, then any pair with multiplication (μ,μ'), is determined by some parameters $c_{ij}^k, c_{ij}'^k \in \mathbb{C}$, called structural constants, such that

$$\mu(e_i,e_j) = \sum_{p=1}^{n} c_{ij}^k e_k \quad \text{and} \quad \mu'(e_i,e_j) = \sum_{p=1}^{n} c_{ij}'^k e_k,$$

which corresponds to a point in the affine space $\mathbb{C}^{2n^3}$. Then a set of bilinear pairs $\mathcal{S}$ corresponds to an algebraic variety, i.e., a Zariski closed set, if there are some polynomial equations in variables c_{ij}^k, $c_{ij}'^k$ with zero locus equal to the set of structural constants of the bilinear pairs in $\mathcal{S}$. Given the identities defining a particular class of compatible Ω-algebras, we can obtain a set of polynomial equations in variables c_{ij}^k, $c_{ij}'^k$. This class of n-dimensional compatible Ω-algebras is a variety. Denote it by $\mathcal{T}_n$. Now, consider the following action of $\mathrm{GL}(\mathbb{V})$ on $\mathcal{T}_n$

$$(g*(\mu,\mu'))(x,y) := \big(g\mu\big(g^{-1}x,g^{-1}y\big), g\mu'\big(g^{-1}x,g^{-1}y\big)\big)$$

for $g \in \mathrm{GL}(\mathbb{V})$, $(\mu, \mu') \in \mathcal{T}_n$ and for any $x, y \in \mathbb{V}$. Observe that the $\mathrm{GL}(\mathbb{V})$-orbit of (μ, μ'), denoted $\mathcal{O}((\mu, \mu'))$, contains all the structural constants of the bilinear pairs isomorphic to the compatible Ω-algebras with structural constants (μ, μ').

A geometric classification of a variety of algebras consists of describing the irreducible components of the variety. Recall that any affine variety can be represented as a finite union of its irreducible components uniquely. Note that describing the irreducible components of $\mathcal{T}_n$ gives us the rigid algebras of the variety, which are those bilinear pairs with an open $\mathrm{GL}(\mathbb{V})$-orbit. This is due to the fact that a bilinear pair is rigid in a variety if and only if the closure of its orbit is an irreducible component of the variety. For this reason, the following notion is convenient. Denote by $\overline{\mathcal{O}((\mu, \mu'))}$ the closure of the orbit of $(\mu, \mu') \in \mathcal{T}_n$.

Definition 3.1. Let T and T' be two n-dimensional compatible Ω-algebras of a fixed class corresponding to the variety $\mathcal{T}_n$ and $(\mu, \mu'), (\lambda, \lambda') \in \mathcal{T}_n$ be their representatives in the affine space, respectively. The algebra T is said to degenerate to T', and we write $\mathrm{T} \to \mathrm{T}'$, if $(\lambda, \lambda') \in \overline{\mathcal{O}((\mu, \mu'))}$. If $\mathrm{T} \not\cong \mathrm{T}'$, then we call it a proper degeneration. Conversely, if $(\lambda, \lambda') \notin \overline{\mathcal{O}((\mu, \mu'))}$ then we say that T does not degenerate to T' and we write $\mathrm{T} \not\to \mathrm{T}'$.

Furthermore, for a parametric family of algebras, we have the following notion.

Definition 3.2. Let $\mathrm{T}(*) = \{\mathrm{T}(\alpha) \colon \alpha \in I\}$ be a family of n-dimensional compatible Ω-algebras of a fixed class corresponding to $\mathcal{T}_n$ and let T' be another algebra. Suppose that $\mathrm{T}(\alpha)$ is represented by the structure $(\mu(\alpha), \mu'(\alpha)) \in \mathcal{T}_n$ for $\alpha \in I$ and T' is represented by the structure $(\lambda, \lambda') \in \mathcal{T}_n$. We say that the family $\mathrm{T}(*)$ degenerates to T', and write $\mathrm{T}(*) \to \mathrm{T}'$, if $(\lambda, \lambda') \in \overline{\{\mathcal{O}((\mu(\alpha), \mu'(\alpha)))\}_{\alpha \in I}}$. Conversely, if $(\lambda, \lambda') \notin \overline{\{\mathcal{O}((\mu(\alpha), \mu'(\alpha)))\}_{\alpha \in I}}$ then we call it a non-degeneration, and we write $\mathrm{T}(*) \not\to \mathrm{T}'$.

Observe that T' corresponds to an irreducible component of $\mathcal{T}_n$ (more precisely, $\overline{\mathrm{T}'}$ is an irreducible component) if and only if $\mathrm{T} \not\to \mathrm{T}'$ for any n-dimensional compatible Ω-algebra T and $\mathrm{T}(*) \not\to \mathrm{T}'$ for any parametric family of n-dimensional compatible Ω-algebras $\mathrm{T}(*)$. To prove a particular algebra corresponds to an irreducible component, we will use the next ideas. Firstly, since $\dim \mathcal{O}((\mu, \mu')) = n^2 - \dim \mathfrak{Der}(\mathrm{T})$, then if $\mathrm{T} \to \mathrm{T}'$ and $\mathrm{T} \not\cong \mathrm{T}'$, we have that $\dim \mathfrak{Der}(\mathrm{T}) < \dim \mathfrak{Der}(\mathrm{T}')$, where $\mathfrak{Der}(\mathrm{T})$ denotes the Lie algebra of derivations of T. Secondly, to prove degenerations, let T and T' be two compatible Ω-algebras represented by the structures (μ, μ') and (λ, λ') from $\mathcal{T}_n$, respectively. Let c_{ij}^k, $c_{ij}'^k$ be the structure constants of (λ, λ') in a basis $e_1, \ldots, e_n$ of $\mathbb{V}$. If there exist n^2 maps $a_i^j(t) \colon \mathbb{C}^* \to \mathbb{C}$ such that $E_i(t) = \sum_{j=1}^n a_i^j(t) e_j$ $(1 \le i \le n)$ form a basis of $\mathbb{V}$ for any $t \in \mathbb{C}^*$ and the structure constants $c_{ij}^k(t)$, $c_{ij}'^k(t)$ of (μ, μ') in the basis $E_1(t), \ldots, E_n(t)$ satisfy $\lim_{t\to 0} c_{ij}^k(t) = c_{ij}^k$ and $\lim_{t\to 0} c_{ij}'^k(t) = c_{ij}'^k$, then $\mathrm{T} \to \mathrm{T}'$. In this case, $E_1(t), \ldots, E_n(t)$ is called a parametrized basis for $\mathrm{T} \to \mathrm{T}'$. In case of $E_1^t, E_2^t, \ldots, E_n^t$ is a parametric basis for $\mathbf{A} \to \mathbf{B}$, it will be denoted by

$$\mathbf{A} \xrightarrow{(E_1^t, E_2^t, \ldots, E_n^t)} \mathbf{B}.$$

Thirdly, to prove non-degenerations we may use a remark that follows from this lemma, see [3].

Lemma 3.3. *Consider two compatible Ω-algebras T and T'. Suppose $\mathrm{T} \to \mathrm{T}'$. Let C be a Zariski closed in $\mathcal{T}_n$ that is stable by the action of the invertible upper (lower) triangular matrices. Then if there is a representation (μ, μ') of T in C, then there is a representation (λ, λ') of T' in C.*

In order to apply this lemma, we will give the explicit definition of the appropriate stable Zariski closed C in terms of the variables c_{ij}^k, $c_{ij}'^k$ in each case.

Remark 3.4. Moreover, let T and T′ be two compatible Ω-algebras represented by the structures (μ,μ') and (λ,λ') from $\mathcal{T}_n$. Suppose $\mathrm{T}\to\mathrm{T}'$. Then if μ, μ', λ, λ' represents algebras T_0, T_1, T'_0, T'_1 in the affine space $\mathbb{C}^{n^3}$ of algebras with a single multiplication, respectively, we have $\mathrm{T}_0\to\mathrm{T}'_0$ and $\mathrm{T}_1\to\mathrm{T}'_1$. So, for example, $(0,\mu)$ can not degenerate in $(\lambda,0)$ unless $\lambda=0$.

Fourthly, to prove $\mathrm{T}(*)\to\mathrm{T}'$, suppose that $\mathrm{T}(\alpha)$ is represented by the structure $(\mu(\alpha),\mu'(\alpha))\in\mathcal{T}_n$ for $\alpha\in I$ and T′ is represented by the structure $(\lambda,\lambda')\in\mathcal{T}_n$. Let c_{ij}^k, $c_{ij}'^k$ be the structure constants of (λ,λ') in a basis $e_1,\dots,e_n$ of $\mathbb{V}$. If there is a pair of maps $\big(f,(a_i^j)\big)$, where $f\colon\mathbb{C}^*\to I$ and $a_i^j\colon\mathbb{C}^*\to\mathbb{C}$ are such that $E_i(t)=\sum_{j=1}^n a_i^j(t)e_j$ $(1\le i\le n)$ form a basis of $\mathbb{V}$ for any $t\in\mathbb{C}^*$ and the structure constants $c_{ij}^k(t)$, $c_{ij}'^k(t)$ of $(\mu(f(t)),\mu'(f(t)))$ in the basis $E_1(t),\dots,E_n(t)$ satisfy $\lim_{t\to0}c_{ij}^k(t)=c_{ij}^k$ and $\lim_{t\to0}c_{ij}'^k(t)=c_{ij}'^k$, then $\mathrm{T}(*)\to\mathrm{T}'$. In this case, $E_1(t),\dots,E_n(t)$ and $f(t)$ are called a parametrized basis and a parametrized index for $\mathrm{T}(*)\to\mathrm{T}'$, respectively. Fifthly, to prove $\mathrm{T}(*)\not\to\mathrm{T}'$, we may use an analogous of Remark 3.4 for parametric families that follows from Lemma 3.5.

Lemma 3.5. *Consider a family of compatible Ω-algebras* $\mathrm{T}(*)$ *and a compatible Ω-algebra* T'. *Suppose* $\mathrm{T}(*)\to\mathrm{T}'$. *Let C be a Zariski closed in $\mathcal{T}_n$ that is stable by the action of the invertible upper (lower) triangular matrices. Then if there is a representation* $(\mu(\alpha),\mu'(\alpha))$ *of* $\mathrm{T}(\alpha)$ *in C for every $\alpha\in I$, then there is a representation* (λ,λ') *of* T' *in C.*

Finally, the following remark simplifies the geometric problem.

Remark 3.6. Let (μ,μ') and (λ,λ') represent two compatible Ω-algebras. Suppose $(\lambda,0)\notin\overline{\mathcal{O}((\mu,0))}$, (resp. $(0,\lambda')\notin\overline{\mathcal{O}((0,\mu'))}$), then $(\lambda,\lambda')\notin\overline{\mathcal{O}((\mu,\mu'))}$. As we construct the classification of a given class of compatible Ω-algebras from a certain class of algebras with a single multiplication that remains unchanged, this remark becomes very useful.

3.2 The geometric classification of compatible commutative associative algebras

The main result of the present section is the following theorem.

Theorem 3.7. *The variety of complex 2-dimensional compatible commutative associative algebras has dimension 7 and it has 2 irreducible components defined by* $\mathcal{O}\big(\mathcal{C}_{39}^{\alpha,\beta}\big)$ *and* $\mathcal{O}\big(\mathcal{C}_{38}^{\alpha,\beta,\gamma}\big)$. *In particular, there are no rigid algebras in this variety.*

Proof. Thanks to Theorem 2.8, we have the algebraic classification of 2-dimensional compatible commutative associative algebras. After carefully checking the dimensions of orbit closures of the more important for us algebras, we have $\dim\mathcal{O}\big(\mathcal{C}_{38}^{\alpha,\beta,\gamma}\big)=7$ and $\dim\mathcal{O}\big(\mathcal{C}_{39}^{\alpha,\beta}\big)=6$. $\mathcal{C}_{38}^{*}\not\to\mathcal{C}_{39}^{\alpha,\beta}$ due to the following relation:

$$\mathcal{R}=\big\{c_{22}^1=c_{12}^1=0,\ c_{22}^2c_{12}'^2+c_{12}^2c_{21}'^1+c_{11}^1c_{22}'^2=c_{11}'^1c_{22}^2+c_{11}^1c_{21}'^1+c_{12}^2c_{22}'^2\big\}.$$

Thanks to [25], we have $\mathcal{C}_{07}\to\big\{\mathcal{C}_{03},\mathcal{C}_{05}^0,\mathcal{C}_{06}^1\big\}$. All necessary degenerations are given by

$$\mathcal{C}_{38}^{0,0,t^{-1}}\xrightarrow{(te_1,te_2)}\mathcal{C}_{07},\qquad \mathcal{C}_{39}^{-it^{-1},it^{-1}}\xrightarrow{(ite_1-ite_2,-t^2e_1-t^2e_2)}\mathcal{C}_{13}^1,$$

$$\mathcal{C}_{38}^{0,0,\alpha}\xrightarrow{(ite_1-ite_2,-t^2e_1-t^2e_2)}\mathcal{C}_{15}^{\alpha},\qquad \mathcal{C}_{38}^{\frac{i}{2t},0,\frac{i}{t}}\xrightarrow{(ite_1-ite_2,-t^2e_1-t^2e_2)}\mathcal{C}_{16}^0,$$

$$\mathcal{C}_{38}^{-\frac{1+\alpha t^2}{4t^2},-\frac{1+\alpha t^2}{4t^2},\frac{\alpha t^2-3}{4t^2}}\xrightarrow{(ite_1-ite_2,-t^2e_1-t^2e_2)}\mathcal{C}_{18}^{\alpha},\qquad \mathcal{C}_{38}^{0,-t,\beta+t}\xrightarrow{(e_1,te_2)}\mathcal{C}_{24}^{0,\beta,0},$$

$$\mathcal{C}_{38}^{0,0,\beta}\xrightarrow{(e_1,te_2)}\mathcal{C}_{25}^{0,\beta,0},\qquad \mathcal{C}_{39}^{\alpha,t^{-1}}\xrightarrow{(e_1,te_2)}\mathcal{C}_{29}^{\alpha},\qquad \mathcal{C}_{39}^{-t+\beta,\beta}\xrightarrow{(e_1+e_2,te_2)}\mathcal{C}_{31}^{1,\beta,\beta},$$

$$\mathcal{C}_{38}^{-\alpha-\beta t+t^{-1},-\beta t,t^{-1}}\xrightarrow{(e_1+e_2,te_2)}\mathcal{C}_{30}^{\alpha,\beta},\qquad \mathcal{C}_{38}^{0,\beta,0}\xrightarrow{(e_1+e_2,te_2)}\mathcal{C}_{32}^{1,\beta,\beta},$$

$$\mathcal{C}_{38}^{-t^{-2},\alpha-(1+\beta t)t^{-2},(1+\beta t)t^{-2}}\xrightarrow{(e_1+e_2,te_2)}\mathcal{C}_{34}^{\alpha,\beta},\qquad \mathcal{C}_{38}^{\frac{1+2\alpha t}{2t},\alpha,0}\xrightarrow{(e_1+e_2,-te_1+te_2)}\mathcal{C}_{35}^{\alpha}.$$

■

3.3 The geometric classification of compatible associative algebras

The main result of the present section is the following theorem.

Theorem 3.8. *The variety of complex 2-dimensional compatible associative algebras has dimension 7 and it has 4 irreducible components defined by* $\overline{\mathcal{O}(\mathcal{C}_{28})}$, $\overline{\mathcal{O}(\mathcal{C}_{33})}$, $\mathcal{O}(\mathcal{C}_{39}^{\alpha,\beta})$ *and* $\mathcal{O}(\mathcal{C}_{38}^{\alpha,\beta,\gamma})$. *In particular, there are two rigid algebras in this variety.*

Proof. Thanks to Theorem 2.10, we have the algebraic classification of 2-dimensional compatible associative algebras. After carefully checking the dimensions of orbit closures of the more important for us algebras, we have

$$\dim \mathcal{O}\big(\mathcal{C}_{38}^{\alpha,\beta,\gamma}\big) = 7, \quad \dim \mathcal{O}\big(\mathcal{C}_{39}^{\alpha,\beta}\big) = 6, \quad \dim \mathcal{O}(\mathcal{C}_{28}) = \dim \mathcal{O}(\mathcal{C}_{34}) = 4.$$

Thanks to Theorem 3.8, we have that $\mathcal{C}_{38}^{\alpha,\beta,\gamma}$ and $\mathcal{C}_{39}^{\alpha,\beta}$ give irreducible components in the variety of 2-dimensional compatible commutative associative algebras. The rest of the necessary degenerations are given by

$$\mathcal{C}_{28} \xrightarrow{(e_2, te_1)} \mathcal{C}_{05}^{1}, \quad \mathcal{C}_{33} \xrightarrow{(e_2, te_1)} \mathcal{C}_{06}^{0}, \quad \mathcal{C}_{28} \xrightarrow{(e_1+\beta e_2, te_2)} \mathcal{C}_{25}^{1,\beta,\beta}, \quad \mathcal{C}_{33} \xrightarrow{(e_1+\beta e_2, te_2)} \mathcal{C}_{32}^{0,\beta,0}. \qquad \blacksquare$$

3.4 The geometric classification of compatible Novikov algebras

The main result of the present section is the following theorem.

Theorem 3.9. *The variety of complex 2-dimensional compatible Novikov algebras has dimension 7 and it has 6 irreducible components defined by* $\overline{\mathcal{O}(\mathcal{C}_{33})}$, $\mathcal{O}\big(\mathcal{C}_{09}^{\alpha,\beta}\big)$, $\mathcal{O}\big(\mathcal{C}_{22}^{\alpha,\beta}\big)$, $\mathcal{O}\big(\mathcal{C}_{39}^{\alpha,\beta}\big)$, $\mathcal{O}\big(\mathcal{C}_{31}^{\alpha,\beta,\gamma}\big)$ *and* $\mathcal{O}\big(\mathcal{C}_{38}^{\alpha,\beta,\gamma}\big)$. *In particular, there is one rigid algebra in this variety.*

Proof. Thanks to Theorem 2.12, we have the algebraic classification of 2-dimensional compatible Novikov algebras. After carefully checking the dimensions of orbit closures of the more important for us algebras, we have

$$\dim \mathcal{O}\big(\mathcal{C}_{38}^{\alpha,\beta,\gamma}\big) = 7, \quad \dim \mathcal{O}\big(\mathcal{C}_{39}^{\alpha,\beta}\big) = \dim \mathcal{O}\big(\mathcal{C}_{09}^{\alpha,\beta}\big) = \dim \mathcal{O}\big(\mathcal{C}_{22}^{\alpha,\beta}\big) = \dim \mathcal{O}\big(\mathcal{C}_{31}^{\alpha,\beta,\gamma}\big) = 6,$$
$$\dim \mathcal{O}(\mathcal{C}_{33}) = 4.$$

Thanks to Theorem 3.9, we have that $\mathcal{C}_{38}^{\alpha,\beta,\gamma}$ and $\mathcal{C}_{39}^{\alpha,\beta}$ give irreducible components in the variety of 2-dimensional compatible commutative associative algebras. Algebras $\mathcal{C}_{09}^{\alpha,\beta}$, $\mathcal{C}_{22}^{\alpha,\beta}$, and $\mathcal{C}_{31}^{\alpha,\beta,\gamma}$ have a one-dimensional subalgebra with zero multiplication (concerning both multiplications), but $\mathcal{C}_{33}$ has not. Hence, $\mathcal{C}_{33}$ is not in the orbit closure of $\mathcal{C}_{09}^{\alpha,\beta}$, $\mathcal{C}_{22}^{\alpha,\beta}$, and $\mathcal{C}_{31}$. The rest of the necessary degenerations are given by

$$\mathcal{C}_{31}^{t,\frac{t+2}{2t},-\frac{1}{2}} \xrightarrow{(te_1+\frac{2t}{3}e_2, \frac{t^2}{2}e_1+t^2e_2)} \mathcal{C}_{01}, \quad \mathcal{C}_{31}^{0,0,t^{-1}} \xrightarrow{(t^2e_1+e_2, te_1-t^2e_2)} \mathcal{C}_{04},$$

$$\mathcal{C}_{31}^{0,t^{-1},\alpha t^{-1}} \xrightarrow{(te_1-\alpha^{-1}t^2e_2, te_2)} \mathcal{C}_{06}^{\alpha\neq 0}, \quad \mathcal{C}_{09}^{\frac{(2\alpha+t)\beta}{2\beta+t},-\frac{\beta t}{2\beta+t}} \xrightarrow{(\frac{2\beta+t}{2\beta}e_1-\frac{(2\beta+t)^2}{2\beta t}e_2, \frac{(2\beta+t)t}{4\beta^2}te_1)} \mathcal{C}_{10}^{\alpha,\beta\neq 0},$$

$$\mathcal{C}_{31}^{1,\alpha t^{-1},t^{-1}} \xrightarrow{(te_1-t^2e_2, -t^3e_2)} \mathcal{C}_{13}^{\alpha}, \quad \mathcal{C}_{31}^{1,t^{-1},0} \xrightarrow{(te_1+\alpha^{-1}te_2, \alpha^{-1}t^2e_2)} \mathcal{C}_{14}^{\alpha\neq 0},$$

$$\mathcal{C}_{31}^{t^{-1}-t,0,-1+\alpha t^{-1}} \xrightarrow{(t^2e_1+\frac{t^2}{1-\alpha t}e_2, te_1+\frac{t^3}{1-\alpha t}e_2)} \mathcal{C}_{20}^{\alpha,0}, \quad \mathcal{C}_{31}^{t^{-1}-t,0,-\alpha t^{-1}} \xrightarrow{(t^2e_1-\alpha^{-1}te_2, te_1-\alpha^{-1}te_2)} \mathcal{C}_{21}^{\alpha,0},$$

$$\mathcal{C}_{31}^{t^{-1}-t,\alpha t^{-1},\beta t^{-1}} \xrightarrow{(t^2e_1-\beta^{-1}te_2, te_1-\beta^{-1}t^2e_2)} \mathcal{C}_{23}^{\alpha,\beta}, \quad \mathcal{C}_{31}^{\alpha,\beta,\gamma} \xrightarrow{(e_1, t^{-1}e_2)} \mathcal{C}_{32}^{\alpha,\beta,\gamma}. \qquad \blacksquare$$

3.5 The geometric classification of compatible pre-Lie algebras

The main result of the present section is the following theorem.

Theorem 3.10. *The variety of complex* 2*-dimensional compatible pre-Lie algebras has dimension* 7 *and it has* 14 *irreducible components defined by*

$$\overline{\mathcal{O}(\mathcal{C}_{28})},\quad \overline{\mathcal{O}(\mathcal{C}_{33})},\quad \overline{\mathcal{O}(\mathcal{C}_{37}^{\alpha})},\quad \overline{\mathcal{O}(\mathcal{C}_{09}^{\alpha,\beta})},\quad \overline{\mathcal{O}(\mathcal{C}_{11}^{\alpha,\beta})},\quad \overline{\mathcal{O}(\mathcal{C}_{22}^{\alpha,\beta})},\quad \overline{\mathcal{O}(\mathcal{C}_{27}^{\alpha,\beta})},$$
$$\overline{\mathcal{O}(\mathcal{C}_{36}^{\alpha,\beta})},\quad \overline{\mathcal{O}(\mathcal{C}_{39}^{\alpha,\beta})},\quad \overline{\mathcal{O}(\mathcal{C}_{41}^{\alpha,\beta})},\quad \overline{\mathcal{O}(\mathcal{C}_{24}^{\alpha,\beta,\gamma})},\quad \overline{\mathcal{O}(\mathcal{C}_{31}^{\alpha,\beta,\gamma})},$$
$$\overline{\mathcal{O}(\mathcal{C}_{38}^{\alpha,\beta,\gamma})}\quad \textit{and}\quad \overline{\mathcal{O}(\mathcal{C}_{40}^{\alpha,\beta,\gamma})}.$$

In particular, there are two rigid algebras in this variety.

Proof. Thanks to Theorem 2.7, we have the algebraic classification of 2-dimensional compatible pre-Lie algebras. After carefully checking the dimensions of orbit closures of the more important for us algebras, we have

$$\begin{aligned}
&\dim \mathcal{O}(\mathcal{C}_{24}^{\alpha,\beta,\gamma}) = \dim \mathcal{O}(\mathcal{C}_{31}^{\alpha,\beta,\gamma}) = \dim \mathcal{O}(\mathcal{C}_{38}^{\alpha,\beta,\gamma}) = \dim \mathcal{O}(\mathcal{C}_{40}^{\alpha,\beta,\gamma}) = 7,\\
&\dim \mathcal{O}(\mathcal{C}_{09}^{\alpha,\beta}) = \dim \mathcal{O}(\mathcal{C}_{11}^{\alpha,\beta}) = \dim \mathcal{O}(\mathcal{C}_{22}^{\alpha,\beta}) = \dim \mathcal{O}(\mathcal{C}_{27}^{\alpha,\beta}) = \dim \mathcal{O}(\mathcal{C}_{36}^{\alpha,\beta}) = \dim \mathcal{O}(\mathcal{C}_{39}^{\alpha,\beta})\\
&\qquad\qquad = \dim \mathcal{O}(\mathcal{C}_{41}^{\alpha,\beta}) = 6,\\
&\dim \mathcal{O}(\mathcal{C}_{37}^{\alpha}) = 5, \quad \dim \mathcal{O}(\mathcal{C}_{28}) = \dim \mathcal{O}(\mathcal{C}_{33}) = 4.
\end{aligned}$$

All necessary reasons for non-degenerations are listed as follows:

Non-degenerations reasons	
$\mathcal{C}_{24}^{*} \not\to \mathcal{C}_{22}^{\alpha,\beta}, \mathcal{C}_{26}^{\alpha}, \mathcal{C}_{27}^{\alpha,\beta}, \mathcal{C}_{28}$	$\mathcal{R} = \left\{ c_{21}'^{1} = c_{21}'^{2} = c_{22}^{1} = c_{22}^{2} = c_{22}'^{1} = c_{22}'^{2} = 0 \right\}$
$\mathcal{C}_{31}^{*} \not\to \mathcal{C}_{33}, \mathcal{C}_{36}^{\alpha,\beta}, \mathcal{C}_{37}^{\alpha}$	$\mathcal{R} = \left\{ c_{22}'^{1} = c_{23}'^{2} = c_{22}^{1} = c_{22}^{2} = 0 \right\}$
$\mathcal{C}_{36}^{*} \not\to \mathcal{C}_{37}^{\alpha}$	$\mathcal{R} = \left\{ c_{22}^{1} = c_{12}^{1} = c_{12}'^{1} = 0 \right\}$
$\mathcal{C}_{40}^{*} \not\to \mathcal{C}_{41}^{\alpha,\beta}$	$\mathcal{R} = \left\{ c_{22}^{1} = c_{12}^{1} = c_{12}'^{1} = 0,\ 2c_{11}'^{1} = c_{12}'^{1} \right\}$

The rest of the non-degenerations between indicated algebras follows from Remark 3.6.

Thanks to Theorem 3.9, we have that $\mathcal{C}_{33}$, $\mathcal{C}_{09}^{\alpha,\beta}$, $\mathcal{C}_{22}^{\alpha,\beta}$, $\mathcal{C}_{39}^{\alpha,\beta}$, $\mathcal{C}_{31}^{\alpha,\beta,\gamma}$, and $\mathcal{C}_{38}^{\alpha,\beta,\gamma}$ give irreducible components in the variety of 2-dimensional compatible Novikov algebras. Hence, each 2-dimensional compatible Novikov algebras is on the orbit closure of these algebras. The rest of the necessary degenerations are given by

$$\mathcal{C}_{24}^{1,1+t^{-1},-1+t^{-1}} \xrightarrow{(te_1+(-1+t)e_2,t^2e_1+te_2)} \mathcal{C}_{02},\quad \mathcal{C}_{24}^{1,t^{-1},\alpha t^{-1}} \xrightarrow{(te_1+\frac{t^2}{1-\alpha}e_2,t^2e_1+e_2)} \mathcal{C}_{05}^{\alpha\neq1},$$
$$\mathcal{C}_{40}^{t^{-1},0,\frac{1}{2t}} \xrightarrow{(te_1,te_2)} \mathcal{C}_{08},\quad \mathcal{C}_{11}^{\alpha,\alpha+t,} \xrightarrow{(e_1+\beta t^{-1}e_2,e_2)} \mathcal{C}_{12}^{\alpha,\beta},$$
$$\mathcal{C}_{24}^{-1+\alpha,t^{-1},\alpha t^{-1}} \xrightarrow{(te_1-\frac{t^2}{\alpha-1}e_2,t^2e_1-t^3e_2)} \mathcal{C}_{16}^{\alpha\neq1},$$
$$\mathcal{C}_{24}^{-2,\alpha+t^{-1},-\alpha+t^{-1}} \xrightarrow{(te_1-\alpha^{-1}te_2,t^2e_1+2\alpha^{-1}t^2e_2)} \mathcal{C}_{17}^{\alpha\neq0},$$
$$\mathcal{C}_{40}^{-\alpha-2t^{-2},-6\alpha-6t^{-2},-\frac{4+\alpha t^2}{6t^2}} \xrightarrow{(te_1-3te_2,\frac{t^2}{2}e_1-3t^2e_2)} \mathcal{C}_{19}^{\alpha},$$
$$\mathcal{C}_{24}^{-1-t^{-2},-(\beta+t)t^{-2},(t-\alpha)t^{-2}} \xrightarrow{(-t^3e_1+\frac{t^4+t^6}{1+\alpha t-\beta t}e_2,-t^2e_1+\frac{t^5}{\beta t-\alpha t-1}e_2)} \mathcal{C}_{20}^{\alpha,\beta},$$
$$\mathcal{C}_{24}^{-1-t^{-2},-\beta t^{-2},-\alpha t^{-2}} \xrightarrow{(-t^3e_1+\frac{t^3+t^5}{\alpha-\beta}e_2,-t^2e_1-\frac{t^4}{\alpha-\beta}e_2)} \mathcal{C}_{21}^{\alpha,\beta},\quad \mathcal{C}_{27}^{\alpha-5t^2,t^{-2}} \xrightarrow{(e_1+2t^2e_2,te_2)} \mathcal{C}_{26}^{\alpha},$$
$$\mathcal{C}_{24}^{\alpha,\beta,\gamma} \xrightarrow{(e_1,t^{-1}e_2)} \mathcal{C}_{25}^{\alpha,\beta,\gamma}.$$

■

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