Symmetry, Integrability and Geometry: Methods and Applications SIGMA **20** (2024), 106, 27 pages

Global Magni4icence, or: 4G Networks

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Received February 20, 2024, in final form November 15, 2024; Published online November 28, 2024
https://doi.org/10.3842/SIGMA.2024.106

Abstract. The global magnificent four theory is the homological version of a maximally supersymmetric $(8+1)$-dimensional gauge theory on a Calabi–Yau fourfold fibered over a circle. In the case of a toric fourfold we conjecture the formula for its twisted Witten index. String-theoretically we count the BPS states of a system of $D0$-$D2$-$D4$-$D6$-$D8$-branes on the Calabi–Yau fourfold in the presence of a large Neveu–Schwarz B-field. Mathematically, we develop the equivariant K-theoretic DT4 theory, by constructing the four-valent vertex with generic plane partition asymptotics. Physically, the vertex is a supersymmetric localization of a non-commutative gauge theory in $8+1$ dimensions.

Key words: vertex; Calabi–Yau fourfold; Donaldson–Thomas; localization

2020 Mathematics Subject Classification: 14N35; 81T30; 81T60

1 Introduction

The discovery [33] of the beautiful connection between the topology of the moduli spaces of instantons [7] and representation theory of Kac–Moody algebras prompted the search for the physical explanation of the connection between the seemingly distinct structures: four-dimensional gauge dynamics on the one hand, and algebraic structures of two-dimensional conformal field theory, on another. In physics, the moduli spaces of instantons show up in the semi-classical evaluation of gauge theory correlation functions. Their moduli space topology plays a more prominent role in $(4+1)$-dimensional theory, where the harmonic differential forms on the moduli space of instantons represent the internal states of solitonic BPS particles. Gauge theories beyond four dimensions require ultraviolet completion, which is provided either by string theory, or, in some cases, by the $(2,0)$ superconformal theory in six dimensions [49]. Computing index of Dirac operator or its equivariant version, of instanton moduli spaces, proved quite beneficial in developing the theory of dualities, both in field theory and in string theory. A useful tool in this endeavour is the equivariant localization with respect to the isometries of four-dimensional space $\mathbb{R}^4$. Equivariant integrals, for example, can be computed exactly as sums of local contributions of the fixed points (or fixed loci more generally). Unfortunately, this is not immediately useful in the context of instantons, as the fixed instantons are singular, i.e., they are not found in the moduli space $\mathcal{M}$ but rather on some compactification $\bar{\mathcal{M}}$. Justifying the use of compactification, e.g., used in [33] led to the noncommutative deformation of gauge theory [45]. In background independent formulation, the latter can be viewed as matrix model or matrix quantum mechanics, with a specific type of infinite matrices [50]. Within this background independent matrix approach, the computation of path integral of gauge theory becomes equivalent to an infinite-dimensional version of a supersymmetric matrix model. Moreover, using various ideas from string theory [22], one can generalize these matrix models so as to describe more non-trivial spatial backgrounds, such as ALE spaces, conifolds or K3 manifolds [35]. One can observe, that the intuition of quantum field theory on curved spacetime, in particular the cluster decomposition, which is a useful method of computations in topological field theories, persists in the matrix model approach [37].

One of the fruitful ideas in mathematics is the program of complexification [1], which prompts an eight-dimensional generalization of instanton enumeration. An early attempt to formulate such a problem was done in [5] in 8 and in [6] in $8+1$ dimensions but, without any understanding of the geometry of the moduli space of generalized instantons let alone its compactification, the progress was minimal. A recent advance in this direction was achieved with the introduction [40] of the extension of the ADHM construction of instanton moduli space, which led to a generalization [41] of instanton partition functions. Enumeration of instantons in 4 or $4+1$ dimensions leads to the computation of sums over Young diagrams, or N-tuples of Young diagrams for rank N theories. The corresponding generalization to 8 or $8+1$ dimensions is a sum over four-dimensional Young diagrams, or solid partitions. The rank N theories, studied in [43] on $\mathbb{R}^8 \times S^1$, reduce to the summations of N-tuples of finite size solid partitions.

The goal of this paper is to analyze the theory [43] in the more global setting. We would like to study the super-Yang–Mills theory in maximal number of space-time dimensions. Classically, super-Yang–Mills theory can be defined in ten dimensions, with sixteen real supercharges generating the corresponding supersymmetry. However, quantum mechanically super-Yang–Mills in ten dimensions suffers from anomalies, and string theory completion in ten dimensions is only possible for a very limited class of gauge groups. In nine dimensions one can use $D8$-branes of IIA string theory to engineer more general theories. We would like to study the states of the corresponding theory when $D8$ branes wrap a Calabi–Yau fourfold. The approach taken in this paper follows the combinatorics [24, 29, 30] of equivariant Donaldson–Thomas ($DT3$) and equivariant K-theoretic Donaldson–Thomas theory ($kDT3$), representing $(6+1)$-dimensional partially twisted maximally supersymmetric Yang–Mills theory on complex threefolds fibered over a circle [42].

The global versions of the magnificent four theory have been recently studied in numerous interesting publications, see [26] for a review. Both mathematical and physical communities explore this terra incognita. The main motivation, from our perspective, is to gain new evidence for M-theory, extend the Gromov–Witten/Donaldson–Thomas correspondence and theory of Kähler gravity [24, 29, 30]. There are of course proper mathematical motivations. The work [9, 11, 12, 16, 46] on mathematical foundations of the four-dimensional version of Donaldson–Thomas theory $DT4$, the concrete proposals for signs [13, 31] (as we recall below, the orientation of the moduli spaces $\mathcal{M}_\zeta$ is a nontrivial issue) are a small selection of recent advances. As in the lower-dimensional cases, the extension to orbifolds is the first step in the physical approach [10, 14, 23, 27, 51]. Another version of the global magnificent four theory is the theory on a union of transversal or intersecting complex surfaces inside a Calabi–Yau fourfold [44], or a similar arrangement of hypersurfaces [48].

Another physically motivated idea is to view the higher-dimensional instantons as holomorphic maps (quasimaps) of complex curves into the moduli spaces of instantons on spaces of two dimension less, cf. [2, 8, 28]. One can also study the four-dimensional analogues of holomorphic maps: the solutions of Seiberg–Witten equations describing BPS configurations in (real) four-dimensional theory with the gauge group $\mathrm{U}(k)$ with the matter fields furnishing the ADHM data for charge k instantons with gauge group $\mathrm{U}(N)$. One can also use the recent studies of the spaces of holomorphic maps of complex surfaces to Kähler manifolds [17]. In our construction below, we combine all these ingredients.

Let X be a toric Calabi–Yau four-fold (smooth quasi-projective toric variety), with Kähler form ω and top holomorphic form Ω. Let F be the curvature of a connection on a complex vector bundle $\mathcal{E}$ over X, with prescribed Chern character $\mathrm{ch}(\mathcal{E})$, satisfying [18]

$$F_{+}^{0,2} := F^{0,2} + \bar{\star}\big(\Omega \wedge F^{0,2}\big) = 0, \qquad F.\omega = 0, \tag{1.1}$$

where we split $\wedge^2(T^*X\otimes\mathbb{C}) = \langle\omega\rangle\oplus\wedge_0^{1,1}\oplus\wedge^{2,0}\oplus\wedge^{0,2}$. In the decomposition $\wedge^2(T^*X) = \wedge_7\oplus\wedge_{21}$ into Spin(7) irreps, equation (1.1) corresponds to the projection $\pi_7(F) = 0$. Let $\mathcal{M}$ be the framed moduli space of solutions to equation (1.1) modulo unitary gauge equivalence. Actually, we work in *non-commutative* gauge theory,[1] and denote the corresponding moduli space by $\mathcal{M}_\zeta$.

Neglecting torsion, the central charge of a bound state of D-branes in type IIA string theory near large radius is

$$Z = \int_X \mathrm{e}^{2\pi\mathrm{i}(B+\mathrm{i}\omega)}\,\mathrm{ch}(\mathcal{E})\hat{\Gamma}_X, \tag{1.2}$$

where the class $\hat{\Gamma}_X = \prod_{i=1}^4\Gamma(1+\delta_i)$ is built out of Chern roots δ_i of TX and it provides a square-root of A-roof and Todd classes [25]. We consider one $D8$-brane, which is infinitely massive and acts as background, see refs. [20, 24, 29, 30, 43], as well as one $\overline{D8}$-brane. We can also turn on flat Ramond–Ramond background forms C_{p+1} with one leg along S^1, provided they are compatible with toric symmetries, and weigh a given instanton configuration by

$$u := \exp\left[-\int_{X\times S^1}\left(\frac{\mathrm{d}s}{g_s}\mathrm{e}^{2\pi\mathrm{i}(B+\mathrm{i}\omega)} + \mathrm{i}\sum_{n=0}^4 C_{2n+1}\right)\mathrm{ch}(\mathcal{E})\hat{\Gamma}_X\right], \tag{1.3}$$

where g_s is the string coupling and $\mathrm{d}s$ a local coordinate on S^1. This u helps keep track of Chern classes $c_i = c_i(\mathcal{E})$. In the following, we only turn on C_1 and define $-p = \exp\int_{S^1}\frac{\mathrm{d}s}{g_s} + \mathrm{i}C_1$.

Definition 1.1. The Donaldson–Thomas partition function is defined in equivariant[2] K-theory as a generating sum of integrals over virtual fundamental classes

$$\mathsf{Z} := \sum_{(c_1,c_2,c_3,c_4)} u\int_{[\mathcal{M}_\zeta]^{\mathrm{vir}}}\hat{A}\,\mathrm{ch}\wedge^\bullet_{\tilde{\mu}}E, \tag{1.4}$$

where the matter bundle E is the kernel of Dirac operator coupled to the gauge bundle and it plays the role of insertions in Gromov–Witten theory $\wedge^\bullet_{\tilde{\mu}}E := \sum_{i=0}^{\mathrm{rk}\,E}(-\tilde{\mu})^i\wedge^i(E)$.

Remark 1.2. Physically, Z is the twisted Witten index of a supersymmetric gauge theory living on the $D8$-brane, up to an overall perturbative factor. Equivalently, it is a sum of twisted Witten indices of supersymmetric quantum mechanical models counting bound states of $D0$-$D2$-$D4$-$D6$-$D8$-branes on X, in the limit of large volume and large B-field.

Remark 1.3. Let $\mathcal{O}^{\mathrm{vir}}$ be the virtual structure sheaf on $\mathcal{M}_\zeta$, which exist thanks to Oh–Thomas [46], and $\mathcal{E}$ the universal sheaf,[3] i.e., the sheaf on $\mathcal{M}_\zeta\times X$, with π the projection to the first factor. The algebro-geometric counterpart of our Z is the holomorphic Euler characteristic $\chi\big(\mathcal{M}_\zeta, \widehat{\wedge^\bullet_{\tilde{\mu}}(\pi_*\mathcal{E})\otimes\mathcal{O}^{\mathrm{vir}}}\big)$ where hat means twist by the square root of determinant as in Nekrasov–Okounkov [42].

The main difficulty is to find an orientation on $\mathcal{M}_\zeta$. One has to provide at least an orientation at the fixed points, so that the integral can be defined via equivariant localization.[4]

Our strategy is to require covariance of every building block, under the assumption that the sign choice is essentially unique (up to an overall sign) and therefore if we can produce a square root that is also covariant, then it must be the correct answer. This is related to the fact that the sign induced by the orientation choice is well defined [46], so that the choice of both a sign and of a square-root of virtual tangent bundle is essentially canonical [31].

[1]Otherwise, Derrick's theorem would apply. Here we mean gauge theory in the sense of Seiberg and Witten, namely we investigate open strings on X, in the limit of large B-field. This is what is expected to produce the higher derivative regularization of PDEs (1.1), with the B-field corrections. The B-field parameters are denoted by ζ.

[2]Equivariant with respect to the maximal torus $\mathrm{U}(1)^3$ of SU(4) holonomy of X, as well as the mass parameter $\tilde{\mu}$.

[3]We previously denoted by the same letter the restriction of $\mathcal{E}$ to $X\times\{m\}$, for some $m\in\mathcal{M}_\zeta$.

[4]The paper [15] mentions the relevance of DT4 theory to square root Euler classes.

2 Background material

2.1 Partitions

A solid partition K is a collection of non-negative integers $K_{i,j,k}$ labeled by triples of positive integers, obeying inequalities

$$K = \left\{ K_{i,j,k} \mid K_{i,j,k} \geq \max(0, K_{i+1,j,k},\, K_{i,j+1,k}, K_{i,j,k+1})\, \forall (i,j,k) \in \mathbb{Z}^3_{>0} \right\}.$$

Its size is $|K| = \sum_{i,j,k} K_{i,j,k}$. Equivalently, we can represent it by its 4d diagram as

$$K = \left\{ (i,j,k,m) \in \mathbb{Z}^4_{>0} \mid m \leq K_{i,j,k} \right\}.$$

Its character is

$$\mathrm{ch}_K(q_1,q_2,q_3,q_4) = \sum_{(k_1,k_2,k_3,k_4)\in K} \prod_{a=1}^{4} q_a^{k_a-1} \in \mathbb{Z}[q_1,q_2,q_3,q_4].$$

In the present setup (local theory) it's unambiguous to identify a partition with its character. (Later in Section 3, when we work on a toric variety (global theory), we must keep track of the variables used to used to compute the character, which depend on the fixed point as defined in Section 2.4.) Given any partition ρ, we denote by ρ^* its character evaluated at the conjugated variables $q_a^* = q_a^{-1}$. Solid partitions are in one-to-one correspondence with finite-codimension monomial ideals $\mathcal{I} \subset \mathbb{C}[z_1,z_2,z_3,z_4]$,

$$K = \left\{ (k_1,\ldots,k_4) \in \mathbb{Z}^4_{>0} \mid \prod_{a=1}^{4} z_a^{k_a-1} \notin \mathcal{I} \right\} \approx \mathbb{C}[z_1,z_2,z_3,z_4]/\mathcal{I}, \tag{2.1}$$

where the $\approx$ symbol means that the set $\left\{ \prod_{a=1}^4 z_a^{k_a-1} \mid (k_1,\ldots,k_4) \in K \right\}$ provides a vector space basis of $\mathbb{C}[z_1,\ldots,z_4]/\mathcal{I}$. The codimension of $\mathcal{I}$ equals the size of K. We define partitions of infinite size by allowing any monomial ideal in (2.1). Their character is a formal power series in $\mathbb{Z}[[q_1,q_2,q_3,q_4]]$.

A similar construction can be performed in any dimension, in particular we will use $\mathbb{C}[q_1,q_2]$ (Young diagrams) and $\mathbb{C}[q_1,q_2,q_3]$ (plane partitions).

A colored partition (plane partition, solid partition) is a vector of partitions (plane partitions, solid partitions).

2.1.1 Taylor resolution[5]

Consider a monomial ideal $\mathcal{I} \subset R = \mathbb{C}[x_1,\ldots,x_n]$, with generators $m_1,\ldots,m_s$ (it is important that s is finite for a monomial ideal). For any ordered $T \subseteq \{1,\ldots,s\}$, let $m_T := \mathrm{lcm}\{m_i | i \in T\}$, the least common multiple of a subset of generators. Consider the simplex $F = (F_t)$, where $F_t = \bigoplus_{|T|=t} R m_T$, for $s \geq t > 1$, has a formal basis $\{e_T \mid |T| = t\}$, and the differential

$$d(e_T) := \sum_{T'=T\setminus\{i_k\}} (-1)^k \frac{m_T}{m_{T'}} e_{T'},$$

where $T = \{i_1,\ldots,i_{|T|}\}$. This gives a free resolution of $\mathcal{I}$, with all the good properties except that it may be non-minimal. For example, the character is

$$\mathrm{ch}\,\mathcal{I} = \sum_{t=1}^{s} (-1)^{t+1}\, \mathrm{ch}\, F_t.$$

[5]This section lies somewhat outside the main line of development of this work.

2.2 Characters and regularizations

Definition 2.1. For some finite set $S \subset \mathbb{Z}^p$, we call a Laurent polynomial

$$A(x_1, \ldots, x_p) = \sum_{\vec{n} \in S} A_{\vec{n}} x_1^{n_1} x_2^{n_2} \cdots x_p^{n_p}$$

movable if it has no constant term, i.e., $A_0 = 0$. (We call A_0 its *unmovable*, a.k.a. fixed, part.) Let us define the *mobility* of A by $\#A = \lim_{\boldsymbol{x} \to 1}(A - A_0)$.

Definition 2.2. The size of a Laurent polynomial $A \in \mathbb{Z}\big[x_1^{\pm 1}, \ldots, x_p^{\pm 1}\big]$ is $|A| = \lim_{\text{all } x \to 1} A$, which can be negative.

Remark 2.3. Clearly we have $|A| = \#A + A_0$, as well as $\#A = \sum_{\vec{n} \neq 0} A_{\vec{n}}$.

Definition 2.4. We call $\chi \in \mathbb{Z}[[q_1, q_2, q_3, q_4]]$ a *pure character* if $\chi \in \mathbb{Z}_{\geq 0}[[q_1, q_2, q_3, q_4]]$.

For $A \subseteq \{1, 2, 3, 4\}$, let $P_A = \prod_{a \in A}(1 - q_a)$.
Let $\tilde{K}$ be a (possibly infinite) solid partition, with finite

$$\tilde{n}_{abc} = \lim_{q_a \to 1, q_b \to 1, q_c \to 1} P_{abc} \tilde{K}, \qquad a < b < c.$$

Let $\tilde{n} = (\tilde{n}_1, \ldots, \tilde{n}_4) := (\tilde{n}_{234}, \ldots, \tilde{n}_{123})$ and $q^{\tilde{n}} := q_1^{\tilde{n}_1} q_2^{\tilde{n}_2} q_3^{\tilde{n}_3} q_4^{\tilde{n}_4}$, where we identify the partition with character $\tilde{n}_a(q_a) = 1 + q_a + \cdots + q_a^{\tilde{n}_a - 1}$ with its size $\tilde{n}_a$.

Change variables to

$$K = q^{-\tilde{n}} \left(\tilde{K} - \frac{1 - q^{\tilde{n}}}{P_{1234}} \right) \tag{2.2}$$

so that $\lim_{q_a, q_b, q_c \to 1} P_{abc} K = 0$, $a < b < c$. Its asymptotics determine partitions

$$\pi_a(q_1, \ldots, \hat{q}_a, \ldots, q_4) = \lim_{q_a \to 1} P_a K(q_1, \ldots, q_4),$$

where hat means removal, and for $a \neq b$

$$\lambda_{ab}(q_1, \ldots, \hat{q}_a, \ldots, \hat{q}_b, \ldots, q_4) = \lim_{q_a, q_b \to 1} P_{ab} K(q_1, \ldots, q_4).$$

Denote $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_4)$.

Remark 2.5. Equivalently, if K is defined by a monomial ideal $\mathcal{I}$, then

$$\pi_a = \left\{ \big(k_1, \ldots, \hat{k}_a, \ldots, k_4\big) \in \mathbb{Z}^3_{>0} \mid \prod_{b=1}^{4} q_b^{k_b - 1} \notin \mathcal{I} \, \forall k_a \in \mathbb{Z}_{>0} \right\}.$$

In identifying the sets π_a and λ_{ab} with their characters, it is crucial to keep track of variables and their ordering correctly

$$\operatorname{ch} \pi_a(q_1, \ldots, \hat{q}_a, \ldots, q_4) = \sum_{(k_1, \ldots, \hat{k}_a, \ldots, k_4) \in \pi_a} \prod_{b \neq a} q_b^{k_b}.$$

Definition 2.6. Introduce regularized partitions

$$K_{\text{reg}} = K - \sum_a \frac{\pi_a}{P_a} + \sum_{a<b} \frac{\lambda_{ab}}{P_{ab}} \tag{2.3}$$

and similarly

$$\pi_{a,\text{reg}} = \pi_a - \sum_{b \neq a} \frac{\lambda_{ab}}{P_b}. \tag{2.4}$$

Remark 2.7. We have

$$K = K_{\mathrm{reg}} + \sum_a \frac{\pi_{a,\mathrm{reg}}}{P_a} + \sum_{a<b} \frac{\lambda_{ab}}{P_{ab}}.$$

Lemma 2.8. *The regularized objects are Laurent polynomials. For example,*

$$\lim_{q_i \to 1} (1-q_i) K_{\mathrm{reg}} = 0.$$

2.3 Plethystics

Definition 2.9. The operator $\hat{a}$ maps a movable Laurent polynomial $\mathscr{P} \in \mathbb{Z}\big[x_1^{\pm}, \dots, x_n^{\pm}\big]$ to a rational function $\hat{a}(\mathscr{P}) \in \mathbb{Z}\big(x_1^{1/2}, \dots, x_n^{1/2}\big)$ such that

- for any monomial $X = x_1^{c_1} \cdots x_n^{c_n}$ and $p \in \mathbb{Z}$, $\hat{a}(pX) = \big(X^{1/2} - X^{-1/2}\big)^{-p}$,
- $\hat{a}(pX_1 + qX_2) = \hat{a}(pX_1) \cdot \hat{a}(qX_2)$ on monomials X_1, X_2 and integers p, q.

Remark 2.10. Such $\hat{a}$ is the representative in localization of the A-roof genus. We will apply it to the ring $\mathbb{Z}\big[q_1^{\pm}, q_2^{\pm}, q_3^{\pm}, \tilde{\mu}^{\pm}\big]$. The definition involves a choice of square roots, which is crucial and we postpone until later, but is unambiguous when operating on perfect squares, namely $\hat{a}(x + x^*)$ is independent of such choice. Note the relation to the plethystic exponent, $\hat{a}\big(x^{-1}\big) = x^{1/2} \exp \sum_{n=1}^{\infty} \frac{x^n}{n}$.

2.4 Toric geometry

Denote the coordinates of $\mathbb{C}^{N+4}$ by Z_A, for $A = 1, \dots, N+4$, and let the index i run from 1 to N. Let $\mathsf{Q} = \big(\mathsf{Q}_i^A\big)$ be an N by $(N+4)$ matrix of integers. This defines a $\mathrm{U}(1)^N$ action on $\mathbb{C}^{N+4}$ (for real parameters α^i)

$$Z_A \mapsto \exp\left(\sqrt{-1} \sum_{i=1}^{N} \alpha^i \mathsf{Q}_i^A\right) Z_A, \qquad A = 1, \dots, N+4.$$

With $m_A = |Z_A|^2$, the momentum map $\mu \colon \mathbb{R}^{N+4} \to \mathbb{R}^N = \big(\mathrm{Lie}\,\mathrm{U}(1)^N\big)^\star$ is

$$\mu_i(m) := \sum_{A=1}^{N+4} \mathsf{Q}_i^A m_A, \qquad i = 1, \dots, N. \tag{2.5}$$

Let the N-tuple of real numbers r be a regular value of μ. The quotient $X_r = \mu^{-1}(r)/\mathrm{U}(1)^N$ is a complex four-manifold with $\dim H_2(X_r, \mathbb{Z}) = N$. It is a Calabi–Yau manifold in a weak sense[6] if

$$\sum_{A=1}^{N+4} \mathsf{Q}_i^A = 0 \qquad \text{for all} \quad i = 1, \dots, N.$$

In this case it is convenient to define a map $p \colon \mathbb{R}^{N+4} \to \mathbb{R}^{N+3} = \big(\mathrm{Lie}\,\mathrm{U}(1)^{N+3}\big)^\star$

$$p(m) = (m_1 - m_2, m_2 - m_3, \dots, m_{N+3} - m_{N+4}).$$

Then μ descends to a map $\mathbb{R}^{N+3} \to \mathbb{R}^N = \big(\mathrm{Lie}\,\mathrm{U}(1)^N\big)^\star$ and the Newton polyhedron is $\Delta(X_r) = \mu^{-1}(r) \cap \mathrm{im}(p \circ m)$. We call its zero-dimensional faces *vertices*, and write $\mathsf{v} \in \Delta_0$.

[6]Strictly speaking, a Calabi–Yau manifold X is a simply-connected compact Kähler manifold X with vanishing $c_1(X)$.

The vertices have valence four, namely each has four legs attached to it (some of which may be non-compact), and they correspond to setting to zero the maximal number of m_A's compatible with equation (2.5). Restricting to the compact skeleta of $\Delta(X_r)$ and regarding them as oriented, we call the one-dimensional faces *edges*, $\mathsf{e} \in \Delta_1$. Every edge $\mathbb{P}^1$ is incident to two vertices. We call two-dimensional faces *faces*, $\mathsf{f} \in \Delta_2$; and three-dimensional faces *cells*, $\mathsf{c} \in \Delta_3$. Denote by $|\Delta_0| = \chi(X_r)$ the number of vertices, and similarly by $|\Delta_i|$, for $i = 1, 2, 3$ the number of edges, faces and cells, respectively.

One associates complex line bundles $\mathcal{L}_i$ $(i = 1, \ldots, N+4)$ to the $\mathrm{U}(1)^N$ action and the cohomology of X is generated by $c_1(\mathcal{L}_i)$. The Kähler form is inherited from the ambient vector space $\omega = \frac{\mathrm{i}}{2\pi} \sum_A \mathrm{d}Z_A \wedge \mathrm{d}\bar{Z}_A$. The Hamiltonian is

$$H = \sum_{A=1}^{N+4} \varepsilon_A m_A, \tag{2.6}$$

where ε_A satisfy $\sum_A \varepsilon_A = 0$ and parametrize the $\mathrm{U}(1)^{N+3}$ action on $\mathbb{C}^{N+4}$ $Z_A \mapsto \mathrm{e}^{\mathrm{i}\varepsilon_A} Z_A$.

At each vertex, we choose four local coordinates z_a built out of $\mathrm{U}(1)^N$-invariant combinations of the Z_A's, such that their product $z_1 z_2 z_3 z_4$ is invariant under $\mathrm{U}(1)^{N+3}$. Two vertices connected by an edge along direction a are related by $z_a \to z_a^{-1}$, while $z_b \to z_b z_a^{d_b}$ for $b \neq a$, for some integers d_b, with $\sum_{b\neq a} d_b = 2$ to preserve the Calabi–Yau condition. The ordering of local coordinates at v is inherited from the orientation of Δ. The quotient $\mathrm{U}(1)^{N+3}/\mathrm{U}(1)^N$ acts on these four coordinates, giving local Ω-background parameters $q_a = \mathrm{e}^{\beta\epsilon_a}$, with $\sum_{a=1}^4 \epsilon_a = 0$. Sometimes we write q_a^{v} to emphasize the local dependence. Two vertices connected by an edge along direction a are related by $\epsilon_a \to -\epsilon_a$, while $\epsilon_b \to \epsilon_b + d_b \epsilon_a$ for $b \neq a$.

Denote by $C_i \in H_2(X, \mathbb{Z})$ the two-cycles of X with volume $\int_{C_i} \omega = r_i$. Dual to them are the (not necessarily compact) divisors D^i, such that $D^j \cdot C_i = \delta_i^j$. We can think of the D^i as generating the Kähler cone in $H^2(X, \mathbb{Z})$, which equals by Poincaré duality $H^6_{\mathrm{cmpct}}(X, \mathbb{Z})$. Denote by D^A the toric divisors $X \cap \{Z_A = 0\}$, which satisfy $\mathsf{Q}_i^A = C_i \cdot D^A$. The Calabi–Yau condition implies trivial canonical bundle, namely $-\sum_A D^A = 0$, which implies $\sum_A \mathsf{Q}_i^A = 0$ for all i. Alternatively, we have $c(TX) = \prod_A \big(1 + D^A\big)$. The compact divisors correspond to $H_6(X, \mathbb{Z})$, which is dual to $H^6(X, \mathbb{Z})$ and Poincaré dual to $H^2_{\mathrm{cmpct}}(X, \mathbb{Z})$.

Since there's a bijection between toric varieties up to isomorphism and fans up to $SL(4, \mathbb{Z})$ action, one can also work in the dual picture. (However, the polyhedron Δ includes more data than the dual fan.) The zero-force condition at each vertex allows to associate to it a tetrahedron. Such tetrahedra triangulate the dual toric diagram (they must have minimal volume $1/3!$ for X to be non-singular). The vectors v_i that generate the one-dimensional cones of the fan satisfy $\sum_A v_A \mathsf{Q}_i^A = 0$ (in fact, given the fan, one can recover Q as the kernel of the matrix of v_A's).

An edge e along direction e, by Duistermaat–Heckman theorem has volume $t_{\mathsf{e}} = \sum_{\mathsf{v}\in\mathsf{e}} \frac{H_{\mathsf{v}}}{\epsilon_e^{\mathsf{v}}}$, where H_{v} is the Hamiltonian (2.6) restricted at vertex v.

For a face f with tangent directions a, b and normal directions c, d, define

$$A_{pqr} := \sum_{\mathsf{v}\in\mathsf{f}} \frac{(\epsilon_c^{\mathsf{v}})^p (\epsilon_d^{\mathsf{v}})^q H_{\mathsf{v}}^r}{\epsilon_a^{\mathsf{v}} \epsilon_b^{\mathsf{v}}}, \tag{2.7}$$

where the non-negative integers p, q, r satisfy $p + q + r = 2$ and the sum runs over vertices in the face. Similarly, define

$$c_{2,\mathsf{f}} := \sum_{\mathsf{v}\in\mathsf{f}} \frac{c_2}{\epsilon_a \epsilon_b}, \tag{2.8}$$

where $c_2 = \sum_{1\leq a<b\leq 4} \epsilon_a \epsilon_b$.

Lemma 2.11. *Let* f *be a compact face. Then* A_{110} *is even.*

Proof. Let S be the toric surface corresponding to f. Since X is toric, we can always split $\mathcal{N}_{S/X} = \mathscr{L}_3 \oplus \mathscr{L}_4$, with $c_1(\mathscr{L}_3) + c_1(\mathscr{L}_4) + c_1(S) = 0$. The line bundles $\mathscr{L}_{3,4}$ realizing the splitting of the normal bundle are closely related to the combinatorial data of the polytope of X, which is the reason why the normal bundle is split in the first place.[7] Then we have $A_{110} = -\big(c_1^2(\mathscr{L}_3) + c_1(\mathscr{L}_3)c_1(S)\big)$. Taking $c_1(S)$ as the characteristic element for the intersection form, we have that $x^2 = c_1(S)x \bmod 2$ for any $x \in H^2(S,\mathbb{Z})$. ■

Lemma 2.12. *For any compact face, denoting tangent directions* 1 *and* 2*, we have* $\sum_{\mathsf{v}\in\mathsf{f}} \frac{1}{P_{12}^*} = 1$.

Proof. For any projective toric variety X, we have $H^m(X,\mathcal{O}_X) = 0$ if $m > 0$. The holomorphic Euler characteristic in the equivariant case is the alternating sum of the characters, which in the simply connected case becomes 1 (the character of the global holomorphic functions, which is one-dim space of constants). ■

Remark 2.13. The cohomological limit of Lemma 2.12 gives $\frac{\chi+\sigma}{4} = \frac{1+b_{2+}}{2} = 1$, which implies $b_{2+} = 1$ for projective toric surfaces.

Remark 2.14. In this notation, the signature of a compact face is

$$\sigma = \frac{1}{3}\sum_{\mathsf{v}\in\mathsf{f}} \frac{(\epsilon_a^{\mathsf{v}})^2 + (\epsilon_b^{\mathsf{v}})^2}{\epsilon_a^{\mathsf{v}}\epsilon_b^{\mathsf{v}}}.$$

3 Statement of the problem

Let us focus on the U(1) theory, i.e., we study a single $D8$-brane. The fixed points of the torus action on $\mathcal{M}_\zeta$ are labeled by collections of solid partitions, possibly of infinite size, $\{\tilde{K}^{\mathsf{v}},\, \mathsf{v}\in\Delta_0\}$, satisfying compatibility conditions. Compatibility means that different $\tilde{K}$'s can be glued together along edges. For example, suppose $\mathsf{e}\in\Delta_1$ joins $\mathsf{v}_1,\mathsf{v}_2\in\Delta_0$ along the first direction, with local equivariant parameters $\big(q_1^{\mathsf{v}_i}, q_2^{\mathsf{v}_i}, q_3^{\mathsf{v}_i}, q_4^{\mathsf{v}_i}\big)$ for $i = 1,2$ related in the standard way. Then compatibility requires that if

$$\lim_{q_1^{\mathsf{v}_1}\to 1} \big(1-q_1^{\mathsf{v}_1}\big)\tilde{K}^{\mathsf{v}_1}\big(q_1^{\mathsf{v}_1}, q_2^{\mathsf{v}_1}, q_3^{\mathsf{v}_1}, q_4^{\mathsf{v}_1}\big) = \pi^{(\mathsf{e})}\big(q_2^{\mathsf{v}_1}, q_3^{\mathsf{v}_1}, q_4^{\mathsf{v}_1}\big),$$

then

$$\lim_{q_1^{\mathsf{v}_2}\to 1} \big(1-q_1^{\mathsf{v}_2}\big)\tilde{K}^{\mathsf{v}_2}\big(q_1^{\mathsf{v}_2}, q_2^{\mathsf{v}_2}, q_3^{\mathsf{v}_2}, q_4^{\mathsf{v}_2}\big) = \pi^{(\mathsf{e})}\big(q_2^{\mathsf{v}_2}, q_3^{\mathsf{v}_2}, q_4^{\mathsf{v}_2}\big).$$

A similar equation holds for double limits and λ's along faces.

Definition 3.1. Let $\tilde{\nu}$ be the character of the framing space, i.e., the Coulomb branch parameter of our U(1) theory.

Remark 3.2. Without loss of generality, we can set $\tilde{\nu} = 1$ in the U(1) theory.

Definition 3.3. The basic object of the local theory is

$$\mathcal{H} := \frac{\tilde{\nu}-\tilde{\mu}}{P_{1234}} - \tilde{\nu}\tilde{K} = q^{\tilde{n}}\left(\frac{\tilde{\nu}-\mu}{P_{1234}} - \tilde{\nu}K\right),$$

[7] We thank an anonymous referee for pointing this out.

where $\tilde{\mu}$ is part of definition (1.4), $\mu = q^{-\tilde{n}}\tilde{\mu}$ and[8] we used (2.2). Define the virtual character

$$T^2[K] := -P_{1234}\mathcal{H}\mathcal{H}^* - T^2_{\text{pert}}(\mu), \tag{3.1}$$

where we subtracted the perturbative part

$$T^2_{\text{pert}}(\mu) := \frac{(\tilde{\nu}-\mu)(\tilde{\nu}-\mu)^*}{P^*_{1234}}. \tag{3.2}$$

This corresponds to an infinite-dimensional space, as it has poles. We can write the perturbative part as the sum of a finite contribution (depending on $\tilde{n}$)

$$T^2_6 := \sum_{\mathsf{v}\in\Delta_0} \left(T^2_{\text{pert}}(\mu) - T^2_{\text{pert}}(\tilde{\mu})\right) \tag{3.3}$$

and an infinite one (independent of $\tilde{n}$). Around $\mathsf{v} \in \Delta_0$, we associate to $\mathsf{f} \in \Delta_2$ along directions a, b the character

$$\mathcal{T}^2_{\mathsf{f}} := T^2\left[\frac{\lambda_{ab}}{P_{ab}}\right]. \tag{3.4}$$

To $\mathsf{e} \in \Delta_1$ along direction e, we associate $\mathcal{T}^2_{\mathsf{e}}$, defined by the formula

$$T^2\left[\frac{\pi_e}{P_e}\right] = \mathcal{T}^2_{\mathsf{e}} + \sum_{\mathsf{f}|\mathsf{e}\in\mathsf{f}} \mathcal{T}^2_{\mathsf{f}}. \tag{3.5}$$

Finally, we define T^2_{v} such that

$$T^2[K] = T^2_{\mathsf{v}} + \sum_{\mathsf{e}|\mathsf{v}\in\mathsf{e}} \mathcal{T}^2_{\mathsf{e}} + \sum_{\mathsf{f}|\mathsf{v}\in\mathsf{f}} \mathcal{T}^2_{\mathsf{f}}. \tag{3.6}$$

Remark 3.4. The perturbative part $T^2_{\text{pert}}(\mu)$ depends on $\tilde{n}$ through μ. However, $\tilde{n}$ does not couple to lower-dimensional partitions.

Remark 3.5. Mutatis mutandis, the same structure appears in complex dimensions $d = 2, 3, 4$

$$T[K] = -P_{1\ldots d}\mathcal{H}\mathcal{H}^* - T_{\text{pert}}$$

with contributions from codimension-one objects $\mathsf{c} \in \Delta_{d-1}$ (almost) tensored away as in (2.2). However, due to the reality of equation (1.1) (real representation of SU(4)), in $d = 4$ for the first time we have to take half of the equations.

Lemma 3.6. *A computation shows that*

$$\begin{aligned}\mathcal{T}^2_{\mathsf{e}} = {} & \left(1 - \mu - P_{1234}\sum_{a\neq e}\frac{\lambda_{ae}}{P_{ae}}\right)\frac{\pi^*_{e,\text{reg}}}{P^*_e} + Q\left(1 - \mu - P_{1234}\sum_{a\neq e}\frac{\lambda_{ae}}{P_{ae}}\right)^*\frac{\pi_{e,\text{reg}}}{P_e} \\ & - P_{1234}\frac{\pi_{e,\text{reg}}}{P_e}\frac{\pi^*_{e,\text{reg}}}{P^*_e} - P_{1234}\sum_{a\neq e,\, b\neq e,\, a\neq b}\frac{\lambda_{ae}}{P_{ae}}\frac{\lambda^*_{be}}{P^*_{be}},\end{aligned} \tag{3.7}$$

where $Q = \prod_{a=1}^4 q_a$, and, if we denote by $\sum'$ the sum over pairwise distinct a, b, c, d,

$$T^2_{\mathsf{v}} = \left(1 - \mu - P_{1234}\left(\sum_a \frac{\pi_{a,\text{reg}}}{P_a} + \sum_{a<b}\frac{\lambda_{ab}}{P_{ab}}\right)\right)K^*_{\text{reg}}$$

[8] We are using the same letter to denote the equivariant mass parameter $\mu^{(\mathsf{v})}$ for $\mathsf{v} \in \Delta_0$ and the vector of moment maps $\mu_i(\cdot)$, for $i = 1, \ldots, N$ in equation (2.5).

$$+ Q\left(1-\mu-P_{1234}\left(\sum_a \frac{\pi_{a,\mathrm{reg}}}{P_a}+\sum_{a<b}\frac{\lambda_{ab}}{P_{ab}}\right)\right)^* K_{\mathrm{reg}} - P_{1234}K_{\mathrm{reg}}K^*_{\mathrm{reg}}$$

$$- P_{1234}\left(\sum_{a\neq b}\frac{\pi_{a,\mathrm{reg}}}{P_a}\frac{\pi^*_{b,\mathrm{reg}}}{P^*_b}+\left(\sum_a \frac{\pi_{a,\mathrm{reg}}}{P_a}\sum_{c<d;c,d\neq a}\frac{\lambda^*_{cd}}{P^*_{cd}}+\mathrm{c.c.}\right)+{\sum_{a<b,c<d}}'\frac{\lambda_{ab}}{P_{ab}}\frac{\lambda^*_{cd}}{P^*_{cd}}\right). \quad (3.8)$$

Here c.c. *means the complex conjugate.*

Proof. The first part is obtained by plugging equations (3.4), (3.1), (3.2) into equation (3.5), and using equation (2.4) as well as the definition of Q in the lemma. Observe that $P^*_{1234}Q = P_{1234}$.

Substituting the first part into equation (3.6), and using equations (2.3) and (2.4), which imply that

$$K = K_{\mathrm{reg}} + \sum_a \frac{\pi_{a,\mathrm{reg}}}{P_a} + \sum_{a<b}\frac{\lambda_{ab}}{P_{ab}},$$

after a similar computation we get the second part. ∎

Globally, each T^2_{v} depends on local coordinates q^{v}_a around $\mathsf{v}\in\Delta_0$ (also possibly through $\mu^{(\mathsf{v})}$); similarly, each summand in $T^2_{\mathsf{e}} := \sum_{\mathsf{v}\in\mathsf{e}}\mathcal{T}^2_{\mathsf{e}}$ (two summands) and $T^2_{\mathsf{f}} := \sum_{\mathsf{v}\in\mathsf{f}}\mathcal{T}^2_{\mathsf{f}}$ (arbitrary number ≥ 3 of summands) depends on its local coordinates. The redistribution

$$\sum_{\mathsf{v}\in\Delta_0} T^2[K^{\mathsf{v}}] = \sum_{\mathsf{v}\in\Delta_0} T^2_{\mathsf{v}} + \sum_{\mathsf{v}\in\Delta_0}\sum_{\mathsf{e}|\mathsf{v}}\mathcal{T}^2_{\mathsf{e}} + \sum_{\mathsf{v}\in\Delta_0}\sum_{\mathsf{f}|\mathsf{v}}\mathcal{T}^2_{\mathsf{f}} = \sum_{\mathsf{v}\in\Delta_0} T^2_{\mathsf{v}} + \sum_{\mathsf{e}\in\Delta_1}\sum_{\mathsf{v}\in\mathsf{e}}\mathcal{T}^2_{\mathsf{e}} + \sum_{\mathsf{f}\in\Delta_2}\sum_{\mathsf{v}\in\mathsf{f}}\mathcal{T}^2_{\mathsf{f}}$$
$$= \sum_{\mathsf{v}\in\Delta_0} T^2_{\mathsf{v}} + \sum_{\mathsf{e}\in\Delta_1} T^2_{\mathsf{e}} + \sum_{\mathsf{f}\in\Delta_2} T^2_{\mathsf{f}}$$

is such that T^2_{v}, T^2_{e}, T^2_{f} (as well as T^2_6) are movable Laurent polynomials, as we discuss below. The equivariant K-theory class of the virtual tangent space to $\mathcal{M}_\zeta$ at a fixed point (including contributions from matter bundle E) is the virtual character

$$I^2 := T^2_6 + \sum_{\mathsf{v}\in\Delta_0} T^2_{\mathsf{v}} + \sum_{\mathsf{e}\in\Delta_1} T^2_{\mathsf{e}} + \sum_{\mathsf{f}\in\Delta_2} T^2_{\mathsf{f}}.$$

Our goal is to apply $\hat{a}$ to I^2, after taking a suitable square root. All these terms are squares in the sense that if they contain a monomial m, then they also contain m^*, and taking square roots means consistently picking only half of these terms. The fact that I^2 is a square has a geometric interpretation in the deformation-obstruction complex of X being self-dual.

Remark 3.7. Consider the pullback $\mathcal{E}_{\mathsf{v}} = \iota^*_{\mathsf{v}}\mathcal{E}$ of the universal sheaf via the inclusion $\iota_{\mathsf{v}}\colon \mathcal{M}\times\{\mathsf{v}\}\to\mathcal{M}\times X$ of a fixed point $\mathsf{v}\in\Delta_0$. With $N = q^{\tilde{n}}\tilde{\nu}$, we have $\mathrm{ch}\,\mathcal{E}_{\mathsf{v}} = N - P_{1234}K$. Then we get

$$\int_X [\mathrm{ch}(\mathcal{E}\otimes\mathcal{E}^*) - \mathrm{ch}(\mathcal{E}\otimes M^*) - \mathrm{ch}(\mathcal{E}^*\otimes M)]\,\mathrm{td}_X$$
$$= \sum_{\mathsf{v}\in\Delta_0}\frac{\mathrm{ch}\,\mathcal{E}_{\mathsf{v}}\otimes\mathcal{E}^*_{\mathsf{v}} - \mathrm{ch}\,\mathcal{E}_{\mathsf{v}}\otimes M^* - \mathrm{ch}\,\mathcal{E}^*_{\mathsf{v}}\otimes M}{P^*_{1234}} = \sum_{\mathsf{v}\in\Delta_0} P_{1234}\mathcal{H}\mathcal{H}^*$$

the last equality up to an irrelevant (non-dynamical) factor MM^*. The gauge theoretic representative of $\mathrm{ch}\,\mathcal{E}_{\mathsf{v}}$ also gives rise to the infinite factors NN^*/P_{1234}, the perturbative part, which we subtract accordingly. The result equals $\sum_{\mathsf{v}\in\Delta_0} T^2[K^{\mathsf{v}}]$. Finally, the map $\hat{a}$ converts this from the equivariant K-theory of the fixed locus in $\mathcal{M}$ to its localization.

4 Computation of fugacities

Each vertex $\mathsf{v} \in \Delta_0$ sits at the intersection of four toric divisors, some of which can be non-compact. Compact toric divisors of X correspond to cells, and to each cell $\mathsf{c} \in \Delta_3$ we associate an integer $\tilde{n}_{\mathsf{c}}$. Let $S_{\mathsf{v}} \subset \{1,2,3,4\}$ be the set of directions that are normal to compact divisors at v, and define a map $i_{\mathsf{v}}\colon S_{\mathsf{v}} \to \Delta_3$ that assigns to each normal direction its cell, so that we can write $\epsilon \cdot \tilde{n} = \sum_{a\in S_{\mathsf{v}}} \epsilon_a^{(\mathsf{v})} \tilde{n}_{i_{\mathsf{v}}(a)}$. Equivalently, we have a line bundle L with equivariant first Chern class $c_1(L) = \sum_{\mathsf{c}\in\Delta_3} \tilde{n}_{\mathsf{c}} c_1(D_{\mathsf{c}})$, where divisor D_{c} corresponds to cell c.

As in Remark 3.7, consider the pullback $\mathcal{E}_{\mathsf{v}} = \iota_{\mathsf{v}}^* \mathcal{E}$ of the universal sheaf via the inclusion $\iota_{\mathsf{v}}\colon \mathcal{M} \times \{\mathsf{v}\} \to \mathcal{M} \times X$ of a fixed point $\mathsf{v} \in \Delta_0$. With $\tilde{\nu} = \epsilon^{\beta\tilde{\alpha}}$ and $N = q^{\tilde{n}}\tilde{\nu}$, its character

$$\operatorname{ch} \mathcal{E}_{\mathsf{v}} = N - P_{1234}K = q^{\tilde{n}}\tilde{\nu} - q^{\tilde{n}}\tilde{\nu}P_{1234}\left(K_{\text{reg}} + \sum_a \frac{\pi_{a,reg}}{P_a} + \sum_{a<b}\frac{\lambda_{ab}}{P_{ab}}\right)$$

can be conveniently written as the tensor product $\operatorname{ch}\mathcal{E}_{\mathsf{v}} = \operatorname{ch}\big(L \otimes \mathcal{E}_{\mathsf{v}}^{(0)}\big)$, where $\mathcal{E}_{\mathsf{v}}^{(0)}$ is simply $\mathcal{E}_{\mathsf{v}}$ with all the fluxes $\tilde{n}$ set to zero, and $\operatorname{ch} L = q^{\tilde{n}} = \mathrm{e}^{\beta\tilde{n}\cdot\epsilon}$.

The equivariant gamma-class of a toric CY 4-fold at a fixed point $\mathsf{v} \in \Delta_0$

$$\hat{\Gamma}_{\mathsf{v}} = \prod_{a=1}^{4} \Gamma\left(1 + \frac{\beta}{2\pi}\epsilon_a\right) = 1 - \frac{1}{24}c_2\beta^2 - \frac{\zeta(3)}{(2\pi)^3}c_3\beta^3 + \frac{\zeta(4)}{(2\pi)^4}\left(\frac{7}{4}c_2^2 - c_4\right)\beta^4 + O\big(\beta^5\big)$$

can be written in terms of elementary symmetric polynomials c_i in variables $\epsilon_1, \ldots, \epsilon_4$.

We want to use equivariant localization to compute equation (1.2) in terms of

$$Z_{2n} := \int_X \frac{(-\omega)^n}{n!}\big(\operatorname{ch}(\mathcal{E})\hat{\Gamma}_X\big)_{4-n} = \sum_{\mathsf{v}\in\Delta_0} \frac{H_{\mathsf{v}}^n}{n!} \frac{1}{\prod_{a=1}^4 \epsilon_a^{\mathsf{v}}}\big(\operatorname{ch}\mathcal{E}_{\mathsf{v}} \cdot \hat{\Gamma}_{\mathsf{v}}\big)_{4-n}, \qquad 0 \le n \le 4, \tag{4.1}$$

where the subscript $4-n$ denotes the power of β, and H_{v} is the Hamiltonian evaluated at v. Recall that Z_0 appears as $(-p)^{-Z_0}$ in the fugacity u.

Definition 4.1. For any edge $\mathsf{e} \in \Delta_1$, we have transition functions d_i defined in Section 2.4 obeying the relation $\sum_i d_i = 2$. Given a (regularized) plane partition π, its box is denoted as $\square = (i-1, j-1, k-1)$, and we introduce the sign $s_\pi(\square) = \pm 1$, depending on whether the box is added or removed (just like the size of a regularized partition can be negative). Let

$$f_{\mathsf{e}}(\pi) := \sum_{\square\in\pi} s_\pi(\square)[1 - id_i - jd_j - kd_k].$$

Given a tuple of integers $(\tilde{n}_{\mathsf{c}})_{\mathsf{c}\in\Delta_3}$, let[9] $f_{\mathsf{e}}(\tilde{n}) := \sum_{\mathsf{v}\in\mathsf{e}} \frac{\epsilon\cdot\tilde{n}}{\epsilon_{\mathsf{e}}}$, where the sum runs over the two vertices in the edge, and ϵ_{e} denotes tangent direction.

For any face $\mathsf{f} \in \Delta_2$, define functions A_{pqr} as in equation (2.7). Given a Young diagram λ, denote its box by $\square = (i-1, j-1)$, and let

$$g_{\mathsf{f}}(\lambda) := \sum_{\square\in\lambda}\left[\left(\frac{i(i-1)}{2} + \frac{1}{6}\right)A_{200} + \left(\frac{j(j-1)}{2} + \frac{1}{6}\right)A_{020} + \left(ij - \frac{i+j}{2} + \frac{1}{4}\right)A_{110}\right].$$

Similarly, let

$$\tilde{g}_{\mathsf{f}}(\lambda) := \sum_{\square\in\lambda}\left[(i - 1/2)A_{101} + (j - 1/2)A_{011}\right].$$

[9] Here and below, we are denoting by the same letter different functions, which are distinguished only by the argument they take.

Given a tuple of integers $(\tilde{n}_{\mathsf{c}})_{\mathsf{c}\in\Delta_3}$, let

$$g_{\mathsf{f}}(\lambda,\tilde{n}) := \sum_{\mathsf{v}\in\mathsf{f}} \frac{(i-1/2)\epsilon_c + (j-1/2)\epsilon_d}{\epsilon_a\epsilon_b}(\epsilon\cdot\tilde{n}),$$

where a, b denote directions tangent to f, c, d denote directions normal to f, and the sum runs over vertices in f. Similarly, let

$$\tilde{g}_{\mathsf{f}}(\tilde{n}) := \sum_{\mathsf{v}\in\mathsf{f}} \frac{(\epsilon\cdot\tilde{n})H_{\mathsf{v}}}{\epsilon_a\epsilon_b} \qquad \text{as well as} \qquad g_{\mathsf{f}}(\tilde{n}) := \sum_{\mathsf{v}\in\mathsf{f}} \frac{(\epsilon\cdot\tilde{n})^2}{\epsilon_a\epsilon_b}.$$

Finally, define global functions

$$c_{p,q,r}(\tilde{n}) := \sum_{\mathsf{v}\in\Delta_0} \frac{c_p(\tilde{n}\cdot\epsilon)^q H_{\mathsf{v}}^r}{\prod_{a=1}^4 \epsilon_a}, \qquad (p,q,r) = (2,1,1),(2,2,0),(3,1,0)$$

as well as

$$h_i(\tilde{n}) := \sum_{\mathsf{v}\in\Delta_0} \frac{H_{\mathsf{v}}^{4-i}}{(4-i)!}\frac{(\tilde{n}\cdot\epsilon)^i}{i!}\frac{1}{\prod_{a=1}^4 \epsilon_a}, \qquad i = 1,2,3,4.$$

Lemma 4.2. *Terms involving $\tilde{\alpha}$ do not contribute to the sum over fixed points in equation (4.1). Up to terms independent of $\mathcal{E}$, which contribute overall factors, the $D0$-brane charge reads*

$$\begin{aligned} Z_0 = &-\sum_{\mathsf{v}\in\Delta_0}|K_{\mathrm{reg}}^{\mathsf{v}}| + \sum_{\mathsf{e}\in\Delta_1} f_{\mathsf{e}}(\pi_{\mathsf{e},\mathrm{reg}}) + \sum_{\mathsf{e}\in\Delta_1}|\pi_{\mathsf{e},\mathrm{reg}}| f_{\mathsf{e}}(\tilde{n}) - \sum_{\mathsf{f}\in\Delta_2} g_{\mathsf{f}}(\lambda_{\mathsf{f}}) + \frac{1}{24}\sum_{\mathsf{f}\in\Delta_2}|\lambda_{\mathsf{f}}| c_{2,\mathsf{f}} \\ &-\sum_{\mathsf{f}\in\Delta_2} g_{\mathsf{f}}(\lambda_{\mathsf{f}},\tilde{n}) - \frac{1}{2}\sum_{\mathsf{f}\in\Delta_2}|\lambda_{\mathsf{f}}| g_{\mathsf{f}}(\tilde{n}) + h_4(\tilde{n}) - \frac{1}{2}\frac{1}{24}c_{2,2,0}(\tilde{n}) - \frac{\zeta(3)}{(2\pi)^3}c_{3,1,0}(\tilde{n}) \end{aligned}$$

with $c_{2\mathsf{f}}$ defined as in equation (2.8). The $D2$-brane charge reads

$$Z_2 = \sum_{\mathsf{e}\in\Delta_1} t_{\mathsf{e}}|\pi_{\mathsf{e},\mathrm{reg}}| - \sum_{\mathsf{f}\in\Delta_2}\tilde{g}_{\mathsf{f}}(\lambda_{\mathsf{f}}) - \sum_{\mathsf{f}\in\Delta_2}|\lambda_{\mathsf{f}}|\tilde{g}_{\mathsf{f}}(\tilde{n}) + h_3(\tilde{n}) - \frac{1}{24}c_{2,1,1}(\tilde{n}).$$

The $D4$-brane charge reads

$$Z_4 = -\frac{1}{2}\sum_{\mathsf{f}\in\Delta_2}|\lambda_{\mathsf{f}}| A_{002} + h_2(\tilde{n}).$$

The $D6$-brane charge reads $Z_6 = h_1(\tilde{n})$.

Proof. Let us drop the suffix v for brevity, and expand $\hat{\Gamma} = 1 + \sum_{i\geq 2}\Gamma_i\beta^i$ and $\operatorname{ch} L = 1 + \sum_{i\geq 1} L_i\beta^i$, where $L_i = \frac{1}{i!}(\tilde{n}\cdot\epsilon)^i$. Let us expand

$$\operatorname{ch}\mathcal{E}^{(0)} = 1 + \sum_{i\geq 1}\mathcal{E}_i^{(0)}\beta^i$$

and compute some terms

$$\begin{aligned} \mathcal{E}_4^{(0)} = &\frac{\tilde{\alpha}^4}{4!} - |K_{\mathrm{reg}}|\prod_{a=1}^4\epsilon_a + \sum_a\sum_{\square\in\pi_{a,\mathrm{reg}}}\left(\square - \frac{1}{2}\epsilon_a + \tilde{\alpha}\right)\prod_{b\neq a}\epsilon_b \\ &-\frac{1}{2}\sum_{a<b}\sum_{\square\in\lambda_{ab}}\left[(\square+\tilde{\alpha})^2 + (\square+\tilde{\alpha})\sum_{c\neq a,b}\epsilon_c + \frac{1}{3}\sum_{c\neq a,b}\epsilon_c^2 + \frac{1}{2}\prod_{c\neq a,b}\epsilon_c\right]\prod_{c\neq a,b}\epsilon_c, \end{aligned}$$

$$\mathcal{E}_3^{(0)} = \frac{\tilde{\alpha}^3}{3!} + \sum_a |\pi_{a,\mathrm{reg}}| \prod_{b\neq a} \epsilon_b - \sum_{a<b} \sum_{\square\in\lambda_{ab}} \left(\square + \tilde{\alpha} + \frac{1}{2}\sum_{c\neq a,b} \epsilon_c\right) \prod_{c\neq a,b} \epsilon_c,$$

$$\mathcal{E}_2^{(0)} = \frac{\tilde{\alpha}^2}{2} - \sum_{a<b} |\lambda_{ab}| \prod_{c\neq a,b} \epsilon_c, \qquad \mathcal{E}_1^{(0)} = \tilde{\alpha}.$$

The relevant product is then

$$\begin{aligned} \operatorname{ch}\mathcal{E}\cdot\hat{\Gamma} = {} & \operatorname{ch}L\cdot\operatorname{ch}\mathcal{E}^{(0)}\cdot\hat{\Gamma} = 1 + \beta\big(L_1 + \mathcal{E}_1^{(0)}\big) + \beta^2\big(\mathcal{E}_2^{(0)} + \Gamma_2 + L_2 + \mathcal{E}_1^{(0)}L_1\big) \\ & + \beta^3\big(\mathcal{E}_3^{(0)} + \Gamma_3 + L_3 + \mathcal{E}_2^{(0)}L_1 + \mathcal{E}_1^{(0)}L_2 + \mathcal{E}_1^{(0)}\Gamma_2 + \Gamma_2 L_1\big) \\ & + \beta^4\big(\mathcal{E}_4^{(0)} + L_4 + \Gamma_4 + \mathcal{E}_3^{(0)}L_1 + \mathcal{E}_2^{(0)}\Gamma_2 + \mathcal{E}_2^{(0)}L_2 + \mathcal{E}_1^{(0)}\Gamma_3 \\ & + \mathcal{E}_1^{(0)}L_3 + L_2\Gamma_2 + L_1\Gamma_3\big). \end{aligned}$$

Summing over fixed points we get the result. ■

5 Vertex

In this section, we introduce the K-theoretic four-vertex. This is where most of the complexity of DT counts lies. First, we introduce some machinery to deal with partitions of infinite size, then take a candidate square-root of the vertex, and fix its sign using residues.

5.1 Euler characteristic

Definition 5.1. Given a quadruple of plane partitions $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4)$, possibly of infinite size, introduce the sets

$$\begin{aligned} \Sigma_1 &= \big\{(\boldsymbol{n}; a) \mid \boldsymbol{n} \in \mathbb{Z}^4_{>},\ 1 \le a \le 4,\ \boldsymbol{n}_a \in \pi_a\big\}, \\ \Sigma_2 &= \big\{(\boldsymbol{n}; a, b) \mid \boldsymbol{n} \in \mathbb{Z}^4_{>},\ 1 \le a < b \le 4,\ \boldsymbol{n}_a \in \pi_a,\ \boldsymbol{n}_b \in \pi_b\big\}, \\ \Sigma_3 &= \big\{(\boldsymbol{n}; a, b, c) \mid \boldsymbol{n} \in \mathbb{Z}^4_{>},\ 1 \le a < b < c \le 4,\ \boldsymbol{n}_a \in \pi_a,\ \boldsymbol{n}_b \in \pi_b,\ \boldsymbol{n}_c \in \pi_c\big\}, \\ \Sigma_4 &= \big\{\boldsymbol{n} \mid \boldsymbol{n} \in \mathbb{Z}^4_{>},\ \boldsymbol{n}_a \in \pi_a\ \forall a = 1, \dots, 4\big\}, \end{aligned}$$

where $\boldsymbol{n}_a$ for $1 \le a \le 4$ means $\boldsymbol{n}$ with a-th entry dropped. As the Σ_i's were defined as sets, we define their characters

$$\operatorname{ch}\Sigma_1 = \sum_{(\boldsymbol{n};a)\in\Sigma_1} \prod_{i=1}^4 q_i^{(\boldsymbol{n})_i - 1}, \qquad \operatorname{ch}\Sigma_2 = \sum_{(\boldsymbol{n};a,b)\in\Sigma_2} \prod_{i=1}^4 q_i^{(\boldsymbol{n})_i - 1},$$

$$\operatorname{ch}\Sigma_3 = \sum_{(\boldsymbol{n};a,b,c)\in\Sigma_3} \prod_{i=1}^4 q_i^{(\boldsymbol{n})_i - 1}, \qquad \operatorname{ch}\Sigma_4 = \sum_{\boldsymbol{n}\in\Sigma_4} \prod_{i=1}^4 q_i^{(\boldsymbol{n})_i - 1},$$

where $(\boldsymbol{n})_i$ denotes the i-th component of $\boldsymbol{n}$. We introduce the set Σ, defined through its character

$$\operatorname{ch}\Sigma := \operatorname{ch}\Sigma_1 - \operatorname{ch}\Sigma_2 + \operatorname{ch}\Sigma_3 - \operatorname{ch}\Sigma_4. \tag{5.1}$$

Given a solid partition K with asymptotics $\boldsymbol{\pi}$, define its 'Euler characteristic'

$$\chi_K := \operatorname{ch}K - \operatorname{ch}\Sigma. \tag{5.2}$$

Remark 5.2. The character Σ is a pure character and the character of some infinite partition. Moreover, $P_{1234}\Sigma$ is a Laurent polynomial.

Remark 5.3. The sets Σ_1, Σ_2, Σ_3, Σ_4 do not depend on K. We have $\operatorname{ch}\Sigma_1 = \sum_a \frac{\pi_a}{P_a}$. One can rewrite

$$\Sigma_2 = \{(\boldsymbol{n}; a, b) \mid (\boldsymbol{n}, a) \in \Sigma_1,\ (\boldsymbol{n}, b) \in \Sigma_1,\ a < b\}.$$

Lemma 5.4. *The character χ_K is a pure character and a Laurent polynomial.*

Proof. Suppose $\lim_{q_i\to 1}(1-q_i)\chi_K = \rho_i$. Then χ_K must contain the whole series $S = \frac{\rho_i}{1-q_i}$ and since $\chi_K \subseteq K$ then $S \subseteq K$. But then it must be $\rho_i \subseteq \pi_i$, therefore $S \subseteq \Sigma_1$. This is in contradiction with the fact that if $\boldsymbol{n} \in \chi_K$ then $\boldsymbol{n} \notin \Sigma_1$. ■

Definition 5.5. Given a quadruple of plane partitions $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4)$, possibly of infinite size, recall the identities

$$\lambda_{ab} := \lim_{q_a\to 1}(1-q_a)\pi_b = \lim_{q_b\to 1}(1-q_b)\pi_a, \qquad 1 \le a < b \le 4,$$

and introduce the set

$$\Sigma_2' = \{(\boldsymbol{n}; a, b) \mid \boldsymbol{n} \in \mathbb{Z}^4_>,\ 1 \le a < b \le 4,\ \boldsymbol{n}_{ab} \in \lambda_{ab}\},$$

where $\boldsymbol{n}_{ab}$ for $1 \le a < b \le 4$ means $\boldsymbol{n}$ with the a-th and b-th entries dropped. Its character is

$$\operatorname{ch}\Sigma_2' = \sum_{(\boldsymbol{n};a,b)\in\Sigma_2'} \prod_{i=1}^4 q_i^{(\boldsymbol{n})_i - 1}.$$

Remark 5.6. The set $\Sigma_2' \subseteq \Sigma_2$ does not depend on K. We have $\operatorname{ch}\Sigma_2' = \sum_{a<b} \frac{\lambda_{ab}}{P_{ab}}$.

Definition 5.7. Define the difference of Laurent polynomials

$$\ell := K_{\mathrm{reg}} - \chi_K = \operatorname{ch}\Sigma - \operatorname{ch}\Sigma_1 + \operatorname{ch}\Sigma_2'. \tag{5.3}$$

Remark 5.8. The Laurent polynomial ℓ only depends on $\boldsymbol{\pi}$ (not on K).

Remark 5.9. A similar construction of the pair (χ, Σ) can be performed in any dimension d. For Young diagrams ($d = 2$), χ is still a Young diagram, while this is not the case in general.

5.2 Square roots

Within this section, we fix $\mathsf{v} \in \Delta_0$.

Definition 5.10. Let us define the virtual character

$$T_{\mathrm{cross}} := -\sum_{a\neq b} P_{d(a,b)} q_b \pi_{a,\mathrm{reg}} \pi^*_{b,\mathrm{reg}} - P_{1234} \sum_a \frac{\pi_{a,\mathrm{reg}}}{P_a} \sum_{\substack{c<d\\ c,d\neq a}} \frac{\lambda^*_{cd}}{P^*_{cd}} - \sum_{\substack{(abcd)=(1234),\\ (1324),(1423)}} \lambda_{ab}\lambda^*_{cd} q_c q_d,$$

where $d(a,b)$ is the smallest integer in $\{1,2,3,4\}$ different from a and b. This T_{cross} contains constant terms (all manifestly movable) from the viewpoint of the vertex.[10]

Remark 5.11. This choice depends on the ordering of the set of edges emanating from a vertex. When gluing vertices, we will need to check the independence of the final result.

[10] We call a virtual character *constant* from the viewpoint of the vertex if it is the same for any two solid partitions K_{v} and K'_{v} that have the same asymptotics $\boldsymbol{\pi}$.

Lemma 5.12. *After imposing $Q = 1$, the virtual character*

$$T_{\mathsf{v}} = \left(1 - \mu - P_{1234}\left(\sum_a \frac{\pi_{a,\mathrm{reg}}}{P_a} + \sum_{a<b} \frac{\lambda_{ab}}{P_{ab}}\right)\right)^* K_{\mathrm{reg}} - P_{123} K_{\mathrm{reg}} K_{\mathrm{reg}}^* + T_{\mathrm{cross}}$$

is a square root of equation (3.8). Both T_{v} and T_{cross} are manifestly Laurent polynomials.

Proof. Use the identities $P_{abc} + P_{abc}^* = P_{1234}$ and $q_d P_d^* = -P_d$, as well as $q_a q_c = (q_d q_b)^*$ for any permutation (a, b, c, d) of $(1, 2, 3, 4)$. ∎

Remark 5.13. The condition $Q = 1$, once imposed at one vertex, is satisfied at every vertex.

Definition 5.14. For a given K, with $\chi := \chi_K$ and Σ defined in equations (5.2) and (5.1), respectively, let us define the non-constant part

$$T := (1 - \mu)^* \chi - P_{123} \chi \chi^* - P_{1234} \Sigma^* \chi.$$

With ℓ as in equation (5.3), let us define the constant part

$$T_{\mathrm{const}} := (1 - \mu - P_{1234}(\Sigma - \ell))^* \ell - P_{123} \ell \ell^*.$$

Lemma 5.15. *Up to conjugation, the Laurent polynomial T_{v} decomposes as $T_{\mathsf{v}} = T + T_{\mathrm{cross}} + T_{\mathrm{const}}$.*

Proof. Recall that $K - K_{\mathrm{reg}} = \sum_a \frac{\pi_{a,\mathrm{reg}}}{P_a} + \sum_{a<b} \frac{\lambda_{ab}}{P_{ab}}$. Plug in the definitions, use the fact that ℓ is a Laurent polynomial, and conjugate the finite term $P_{123} \ell \chi^*$. ∎

Lemma 5.16. *The Laurent polynomial T is movable.*

Proof. Recall that Σ is a pure character and the character of some (infinite) solid partition. We use induction on the size of χ. For $\chi = 0$, $T = 0$ is movable. Denote the new box by ξ. We work up to movable terms, i.e., $(\cdot)_0$ is understood everywhere in the proof. We have

$$\delta T := T[\chi + \xi] - T[\chi] = \xi + 1 - P_{123}(\chi + \xi)^* \xi - P_{123}(\chi + \xi)\xi^* - P_{1234} \Sigma^* \xi.$$

Let $\Pi = \Pi(\xi)$ be the parallelepiped generated by ξ, and $\Pi_1 = (\chi + \xi) \cap \Pi$. Then $\chi + \xi = \Pi_1 + (\chi + \xi) \setminus \Pi_1$. By the same arguments as the finite case [43, Section 2.4.1], $P_{123}((\chi + \xi) \setminus \Pi_1)^* \xi = 0$ and $P_{123}((\chi + \xi) \setminus \Pi_1)\xi^* = 0$. Then

$$\delta T = 1 + \xi - P_{123} \Pi_1^* \xi - P_{123} \Pi_1 \xi^* - P_{1234} \Sigma^* \xi.$$

Now we claim that $P_{1234}(\Sigma \setminus \Pi_2)^* \xi = 0$, where $\Pi_2 = \Sigma \cap \Pi$. Then

$$\delta T = 1 + \xi - P_{1234}(\Pi_1 + \Pi_2)^* \xi$$

but $\Pi_1 + \Pi_2 = \Pi$ and $P_{1234} \Pi^* \xi = 1 + \xi$ is literally the finite case result. ∎

Definition 5.17. Let $k = |\chi|$. By replacing $\chi \to \sum_{i=1}^k x_i$ in T, let

$$T_{\mathrm{formal}} = (1 - \mu - P_{1234}\Sigma)^* \sum_i x_i - P_{123} \sum_{i \neq j} x_i x_j^{-1} - k P_{123},$$

where the dependence on $\boldsymbol{\pi}$ is encoded in Σ. We define the K-theoretic 4-vertex

$$\mathsf{V}_{\mathsf{v}}(\boldsymbol{\pi}) := \hat{a}(T_{\mathrm{cross}}) \hat{a}(T_{\mathrm{const}}) \sum_{k=0}^{\infty} (-p)^{|K_{\mathrm{reg}}|} \oint \hat{a}(T_{\mathrm{formal}} + k) \prod_{i=1}^{k} \frac{\mathrm{d}x_i}{x_i},$$

where the subscript v keeps track of local coordinates and the integral is defined using Jeffrey–Kirwan prescription. In computing iterated residues, we order $\chi = \chi_1 + \cdots + \chi_k$ as in [43]. The fugacity $(-p)^{|K_{\mathrm{reg}}|}$ comes from equation (1.3), and it is equal to $(-p)^k$, up to a constant factor. Here we neglect an overall sign, relevant only for the global picture.

Theorem 5.18. *For a given k, the admissible poles are in one-to-one correspondence with the Euler characteristics χ of solid partitions K with asymptotics $\boldsymbol{\pi}$, such that $|\chi| = k$.*

Proof. The relevant measure (neglecting μ terms) is

$$m_k = \prod_{i=1}^{k} \hat{a}(-P_{1234}\Sigma^* x_i) \prod_{i=1}^{k} \hat{a}(x_i) \prod_{k\geq i>j\geq 1} \hat{a}(-P_{1234}x_i/x_j). \tag{5.4}$$

Notice that, except for the Σ term, this is the same as the finite case [43, Section 2.3]. Let us assume by induction that the statement is true for $k-1$. The tree argument from finite case still works, so the new pole u_k can only grow by $e_k - e_i$, with $i < k$, from some u_i belonging to some admissible χ_{k-1}. (Here, as in [43], $e_k - e_i$ is a positive root of $\mathfrak{sl}(k)$ corresponding to the last factor in equation (5.4).) Let us work with Q unconstrained first. There's one new case, which is when the new box borders Σ. In this case there are two poles (one from $u_i \in \chi_{k-1}$ and one from some $\sigma \in \Sigma$) and one zero (from $q_a^{-1}\sigma$ for some $a \in \{1,2,3,4\}$), therefore we still get a simple pole and the case is admissible. This means that χ_k is a partition in the background of Σ, which is what we wanted to prove. ∎

Theorem 5.19. *With the ordering for χ as in Definition 5.17, let us define the sign*

$$s_{\mathsf{v}}(\chi) := \Bigg(P_{123}\sum_{i<j}\chi_i\chi_j^*\Bigg)_0,$$

where the subscript 0 denotes unmovable part. Define the measure

$$M_{\mathsf{v}}(K) := (-1)^{s_{\mathsf{v}}(\chi)}\hat{a}(T + T_{\text{cross}} + T_{\text{const}}).$$

Then the following holds:

$$\mathsf{V}_{\mathsf{v}}(\boldsymbol{\pi}) = \sum_{K \text{ ending on } \boldsymbol{\pi}} (-p)^{|K_{\text{reg}}|} M_{\mathsf{v}}(K).$$

Proof. We can prove the relation at each pole, by analyzing the terms contributing a sign in the residue, as in the finite case [43, Section 2.4.2]. The only term in the measure involving x^{-1} is $-P_{123}x_i/x_j$; by applying induction on the size of χ, we get the thesis. ∎

5.3 Examples

5.3.1 Vertex with one leg

Consider the vertex with one non-trivial leg π_1. Let K be any solid partition with asymptotics $\pi_1 = \pi_1(q_2, q_3, q_4)$ along direction q_1, and trivial asymptotics otherwise. Let $\chi = K - \frac{\pi_1}{P_1}$. Then we have

$$T = \left(1 - \mu - P_{1234}\frac{\pi_1}{P_1}\right)^* \chi - P_{123}\chi\chi^*.$$

This case can be checked against K-theoretic quasimap counts, and this is done in a companion paper [47]. Here we only write a formula for the simplest example.

Conjecture 5.20. *The one-leg vertex with one-box asymptotics ($\pi_1 = 1$) takes the form*

$$\frac{\mathsf{V}_{\mathsf{v}}(\square, \varnothing, \varnothing, \varnothing)}{\mathsf{V}_{\mathsf{v}}(\varnothing, \varnothing, \varnothing, \varnothing)} = \sum_{n\geq 0} p^n \mu^{-n/2} \prod_{k=1}^{n} \frac{1 - \mu q_1^k}{1 - q_1^k}.$$

See also [32, *Lemma* 6.5.1].

6 Theory on the edge

Let Σ be a two-dimensional Riemann surface, with metric g_Σ and local complex coordinates z, $\bar{z}$, in which the metric g_Σ is Hermitian, and, therefore, Kähler. We also fix a triplet of line bundles L_a, such that the tensor product of all three bundles equals the canonical bundle

$$L_1 \otimes L_2 \otimes L_3 \approx K_\Sigma. \tag{6.1}$$

We endow each of these bundles with the Hermitian connections $\varpi_a \mathrm{d}z + \bar{\varpi}_a \mathrm{d}\bar{z}$, compatible with (6.1) and the metric on Σ. In a local trivialization, a section s_a of L_a has the norm $\rho_a(z,\bar{z})|s_a|^2$ so that $\rho_1\rho_2\rho_3 = g^{z\bar{z}}$, and $\varpi_a = \frac{1}{2}\rho_a^{-1}\partial\rho_a$, $\bar{\varpi}_a = \frac{1}{2}\rho_a^{-1}\bar{\partial}\rho_a$.

We are going to define a two-dimensional cohomological field theory, whose fields are the $\mathrm{U}(k)$-gauge field $A_z \mathrm{d}z + A_{\bar{z}}\mathrm{d}\bar{z}$, a triplet of L_a-twisted complex adjoint scalars B_a and their $L_a^{-1} = \bar{L}_a$-twisted conjugates $B_a^\dagger$, and a scalar field I valued in the fundamental representation of $\mathrm{U}(k)$ (i.e., a section of a rank k complex vector bundle associated with the principal $\mathrm{U}(k)$ bundle in which A is a connection).

These fields are constrained by the equations, which are the two-dimensional generalization of the equations describing the Hilbert scheme of points on $\mathbb{C}^3$ [36, 38, 39, 41]

$$\begin{aligned} &D_{\bar{z}}B_a + \varepsilon_{abc}\big[B_b^\dagger, B_c^\dagger\big] = 0, \qquad a = 1,2,3, \qquad D_{\bar{z}}I = 0, \\ &-g^{z\bar{z}}F_{z\bar{z}} + II^\dagger + \sum_{a=1}^{3}\rho_a^{-1}\big[B_a, B_a^\dagger\big] = r\cdot 1. \end{aligned} \tag{6.2}$$

Here $D_{\bar{z}}B_a = \partial_{\bar{z}}B_a + \bar{\varpi}_a B_a + [A_{\bar{z}}, B_a]$, and $D_{\bar{z}}I = \partial_{\bar{z}}I + A_{\bar{z}}I$. The space of solutions to equations (6.2) is to be modded out by the group of $\mathrm{U}(k)$ gauge transformations. The corresponding moduli space is a disjoint union of spaces $\mathcal{M}(k) = \coprod_{p\in\mathbb{Z}}\mathcal{M}_p(k)$ with p being the first Chern class of the gauge bundle

$$p = \frac{1}{2\pi\mathrm{i}}\int_\Sigma \mathrm{tr}\, F_A.$$

For the purposes of this paper, we shall only need to consider the case of $\Sigma = S^2$, with the metric g_Σ having a $\mathrm{U}(1)$ isometry (the round metric on a sphere is one such example). Imagine the geometry of a long cylinder that is capped at the ends by two hemispheres. In the long flat region, where $\varpi = 0$, we can look for specific solutions of equations (6.2), namely, the z,$\bar{z}$-independent B_1, B_2, B_3 and $B_4 \equiv A_{\bar{z}}$. Then equations (6.2) reduce to the familiar equations

$$[B_4, B_a] + \varepsilon_{abc}\big[B_b^\dagger, B_c^\dagger\big] = 0, \qquad a = 1,2,3, \qquad B_4 I = 0, \qquad II^\dagger + \sum_{i=1}^{4}\big[B_i, B_i^\dagger\big] = r\cdot 1$$

describing the Hilbert scheme of points on $\mathbb{C}^3$. Towards the caps, the derivative terms become important.

The fields and the equations, together with the gauge symmetry, make up the field content of twisted $\mathcal{N} = (2,2)$ supersymmetric gauge theory in two-dimensions. Unlike the generic theory with four supercharges, which has only A-type or B-type twists, this theory admits a variety of twists due to the extended supersymmetry of its core component. Namely, our theory is $\mathcal{N} = (8,8)$ $\mathrm{U}(k)$ super-Yang–Mills theory, whose symmetry is broken down to $\mathcal{N} = (2,2)$ by coupling to the fundamental chiral multiplet, whose complex scalar component is simply I. The $\mathcal{N} = (8,8)$ part has an $\mathrm{SO}(8)$ R-symmetry group, which partly survives the coupling to the fundamental chiral, allowing for a variety of twisting R-symmetries.

Our theory can be canonically lifted to a three-dimensional theory with the same field content, except that one of the real scalars in the vector multiplet becomes the third component of the gauge field. It is this three-dimensional theory that we use in this paper in defining the edge contributions.

Let us now analyze the solutions to equations (6.2) in the case of $\Sigma = S^2$. First, the norm squared of the first equations

$$\begin{aligned}0 &= \sum_{a=1}^{3} \int_{S^2} \mathrm{d}^2 z \rho_a^{-1} \operatorname{tr}\big(D_{\bar z} B_a + \varepsilon_{abc}\big[B_b^\dagger, B_c^\dagger\big]\big)\big(D_z B_a^\dagger - \bar\varepsilon_{abc}[B_b, B_c]\big) \\ &= \sum_{a=1}^{3} \int_{S^2} \mathrm{d}^2 z \rho_a^{-1} \operatorname{tr}\big(D_{\bar z} B_a D_z B_a^\dagger\big) + \sum_{b<c} \int_{S^2} \mathrm{d}^2 z \sqrt{g} \rho_b \rho_c \operatorname{tr}[B_b, B_c][B_b, B_c]^\dagger,\end{aligned}$$

where we used the identity $\rho_1\rho_2\rho_3\sqrt{g} = 1$, and $\int_{S^2} \bar\partial \operatorname{tr}(B_1[B_2, B_3]) = 0$ with the understanding that $\operatorname{tr}(B_1[B_2, B_3])$ is a $(1,0)$-form on Σ (being a section of the canonical bundle). Thus, equations (6.2) imply

$$D_{\bar z} B_a = 0, \quad D_{\bar z} I = 0, \qquad a = 1, 2, 3, \tag{6.3}$$

and $[B_b, B_c] = 0$, $1 \le b < c \le 3$. The operator $\nabla_{\bar z} = \bar\partial_{\bar z} + A_{\bar z}$ defines the structure of a holomorphic rank k bundle $\mathcal{E}$ over $\mathbb{P}^1$, of which I is a holomorphic section, while the operators B_a are the commuting holomorphic twisted Higgs fields

$$I \in H^0\big(\mathbb{P}^1, \mathcal{E}\big), \qquad B_a \in H^0\big(\mathbb{P}^1, \mathcal{E} \otimes \mathcal{E}^* \otimes L_a\big).$$

Now, to proceed algebro-geometrically we would like to replace the last equation in (6.2) by an r-dependent stability condition, so that instead of solving the last equation in (6.2) we divide the space of stable solutions to equations (6.3) by the group $\mathcal{G}_{\mathbb{C}}$ of $\mathrm{GL}(k, \mathbb{C})$ gauge transformations

$$(B_1, B_2, B_3, A_{\bar z}, I) \mapsto \big(g^{-1} B_1 g, g^{-1} B_2 g, g^{-1} B_3 g, g^{-1} A_{\bar z} g + g^{-1} \bar\partial_{\bar z} g, g^{-1} I\big).$$

We can now fix the gauge $A_{\bar z} = 0$ on the northern and on the southern hemispheres $H_\pm$ with the transition function $h(z)$ being holomorphic on the intersection $H_+ \cap H_- = \mathbb{C}^\times$. Grothendieck's theorem allows us to find a conjugacy class of h in the form of a diagonal matrix

$$h(z) = \operatorname{diag}\big(z^{p_1}, \dots, z^{p_k}\big),$$

where $p_i \in \mathbb{Z}$, $p_1 \ge p_2 \ge \dots \ge p_k$, and $p_1 + p_2 + \dots + p_k = p$. In this gauge the rest of the fields are holomorphic on $H_\pm$ respectively, with the identifications

$$I_+(z) = h(z) I_-(z), \qquad B_{a,+}(z) = \ell_a(z) h(z) B_{a,-}(z) h(z)^{-1}, \qquad a = 1, 2, 3,$$

where $\ell_a(z) = z^{-l_a}$ are the transition functions of the holomorphic bundles L_a. The isomorphism (6.1) implies $l_1 + l_2 + l_3 = -2$. We see that the non-trivial solutions for I lie in the subspace of $\mathbb{C}^k$ spanned by the eigenvectors e_i of h with $p_i \ge 0$. The corresponding components $v^i(z)$,

$$I_+(z) = \sum_{i=1}^{k} v^i(z) e_i$$

are simply some degree p_i polynomials in z, so that $z^{-p_i} v^i(z)$ is a degree p_i polynomial in z^{-1}. The classification of possible solutions for B_a's is more involved.

6.1 ADHM-like model for the one-leg theory

Let us present the matrix quantum mechanics describing the moduli space of solutions to the vortex equations (6.2). We fix two complex vector spaces R, P, of dimensions r and p, respectively, endowed with Hermitian metrics. The fields of our model are

$$\begin{aligned} &I \in \mathrm{Hom}(\mathbb{C}, R), && B_A \in \mathrm{End}(R), && A = 1,2,3,4,\\ &\gamma \in \mathrm{Hom}(R, P), && \beta_a \in \mathrm{Hom}(P, R), && a = 1,2,3. \end{aligned}$$

They are subject to the equations

$$\begin{aligned} &[B_a, B_b] + \varepsilon_{abc}([B_c, B_4] + \beta_c\gamma)^\dagger = 0, \qquad B_a\beta_b - B_b\beta_a + \varepsilon_{abc}(\gamma B_c)^\dagger = 0,\\ &\sum_{a=1}^{3} B_a^\dagger \beta_a + B_4\gamma^\dagger = 0, \qquad \sum_{A=1}^{4}[B_A, B_A^\dagger] + II^\dagger + \sum_{a=1}^{3}\beta_a\beta_a^\dagger - \gamma^\dagger\gamma = \zeta_{\mathbb{R}}\cdot 1_R \end{aligned}$$

and to the equivalence relation

$$(I; B_A; \beta_a; \gamma) \mapsto \big(g^{-1}I; g^{-1}B_A g; g^{-1}\beta_a; \gamma g\big), \qquad g \in \mathrm{U}(r).$$

6.2 Connection to cigar partition function

One can interpret the 1-leg vertex function of the magnificent four theory as the cigar partition function [19] of the (2+1)-dimensional gauged linear sigma model with the field content described in the previous subsections. The latter is expected to obey a system of difference equations, forming part of the rich algebraic structure hidden in the full 4-vertex.

7 Edge

Recall that the product P_{1234}, just like q_a for $a = 1, 2, 3, 4$, depends on the choice of $\mathsf{v} \in \Delta_0$.

Definition 7.1. Fix a reference v and direction $e = 1$. Let

$$\mathcal{T}_{\mathsf{e}} = \left(1 - \mu - P_{1234}\sum_{a\neq e}\frac{\lambda_{ae}}{P_{ae}}\right)^{*}\frac{\pi_{e,\mathrm{reg}}}{P_e} - P_{123}\frac{\pi_{e,\mathrm{reg}}}{P_e}\frac{\pi^{*}_{e,\mathrm{reg}}}{P^{*}_e} - \sum_{\substack{a\neq e, b\neq e\\ a\neq b}} P_{abe}\frac{\lambda_{ae}}{P_{ae}}\frac{\lambda^{*}_{be}}{P^{*}_{be}}. \tag{7.1}$$

Lemma 7.2. *$\mathcal{T}_{\mathsf{e}}$ is a square root of equation (3.7). $T_{\mathsf{e}} = \sum_{\mathsf{v}\in\mathsf{e}}\mathcal{T}_{\mathsf{e}}$ is a movable Laurent polynomial.*

Proof. We take for simplicity $e = 1$ and focus on the non-constant part of equation (7.1)

$$\mathcal{T} = \left(1 - \mu - P_{1234}\frac{\Sigma}{P}\right)^{*}\frac{\chi}{P} - P_{123}\frac{\chi}{P}\frac{\chi^{*}}{P^{*}},$$

where $\chi = \chi_{\pi_e}$, $\Sigma = \Sigma_e = \pi_{e,1} - \pi_{e,2} + \pi_{e,3}$ and $P = P_e$. By induction, add a box ξ to χ. We have $\delta T = \delta\mathcal{T} + \delta\bar{\mathcal{T}}$, where bar means 'evaluated at the other vertex', and

$$\delta\mathcal{T} = \left(1 - P_{1234}\frac{\Sigma}{P}\right)^{*}\frac{\xi}{P} - P_{123}\frac{1}{PP^{*}} - P_{1234}\frac{\chi}{P}\frac{\xi^{*}}{P^{*}}.$$

We work up to movable terms, and compute

$$\frac{P_{123} + \bar{P}_{123}}{(1-q_1)(1-q_1^{*})} = 1.$$

For any $m = q_2^{\alpha-1} q_3^{\beta-1} q_4^{\gamma-1}$, we define $[m] := \frac{1}{1-q_1}(\bar{m} - q_1 m)$. By introducing the parallelepiped $\Pi = \Pi(\xi)$ and decomposing, we get

$$\delta T = -P_{1234} \frac{\Sigma \setminus \Pi_2}{P} \frac{\xi^*}{P^*} - P_{1234} \frac{\chi \setminus \Pi_1}{P} \frac{\xi^*}{P^*} - P_{1234} \frac{\Pi}{P} \frac{\xi^*}{P^*} + \text{c.c.} - 1 + [\xi],$$

where c.c. means bar. We compute $[P_{234}(\Sigma \setminus \Pi_2)\xi^*] = 0 = [P_{234}(\chi \setminus \Pi_1)\xi^*]$, so that

$$\delta T = -1 + [\xi] - [P_{234} \Pi \xi^*],$$

and since $[P_{234} \Pi \xi^*] = -[\xi] + 1$, we have $\delta T = 0$. ■

Definition 7.3. The edge measure is

$$\mathsf{E}_{\mathsf{e}}(\lambda) := (-p)^{-f_{\mathsf{e}}(\pi_{\mathsf{e},\mathrm{reg}}) - |\pi_{\mathsf{e},\mathrm{reg}}| f_{\mathsf{e}}(\tilde{n})} \mathrm{e}^{-t_{\mathsf{e}} |\pi_{\mathsf{e},\mathrm{reg}}|} \hat{a}(T_{\mathsf{e}})$$

with fugacities determined by the relevant summands in Lemma 4.2.

Remark 7.4. The unmovable part of $\frac{q_1 P_{23} - \bar{P}_{23}}{1-q_1}$ is -1, so that

$$\text{const} := k \frac{q_1 P_{23} - \bar{P}_{23}}{1 - q_1} + k$$

is a movable Laurent polynomial.

8 Theory on a face

Now let us do a similar exercise in four and five dimensions. We start with a complex Kähler surface S, with Kähler metric g_S and local holomorphic coordinates z, w. We also need to fix the line bundles L_1, L_2 over S, such that $L_1 \otimes L_2 = K_S$. This choice is similar [52] to the choice of the so-called "basic classes" of Donaldson–Kronheimer–Mrowka.

The fields of our theory are the gauge field A, the two adjoint-valued complex scalars B_1, B_2, twisted by the line bundles L_1 and L_2, respectively, a pair (I, J) of fundamental and anti-fundamental scalar fields, with J twisted by K_S, and a pair (Υ, Ψ) of fermionic fundamental and anti-fundamental scalar fields, with Ψ twisted by K_S.

Our fields are again constrained by a set of elliptic (modulo gauge symmetry) equations

$$\begin{gathered} D_{\bar{z}} B_1 + D_w B_2^\dagger = 0, \qquad D_{\bar{w}} B_1 - D_z B_2^\dagger = 0, \qquad D_{\bar{z}} \Upsilon + D_w \Psi^\dagger = 0, \\ D_{\bar{w}} \Upsilon - D_z \Psi^\dagger = 0, \qquad D_{\bar{z}} I + D_w J^\dagger = 0, \qquad D_{\bar{w}} I - D_z J^\dagger = 0, \\ F_{zw} + [B_1, B_2] + IJ = 0, \\ -F_{z\bar{z}} - F_{w\bar{w}} + II^\dagger - J^\dagger J + \Upsilon\Upsilon^\dagger - \Psi^\dagger \Psi + \sum_{a=1}^{2} \big[B_a, B_a^\dagger\big] = r \cdot 1. \end{gathered} \tag{8.1}$$

The middle equations in (8.1) can be more invariantly stated as $\bar{\partial}_{\bar{A}} I + \bar{\partial}^\dagger_{\bar{A}} J^\dagger = 0$ where $J^\dagger$ is naturally viewed as a $(0,2)$-form valued in the same vector bundle, as I.

Our theory is, naturally, a twisted $\mathcal{N} = 2$ theory in four dimensions, which is obtained from a twisted $\mathcal{N} = 4$ theory by coupling it to a (twisted) hypermultiplet in the fundamental representation, and reversed statistics twisted hypermultiplet in the fundamental representation (such hypermultiplets naturally occur in the theories with negative branes as in [21]).

Of course, four-dimensional theory with $\mathcal{N} = 2$ supersymmetry with matter hypermultiplets both in the adjoint and fundamental representations is strongly coupled in the ultraviolet, and the localization computations reducing path integral to the semi-classical analysis are not valid.

Fortunately, the reversed statistics hypermultiplet cancels the contribution of the fundamental hypermultiplet to the beta function. It is this reversed statistics hypermultiplet which is coupled to the μ-parameter of the magnificent four theory.

The gauge bundle now can have both c_1 and c_2 and these result in the nontrivial edge and vertex contributions in the localization approach.

Our theory canonically lifts to five dimensions, which is the version used in our paper.

9 Face

Definition 9.1. Given a Young diagram λ and a box (i,j) in it, we define its arm and leg lengths as $\ell = \lambda_j - i$, $\mathrm{a} = \lambda_i^t - j$, where λ^t denotes the transposed diagram.

Definition 9.2. Fix reference directions $a = 1$ and $b = 2$ for $\mathsf{f} \in \Delta_2$. Following [34], let

$$\mathcal{T}_{\mathsf{f}} = -\mu^* \frac{\lambda_{12}}{P_{12}} + \sum_{\square \in \lambda_{12}} \frac{q_3^{\ell+1} q_4^{-\mathrm{a}}}{P_{12}^*}, \tag{9.1}$$

where ℓ and a are the leg and arm lengths of $\square \in \lambda$.

Lemma 9.3. *$\mathcal{T}_{\mathsf{f}}$ is a square root of equation (3.4). $T_{\mathsf{f}} = \sum_{\mathsf{v}\in\mathsf{f}} \mathcal{T}_{\mathsf{f}}$ is a movable Laurent polynomial.*

Proof. The function

$$s_{\alpha,\beta} := \sum_{\mathsf{v}\in\mathsf{f}} \frac{q_3^{\alpha} q_4^{\beta}}{\left(1-q_1^{-1}\right)\left(1-q_2^{-1}\right)}$$

is a Laurent polynomial for all $\alpha, \beta \in \mathbb{Z}$ because it is the equivariant Euler characteristic of the product of two line bundles over a compact surface. Then

$$f_2(\mathrm{a},\ell) = \sum_{\mathsf{v}\in\mathsf{f}} \frac{q_3^{\mathrm{a}+\ell+1}(q_1q_2)^{\mathrm{a}}}{\left(1-q_1^{-1}\right)\left(1-q_2^{-1}\right)}$$

is a Laurent polynomial, being of the form $s_{\alpha,\beta}$. It is also movable for all $\mathrm{a}, \ell \geq 0$, since $q_3^{(v)} = q_3 q_1^{p(v)} q_2^{q(v)}$ and $h = \mathrm{a} + \ell + 1 > 0$. ∎

Definition 9.4. The face measure is

$$\mathsf{F}_{\mathsf{f}} := (-p)^{g_{\mathsf{f}}(\lambda_{\mathsf{f}}) - \frac{1}{24}|\lambda_{\mathsf{f}}|c_{2,\mathsf{f}} + g_{\mathsf{f}}(\lambda_{\mathsf{f}},\tilde{n}) + \frac{1}{2}|\lambda_{\mathsf{f}}|g_{\mathsf{f}}(\tilde{n})} e^{\tilde{g}_{\mathsf{f}}(\lambda_{\mathsf{f}}) + |\lambda_{\mathsf{f}}|\tilde{g}_{\mathsf{f}}(\tilde{n}) + \frac{1}{2}|\lambda_{\mathsf{f}}|A_{002}} \hat{a}(\mathcal{T}_{\mathsf{f}})$$

with fugacities determined by the corresponding summands in Lemma 4.2.

Remark 9.5. Upon enforcing the CY condition, the index of Dirac operator gives

$$\begin{aligned} \# s_{\alpha,\beta} &= \frac{1}{2}\int_S \left[\left(\alpha - \frac{1}{2}\right)c_1(\mathscr{L}_3) + \left(\beta - \frac{1}{2}\right)c_1(\mathscr{L}_4)\right]^2 - \frac{\sigma}{8} \\ &= \frac{1}{8}(S.S - \sigma) + \frac{1}{2}\beta(\beta-1)S.S + \frac{1}{2}(\beta-\alpha)^2 \mathscr{L}_3.\mathscr{L}_3 + S.\mathscr{L}_3\left(\beta - \frac{1}{2}\right)(\beta - \alpha), \end{aligned}$$

whose integrality implies

$$S.S - \sigma = 0 \mod 8, \qquad S.\mathscr{L}_3 + \mathscr{L}_3.\mathscr{L}_3 = 0 \mod 2. \tag{9.2}$$

Here $S.S = \int_S c_1(S)^2$, $S.\mathscr{L}_3 = \int_S c_1(S)c_1(\mathscr{L}_3)$, and $\mathscr{L}_3.\mathscr{L}_3 = \int_S c_1(\mathscr{L}_3)^2$. The first equation in (9.2) follows from Hirzebruch signature theorem $2\chi + 3\sigma = S.S$, while the second is equivalent to A_{110} and $\int_S e(\mathcal{N})$ being even (see Lemma 2.11, where $\mathcal{N}$ is defined).

9.1 Virtual dimension

Recently, there has been some progress [3, 4] on surface counting in the CY4 setting. In order to facilitate comparison and future work, let us compute the number of monomials in the (movable) Laurent polynomial T_{f}.

Lemma 9.6. *With definitions as in Lemma 4.2, if we decompose the Laurent polynomial T_{f} as $P_1 + \tilde{\mu} P_2$, then the identities*

$$\#P_1 =: \mathrm{vdim} = g(\lambda) - \frac{1}{2}|\lambda|^2 A_{110} - \frac{1}{8}|\lambda|(\sigma + \tilde{\sigma})$$

as well as

$$\#P_2 = -\frac{1}{2}|\lambda| g(\tilde{n}) - g(\lambda, \tilde{n}) - g(\lambda) + \frac{1}{8}|\lambda|(\sigma + \tilde{\sigma})$$

hold true for any compact face f in a toric CY4.

Proof. Up to conjugation, we can write equation (9.1) as

$$\mathcal{T}_{\mathsf{f}} = \sum_{\square \in \lambda} \frac{q_3^{\ell+1} q_4^{-\mathrm{a}} - \tilde{\mu} q^{-\tilde{n}} q_3^{1-i} q_4^{1-j}}{P_{12}^*}.$$

We apply twice the remark above,

$$\#P_1 = \frac{1}{2} \sum_{\square \in \lambda} \left[\left(\ell + \frac{1}{2} \right) \mathscr{L}_3 - \left(\mathrm{a} + \frac{1}{2} \right) \mathscr{L}_4 \right]^2 - \frac{\sigma|\lambda|}{8}.$$

The integrand instead depends on μ, and therefore on the the restriction to S of the line bundle $L = \sum_{\mathsf{c} \in \Delta_3} \tilde{n}_{\mathsf{c}} L_{\mathsf{c}}$, with L_{c} the line bundle associated to cell c

$$\#P_2 = -\frac{1}{2} \sum_{\square \in \lambda} \left[\left(i - \frac{1}{2} \right) \mathscr{L}_3 + \left(j - \frac{1}{2} \right) \mathscr{L}_4 + L \right]^2 + \frac{\sigma|\lambda|}{8}.$$

For the purpose of this proof, we define $\tilde{\sigma}_{\mathsf{f}} := \frac{1}{3}(A_{200} + A_{020})$, and

$$\hat{g}(\lambda) := \sum_{\square \in \lambda} \left[\left(\frac{\ell(\ell+1)}{2} + \frac{1}{6} \right) A_{200} + \left(\frac{\mathrm{a}(\mathrm{a}+1)}{2} + \frac{1}{6} \right) A_{020} - \left(\ell \mathrm{a} + \frac{1}{2}(\mathrm{a} + \ell) + \frac{1}{4} \right) A_{110} \right].$$

We claim that $\hat{g}(\lambda) - g_{\mathsf{f}}(\lambda) = -\frac{1}{2}|\lambda|^2 A_{110}$.

We immediately recognize various terms $-\frac{1}{2}|\lambda| L^2 = -\frac{1}{2}|\lambda| g_f(\tilde{n})$ as well as

$$-\sum_{\square \in \lambda} \left[\left(i - \frac{1}{2} \right) \mathscr{L}_3 + \left(j - \frac{1}{2} \right) \mathscr{L}_4 \right] \cdot L = -g_f(\lambda, \tilde{n})$$

and

$$-\frac{1}{2} \sum_{\square \in \lambda} \left[\left(i - \frac{1}{2} \right) \mathscr{L}_3 + \left(j - \frac{1}{2} \right) \mathscr{L}_4 \right]^2 = -g_f(\lambda) + \frac{1}{8} \tilde{\sigma}_{\mathsf{f}} |\lambda|.$$

From this, we can read off the virtual dimension of a face

$$\mathrm{vdim} = g(\lambda) - \frac{1}{2}|\lambda|^2 A_{110} - \frac{1}{8}|\lambda|(\sigma + \tilde{\sigma})$$

and of the integrand

$$\#P_2 = -\frac{1}{2}|\lambda| g(\tilde{n}) - g(\lambda, \tilde{n}) - g(\lambda) + \frac{1}{8}|\lambda|(\sigma + \tilde{\sigma}).$$

Finally, we get the equality

$$\#T_{\mathsf{f}} = -\frac{1}{2}|\lambda|^2 A_{110} - \frac{1}{2}|\lambda| g(\tilde{n}) - g(\lambda, \tilde{n}).$$

The last step is to show that $\hat{g}(\lambda) - g_{\mathsf{f}}(\lambda) = -\frac{1}{2}|\lambda|^2 A_{110}$. We do this in three steps. First, the identity $\sum_{\square\in\lambda} \ell(\ell+1) - i(i-1) = 0$ follows from the fact that, for every j, we have

$$\sum_{\ell=0}^{\lambda_j - 1} \ell(\ell+1) = \sum_{i=1}^{\lambda_j} i(i-1).$$

Taking the transpose of the diagram, we get a similar identity for the arm-length. Second, the identity

$$\frac{1}{2}\sum_{\square\in\lambda} \mathrm{a} + \ell - (i+j) = -|\lambda|$$

follows from computing the terms: $\sum_{\square\in\lambda} \lambda_j = \sum_{j=1}^{h} \lambda_j^2$, where h is the height of first column;

$$\sum_{\square\in\lambda} j = \sum_{i=1}^{h^t} \frac{1}{2}\lambda_i^t(\lambda_i^t + 1),$$

where t denotes transpose and h^t is the length of first row of λ;

$$\sum_{\square\in\lambda} \lambda_i^t = \sum_{i=1}^{h^t} (\lambda_i^t)^2, \qquad \sum_{\square\in\lambda} i = \sum_{j=1}^{h} \frac{1}{2}\lambda_j(\lambda_j + 1).$$

Finally, we claim the identity

$$\sum_{\square\in\lambda} \mathrm{a}\ell + ij = \frac{1}{2}|\lambda|(|\lambda| + 1).$$

These three steps together imply $\hat{g} - g = -A_{110}\frac{1}{2}|\lambda|^2$.

Let us prove the claim by induction on the size of λ. It is true for $\lambda = 1$. Assume it is true for $|\lambda| = k$. Let μ be the partition obtained from adding a box (p, q) to λ. Let us consider two types of boxes $(i, j) \in \lambda$: first, the boxes with $j = q$ have $\ell_\mu = \ell_\lambda + 1$; second, the boxes with $i = p$ have $\mathrm{a}_\mu = \mathrm{a}_\lambda + 1$. Correspondingly, we can write

$$\sum_{\square\in\mu} \mathrm{a}_\mu \ell_\mu + ij = \sum_{\square\in\lambda} \mathrm{a}_\lambda \ell_\lambda + ij + \sum_{j=q, i=1,\dots,\lambda_q} \mathrm{a}_\lambda + \sum_{i=p, j=1,\dots,\lambda_p^t} \ell_\lambda + (\lambda_q + 1)\big(\lambda_p^t + 1\big),$$

where the last term comes from the box (p, q) itself. By the induction hypothesis, the first term equals $\frac{1}{2}k(k+1)$, while the sum of the last three terms equals $k+1$. A way to see this is to decompose λ as the sum of three terms: the rectangle generated by (p, q), which has $\lambda_q \lambda_p^t$ boxes; anything above it, which has $\lambda_1^t + \cdots + \lambda_{\lambda_q}^t - (q-1)\lambda_q$ boxes; and anything to its right, which has $\lambda_1 + \cdots + \lambda_{\lambda_p^t} - (p-1)\lambda_p^t$ boxes. ∎

10 Perturbative matters

Definition 10.1. Define

$$T_6 := \tilde{\mu}/\tilde{\nu} \sum_{\mathsf{v}\in\Delta_0} \frac{1-q^{-\tilde{n}}}{P_{1234}}.$$

Lemma 10.2. *Fix any reference vertex* $\mathsf{v}\in\Delta_0$*. Then* T_6 *is a Laurent polynomial in* $\tilde{\mu}$ *and the local variables* q_a*'s at* v*. It is movable since it is multiplied by* $\tilde{\mu}$*. By the condition* $Q=1$*, it is a square root of* T_6^2 *in equation* (3.3)*.*

Proof. Set $\tilde{\nu}=1$ without loss of generality. Up to complex conjugation, let us write

$$T_6 = -\tilde{\mu}^* \sum_{\mathsf{v}\in\Delta_0} \big(\tilde{K} - q^{\tilde{n}} K\big)$$

and observe that in the right-hand side of

$$\begin{aligned}\sum_{\mathsf{v}\in\Delta_0} \big(\tilde{K} - q^{\tilde{n}} K\big) = &\sum_{\mathsf{v}\in\Delta_0} \big(\tilde{K}_{\mathrm{reg}} - q^{\tilde{n}} K_{\mathrm{reg}}\big) + \sum_{\mathsf{e}\in\Delta_1}\sum_{\mathsf{v}\in\mathsf{e}} \frac{\tilde{\pi}_{e,\mathrm{reg}} - q^{\tilde{n}}\pi_{e,\mathrm{reg}}}{P_e} \\ &+ \sum_{\mathsf{f}\in\Delta_2}\sum_{\mathsf{v}\in\mathsf{f}} \frac{\tilde{\lambda}_{ab,\mathrm{reg}} - q^{\tilde{n}}\lambda_{ab}}{P_{ab}} + \sum_{\mathsf{c}\in\Delta_3}\sum_{\mathsf{v}\in\mathsf{c}} \frac{\tilde{n}_{abc}}{P_{abc}}\end{aligned} \tag{10.1}$$

the first sum is clearly finite, in the second and third sums each summand in the differences is finite once summed over vertices in the appropriate edge or face, while the last term is a sum of integrals over compact cells in the $\beta\to 0$ limit. ■

Remark 10.3. One can reabsorb all the terms in equation (10.1) except the last one by redefining the vertex, edge and face, replacing schematically the term $\mu^* X_{\mathrm{reg}}/P$ by $\tilde{\mu}^*\tilde{X}_{\mathrm{reg}}/P$ for X a solid, plane or ordinary partition respectively and P the appropriate denominator.

Definition 10.4. Let

$$\mathsf{C} := (-p)^{-h_4(\tilde{n}) + \frac{1}{2}\frac{1}{24}c_{2,2,0}(\tilde{n}) + \frac{\zeta(3)}{(2\pi)^3}c_{3,1,0}}\, \mathrm{e}^{-h_3(\tilde{n}) + \frac{1}{24}c_{2,1,1}(\tilde{n}) - h_2(\tilde{n}) - h_1(\tilde{n})}\,\hat{a}(T_6),$$

where the fugacity comes from the relevant summands in Lemma 4.2.

Define through ζ-function regularization

$$\hat{a}\left(\frac{1-\tilde{\mu}}{P_{1234}}\right) := \tilde{\mu}^{-\frac{1}{2}\zeta_4} PE\left(\frac{1-\tilde{\mu}^*}{P_{1234}}\right),$$

where $\zeta_4 = \sum_{n_1,n_2,n_3,n_4=0}^{\infty} 1$, and let

$$Z_{\mathrm{pert}} := \prod_{\mathsf{v}\in\Delta_0} \hat{a}\left(\frac{1-\tilde{\mu}}{P_{1234}}\right).$$

This is the only infinite product left after reshuffling terms. We don't need to worry about its sign, as it multiplies the whole partition function. The rest of the instanton configuration is given by the movable Laurent polynomial

$$I = T_6 + \sum_{\mathsf{v}\in\Delta_0} T_{\mathsf{v}} + \sum_{\mathsf{e}\in\Delta_1} T_{\mathsf{e}} + \sum_{\mathsf{f}\in\Delta_2} T_{\mathsf{f}}.$$

11 Conclusions and a conjecture

We presented a construction, motivated by gauge theory, which allows us to define the K-theoretic vertex of fourfolds for arbitrary asymptotics, in a combinatorial fashion and with concrete signs.

Conjecture 11.1. *There exist sign choices* $s_{\mathsf{f}}\big(\lambda^{(\mathsf{f})}\big)$ *and* $s_{\mathsf{e}}\big(\pi^{(\mathsf{e})}\big)$ *such that* (1.4) *reads*

$$\mathsf{Z} = \sum_{P\in\mathsf{P}} \mathsf{C} \prod_{\mathsf{v}\in\Delta_0} \mathsf{V}_{\mathsf{v}} \prod_{\mathsf{e}\in\Delta_1} (-1)^{s_{\mathsf{e}}(\pi^{(\mathsf{e})})} \mathsf{E}_{\mathsf{e}} \prod_{\mathsf{f}\in\Delta_2} (-1)^{s_{\mathsf{f}}(\lambda^{(\mathsf{f})})} \mathsf{F}_{\mathsf{f}}, \tag{11.1}$$

where the sum is over all collections of partitions

$$\mathsf{P} = \Big\{ P = \big(\tilde{n}^{(1)}, \dots, \tilde{n}^{(|\Delta_3|)}; \lambda^{(1)}, \dots, \lambda^{(|\Delta_2|)}; \pi^{(1)}_{\mathrm{reg}}, \dots, \pi^{(|\Delta_1|)}_{\mathrm{reg}}\big) \;\Big|\; P \textit{ is the profile of a fixed point } \big\{\tilde{K}^{\mathsf{v}}, \mathsf{v}\in\Delta_0\big\}\Big\}.$$

While the signs for the vertex V are determined in the present work, and the corresponding part of the partition function is completely fixed, the interaction terms between edges and faces, which are overall from the viewpoint of a vertex, as well as the signs for edge and face are not. We plan to address this shortcoming, as well as present some examples, in part II.

Once we fix X, the partition function Z depends on the fugacities (see (1.3)), the mass $\tilde{\mu}$, and four Ω background parameters q_1, q_2, q_3, q_4 with product $Q = 1$ (one can choose a reference vertex and express the local variables at the other vertices in terms of it). It may be possible to prove (11.1) using derived algebraic geometry [11].

Acknowledgements

NP thanks M. de Cataldo, L. Cassia, E. Diaconescu, S. Franco, M. Kool, V. Saxena for discussions. The research of NP was supported by the US Department of Energy under grant DE-SC0010008. We also thank the anonymous referees for their detailed feedback.

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