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Scale Invariant Scattering and Bernoulli Numbers

Thomas L. CURTRIGHT

Department of Physics, University of Miami, Coral Gables, FL 33124, USA
E-mail: *curtright@miami.edu*
URL: https://people.miami.edu/profile/7884b39ba94cfdbc1698d114121a27f5

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Abstract. Non-relativistic quantum mechanical scattering from an inverse square potential in two spatial dimensions leads to a novel representation of the Bernoulli numbers.

Key words: scale invariance; Bernoulli numbers; Riemann hypothesis

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1 Introduction

In this note, I summarize some observations about Bernoulli numbers as obtained in the context of computing the scattering amplitude for a scale invariant potential.

In two spatial dimensions non-relativistic scattering by a repulsive potential $V = \kappa/r^2$, with $\kappa > 0$, results in a surprisingly simple form for the integrated cross section, $\sigma = \int_0^{2\pi} \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\theta}\right) \mathrm{d}\theta$, when computed using quantum mechanics. The result for a mono-energetic beam of mass m particles is [2, 3] $\sigma = \frac{2\pi^2 m\kappa}{\hbar^2 k}$ where the incident energy is $E = \hbar^2 k^2/(2m)$. This result follows from a straightforward application of phase-shift analysis for the potential $V = \kappa/r^2$ upon realizing a remarkable identity involving the sinc function, $\operatorname{sinc}(z) \equiv \sin(z)/z$, as shown in [2]. For real x, a succinct form of the identity in question is[1]

$$1 = \frac{\sin(\pi x)}{\pi x} + 2\sum_{l=1}^{\infty} \frac{(-1)^l \sin\left(\pi\sqrt{l^2+x^2}\right)}{\pi\sqrt{l^2+x^2}}. \tag{1.1}$$

All higher powers of x cancel when terms on the right-hand side are expanded as series in x^2, as a consequence of familiar $\zeta(2n)$ exact values for integer $n > 0$.

2 Bessel functions and Bernoulli numbers

Upon expressing the sinc function in terms of spherical Bessel functions,

$$j_n(z) = \sqrt{\frac{\pi}{2z}} J_{n+1/2}(z) = (-z)^n \left(\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}\right)^n \frac{\sin z}{z}$$

one may expand the summand of (1.1) in powers of x^2 to obtain

$$\frac{\sin\left(\pi\sqrt{l^2+x^2}\right)}{\pi\sqrt{l^2+x^2}} = \sum_{n=1}^{\infty} \frac{\left(-\pi x^2\right)^n}{n! 2^n} \frac{1}{l^n} \sqrt{\frac{1}{2l}} J_{n+1/2}(\pi l).$$

[1] A selection of other identities of this type, but not exactly (1.1) so far as I can tell, can be found in [5].

Performing the sum over l in (1.1) before the sum over n then leads to[2]

$$\frac{\pi^n}{\sqrt{2}} = -\frac{(2n+1)!}{2^n n!}\sum_{l=1}^{\infty}\frac{(-1)^l}{l^{n+1/2}}\ J_{n+1/2}(\pi l) \qquad \text{for integer} \quad n \geq 1. \tag{2.1}$$

Note the prefactor on the right-hand side can be expressed in terms of the double factorial: $(2n+1)!! = (2n+1)!/\bigl(2^n n!\bigr)$.

Implicit in (2.1) is an identity involving the Bernoulli numbers, as follows from using series representations for the Bessel functions [1, Chapter 10, in particular, equation (10.1.8)] and interchanging summations. For even n,

$$\sum_{l=1}^{\infty}\frac{(-1)^l}{l^{n+1/2}} J_{n+1/2}(l\pi)$$
$$= (-1)^{\lfloor n/2\rfloor}\sum_{k=0}^{\lfloor (n-1)/2\rfloor}(-1)^k\frac{\sqrt{2}(2\lfloor n/2\rfloor+2k+1)!}{(2k+1)!\Gamma(2\lfloor n/2\rfloor-2k)}\frac{\zeta(2k+2\lfloor n/2\rfloor+2)}{2^{2k+1}\pi^{2k+2}},$$

while for odd n

$$\sum_{l=1}^{\infty}\frac{(-1)^l}{l^{n+1/2}} J_{n+1/2}(l\pi)$$
$$= -(-1)^{\lfloor n/2\rfloor}\sum_{k=0}^{\lfloor (n-1)/2\rfloor}(-1)^k\frac{\sqrt{2}(2\lfloor n/2\rfloor+2k+1)!}{(2k)!\Gamma(2\lfloor n/2\rfloor+2-2k)}\frac{\zeta(2k+2\lfloor n/2\rfloor+2)}{2^{2k}\pi^{2k+1}}.$$

But then $B_{2n+1} = 0$ for $n = 1, 2, \ldots$ and $\zeta(2n) = \frac{(2\pi)^{2n}}{2(2n)!}|B_{2n}|$ for $n = 1, 2, \ldots$ with the usual phases given by $B_{2n} = (-1)^{n+1}|B_{2n}|$. Thus from (2.1) we are led to another, rather remarkable identity

$$1 = (-1)^{n+1}(4n+2)\sum_{k=0}^{n}\frac{(2n)!}{n!k!(n-k)!}\left(\frac{B_{n+k+1}}{n+k+1}\right) \qquad \text{for integer} \quad n \geq 1. \tag{2.2}$$

Here the sum involves trinomial coefficients as well as divided Bernoulli numbers, $\beta_m \equiv B_m/m$. It is not difficult to check the validity of (2.2) using various expressions of the Bernoulli numbers as finite, *alternating* sums, e.g., as sums of Worpitzky numbers weighted by the harmonic sequence.[3]

3 A novel representation of Bernoulli numbers

If encountered as graffiti on the stones of a bridge, e.g., in Ireland, either (1.1) or its companion identity (2.2) might cause nothing more than a raised eyebrow in passing. Perhaps justifiably so.

However, upon inverting the linear relations in (2.2) to obtain expressions for each individual Bernoulli number, the results are more striking: The unsigned Bernoulli numbers $|B_{2n}|$ for $n \geq 2$ are given by interesting sums of $n-1$ monotonically decreasing positive rational numbers. Unlike many other such representations [11], here the terms in the finite sums that represent B_{2n} do *not* alternate in sign.

[2] Truncating the right-hand side of (2.1) to $\sum_{l=1}^{N}$ produces an $O(1/N)$ error which is largest at $n = 1$.

[3] An excellent introduction to the extensive literature on the Bernoulli numbers can be found in [7, 8].

For example,

$$\begin{pmatrix} |B_2| \\ |B_4| \\ |B_6| \\ |B_8| \\ |B_{10}| \\ |B_{12}| \\ |B_{14}| \\ |B_{16}| \\ |B_{18}| \\ |B_{20}| \end{pmatrix} = \begin{pmatrix} \frac{1}{6} \\ \frac{1}{30} \\ \frac{1}{42} \\ \frac{1}{30} \\ \frac{5}{66} \\ \frac{691}{2730} \\ \frac{7}{6} \\ \frac{3617}{510} \\ \frac{43867}{798} \\ \frac{174611}{330} \end{pmatrix} = \begin{pmatrix} \frac{1}{6} \\ \frac{1}{30} \\ \frac{1}{60} + \frac{1}{140} \\ \frac{1}{45} + \frac{1}{105} + \frac{1}{630} \\ \frac{1}{20} + \frac{3}{140} + \frac{1}{252} + \frac{1}{2772} \\ \frac{1}{6} + \frac{1}{14} + \frac{17}{1260} + \frac{1}{693} + \frac{1}{12012} \\ \frac{691}{900} + \frac{691}{2100} + \frac{59}{945} + \frac{41}{5940} + \frac{5}{10296} + \frac{1}{51480} \\ \frac{14}{3} + 2 + \frac{359}{945} + \frac{8}{189} + \frac{4}{1287} + \frac{1}{6435} + \frac{1}{218790} \\ \frac{3617}{100} + \frac{10851}{700} + \frac{1237}{420} + \frac{217}{660} + \frac{293}{12012} + \frac{1}{780} \\ \quad + \frac{7}{145860} + \frac{1}{923780} \\ \frac{43867}{126} + \frac{43867}{294} + \frac{750167}{26460} + \frac{6583}{2079} + \frac{943}{4004} + \frac{1129}{90090} \\ \quad + \frac{217}{437580} + \frac{2}{138567} + \frac{1}{3879876} \end{pmatrix} . \tag{3.1}$$

For these examples, in the finite sequence of terms that sum to give $|B_{2n}|$ obviously the second number in the sequence is just 3/7 times the first. Less obviously, each term in the sequence for $|B_{2n}|$ is greater than the subtotal of all the smaller terms in that same sequence.

The general result for $|B_{2n}|$ is obtained by writing (2.2) as an infinite matrix equation, $\mathbf{1} = \boldsymbol{M} \cdot \boldsymbol{B}$, where $\boldsymbol{B}$ is an infinite column of the even index Bernoulli numbers, $\mathbf{1}$ is an infinite column of 1 s, and $\boldsymbol{M}_{m,n} = 2(-1)^{m+1}\binom{2n-1}{m}\binom{2m+1}{2n}$. Computing the inverse for the triangular matrix $\boldsymbol{M}$ then gives $\boldsymbol{B} = \boldsymbol{M}^{-1} \cdot \mathbf{1}$. The ordered terms in the sums of (3.1) are just the unsigned entries in the corresponding columns of $\boldsymbol{M}^{-1}$ for the n-th row. All terms in a given row of $\boldsymbol{M}^{-1}$ have the same sign. Some straightforward algebra then leads to

$$|B_{2n}| = \frac{(n!)^2}{(2n+1)!} + \sum_{k=2}^{n-1} n!\ q_{n-k-1}(n) \frac{k!(k-1)}{(2k+1)!},$$

where the l-th order polynomials $q_l(n)$, with $l \geq 0$ and $n \geq l+3$, may be obtained sequentially from

$$q_l(n) = \frac{(-1)^l (n-l-3)!}{(2l+3)!(n-2l-3)!} + \sum_{j=0}^{l-1} \frac{(-1)^{l+j+1}(n-l-1+j)!}{(2l+1-2j)!(n-2l-1+2j)!}\ q_j(n+j-l).$$

For example,

$$q_0(n) = \frac{1}{6}, \qquad q_1(n) = \frac{7}{360} n - \frac{1}{45}, \qquad q_2(n) = \frac{31}{15120} n^2 - \frac{89}{15120}\ n + \frac{1}{315}, \qquad \text{etc.}$$

4 Conclusions

Many things can be said about the entries in $\boldsymbol{M}^{-1}$, such as those along the diagonal, i.e.,

$$\left|\boldsymbol{M}^{-1}_{n,n}\right| = \frac{(n!)^2}{(2n+1)!},$$

the first sub-diagonal, i.e.,

$$\left|\boldsymbol{M}^{-1}_{n\geq 2,n-1}\right| = \frac{1}{6}(n-2)\frac{n!(n-1)!}{(2n-1)!},$$

the second sub-diagonal, i.e.,

$$\left|\boldsymbol{M}^{-1}_{n\geq 3,n-2}\right| = \frac{7}{360}\left(n-\frac{8}{7}\right)(n-3)\frac{n!(n-2)!}{(2n-3)!},$$

etc. But those things remain to be said later, and not here.[4]

It is also possible to relate the terms in the finite monotonic series for $|B_{2n}|$ to various partitions of the *infinite* monotonic series obtain from

$$|B_{2n}| = \frac{2(2n)!}{(2\pi)^{2n}}\zeta(2n) = \frac{2(2n)!}{(2\pi)^{2n}}\sum_{k=1}^{\infty} 1/k^{2n}.$$

But that too remains to be discussed later, and not here.

Suffice it to say here that this monotonic finite series representation of $B_{2n} = (-1)^{n+1}|B_{2n}|$ clearly gives a series of progressively better bounds on B_{2n}. Such a series of constraints on B_{2n} might be useful to establish bounds on functions defined as infinite series whose coefficients involve the Bernoulli numbers [10]. But that remains to be shown. Finally, as suggested by an anonymous reviewer, the identity (2.2) may have an implicit connection to various multi-linear identities for Bernoulli numbers [6, 9], perhaps due in part to the fact that such "convolution" identities also appear in the context of theoretical physics models [4].

These pending developments notwithstanding, in closing it seems appropriate to note the results presented here have already led to the addition of three entries to the Online Encyclopedia of Integer Sequences (see A368846, A368847 and A368848).

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[4]Some additional details have been worked out explicitly in T.S. Van Kortryk, On Bernoulli numbers (private communication).