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Algebraic Complete Integrability of the $a_4^{(2)}$ Toda Lattice

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Abstract. The aim of this work is focused on the investigation of the algebraic complete integrability of the Toda lattice associated with the twisted affine Lie algebra $a_4^{(2)}$. First, we prove that the generic fiber of the momentum map for this system is an affine part of an abelian surface. Second, we show that the flows of integrable vector fields on this surface are linear. Finally, using the formal Laurent solutions of the system, we provide a detailed geometric description of these abelian surfaces and the divisor at infinity.

Key words: Toda lattice; integrable system; algebraic integrability; abelian surface

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1 Introduction

The study of integrable Hamiltonian systems has been motivated by several factors, including the development of powerful and beautiful mathematical theories and the application of integration concepts to various physical, biological and chemical systems. However, it remains challenging to describe or recognize integrable Hamiltonian systems with ease, as they are exceptional cases. The Korteweg–de Vries equation has generated numerous new ideas in the field of completely integrable Hamiltonian systems, leading to unexpected connections between mechanics, spectral theory, Lie algebra theory, algebraic geometry, and even differential geometry. Some interesting integrable systems also appear as coverings of algebraic complete integrable systems. These systems are sometimes also called algebraic complete integrable.

An algebraic complete integrable system can be linearized on a complex torus, and its invariant functions (often called first integrals or constants) are polynomial maps and their restrictions to an invariant complex variety are meromorphic functions on a complex abelian variety. The fluxes generated by the constants of motion are straight lines in this complex abelian variety.

Several nonlinear completely integrable systems were known in the 19th century, among them the geodesic flow on the ellipsoid, Neumann's system or the Kowalevski top. The Toda lattice, introduced by Morikazu Toda in 1967, is a simple model for a one-dimensional crystal in solid-state physics. It is famous because it is one of the first integrable systems for which a Lax pair was discovered by Flaschka. The classical Toda lattice is a system of particles with unit mass, connected by exponential springs. Its equations of motion derived from the Hamiltonian

$$H = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \sum_{j=1}^{n-1} \mathrm{e}^{q_j - q_{j+1}}, \tag{1.1}$$

where q_j is the position of the j-th particle and p_j is its amount of movement. This type of Hamiltonian was considered first by Morikazu Toda [12, 13]. The equation (1.1) is known as the finite classic non-periodic Toda lattice to distinguish other versions of various forms of the system. The periodic version of (1.1) is given by

$$H = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \sum_{j=1}^{n} \mathrm{e}^{q_j - q_{j+1}}, \qquad q_{n+1} = q_1,$$

where the equations of motion are given by

$$\dot{p}_j = -\frac{\partial H}{\partial q_j} = \mathrm{e}^{(q_{j-1}-q_j)} - \mathrm{e}^{(q_j - q_{j+1})}, \qquad \dot{q}_j = \frac{\partial H}{\partial p_j} = p_j, \qquad 1 \leq j \leq n.$$

The integrability of the periodic Toda lattice was established by Henon [10] and Flaschka [8] using the Lax pairs method. In 1976, Bogoyavlensky [5] introduced a generalization of the classical Toda periodic lattice to arbitrary Lie algebras.

Adler, van Moerbeke and Vanhaecke in [4] give the explicit Hamiltonians for the periodic Toda lattices that involve precisely three (connected) particles. In two-dimensional, there are precisely six cases of them, going with the extended root systems $a_2^{(1)}$, $a_4^{(2)}$, $c_2^{(1)}$, $d_3^{(2)}$, $g_2^{(1)}$ and $d_4^{(3)}$. They prove in [1, 4] that the case $a_2^{(1)}$ is algebraic completely integrable. In his case, Dehainsala prove in [6] that the two cases $c_2^{(1)}$ and $d_3^{(2)}$ are algebraic complete integrable.

In this work, we consider that $a_4^{(2)}$ is a two-dimensional integrable system. This system satisfies the linearization criterion [4, Theorem 6.41]. We prove that this system is an algebraic completely integrable in the Adler–van Moerbeke sense.

To prove this, with respect to the complex Liouville theorem, firstly, by the indicial locus and the Kowalevski matrix, we have shown that our system has three distinct families of homogeneous Laurent solutions with weights depending on four free parameters and find the Zariski open set Ω and the fiber $\mathbb{F}_c$, $c \in \Omega$. Secondly, with these Laurent solutions, we determined the Painlevé divisor and their arithmetic genus. We have obtain three divisors, for $c \in \Omega$, the Painlevé divisor $\Gamma_c^{(0)}$ is a smooth genus three hyperelliptic curve, the Painlevé divisor $\Gamma_c^{(1)}$ is a smooth genus four curve and the Painlevé divisor $\Gamma_c^{(2)}$ is a smooth genus two hyperelliptic curve. Thirdly, to compact the fiber, we determined the projective space to embedding the divisor at infinity to find the singularities and the intersection between the different curves. We have obtain twenty-five (25) functions which forms the basis of the projective space and determined the intersections points. To end our prove, we show that the vector field $(\varphi_c)_*\mathcal{V}_1$ extends to an holomorphic vector fields on $\mathbb{P}^{24}$. To show that the vector field $(\varphi_c)_*\mathcal{V}_1$ is holomorphic on two chart of $\mathbb{P}^{24}$, we have established that this vector field can be written as a quadratic vector field in two appropriate chart.

This paper is organized as follows. In Section 2, we review the basic notions of algebraic integrability in sense of Adler–van Moerbeke. Section 3 contains the main part of the paper, we verify that the $a_4^{(2)}$ Toda lattice is Liouville integrable, we do the Painlevé analysis of the system. This analysis shows that our integrable system admits three principal balances, i.e., three families of Laurent solutions depending on the maximal number of free parameters,four in our case. Thus, by confining each family of Laurent solutions to the invariant manifolds we calculate the Painlevé divisors associated to these principal balances and, for $c \in \Omega$, we give an explicit embedding of the invariant manifold $\mathbb{F}_c$ in the projective space $\mathbb{P}^{24}$. Finally, in Section 4, we determine the holomorphic differentials forms on the abelian surface.

2 Preliminaries

In this section, we also recall some basics notions. For more comprehension just read the book [4].

Let $v = (v_1, \dots, v_n)$ be a collection of positive integers without a common divisor. Such a v is called a weight vector.

Definition 2.1 ([4])**.** A polynomial $f \in \mathcal{F}(\mathbb{C}^n) := \mathbb{C}[x_1, \dots, x_n]$ is a weight homogeneous polynomial of weight k (with respect to v) if $f(t^{v_1}x_1, \dots, t^{v_n}x_n) = t^k f(x_1, \dots, x_n)$, $\forall\,(x_1, \dots, x_n) \in \mathbb{C}^n$ and $t \in \mathbb{C}$.

When the weight of f is $k \in \mathbb{N}$, we denote

$$\mathcal{F}^{(k)} := \{F \in \mathcal{F}(\mathbb{C}^n) \mid \varpi(F) = k\}.$$

Definition 2.2 ([4])**.** A polynomial vector field on $\mathbb{C}^n$

$$\begin{cases} \dot{x}_1 = f_1(x_1, \dots, x_n), \\ \dots\dots\dots\dots\dots\dots \\ \dot{x}_n = f_n(x_1, \dots, x_n) \end{cases} \tag{2.1}$$

is called a weight homogeneous vector field of weight k (with respect to v) if each of the polynomials $f_1, \dots, f_n$ is weight homogeneous (with respect to v) and if $\varpi(f_i) = v_i + k = \varpi(x_i) + k$ for $i = 1, \dots, n$.

According to [4, Proposition 7.6], if (2.1) is a weight homogeneous vector field, then Laurent solutions have the form

$$x_i(t) = \frac{1}{t^{v_i}} \sum_{k=0}^{\infty} x_i^{(k)} t^k, \qquad i = 1, \dots, n, \quad \text{with} \quad x^{(0)} = \big(x_1^{(0)}, \dots, x_n^{(0)}\big) \neq 0,$$

are called weight homogeneous Laurent solution. We will say that a formal Laurent solution is a principal balance if it depends on $n-1$ free parameters; otherwise, it will be called the lower balance.

The positive integers v_i being the weights of the phase variables, then the leading coefficients $x_i^{(0)}$ satisfy the nonlinear algebraic equations

$$v_i x_i^{(0)} + f_i\big(x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)}\big) = 0, \qquad i = 1, \dots, n, \tag{2.2}$$

while the other coefficients $x_i^{(k)}$, $k \geq 1$, satisfy

$$\big(k \operatorname{Id}_n - \mathcal{K}\big(x^{(0)}\big)\big)x^{(k)} = P^{(k)},$$

where

$$x^{(k)} = \big(x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}\big)^{\mathsf{T}}, \qquad P^{(k)} = \big(P_1^{(k)}, P_2^{(k)}, \dots, P_n^{(k)}\big)^{\mathsf{T}}$$

and each $P_i^{(k)}$ is a polynomial which depends only on the variables $x_1^{(l)}, x_2^{(l)}, \dots, x_n^{(l)}$ with $0 \leq l \leq k$. The matrix $\mathcal{K}$ of order n is defined by $\mathcal{K}_{ij} = \frac{\partial f_i}{\partial x_j} + v_i \delta_{ij}$ is called the Kowalevski matrix.

Let Λ be a discrete subgroup of rank k of $\mathbb{C}^n$. The quotient group $\mathbb{C}^n/\Lambda$ has a smooth complex manifold structure induced by the projection $\pi\colon \mathbb{C}^n \longrightarrow \mathbb{C}^n/\Lambda$. This complex manifold is compact if and only if $k = 2n$. In this case, $\mathbb{T}^n = \mathbb{C}^n/\Lambda$ is a complex torus. If such a complex torus is also an algebraic variety, i.e., admits a projective embedding, we then say that it is an abelian variety. If $k = 2n = 4$, we say that it is an abelian surface.

To define an algebraic complete integrable system, we need the following definitions.

Definition 2.3 ([4])**.** A fiber of $\mathbf{F}$ is a common level set of the functions $\mathbf{F_i}$. The fiber of $\mathbf{F}$ that passes through $m \in M$ will be denoted by

$$\mathbf{F_m} = \{p \in M \mid \mathbf{F_i}(p) = \mathbf{F_i}(m),\ \forall i \in \{1, \dots, s\}\}.$$

For $c \in \mathbb{R}^s$ (resp. $\mathbb{C}^s$), we will note the fiber $\mathbf{F}^{-1}(c)$ above c by $\mathbf{F}_c$. So, we have

$$\mathbf{F}_m = \mathbf{F}^{-1}(\mathbf{F}(m)) = \mathbf{F}_{\mathbf{F}(m)} \qquad \forall m \in M.$$

The set of regular values of $\mathbf{F}$ is a residual subset (hence a dense subset) of $\mathbb{R}^s$ (resp. $\mathbb{C}^s$). By the inverse function theorem, the fiber $\mathbf{F}_c$ over each regular value c that lies in the image of $\mathbf{F}$ is non-singular. Hence, when $\mathbf{F} = (F_1, \dots, F_s)$ is involutive, the Hamiltonian vector fields $\mathcal{X}_{\mathbf{F}_i}$, $1 \le i \le s$, commute and for any point m they are tangent to the non-singular affine part of $\mathbf{F}(m)$.

Definition 2.4. An abelian variety is a complex torus $\mathbb{C}^r/\Lambda$ (Λ a lattice in $\mathbb{C}^r$) which is projective, which means that it admits an embedding in a project space $\mathbb{P}^N$.

An abelian variety $\mathbb{T}^r$ will be called an *irreducible abelian variety*, when $\mathbb{T}^r$ does not contain any abelian subvariety, otherwise it will be called a reducible abelian variety.

Definition 2.5. Let $(M, \{\cdot,\cdot\}, \mathbf{F})$ be a complex integrable system, where M is a non-singular affine variety and where $\mathbf{F} = (F_1, \dots, F_s)$. We say that $(M, \{\cdot,\cdot\}, \mathbf{F})$ is an algebraic completely integrable system if for generic $c \in \mathbb{C}^s$ the fiber $\mathbf{F}_c$ is an affine part of an abelian variety and if the Hamiltonian vector fields $\mathcal{X}_{\mathbf{F}_i}$ are translation invariant, when restricted to these fibers. In the particular case in which M is an affine space $\mathbb{C}^n$, we will call $(\mathbb{C}^n, \{\cdot,\cdot\}, \mathbf{F})$ a polynomial algebraic complete integrable system. When the generic abelian variety of the algebraic complete integrable system is irreducible, we speak of an irreducible algebraic complete integrable system.

An integrable system is said to be algebraic completely integrable if the fibers of the momentum map are affine parts of abelian varieties and the integrable fields are linear.

The following theorem gives a necessary condition for the algebraic integrability of an integrable system. It is inspired by the Kowalevski work [11], and is based on the fact that the phase space of an algebraic complete integrable system admits a partial compactification on which the integrable vector fields extend into complete vector fields. This means that each of the integrable vector fields of an irreducible algebraic complete integrable system on $\mathbb{C}^n$ admits one or several families of Laurent solutions (called balances), which will lead to a necessary condition for algebraic complete integrability, which we call the Kowalevski–Painlevé criterion.

Theorem 2.6 (Kowalevski–Painlevé criterion [4])**.** *Let $(\mathbb{C}^n, \{\cdot,\cdot\}, \mathbf{F})$ be an irreducible polynomial algebraic complete integrable system, where $\mathbf{F} = (F_1, \dots, F_s)$ is a family of polynomials and $(x_1, \dots, x_n)$ is a system of linear coordinates on $\mathbb{C}^n$. Let $\mathcal{V}$ be any one of the integrable vector fields $\mathcal{X}_{F_1}, \dots, \mathcal{X}_{F_s}$. For each $1 \le i \le n$ such that x_i is not constant along the integral curve of $\mathcal{V}$, i.e., $\dot{x}_i := \mathcal{V}[x_i] \neq 0$, there exists a principal balance $x(t) = (x_1(t), \dots, x_n(t))$, depending on $n-1$ free parameters for which $x_i(t)$ has a pole.*

Let $\mathcal{A}$ an affine variety, $\varphi\colon \mathcal{A} \longrightarrow \mathbb{P}^N$ a regular map, let $\overline{\varphi(\mathcal{A})}$ be the closure of the image of $\mathcal{A}$ in $\mathbb{P}^N$.

The following theorem gives the sufficient conditions to be satisfied by the fibers of the momentum map of an integrable system to be algebraic complete integrable.

Theorem 2.7 (complex Liouville theorem [4])**.** *Let $\mathcal{A} \in \mathbb{C}^s$ be a non-singular affine variety of dimension r which supports r holomorphic vector fields $\mathcal{V}_1, \dots, \mathcal{V}_r$ and let $\varphi\colon \mathcal{A} \longrightarrow \mathbb{C}^N \subset \mathbb{P}^N$ be a regular map; here $\mathbb{C}^N \subset \mathbb{P}^N$ is the usual inclusion of $\mathbb{C}^N$ as the complement of a hyperplane $\mathbf{H}$ in $\mathbb{P}^N$. We define $\Delta := \overline{\varphi(\mathcal{A})} \setminus \varphi(\mathcal{A})$ and we decompose the analytic subset Δ as $\Delta = \Delta' \cup \Delta''$, where Δ' is the union of the irreducible components of Δ of dimension $r-1$ and Δ'' is the union of the other irreducible components of Δ. The following conditions are assumed to be verified:*

1. *$\varphi\colon \mathcal{A} \longrightarrow \mathbb{C}^N$ is an isomorphic embedding.*
2. *The vector fields commute pairwise, $[\mathcal{V}_i, \mathcal{V}_j] = 0$ for $1 \le i, j \le r$.*
3. *At every point $m \in \mathcal{A}$, the vector fields $\mathcal{V}_1, \dots, \mathcal{V}_r$ are independent.*
4. *The vector field $\varphi_*\mathcal{V}_1$ extends to a vector field $\overline{\mathcal{V}}_1$ which is holomorphic on a neighborhood of Δ' in $\mathbb{P}^N$.*
5. *The integral curves of $\overline{\mathcal{V}}_1$ that start at points $m \in \Delta'$ go immediately into $\varphi(\mathcal{A})$.*

Then $\overline{\varphi(\mathcal{A})}$ is an abelian variety of dimension r and $\Delta'' = \varnothing$, so that $\overline{\varphi(\mathcal{A})} = \varphi(\mathcal{A}) \cup \Delta'$. Moreover, the vector fields $\varphi_*\mathcal{V}_1, \dots, \varphi_*\mathcal{V}_r$ extends to holomorphic vector fields on $\overline{\varphi(\mathcal{A})}$.

3 Algebraic integrability of the $a_4^{(2)}$ Toda lattice

The aims of this section is to prove, by following the strategy developed by Adler, van Moerbeke and Vanhaecke in [4, Section 9.4] to establish the algebraic integrability of the $a_2^{(1)}$ Toda lattice, the algebraic complete integrability of the $a_4^{(2)}$ Toda lattice.

3.1 Liouville integrability of the $a_4^{(2)}$ Toda lattice

The differential equations of the periodic $a_4^{(2)}$ Toda lattice are given on the five dimensions hyperplane $\mathcal{H} = \left\{(x_0, x_1, x_2, y_0, y_1, y_2) \in \mathbb{C}^6 \mid y_0 + 2y_1 + 2y_2 = 0\right\}$ of $\mathbb{C}^6$ by

$$\begin{cases} \dot{x} = x.y, \\ \dot{y} = Ax, \end{cases}$$

where $x = (x_0, x_1, x_2)^\mathsf{T}$, $y = (y_0, y_1, y_2)^\mathsf{T}$ and A is the Cartan matrix of the twisted affine Lie algebra $a_4^{(2)}$ given in [4] by

$$\begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}$$

and $\varepsilon = (1, 2, 2)^\mathsf{T}$ is the normalized null vector of A^T. The equations of motion of the $a_4^{(2)}$ Toda lattice are given in [4] by

$$\begin{aligned} \dot{x}_0 &= x_0 y_0, & \dot{y}_0 &= 2x_0 - 2x_1, \\ \dot{x}_1 &= x_1 y_1, & \dot{y}_1 &= -x_0 + 2x_1 - 2x_2, \\ \dot{x}_2 &= x_2 y_2, & \dot{y}_2 &= -x_1 + 2x_2. \end{aligned} \tag{3.1}$$

We denote by $\mathcal{V}_1$ the vector field defined by the above differential equations (3.1). Then $\mathcal{V}_1$ is the Hamiltonian vector field, with Hamiltonian function $F_2 = y_0^2 + 4y_2^2 - 4x_0 - 8x_1 - 16x_2$ with respect to the Poisson structure $\{\cdot, \cdot\}$ defined by the following skew-symmetric matrix:

$$J = \frac{1}{8} \begin{pmatrix} 0 & 0 & 0 & 4x_0 & -2x_0 & 0 \\ 0 & 0 & 0 & -2x_1 & 2x_1 & -x_1 \\ 0 & 0 & 0 & 0 & -x_2 & x_2 \\ -4x_0 & 2x_1 & 0 & 0 & 0 & 0 \\ 2x_0 & -2x_1 & x_2 & 0 & 0 & 0 \\ 0 & x_1 & -x_2 & 0 & 0 & 0 \end{pmatrix}. \tag{3.2}$$

This Poisson structure is given on $\mathbb{C}^6$; the function $F_0 = y_0 + 2y_1 + 2y_2$ is a Casimir, so that the hyperplane $\mathcal{H}$ is a Poisson subvariety. The rank of this Poisson structure $\{\cdot, \cdot\}$ is 0 on the three-dimensional subspace $\{x_0 = x_1 = x_2 = 0\}$; the rank is 2 on the three four-dimensional subspaces: $\{x_0 = x_1 = 0\}$, $\{x_0 = x_2 = 0\}$ and $\{x_1 = x_2 = 0\}$. Thus, for all points of $\mathcal{H}$ except the four subspaces above the rank is 4. The vector field $\mathcal{V}_1$ admits also the following three constants of motion:

$$\begin{aligned} &F_1 = x_0x_1^2x_2^2, \\ &F_2 = y_0^2 + 4y_2^2 - 4x_0 - 8x_1 - 16x_2, \\ &F_3 = \big(y_0^2 - 4x_0\big)\big(y_2^2 - 4x_2\big) - 4x_1(y_0y_2 - 4x_2 - x_1). \end{aligned} \tag{3.3}$$

F_1 is a Casimir for $\{\cdot,\cdot\}$, and the function F_3 generates a second Hamiltonian vector field $\mathcal{V}_2$, which commutes with $\mathcal{V}_1$, given by the differential equations

$$\begin{aligned} &x_0' = x_0y_2(y_0y_2 - 2x_1) - 4x_0x_2y_0, \\ &x_1' = -x_1y_1y_2(y_1 + y_2) - x_1^2y_1 + x_1(x_0y_2 + 2x_2y_0), \\ &x_2' = x_2(y_1 + y_2)((y_1 + y_2)y_2 + x_1) + x_0x_2y_0, \\ &y_0' = 2\big(2x_1x_2 + x_0y_2^2\big) + x_1(2x_1 - y_0y_2) - 8x_0x_2, \\ &y_1' = -x_0y_2^2 + 2x_2(3x_0 - x_1) + y_0y_2(x_1 + x_2) - 2x_1^2 + x_2y_0y_1, \\ &y_2' = x_1y_2(y_1 + y_2) + x_1^2 - x_2(y_1 + y_2) - 2x_2x_0. \end{aligned} \tag{3.4}$$

Hence, the system (3.1) is completely integrable in the Liouville sense. It can be written as a Hamiltonian vector fields

$$\dot{z} = J\frac{\partial H}{\partial z}, \qquad z = (z_1, \dots, z_6)^\intercal = (x_0, x_1, x_2, y_0, y_1, y_2)^\intercal,$$

where $H = F_2$. The Hamiltonian structure is defined by the following Poisson bracket:

$$\{F, H\} = \left\langle \frac{\partial F}{\partial z}, J\frac{\partial H}{\partial z} \right\rangle = \sum_{i,k=1}^{6} J_{ik}\frac{\partial F}{\partial z_i}\frac{\partial H}{\partial z_k},$$

where $\frac{\partial H}{\partial z} = \big(\frac{\partial H}{\partial x_0}, \frac{\partial H}{\partial x_1}, \frac{\partial H}{\partial x_2}, \frac{\partial H}{\partial y_0}, \frac{\partial H}{\partial y_1}, \frac{\partial H}{\partial y_2}\big)^\intercal$ and J is an antisymmetric matrix.

The vector field $\mathcal{V}_2$ admits the same constants of motion (3.3) and is in involution with $\mathcal{V}_1$ therefore $\{F_2, F_3\} = 0$. The involution σ defined on $\mathbb{C}^6$ by

$$\sigma(x_0, x_1, x_2, y_0, y_1, y_2) = (x_0, x_1, x_2, -y_0, -y_1, -y_2)$$

preserves the constants of motion F_1, F_2 and F_3, hence leave the fibers of the momentum map F invariant. This involution can be restricts to the hyperplane $\mathcal{H}$.

Let $\mathbf{F} = (F_1, F_2, F_3)\colon \mathcal{H} \to \mathbb{C}^3$ be the momentum map; functions $\mathbf{F}_i$ being two by two in involution, $\mathbf{F}$ is involutive. The Jacobian matrix of $\mathbf{F}$ is given by

$$\mathrm{Jac} := \begin{pmatrix} x_1^2x_2^2 & 2x_0x_1x_2^2 & 2x_0x_1^2x_2 & 0 & 0 \\ -4 & -8 & -16 & 2y_0 & 8y_2 \\ 16x_2 - 4y_2^2 & \bigstar & -4y_0^2 & \blacklozenge & \vartriangle \end{pmatrix}$$

$$\text{with} \begin{cases} \bigstar = -4y_0y_2 + 16x_2 + 8x_1, \\ \blacklozenge = 2y_0\big(y_2^2 - 4x_2\big) - 4x_1y_2, \\ \vartriangle = 2y_2\big(y_0^2 - 4x_0\big) - 4x_1y_0. \end{cases}$$

Let $p_0\big(1, 1, 1, 1, -\frac{3}{2}, 1\big)$ be a point of $\mathcal{H}$. The Jacobian matrix of $\mathbf{F}$ at p_0 is given by

$$\mathrm{Jac}(p_0) = \begin{pmatrix} 1 & 2 & 2 & 0 & 0 \\ -4 & -8 & -16 & 2 & 8 \\ 12 & 20 & 28 & -10 & -10 \end{pmatrix}.$$

The rank of matrix $\mathrm{Jac}(p_0)$ is 3, so the differentials $\mathrm{d}\mathbf{F}_i$, $i=1,2,3$, are independent at p_0 and since the functions $\mathbf{F}_i$ are polynomials, $\mathbf{F}$ is independent on a dense open subset $\mathcal{U}_{\mathbf{F}}$ of $\mathcal{H}$. We can deduce that $(\mathcal{H},\{\cdot,\cdot\},\mathbf{F})$ is an integrable system in the sense of Liouville.

Let $\mathcal{S}$ be the set of points in $\mathcal{H}$ where the determinants of all 3×3 minors of the matrix Jac cancel. By direct computation, we prove that S is the union of following subvarieties:

$$\mathcal{S}_1 := \{x_1 = 0\}, \qquad \mathcal{S}_2 := \{x_2 = 0\}, \qquad \mathcal{S}_3 := \big\{x_1 = 2y_2^2,\, x_0 = 4x_2,\, y_0 = 4y_2\big\},$$
$$\mathcal{S}_4 := \left\{y_0 = y_2 = 0,\, x_1 = \frac{2x_2^2 + x_0x_2 + \alpha_1}{x_0},\, x_1 = \frac{2x_2^2 + x_0x_2 - \alpha_1}{x_0}\right\},$$
$$\mathcal{S}_5 := \left\{x_0 = \frac{1}{4}\beta,\, x_2 = \frac{1}{2}\frac{y_2\big(4x_1y_2 - 2x_1y_0 - 2y_0y_2^2 + y_0^2y_2\big)}{y_0(-4y_2+y_0)}\right\}$$

with

$$\alpha_1 = \sqrt{4x_2^4 + 4x_2^3x_0 - 7x_0^2x_2^2 + 2x_2x_0^3},$$
$$\beta = \frac{8x_1y_0y_2 - 8x_1y_2^2 + 4y_0y_2^3 - 4y_0^2y_2^2 - 2x_1y_0^2 + y_0^3y_2}{y_2(-4y_2+y_0)}.$$

The images under $\mathbf{F}$ of S_1 and S_2 are contained in the subset $F_1 = c_1 = 0$. By substituting $x_1 = \frac{1}{8}y_0^2$, $x_2 = \frac{1}{4}x_0$ and $y_0 = 4y_2$ in the three constants of motion $F_i = c_i$, $i=1,2,3$, by direct computation with Maple, we obtain

$$c_1 = \frac{1}{4}x_0^3y_2^4, \qquad c_2 = 4y_2^2 - 8x_0, \qquad c_3 = 4x_0\big(-3y_2^2 + x_0\big).$$

Using $F_i = c_i$, $i=1,2,3$, and eliminating the variables x_i, y_i, $i=1,2,3$, in the above expresions, we deduce that the image under F of S_3 is contained in the subset

$$256\big(3200000c_1^2 + 2000c_3^2c_2c_1 - 225c_3c_2^3c_1 + c_3^5\big) + 1728c_2^5c_1 - 32c_3^4c_2^2 + c_3^3c_2^4 = 0,$$

and the set of regular values of the momentum map $\mathbf{F}$ is the Zariski open subset Ω defined by

$$\Omega = \big\{c = (c_1,c_2,c_3) \in \mathbb{C}^3 \mid c_1 \neq 0 \text{ and}$$
$$256\big(3200000c_1^2 + 2000c_3^2c_2c_1 - 225c_3c_2^3c_1 + c_3^5\big) + 1728c_2^5c_1 - 32c_3^4c_2^2 + c_3^3c_2^4 \neq 0\big\}.$$

At a generic point $c = (c_1,c_2,c_3) \in \mathbb{C}^3$, the fiber on $c \in \Omega$ of $\mathbf{F}$ is therefore

$$\mathbb{F}_c := \mathbb{F}^{-1}(c) = \bigcap_{i=1}^{3}\{m \in \mathcal{H} \mid \mathbf{F}_i(m) = c_i\}.$$

Hence, we have the following result which prove that $a_4^{(2)}$ Toda lattice is a completely integrable system in the Liouville sense.

Proposition 3.1. *For $c \in \Omega$, the fiber $\mathbb{F}_c$ over c of the momentum F is a smooth affine variety of dimension 2 and the rank of the Poisson structure (3.2) is maximal and equal to 4 at each point of $\mathbb{F}_c$; moreover, the vector fields $\mathcal{V}_1$ and $\mathcal{V}_2$ are independent at each point of the fiber $\mathbb{F}_c$.*

Proposition 3.2. $(\mathcal{H},\{\cdot,\cdot\},\mathbf{F})$ *is a completely integrable system describing the $a_4^{(2)}$ Toda lattice, where $\mathbf{F} = (F_1,F_2,F_3)$ and $\{\cdot,\cdot\}$ are given respectively by (3.3) and (3.2) with commuting vector fields (3.1) and (3.4).*

3.2 Algebraic integrability of $a_4^{(2)}$ Toda lattice

To show that the $a_4^{(2)}$ Toda lattice is algebraically completely integrable, we show that for all $c = (c_1, c_2, c_3) \in \Omega$, the fiber $\mathbf{F_c}$ is the affine part of an abelian surface on which the two vector fields $\mathcal{V}_1$ and $\mathcal{V}_2$ are linear [6]. For that it is necessary to verify that $\mathbf{F_c}$ satisfies the conditions of the Liouville complex theorem (Theorem 2.7).

Observe that the vector field $\mathcal{V}_1$ is homogeneous with respect to the following weights:

$$\varpi(x_0, x_1, x_2, y_0, y_1, y_2) = (2, 2, 2, 1, 1, 1).$$

We can also verify that the constants of motion F_i are weight homogeneous and they have the following weights: $\varpi(F_1, F_2, F_3) = (10, 2, 4)$.

3.2.1 Laurent solutions

According to [4], the vector fields of an algebraic complete integrable system have good properties at infinity; after all, a linear vector field on a complex torus is the same along the divisor, which will happen to be absent in phase space, as on the rest of the torus. In fact, since every holomorphic function on a complex torus can be written as a quotient of theta functions, the integral curves (solutions) to any of the vector fields of an algebraic complete integrable system can be written as a quotient of holomorphic functions. Intuitively speaking this means that we can consider not only Taylor solutions to the differential equations that describe these vector fields but also Laurent solutions, which will correspond to initial conditions at infinity (precisely: on the divisor that needs to be adjoined to the fibers of the momentum map to complete them into abelian varieties). Moreover, these Laurent solutions must depend on dim $\mathcal{H} - 1$ free parameters, which corresponds to the freedom of choice of the initial condition at infinity.

Let us look for weights homogeneous Laurent solutions associated to the vector field $\mathcal{V}_1$. The solution form of vector field $\mathcal{V}_1$ is

$$x_i(t) = \frac{1}{t^2} \sum_{k=0}^{\infty} x_i^{(k)} t^k \qquad \text{and} \qquad y_i(t) = \frac{1}{t} \sum_{k=0}^{\infty} y_i^{(k)} t^k, \qquad i = 0, 1, 2,$$

that is to say that $\varpi(x_i) = 2\varpi(y_i) = 2$ for $i = 0,1,2$. By substituting these solutions into the differential equations (3.1) associated with the vector field $\mathcal{V}_1$, the indicial locus is the subset of $\mathcal{H}$ given according to (2.2) by

$$\begin{aligned}
0 &= x_0^{(0)}\big(2 + y_0^{(0)}\big),\\
0 &= x_1^{(0)}\big(2 + y_1^{(0)}\big),\\
0 &= x_2^{(0)}\big(2 + y_2^{(0)}\big),\\
0 &= y_0^{(0)} + 2x_0^{(0)} - 2x_1^{(0)},\\
0 &= y_1^{(0)} - x_0^{(0)} + 2x_1^{(0)} - 2x_2^{(0)},\\
0 &= y_2^{(0)} - x_1^{(0)} + 2x_2^{(0)}.
\end{aligned}$$

These equations are easily solved and they yield the following (non-zero) solutions:

$$\begin{aligned}
m_0 &= (1, 0, 0, -2, 1, 0), &\qquad m_3 &= (0, 4, 3, 8, -2, -2),\\
m_1 &= (0, 1, 0, 2, -2, 1), &\qquad m_4 &= (1, 0, 1, -2, 3, -2),\\
m_2 &= (0, 0, 1, 0, 2, -2), &\qquad m_5 &= (4, 3, 0, -2, -2, 3).
\end{aligned}$$

The Kowalevski matrix at an arbitrary point $\left(x^{(0)}, y^{(0)}\right)$, solution of the indicial equation is given by

$$\mathcal{K}\left(x^{(0)}, y^{(0)}\right) = \begin{pmatrix} 2+y_0^{(0)} & 0 & 0 & x_0^{(0)} & 0 \\ 0 & 2-\frac{1}{2}y_0^{(0)}-y_2^{(0)} & 0 & -\frac{1}{2}x_1^{(0)} & -x_1^{(0)} \\ 0 & 0 & 2+y_2^{(0)} & 0 & x_2^{(0)} \\ 2 & -2 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 1 \end{pmatrix}.$$

Lemma 3.3. *The system of differential equation* (3.1) *of the vector field* $\mathcal{V}_1$ *has three distinct families of homogeneous Laurent solutions with weights depending on four* $(\dim \mathcal{H} - 1)$ *free parameters.*

Proof. The characteristic polynomial of the Kowalevski matrix at m_0 is given by

$$\mathcal{X}(\lambda; m_0) = (\lambda-1)(\lambda-3)(\lambda+1)(\lambda-2)^2.$$

It admits 4 non-negative eigenvalues which leads to a Laurent solution depending on four free parameters, whose the five leading terms (going with steps 1, 2, 2, 3, respectively, are denoted by a, c, d and e). The first four terms of this balance which we note $x(t, m_0)$:

$$\begin{aligned}
x_0(t,m_0) &= \frac{1}{t^2} + c - \frac{1}{2}ed + O\left(t^2\right), \\
x_1(t,m_0) &= et + O\left(t^2\right), \\
x_2(t,m_0) &= d - adt + O\left(t^2\right), \\
y_0(t,m_0) &= -\frac{2}{t} + 2ct - \frac{3}{2}et^2 + O\left(t^3\right), \\
y_1(t,m_0) &= \frac{1}{t} + a - (c+2d)t + \left(ad + \frac{5}{4}e\right)t^2 + O\left(t^3\right), \\
y_2(t,m_0) &= -a + 2dt - \left(ad + \frac{1}{2}e\right)t^2 + O\left(t^3\right).
\end{aligned} \tag{3.5}$$

The characteristic polynomial of the Kowalevski matrix at m_1 is given by

$$\mathcal{X}(\lambda; m_1) = (\lambda-4)(\lambda-1)(\lambda-2)(\lambda-3)(\lambda+1).$$

It admits 4 non-negative eigenvalues which leads to a Laurent solution depending on four free parameters, whose the five leading terms (going with steps 1, 2, 3, 4, respectively, are denoted by a, c, d and e). The first four terms of this balance which we note $x(t, m_1)$:

$$\begin{aligned}
x_0(t,m_1) &= et^2 + O\left(t^3\right), \\
x_1(t,m_1) &= \frac{1}{t^2} + c - \frac{1}{2}dt + \frac{1}{10}\left(6c^2 + ad - e\right)t^2 + O\left(t^3\right), \\
x_2(t,m_1) &= dt - \frac{1}{2}adt^2 + O\left(t^3\right), \\
y_0(t,m_1) &= \frac{2}{t} + a - 2ct + \frac{1}{2}dt^2 + \frac{1}{15}\left(11e - 6c^2 - ad\right)t^3 + O\left(t^4\right), \\
y_1(t,m_1) &= -\frac{2}{t} + 2ct - \frac{3}{2}dt^2 + \frac{2}{5}\left(c^2 + ad - e\right)t^3 + O\left(t^4\right), \\
y_2(t,m_1) &= \frac{1}{t} - \frac{1}{2}a - ct + \frac{5}{4}dt^2 + \frac{1}{30}\left(e - 6c^2 - 11ad\right)t^3 + O\left(t^4\right).
\end{aligned} \tag{3.6}$$

The characteristic polynomial of the Kowalevski matrix at m_2 is given by

$$\mathcal{X}(\lambda; m_2) = (\lambda+1)(\lambda-4)(\lambda-1)(\lambda-2)^2.$$

It admits 4 non-negative eigenvalues which leads to a Laurent solution depending on four free parameters, whose the five leading terms (going with steps 1, 2, 2, 4, respectively, are denoted by a, c, d and e). The first four terms of this balance which we note $x(t, m_2)$:

$$\begin{aligned}
x_0(t, m_2) &= c + act + \frac{1}{2}c\big(2c + a^2\big)t^2 + O\big(t^3\big),\\
x_1(t, m_2) &= et^2 + O\big(t^3\big),\\
x_2(t, m_2) &= \frac{1}{t^2} + d + \frac{1}{10}\big(6d^2 - e\big)t^2 + O\big(t^3\big),\\
y_0(t, m_2) &= a + 2ct + act^2 + \frac{1}{3}\big(2c^2 + a^2c - 2e\big)t^3 + O\big(t^4\big),\\
y_1(t, m_2) &= \frac{2}{t} - \frac{1}{2}a - (c + 2d)t - \frac{1}{2}act^2 - \frac{1}{30}\big(10c^2 + 5a^2c + 12d^2 - 22e\big)t^3 + O\big(t^4\big), \qquad (3.7)\\
y_2(t, m_2) &= -\frac{2}{t} + 2dt + \frac{2}{5}\big(d^2 - e\big)t^3 + O\big(t^4\big).
\end{aligned}$$

■

According to Adler and van Moerbeke [3], since the Toda problem is algebraic integrable, [2, Theorem 1] implies the existence of a coherent tree of Laurent solutions, satisfying all the conditions described in [4]. The existence of a coherent tree of Laurent solutions is crucial for obtaining the complete structure of the divisor $\mathcal{D}$ at infinity in [3]. In this work, we do not use a coherent tree of Laurent solutions, but only the so-called principal balances depending on 4 free parameters corresponding to m_0, m_1, m_2 and that m_3, m_4, m_5 correspond to lower balances depending on 3 free parameters.

3.2.2 Painlevé divisors of $a_4^{(2)}$ Toda lattice

We now search the formal Painlevé divisors, i.e., the algebraic curves defined by the three different principal balances $x(t, m_i)_{i=0,1,2}$, confined to a fixed affine invariant surface $\mathbb{F}_c$, $c \in \Omega$. We have the following assertion.

Proposition 3.4. *For the Laurent solution $x(t; m_0)$ restricted to the invariant surface, for $c \in \Omega$, the Painlevé divisor $\Gamma_c^{(0)}$ is a smooth genus three hyperelliptic curve. It is given by*

$$\begin{aligned}
\Gamma_c^{(0)}\colon\ & 16d^2a^8 - \big(256d^3 + 8d^2c_2\big)a^6 + \big(1536d^2 + 96dc_2 + 8c_3 + c_2^2\big)d^2a^4\\
&- \big(\big(8\big(8c_3 + 48dc_2 + c_2^2 + 512d^2\big)d + 2c_2c_3\big)d^2 + 64c_1\big)a^2\\
&+ \big(8d\big(c_2c_3 + 16dc_3 + 64d^2c_2 + 512d^3 + 2dc_2^2\big) + c_3^2\big)d^2 = 0.
\end{aligned}$$

It is completed in a Riemann surface, denoted $\overline{\Gamma}_c^{(0)}$ by adding 8 points to infinity.

Proof. Consider Laurent's solution $x(t; m_0)$ in (3.5). When it is substituted in the equations $F_i = c_i$, $i = 1, 2, 3$, where $c_i = (c_1, c_2, c_3) \in \Omega$, we get

$$c_1 = e^2d^2, \qquad c_2 = 4a^2 - 12c - 16d, \qquad c_3 = -12ca^2 + 48cd - 8ae.$$

By eliminating the parameters c and e, we obtain an algebraic relation between a and d which is the equation of an affine curve in $\mathbb{C}^2$ defined by

$$\begin{aligned}
\Gamma_c^{(0)}\colon\ & 16d^2a^8 - \big(256d^3 + 8d^2c_2\big)a^6 + \big(1536d^2 + 96dc_2 + 8c_3 + c_2^2\big)d^2a^4\\
&- \big(\big(8\big(8c_3 + 48dc_2 + c_2^2 + 512d^2\big)d + 2c_2c_3\big)d^2 + 64c_1\big)a^2\\
&+ \big(8d\big(c_2c_3 + 16dc_3 + 64d^2c_2 + 512d^3 + 2dc_2^2\big) + c_3^2\big)d^2 = 0.
\end{aligned}$$

The affine curve $\Gamma_c^{(0)}$ is smooth for $c \in \Omega$. It is completed in a Riemann surface, denoted $\overline{\Gamma}_c^{(0)}$ by adding 8 points to infinity denoted by ∞_ϵ, $\infty_{\epsilon_1\epsilon_2}$ and $\infty_{\epsilon_3}^3$ with $\epsilon^2 = \epsilon_1^2 = \epsilon_2^2 = \epsilon_3^2 = 1$ and $\delta = \sqrt{c_2^2 - 16c_3}$. A neighborhood of each of these points is described according to a local parameter ς by

$$\infty_\epsilon: \qquad a = \varsigma^{-1}, \qquad d = \frac{1}{2}\epsilon c_2\sqrt{c_1}\varsigma^5 + 2\epsilon\sqrt{c_1}\varsigma^3 + O(\varsigma^5), \tag{3.8}$$

$$\infty_{\epsilon_1\epsilon_2}: \quad a = \varsigma^{-1}, \qquad d = \frac{\frac{1}{4} + \left(-\frac{1}{32}c_2 + \frac{1}{32}\epsilon_1\delta\right) + 8\epsilon_2\varsigma^3\sqrt{\frac{c_1}{c_2^2-16c_3}}}{\varsigma^2} + O(\varsigma), \tag{3.9}$$

$$\infty_{\epsilon_3}^3: \quad a = \varsigma, \qquad d = -\frac{256c_1c_2\varsigma^2}{c_3^3} + 8\epsilon_3\varsigma\sqrt{\frac{c_1}{c_3^2}} + O(\varsigma^3), \tag{3.10}$$

This completes the proof of Proposition 3.4. ■

Proposition 3.5. *For the Laurent solution $x(t; m_1)$ restricted to the invariant surface, for $c \in \Omega$, the Painlevé divisor $\Gamma_c^{(1)}$ is a smooth genus four curve. It is given by*

$$\Gamma_c^{(1)}: \ 256ad^3 - \left(\left(4a^2 - c_2\right)^2 - 16c_3\right)d^2 + 64c_1 = 0.$$

It is completed in a Riemann surface, denoted $\overline{\Gamma}_c^{(1)}$ by adding 4 points to infinity.

Proof. Consider Laurent's solution $x(t; m_1)$ in (3.6). When it was substituted in the equations $F_i = c_i$, $i = 1, 2, 3$, where $c_i = (c_1, c_2, c_3) \in \Omega$, we obtain

$$c_1 = ed^2, \qquad c_2 = 2a^2 - 24c, \qquad c_3 = \frac{1}{4}a^4 + 6a^2c - 16ad + 36c^2 - 4e$$

by eliminating the parameters c and e, we obtain an algebraic relation between a and d which is the equation of an affine curve in $\mathbb{C}^2$ defined by

$$\Gamma_c^{(1)}: \ 256ad^3 - \left(\left(4a^2 - c_2\right)^2 - 16c_3\right)d^2 + 64c_1 = 0.$$

The affine curve $\Gamma_c^{(1)}$ is smooth for $c \in \Omega$. Indeed, let

$$g(a, d) = 256ad^3 - \left(\left(4a^2 - c_2\right)^2 - 16c_3\right)d^2 + 64c_1,$$

we have

$$\frac{\partial g}{\partial a}(a, d) = 16d^2\left(4a^3 - ac_2 - 16d\right),$$
$$\frac{\partial g}{\partial d}(a, d) = 2d\left(16a^4 - 8a^2c_2 - 384ad + c_2^2 - 16c_3\right).$$

Thus, a point (a, d) is singular for the affine curve $\Gamma_c^{(1)}$ if

$$g(a, d) = \frac{\partial g}{\partial a}(a, d) = \frac{\partial g}{\partial d}(a, d) = 0$$

as $d \neq 0$ since $c_1 \neq 0$, then

$$\begin{cases} 4a^3 - ac_2 - 16d = 0, \\ 16a^4 - 8a^2c_2 - 384ad + c_2^2 - 16c_3 = 0, \\ -256ad^3 + \left(16a^4 - 8a^2c_2 - 384ad + c_2^2 - 16c_3\right)d^2 - 64c_1 = 0, \end{cases}$$

$$\Leftrightarrow \begin{cases} 4a^3 - ac_2 - 16d = 0, \\ 16a^4 - 8a^2c_2 - 384ad + c_2^2 - 16c_3 = 0, \\ 2ad^3 - c_1 = 0. \end{cases}$$

By expressing a as a function of d in the last equation of the system and substituting the expression into both first equations, we obtain

$$a = \frac{c_1}{2d^3}, \qquad -c_1^3 + c_1c_2d^6 + 32d^{10} = 0,$$
$$c_1 - 2c_1^2c_2d^6 + c_2^2d^{12} - 192c_1d^{10} - 16c_3d^{12} = 0.$$

The resultant of these two last polynomials in d is given by the following polynomial in terms of c_1, c_2 and c_3 up to a constant

$$c_1^{36}\big(819200000c_1^2 + 512000c_3^2c_2c_1 - 57600c_3c_2^3c_1 - 32c_3^4c_2^2 + c_3^3c_2^4 + 256c_3^5 + 1728c_2^5c_1\big)^2.$$

This expression is not zero for $c \in \Omega$. We deduce that $\Gamma_c^{(1)}$ is a smooth affine curve for $c \in \Omega$. It is complete into a Riemann surface, denoted $\overline{\Gamma}_c^{(1)}$ by adding four points at infinity denoted by ∞^1, ∞^2 and ∞_ϵ. A neighborhood of each of these points is described according to a local parameter ς by

$$\infty_\epsilon\colon\ d = \varepsilon\sqrt{4}\sqrt{c_1}\varsigma^2, \qquad a = \frac{\big(\frac{c_2^2}{128} - \frac{c_3}{4}\big)\varsigma^5 + \frac{c_2\varsigma^3}{8} + \varsigma}{\varsigma^2} + O\big(\varsigma^3\big), \tag{3.11}$$

$$\infty^1\colon\ d = \varsigma^{-1}, \qquad a = \left(\frac{c_2^2}{256} - \frac{c_3}{16}\right)\varsigma + O\big(\varsigma^2\big), \tag{3.12}$$

$$\infty^2\colon\ d = \frac{1}{16\varsigma^3}, \qquad a = \frac{\frac{1}{3}\big(16c_3 - c_2^2\big)\varsigma^6 + \frac{8c_2}{3}\varsigma^4 + 16\varsigma^2}{16\varsigma^3} + O\big(\varsigma^3\big), \tag{3.13}$$

This completes the proof of Proposition 3.5. ■

Proposition 3.6. *For the Laurent solution $x(t; m_2)$ restricted to the invariant surface, for $c \in \Omega$, the Painlevé divisor $\Gamma_c^{(2)}$ is a smooth genus two hyperelliptic curve. It is given by*

$$\Gamma_c^{(2)}\colon\ e^4a^4 - \big(8c_1 + c_2e^2\big)a^2e^2 - 64e^5 + 4e^2c_1c_2 + 4c_3e^4 + 16c_1^2 = 0.$$

It is completed in a Riemann surface, which is a double covering of $\mathbb{P}^1$ ramified into 5 points.

Proof. For a point $c = (c_1, c_2, c_3) \in \mathbb{C}^3$, by substituting the Laurent's solution $x(t; m_2)$ of (3.7) inside equations $F_i = c_i$, $i = 1, 2, 3$, we find the independent algebraic expressions of t, namely the three algebraic relations between the parameters a, c, d and e below

$$c_1 = ce^2, \qquad c_2 = a^2 - 4c - 48d, \qquad c_3 = -12a^2d + 48cd + 16e.$$

For $c \in \Omega$, $c_1 \neq 0$ therefore the parameters c, d and e are not zero. The first two equations are linear in the parameters c and d, and can be solved linearly in these parameters as a function of the constants of motion thus giving

$$c = \frac{c_1}{e^2} \qquad \text{and} \qquad d = \frac{1}{48}\left(a^2 - c_2 - \frac{4c_1}{e^2}\right).$$

The third equation then reduces to the following equation of an affine curve in $\mathbb{C}^2$.

$$\Gamma_c^{(2)}\colon\ e^4a^4 - \big(8c_1 + c_2e^2\big)a^2e^2 - 64e^5 + 4e^2c_1c_2 + 4c_3e^4 + 16c_1^2 = 0. \tag{3.14}$$

The affine curve $\Gamma_c^{(2)}$ is smooth for $c \in \Omega$. In fact, let

$$f(a,e) = e^4a^4 - \big(8c_1 + c_2e^2\big)a^2e^2 - 64e^5 + 4e^2c_1c_2 + 4c_3e^4 + 16c_1^2,$$

we have

$$\frac{\partial f}{\partial a}(a,e) = 4a^3e^4 + 2a\big(-8c_1e^2 - c_2e^4\big),$$
$$\frac{\partial f}{\partial e}(a,e) = 4a^4e^3 + \big(-16c_1e - 4c_2e^3\big)a^2 - 320e^4 + 8c_1c_2e + 16c_3e^3.$$

So, a point (a,e) is singular for the affine curve $\Gamma_c^{(2)}$ if $f(a,e) = \frac{\partial f}{\partial a}(a,e) = \frac{\partial f}{\partial e}(a,e) = 0$ as $e \neq 0$ so either $a = 0$, or $a^2 = \frac{8c_1+c_2e^2}{2e^2}$.

- If $a^2 = \frac{8c_1+c_2e^2}{2e^2}$, after substitution in the equations $f(a,e) = 0$ and $\frac{\partial f}{\partial e}(a,e) = 0$, we then obtain

$$-\frac{1}{4}e^4\big(c_2^2 + 256e - 16c_3\big) = 0 \qquad \text{and} \qquad -e^3\big(c_2^2 + 320e - 16c_3\big) = 0.$$

  As $e \neq 0$, we have $c_2^2 - 16c_3 = -256e = -320e$, which implies that $e = 0$, which is absurd!
- If $a = 0$, this leads to the following system:

$$-64e^5 + 4c_1c_2e^2 + 4c_3e^4 + 16c_1^2 = 0,$$
$$-320e^4 + 8c_1c_2e + 16c_3e^3 = 0.$$

  The resultant of these two polynomials in e composing the system is given up to a constant by the following polynomial in terms of c_1, c_2 and c_3:

$$c_1\big(819200000c_1^2 + 512000c_3^2c_2c_1 - 57600c_3c_2^3c_1 - 32c_3^4c_2^2 + c_3^3c_2^4 + 256c_3^5 + 1728c_2^5c_1\big).$$

This expression is not zero for $c \in \Omega$. We deduce that $\Gamma_c^{(2)}$ is a smooth affine curve for $c \in \Omega$.

The equation (3.14) of the affine curve $\Gamma_c^{(2)}$ can be written as

$$\left(a^2 - \frac{4c_1}{e^2}\right)\left(a^2 - \frac{4c_1}{e^2} - c_2\right) = 64e - 4c_3.$$

We deduce that $\Gamma_c^{(2)}$ is a double cover of the rational affine curve

$$\varepsilon_c^{(2)}\colon\ u(u - c_2) - 64e + 4c_3 = 0,$$

the map which links the two curves is explicitly given by

$$\psi\colon\ \Gamma_c^{(2)} \longrightarrow \varepsilon_c^{(2)}, \qquad (a,e) \longmapsto (u,e) = \left(a^2 - \frac{4c_1}{e^2}, e\right).$$

$\varepsilon_c^{(2)}$ being irreducible curve of genus 0, a parametrization of $\varepsilon_c^{(2)}$ is given by

$$\varepsilon_c^{(2)} = \left\{(u,e) = \left(t, \frac{1}{64}\big(t^2 - c_2t + 4c_3\big)\right),\ t \in \mathbb{C}\right\}.$$

If $\psi(a,e) = (u,e)$, then we have

$$a = \pm\sqrt{\frac{t^5 - 2c_2t^4 + \big(8c_3 + c_2^2\big)t^3 - 8c_2c_3t^2 + 16c_3^2t + 16384c_1}{\big(t^2 - c_2t + 4c_3\big)^2}}, \qquad e = \frac{1}{64}\big(t^2 - c_2t + 4c_3\big).$$

The branch points of the cover $\psi\colon \Gamma_c^{(2)} \longrightarrow \varepsilon_c^{(2)}$ are the points (u,e) of the curve $\varepsilon_c^{(2)}$ for which $a=0$, i.e., where u is a root of the polynomial of degree 5 given by

$$P(t) = t^5 - 2c_2t^4 + \left(8c_3 + c_2^2\right)t^3 - 8c_2c_3t^2 + 16c_3^2t + 16384c_1,$$

we verify that the discriminant of $P(t)$ is given, up to a constant, by

$$c_1^2\left(819200000c_1^2 + 512000c_1c_2c_3^2 - 57600c_1c_2^3c_3 + 1728c_1c_2^5 + 256c_3^5 - 32c_2^2c_3^4 + c_2^4c_3^3\right),$$

which for $c \in \Omega$ is not zero. Therefore, these five branch points are distinct. Thus, the map ψ admits five ramification points on $\Gamma_c^{(2)}$.

The curve $\Gamma_c^{(2)}$ can be completed into a compact Riemann surface, denoted $\overline{\Gamma}_c^{(2)}$ by adding to it five points to infinity denoted by ∞, $\infty_{\epsilon_1\epsilon_2}$ where $\epsilon_1^2 = \epsilon_2^2 = 1$ and $\delta = \sqrt{c_2^2 - 16c_3}$. Neighborhoods of these points are described according to a local parameter ς by

$$\infty_{\epsilon_1\epsilon_2}\colon\ a = \varsigma^{-1}, \qquad e = 2\epsilon_1\sqrt{c_1}\varsigma\left(1 + \frac{1}{4}(c_2 + \epsilon_2\delta)\varsigma^2 + O\left(\varsigma^4\right)\right), \tag{3.15}$$

$$\infty\colon\qquad a = \varsigma^{-1}, \qquad e = \frac{1}{64}\left(\varsigma^{-4} - c_2\varsigma^{-2} + 4c_3 + O\left(\varsigma^6\right)\right). \tag{3.16}$$

When the application ψ is extended in application $\overline{\psi}\colon \overline{\Gamma}_c^{(2)} \longrightarrow \overline{\varepsilon}_c^{(2)}$, there is another branching point. Indeed, if we write $t = \frac{1}{\varsigma^2}$ depending on a local parameter ς, we obtain the point at infinity $\infty \in \overline{\Gamma}_c^{(2)} \backslash \Gamma_c^{(2)}$ given by

$$(a,e) = \left(\varsigma^{-1}, \frac{1}{64}\left(\varsigma^{-4} - c_2\varsigma^{-2} + 4c_3 + O\left(\varsigma^6\right)\right)\right).$$

There is no other branching point. Indeed, noting $t = t_1 + \varsigma^2$ and $t = t_2 + \varsigma^2$ local parameterizations in the neighborhood respectively of t_1 and t_2 the roots of $t^2 - c_2t + 4c_3$, it follows that

$$a = \pm\frac{128}{\varsigma}\sqrt{\frac{{c_1}^2}{c\ _2} - 16c_3} + O(1),$$

then respectively

$$e = \frac{1}{64}c_2(c_2 + \delta) + O\left(\varsigma^2\right) \qquad \text{and} \qquad e = \frac{1}{64}c_2(c_2 - \delta) + O\left(\varsigma^2\right),$$

which shows that above from the points t_1 and t_2, the map $\overline{\psi}$ is not ramified. We then conclude that the application $\overline{\psi}\colon \overline{\Gamma}_c^{(2)} \longrightarrow \overline{\varepsilon}_c^{(2)}$ is a double covering of $\mathbb{P}^1$ branched into 5 points. We deduce that the genus of $\overline{\Gamma}_c^{(2)}$ is equal to 2 according to the Riemann–Hurwitz formula. ∎

Remark 3.7. The above propositions only compute an affine part of the divisor $\mathcal{D}$, since the lower balances depending on 3 free parameters are not used. The completed non singular curves of respective genus 3, 4 and 2 mentioned in these propositions correspond to blowing up the singular points of the three irreducible components of $\mathcal{D}$.

3.3 Abelian surface

According to [7], in order to embed the three Riemann surfaces $\overline{\Gamma}_c^{(0)}$, $\overline{\Gamma}_c^{(1)}$ and $\overline{\Gamma}_c^{(2)}$ into some projective space, one of the key underlying principles used the Kodaira embedding theorem, which states that a smooth complex manifold can be smoothly embedded into projective space $\mathbb{P}^N(\mathbb{C})$ with the set of functions having a pole of order k along positive divisor on the manifold, provided k is large enough; fortunately, for abelian surfaces, k need not be larger than three according to Lefschetz theorem. These functions are easily constructed from the three Laurent solutions by looking for polynomials in the phase variables which in the expansions have at most a k-fold pole. The nature of the expansions and some algebraic properties of abelian varieties provide a recipe for when to terminate our search for such functions, thus making the procedure implementable. Precisely, we wish to find a set of polynomial functions $\{z_0, \dots, z_N\}$, of increasing degree in the original variables $x_0, \dots, y_2$ having the property that the embedding $\mathcal{D}$ of $\overline{\Gamma}_c^{(i)}$, $i = 0, 1, 2$, into $\mathbb{P}^N(\mathbb{C})$ via those functions satisfies the relation: $g(\mathcal{D}) = N + 2$ where $g(\mathcal{D})$ is the arithmetic genus of $\mathcal{D}$. If the $a_4^{(2)}$ Toda lattice is an irreducible algebraic complete integrable system, then as we have seen above a divisor $\mathcal{D}$ can be added to a Zariski open subset of $\mathcal{H}$, having the effect of compacting all fibers $\mathbb{F}_c$, where $c \in \Omega$. The divisor that is added to $\mathbb{F}_c$ will be denoted by $\mathcal{D}_c$ and the resulting torus by $\mathbb{T}_c^2$. The vector fields $\mathcal{V}_1$ and $\mathcal{V}_2$ extend to linear (hence holomorphic) vector fields on this partial compactification of $\mathcal{H}$, hence we may consider the integral curves of $\mathcal{V}_1$, starting from any component $\mathcal{D}_c^{(i)}$. Since the third power of an ample divisor on an abelian variety is very ample, we look for all polynomials which have a simple pole at most when any of the three principal balances are substituted in them. Precisely, we look for a maximal independent set of functions which are independent when restricted to $\mathbb{F}_c$. By direct computation, we obtain twenty-five (25) weight homogeneous polynomials of weight at most 13.

These twenty-five (25) functions are defined by

$$
\begin{aligned}
&z_0 = 1, && z_{11} = x_1x_2(y_0 - 2y_2),\\
&z_1 = y_0, && z_{12} = -x_1x_2z_4,\\
&z_2 = y_2, && z_{13} = x_0x_1\big(x_1 - y_2^2\big),\\
&z_3 = y_2^2 - x_1 - 4x_2, && z_{14} = x_0x_1x_2,\\
&z_4 = y_0y_2 - 2x_1, && z_{15} = x_0x_1x_2y_2,\\
&z_5 = y_2(y_0y_2 - 2x_1) - 4x_2y_0, && z_{16} = x_0x_1(-y_0z_4 + 4x_0y_2),\\
&z_6 = 2y_2\big(y_2^2 - 4x_2\big) + x_1(y_0 - 4y_2), && z_{17} = x_1x_2((y_0 - 2y_2)z_4 + 8x_2y_0),\\
&z_7 = x_0x_1, && z_{18} = x_0x_1x_2\big(4x_2 - y_2^2\big),\\
&z_8 = x_1x_2, && z_{19} = x_0x_1x_2\big(4x_0 - y_0^2\big),\\
&z_9 = -y_0y_2z_3 + 2x_1\big(y_2^2 - x_1\big), && z_{20} = -x_0x_1x_2(-y_0z_4 + 4x_0y_2),\\
&z_{10} = x_0x_1y_2, && z_{21} = -x_0x_1^2x_2(y_0 - 2y_2),
\end{aligned}
$$
$$
\begin{aligned}
&z_{22} = -x_0x_1x_2\big(y_0y_2(y_0y_2 - 2x_1) + 4x_0\big(4x_2 - y_2^2\big) - 4x_2y_0^2\big),\\
&z_{23} = x_0x_1(4y_2x_0 + y_0(2x_1 - y_0y_2))\big(4x_0y_2^2 - y_0^2y_2^2 + 4x_1y_0y_2 - 4x_1^2\big),\\
&z_{24} = x_2x_1^3x_0^2.
\end{aligned}
\tag{3.17}
$$

For $c = (c_1, c_2, c_3) \in \Omega$, we consider the regular map

$$
\varphi_c\colon\ \mathbb{F}_c \subset \mathcal{H} \longrightarrow \mathbb{P}^{24}, \qquad (x_0, \dots, y_2) \longmapsto (1, z_1, \dots, z_{24}), \tag{3.18}
$$

where the functions z_i are given by (3.17); this map is embedding of $\mathbb{F}_c$ in the projective space $\mathbb{P}^{24}(\mathbb{C})$.

Notice that in this section, the strategy is to embed $\mathbb{F}_c \cup (\mathcal{D}\ \{\text{singular points}\}) \longrightarrow \mathbb{P}^{24}$, then show that the two independent vector fields $\mathcal{V}_1$, $\mathcal{V}_2$ extend holomorphically to the closure of this embedding in $\mathbb{P}^{24}$, therefore proving the algebraic complete integrability of the $a_4^{(2)}$ Toda lattice, by the complex Liouville theorem.

3.3.1 Embedding in the projective space $\mathbb{P}^{24}$ and singularities of the divisor at infinity

In order to show that the $a_4^{(2)}$ Toda lattice is algebraic completely integrable, we show that, for $c \in \Omega$, the fiber

$$\mathbb{F}_c := \mathbb{F}^{-1}(c) = \bigcap_{i=1}^{3} \{m \in \mathcal{H} \mid F_i(m) = c_i\}$$

is an affine part of abelian surface, on which the vector fields $\mathcal{V}_1$ and $\mathcal{V}_2$ restrict to linear vector fields. To do this, we must check that $\mathbb{F}_c$: satisfies the conditions of the complex Liouville theorem (Theorem (2.7)). Using (3.18) and let $t \to 0$, we see that the coefficients of t^{-1} of series $z_i(t; m_0)$ define an application

$$\varphi_c^{(0)} \colon \; \Gamma_c^{(0)} \longrightarrow \mathbb{P}^{24}$$

given by

$$\begin{aligned} \varphi_c^{(0)} \colon \; (a, e) \mapsto (&0 : -2 : 0 : 0 : 2a : 8d - 2a^2 : 0 : 0 : e : -2a(-4d + a^2) : -ae : 0 : \\ & - a^2 e : 0 : de : -ade : -4e(e + 3ac) : -4ade : de(4d - a^2) : 12cde : \\ & 4ed(e + 3ac) : 2de^2 : 4de(3a^2 c + ae - 12cd) : \\ & - 16ae(3ac + 2e)(3ac + e) : e^3 d), \end{aligned} \tag{3.19}$$

which is, for $c \in \Omega$, an embedding of the affine curve Γ_c. Similarly, the series $z_i(t; m_1)$ and $z_i(t; m_2)$ define two embedding $\varphi_c^{(1)}$ and $\varphi_c^{(2)}$ of the affine curves $\Gamma_c^{(1)}$ and $\Gamma_c^{(2)}$ respectively in the projective space $\mathbb{P}^{24}$ given by

$$\begin{aligned} \varphi_c^{(1)} \colon \; (a, e) \mapsto \Big(&0 : 2 : 1 : -a : 0 : -6c - \frac{a^2}{2} : -6c + \frac{3a^2}{2} : d : -8d + 6ac + \frac{a^3}{2} : \\ & 0 : e : 2ad : -8ea : \frac{d(a^2 + 12c)}{2} : 0 : 0 : e(a^2 + 12c) : \\ & - d(-16d + 12ca + a^3) : -ed : ed : 0 : -2aed : de(12c + a^2) : \\ & - \frac{1}{4} e(12c + a^2)(a^4 + 24ca^2 - 16e + 144c^2) : de^2 \Big), \end{aligned} \tag{3.20}$$

$$\begin{aligned} \varphi_c^{(2)} \colon \; (a, e) \mapsto (&0 : 0 : -2 : 0 : -2a : 0 : 48d : 0 : 24ad : 0 : 0 : 4e : 0 : 2ae : \\ & 0 : -2ce : 0 : -2a^2 e : 0 : 0 : -2ce(-4c + a^2) : 0 : 0 : \\ & - 8ce(-4c + a^2)^2 : 0). \end{aligned} \tag{3.21}$$

Remind that in (3.21) $c = \frac{c_1}{e^2}$ and $d = \frac{1}{48}\left(a^2 - c_2 - \frac{4c_1}{e^2}\right)$, in (3.20) $c = \frac{1}{24}\left(2a^2 - c_2\right)$ and $e = \frac{c_1}{d^2}$, in (3.19) $c = \frac{1}{12}\left(4a^2 - 16d - c_2\right)$ and $e^2 = \frac{c_1}{d^2}$.

Looking at the first three coordinates, we observe that the image curves by the embedding $\varphi_c^{(i)}$ are distinct. However, they are not complete, so we check if maybe their closures intersect.

Let us determine the singularities of the divisor at infinity. Let us denote by $\mathcal{D}_c^{(0)}$, $\mathcal{D}_c^{(1)}$ and $\mathcal{D}_c^{(2)}$, respectively, the closures of

$$\overline{\varphi_c^{(0)}(\Gamma_c^{(0)})}, \quad \overline{\varphi_c^{(1)}(\Gamma_c^{(1)})} \quad \text{and} \quad \overline{\varphi_c^{(2)}(\Gamma_c^{(2)})},$$

and let $\mathcal{D}_c = \bigcup_{i=0}^{2} \mathcal{D}_c^{(i)}$. Let us determine the singularity of the divisor $\mathcal{D}_c$. To do this, let us substitute the local parametrization ς around each point at infinity in the corresponding $\varphi_c^{(i)}$ embedding and let $\varsigma \to 0$, we find the following leading terms, where $\epsilon = \epsilon_1 = \epsilon_2 = \epsilon_3 = \pm 1$.

By substituting (3.8), (3.9) and (3.10) in (3.19) and considering the first two terms, we have

$$\begin{aligned}
&\varphi_c^{(0)}(\infty_\epsilon) \sim \left(0:\cdots:0:1:0:0:0:0:0:0:-c_3:\frac{\epsilon\sqrt{c_1}}{4}\right),\\
&\varphi_c^{(0)}(\infty_{\epsilon_1\epsilon_2}) \sim \Bigg(0:0:0:0:2:0:\cdots:0:\frac{-c_2+\epsilon_2\delta}{4}:0:\cdots:0:\epsilon_1\epsilon_2\sqrt{c_1}:0:4\epsilon_1\epsilon_2\sqrt{c_1}\\
&\qquad 0:0:\frac{\epsilon_2\sqrt{c_1}(\epsilon_1 c_2+\epsilon_1\epsilon_2\delta)}{2}:0:0:\epsilon_1\epsilon_2\sqrt{c_1}(\epsilon_1\epsilon_2 c_2+\epsilon_1\delta)^2:0\Bigg),\\
&\varphi_c^{(0)}(\infty_{\epsilon_3}^3) \sim \left(0:\cdots:0:1:0:0:0:0:0:0:-c_3:\frac{\epsilon_3\sqrt{c_1}}{4}\right).
\end{aligned} \tag{3.22}$$

Substituting $\epsilon=\epsilon_1=\epsilon_2=\epsilon_3=\pm1$ in (3.22), we have

$$\begin{aligned}
&P_+ := \lim_{p\to\infty_+}\varphi_c^{(0)}(p) = \lim_{p\to\infty_+^3}\varphi_c^{(0)}(p) = \left(0:0:0:\cdots:0:1:0:\cdots:0:-c_3:\frac{\sqrt{c_1}}{4}\right),\\
&P_- := \lim_{p\to\infty_-}\varphi_c^{(0)}(p) = \lim_{p\to\infty_-^3}\varphi_c^{(0)}(p) = \left(0:0:0:\cdots:0:1:0:\cdots:0:-c_3:-\frac{\sqrt{c_1}}{4}\right),\\
&Q_{++} := \lim_{p\to\infty_{++}}\varphi_c^{(0)}(p) = \Bigg(0:0:0:0:2:0:0:0:0:0:\frac{-c_2+\delta}{4}:0:0:0:0:0:0:\sqrt{c_1}:0:\\
&\qquad 4\sqrt{c_1}:0:0:\frac{\sqrt{c_1}(c_2+\delta)}{2}:0:0:\sqrt{c_1}(c_2+\delta)^2:0\Bigg),\\
&Q_{-+} := \lim_{p\to\infty_{-+}}\varphi_c^{(0)}(p) = \Bigg(0:0:0:0:2:0:0:0:0:0:\frac{-c_2+\delta}{4}:0:0:0:0:0:0:-\sqrt{c_1}:\\
&\qquad 0:-4\sqrt{c_1}:0:0:\frac{\sqrt{c_1}(-c_2-\delta)}{2}:0:0:-\sqrt{c_1}(c_2+\delta)^2:0\Bigg),\\
&Q_{+-} := \lim_{p\to\infty_{+-}}\varphi_c^{(0)}(p) = \Bigg(0:0:0:0:2:0:0:0:0:0:\frac{-c_2-\delta}{4}:0:0:0:0:0:0:-\sqrt{c_1}:\\
&\qquad 0:-4\sqrt{c_1}:0:0:\frac{\sqrt{c_1}(-c_2+\delta)}{2}:0:0:-\sqrt{c_1}(-c_2+\delta)^2:0\Bigg),\\
&Q_{--} := \lim_{p\to\infty_{--}}\varphi_c^{(0)}(p) = \Bigg(0:0:0:0:2:0:0:0:0:0:\frac{-c_2-\delta}{4}:0:0:0:0:0:0:\sqrt{c_1}:0:\\
&\qquad 4\sqrt{c_1}:0:0:\frac{\sqrt{c_1}(c_2-\delta)}{2}:0:0:\sqrt{c_1}(-c_2-\delta)^2:0\Bigg).
\end{aligned}$$

The point P_+ is the image of the two points at infinity ∞_+ and ∞_+^3 while P_- is the image of two points at infinity ∞_- and ∞_-^3. We deduce that $\varphi_c^{(0)}$ does not extend into an embedding of $\overline{\varphi_c^{(0)}}$. The curve $\mathcal{D}_c^{(0)}$ is therefore singular at the points P^ϵ.

By substituting (3.11), (3.12) and (3.13) in (3.20) and consider the two first terms, we obtain

$$\begin{aligned}
&\varphi_c^{(1)}(\infty^1) \sim (0:0:0:\cdots:0:-\varsigma:0:0:0:1:0:\cdots:0),\\
&\varphi_c^{(1)}(\infty^2) \sim \left(0:0:0:\cdots:0:\frac{1}{16}\varsigma:-\frac{1}{2}\varsigma:0:0:0:0:-\frac{c_2}{64}\varsigma:0:0:0:1:0:\cdots:0\right),\\
&\varphi_c^{(1)}(\infty_\epsilon) \sim \left(0:0:0:\cdots:0:-4\varsigma:0:0:0:1:0:0:0:0:0:0:-c_3:\varepsilon\frac{\sqrt{c_1}}{4}\right).
\end{aligned}$$

Letting $\varsigma\to0$ and $\epsilon=\pm1$, we obtain the points in projective space $\mathbb{P}^{24}$

$$P_+ := \lim_{p\to\infty_+}\varphi_c^{(1)}(p) = \left(0:0:0:\cdots:0:1:0:0:0:0:0:0:-c_3:\frac{\sqrt{c_1}}{4}\right),$$

$$P_{-} := \lim_{p\to\infty_-} \varphi_c^{(1)}(p) = \left(0:0:0:\cdots:0:1:0:0:0:0:0:0:-c_3:-\frac{\sqrt{c_1}}{4}\right),$$
$$T := \lim_{p\to\infty^1} \varphi_c^{(1)}(p) = \lim_{p\to\infty^2} \varphi_c^{(1)}(p) = (0:0:0:0:\cdots:1:0:0:0:0:0:0:0).$$

The points P_+ and P_- are distinct but T is the image of two points at infinity ∞_1 and ∞_2. We then deduce that $\varphi_c^{(1)}$ does not extend into an embedding of $\overline{\varphi}_c^{(1)}$. The curve $\mathcal{D}_c^{(1)}$ is therefore singular at the point T. By substituting (3.15) and (3.16) in (3.21) and considering that the first three terms and the fact that $\epsilon_1^2 = \epsilon_2^2 = 1$, we obtain the following points in the projective space $\mathbb{P}^{24}$:

$$\varphi_c^{(2)}(\infty_{++}) \sim \Big(0:0:0:0:2:0:0:0:\frac{-c_2+\delta}{4}:0:0:0:0:0:0:\sqrt{c_1}:0:4\sqrt{c_1}:$$
$$0:0:\frac{\sqrt{c_1}(c_2+\delta)}{2}:0:0:\sqrt{c_1}(c_2+\delta)^2:0\Big),$$
$$\varphi_c^{(2)}(\infty_{-+}) \sim \Big(0:0:0:0:2:0:0:0:\frac{-c_2+\delta}{4}:0:0:0:0:0:0:-\sqrt{c_1}:0:-4\sqrt{c_1}:$$
$$0:0:\frac{\sqrt{c_1}(-c_2-\delta)}{2}:0:0:-\sqrt{c_1}(c_2+\delta)^2:0\Big),$$
$$\varphi_c^{(2)}(\infty_{+-}) \sim \Big(0:0:0:0:2:0:0:0:\frac{-c_2-\delta}{4}:0:0:0:0:0:0:-\sqrt{c_1}:0:-4\sqrt{c_1}:$$
$$0:0:\frac{\sqrt{c_1}(-c_2+\delta)}{2}:0:0:-\sqrt{c_1}(-c_2+\delta)^2:0\Big),$$
$$\varphi_c^{(2)}(\infty_{--}) \sim \Big(0:0:0:0:2:0:0:0:\frac{-c_2-\delta}{4}:0:0:0:0:0:0:\sqrt{c_1}:0:4\sqrt{c_1}:$$
$$0:0:\frac{\sqrt{c_1}(c_2-\delta)}{2}:0:0:\sqrt{c_1}(-c_2-\delta)^2:0\Big),$$
$$\varphi_c^{(2)}(\infty) \sim (0:0:0:\cdots:0:-\varsigma:0:0:0:1:0:0:0:0:0:0:0).$$

Hence, making $\varsigma \to 0$, we obtain

$$Q_{++} := \lim_{p\to\infty_{++}} \varphi_c^{(2)}(p) = \Big(0:0:0:0:2:0:0:0:\frac{-c_2+\delta}{4}:0:0:0:0:0:0:\sqrt{c_1}:0:$$
$$4\sqrt{c_1}:0:0:\frac{\sqrt{c_1}(c_2+\delta)}{2}:0:0:\sqrt{c_1}(c_2+\delta)^2:0\Big),$$
$$Q_{-+} := \lim_{p\to\infty_{-+}} \varphi_c^{(2)}(p) = \Big(0:0:0:0:2:0:0:0:\frac{-c_2+\delta}{4}:0:0:0:0:0:0:-\sqrt{c_1}:$$
$$0:-4\sqrt{c_1}:0:0:\frac{\sqrt{c_1}(-c_2-\delta)}{2}:0:0:-\sqrt{c_1}(c_2+\delta)^2:0\Big),$$
$$Q_{+-} := \lim_{p\to\infty_{+-}} \varphi_c^{(2)}(p) = \Big(0:0:0:0:2:0:0:0:\frac{-c_2-\delta}{4}:0:0:0:0:0:0:-\sqrt{c_1}:$$
$$0:-4\sqrt{c_1}:0:0:\frac{\sqrt{c_1}(-c_2+\delta)}{2}:0:0:-\sqrt{c_1}(-c_2+\delta)^2:0\Big),$$
$$Q_{--} := \lim_{p\to\infty_{--}} \varphi_c^{(2)}(p) = \Big(0:0:0:0:2:0:0:0:\frac{-c_2-\delta}{4}:0:0:0:0:0:0:\sqrt{c_1}:0:$$
$$4\sqrt{c_1}:0:0:\frac{\sqrt{c_1}(c_2-\delta)}{2}:0:0:\sqrt{c_1}(-c_2-\delta)^2:0\Big),$$
$$T := \lim_{p\to\infty} \varphi_c^{(2)}(p) = (0:0:0:\cdots:0:1:0:\cdots:0).$$

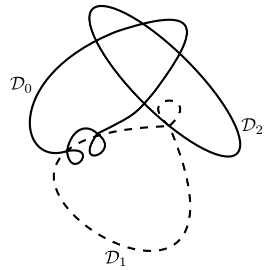

Figure 1. Curves completing the invariant surfaces $\mathbb{F}_c$ of the $a_4^{(2)}$ Toda lattice in abelian surfaces, where $\mathcal{D}_i$ is the curve $\mathcal{D}_c^{(i)}$.

The points $Q_{\epsilon_1\epsilon_2}$, T are all distinct. We then deduce that $\varphi_c^{(2)}$ extends into an embedding of $\overline{\varphi}_c^{(2)}$. The curve $\mathcal{D}_c^{(2)}$ is therefore not singular. The singularities of the curves can be seen from Figure 1. Notice that Figure 1 shows the divisor and coincides with that found by M. Adler and P. van Moerbeke (see [3, Table 2]). Hence the conjecture is verify.

3.3.2 The quadratic differential equations and holomorphic

Now we want to show that the vector field $(\varphi_c)_*\mathcal{V}_1$ extends into a holomorphic vectors field on $\mathbb{P}^{24}$. This is done by exhibiting the quadratic differential equations in two of the charts. We will use the following

Lemma 3.8 ([4])**.** *Let $\mathcal{X}$ be a vector field on $\mathbb{P}^N$ holomorphic in two different maps $(Z_i \neq 0)$ and $(Z_j \neq 0)$. Then $\mathcal{X}$ is holomorphic over $\mathbb{P}^N$. That is to say on any map $(Z_j \neq 0)$.*

Proposition 3.9. *The vector field $(\varphi_c)_*\mathcal{V}_1$ extends into a holomorphic vectors field on $\mathbb{P}^{24}$.*

Proof. It suffices to show that the vector field $(\varphi_c)_*\mathcal{V}_1$ is holomorphic on two chart of $\mathbb{P}^{24}$. To do this, let us establish that this vector field can be written as a quadratic vector field in the chart $Z_0 \neq 0$ and $Z_1 \neq 0$. In the chart $Z_0 \neq 0$, we obtain the following result:

$$
\begin{aligned}
\dot{z}_1 &= -\frac{1}{2}(c_2 - 4(z_3 + z_4) + z_1(4z_2 - z_1)),\\
\dot{z}_2 &= \frac{1}{4}(3(z_4 - z_1z_2) + 2(z_2^2 - z_3)),\\
\dot{z}_3 &= \frac{1}{2}z_6 - z_2z_3,\\
\dot{z}_4 &= \frac{1}{2}(z_4(z_1 + z_2) - c_2z_2 + 2z_6 - z_5),\\
\dot{z}_5 &= \frac{1}{2}(z_1z_5 - c_3) + 2(6z_7 - z_9 - z_2z_5),\\
\dot{z}_6 &= -\frac{1}{2}(z_2z_5 + z_6(z_1 - z_2) + 3z_9) + 2(4z_7 + z_8) - z_3^2,\\
\dot{z}_7 &= -\frac{1}{2}(z_{11} + 2z_2z_7),\\
\dot{z}_8 &= \frac{1}{2}z_1z_8 - z_{10},\\
\dot{z}_9 &= \frac{1}{2}(z_2(c_2z_3z + z_9 - 24z_7) + z_5(z_3 - z_4)) - 4z_{11} - z_3z_6,\\
\dot{z}_{10} &= \frac{1}{2}(-2z_2z_{10} + z_4z_8 + 4z_{14}),\\
\dot{z}_{11} &= \frac{1}{2}(z_{13} + 2(z_3z_7 + 2z_{14}) + z_{11}(z_2 - z_1)),
\end{aligned}
$$

$$
\begin{aligned}
\dot{z}_{12} &= \frac{1}{2}(2z_3z_{10} - z_5z_8 - 4z_1z_{14}),\\
\dot{z}_{13} &= \frac{1}{2}(z_{13}(z_2 - z_1) + z_5z_7 - 4z_{15}),\\
\dot{z}_{14} &= \frac{1}{2}z_1z_{14},\\
\dot{z}_{15} &= \frac{1}{2}(z_{18} - 2z_7z_8 + z_{15}(z_1 + z_2)),\\
\dot{z}_{16} &= *,\\
\dot{z}_{17} &= \Delta,\\
\dot{z}_{18} &= -\frac{1}{2}z_5z_{14} + z_7z_{10},\\
\dot{z}_{19} &= 4z_7z_{10} - \frac{1}{2}z_{14}(z_1 - 4z_5),\\
\dot{z}_{20} &= \frac{1}{2}(z_{22} + z_2z_{20} - z_4z_{19} + 4z_8(z_{13} - 2z_{14})),\\
\dot{z}_{21} &= 2z_7(z_{12} + 2z_{14}) - z_8(z_{13} + 2z_{14}),\\
\dot{z}_{22} &= \frac{1}{2}(z_1z_{22} - z_7z_{16}) - 2(z_{10}z_{13} + 6z_1c_1),\\
\dot{z}_{23} &= \star,\\
\dot{z}_{24} &= \frac{1}{2}(z_{24}(z_1 - 4z_2))
\end{aligned}
$$

with

$$
\begin{aligned}
* &= \frac{1}{4}(z_{16}(z_1 - 4z_2) + z_8(32z_7 + 2z_8 - c_2z_4) + 4(2z_{19} + z_5z_{10})),\\
\star &= \frac{1}{2}(z_{23}(z_1 + z_2) + 5z_{16}(8z_{11} + c_3z_2)) + 128z_{14}(z_{13} + 2z_{14}) + c_3z_8(c_3 - 32z_7)\\
&\quad + 64c_1z_1(2z_2 - z_1) - 4c_3z_{19},\\
\Delta &= \frac{1}{2}(4(3z_{18} + 2z_1z_{15}) - z_1z_{17} - z_4z_{13} - z_7(8z_8 + c_3)).
\end{aligned}
$$

For the second chart $Z_1 \neq 0$, we put $s_i = z_i/z_1$, for all $i = 0, \ldots, 24$. Then the quadratic differential equations take the following form:

$$
\begin{aligned}
\dot{s}_0 &= \frac{1}{2}(s_0(c_2s_0 - 4(s_3 + s_4)) + 4s_2 - 1),\\
\dot{s}_1 &= 0,\\
\dot{s}_2 &= \frac{1}{2}(s_0(c_2s_2 - s_5 - 4s_6) + s_2(4s_3 + s_4) - s_4),\\
\dot{s}_3 &= \frac{1}{2}(s_0(c_3s_0 - 32s_7 + 4s_9) - s_5 + s_6 + 2s_2s_5),\\
\dot{s}_4 &= \frac{1}{2}\big(s_0(8s_8 + 6s_9) + s_5(6s_2 - 1) + s_4^2\big),\\
\dot{s}_5 &= 4s_0(s_{10} + s_{11}) - s_9 + 8s_2s_7 - s_4s_5,\\
\dot{s}_6 &= \frac{1}{2}s_0(16s_{10} - 24s_{11} - c_2s_6) + s_2(2c_2s_3 - 16s_7 - s_9) - s_3(s_5 + s_6) - 2s_8 + s_9,\\
\dot{s}_7 &= s_0(s_{13} - 2s_{14}) - \frac{1}{2}s_{11},\\
\dot{s}_8 &= \frac{1}{2}(s_0(c_2s_8 - 32s_{14})) + s_{10}(4s_2 - 1) - 6s_3s_8,
\end{aligned}
$$

$$
\begin{aligned}
\dot{s}_9 &= \frac{3}{4}(s_0(c_3s_4 + 32s_{14} + 16s_{13}) - c_3s_2)\\
&\quad + \frac{1}{2}\big(s_2(2(c_3s_2 + 8s_{11} - 2s_{10}) - c_2s_6) + s_4(s_9 - 4s_8) + s_5^2 + 2s_6^2 - 16s_{11}\big),\\
\dot{s}_{10} &= -\frac{1}{2}s_0s_{16} - 2s_{14} - s_5s_8,\\
\dot{s}_{11} &= -2s_{14} + 4s_0s_{15} + s_{13}(s_2 - 1) - \frac{1}{2}s_7(c_2 + 6s_5),\\
\dot{s}_{12} &= \frac{1}{8}(c_2s_4s_8 + s_{16}(4s_2 + 1)) - s_8(4s_7 + 3s_8),\\
\dot{s}_{13} &= \frac{1}{2}(s_4s_{13} + s_{17} + s_7(3c_3s_0 - 48s_7 - 4s_8)) - 6s_0s_{17},\\
\dot{s}_{14} &= -\frac{1}{2}(s_0s_{19} - 4s_7s_8),\\
\dot{s}_{15} &= \frac{1}{2}(s_0s_{20} + s_4s_{15} - s_5s_{14} + 4s_7s_{10}),\\
\dot{s}_{16} &= 2(8s_{18} + 3s_{19}) + \frac{1}{2}s_{16}(c_2s_0 - 4s_4) + 2s_{10}(8s_8 + c_3s_0 + 32s_7)\\
&\quad - s_5(c_2s_8 + 4s_{12}) - 32s_0s_{21},\\
\dot{s}_{17} &= 2s_{14}(s_5 - 2s_6) + s_{11}(4s_8 - c_2s_3) + \frac{1}{2}s_7(c_2(2s_5 - c_2s_2) + c_3 + 8s_{10}) - 3s_5s_{13}\\
&\quad - 2s_0s_{20} + \frac{1}{2}s_4s_{17},\\
\dot{s}_{18} &= \frac{1}{2}(s_3s_{19} + s_{21}) + 8c_1s_0^2 + 2s_7s_{12} + s_8(2s_{14} - s_{13}),\\
\dot{s}_{19} &= \frac{1}{2}c_2s_0s_{19} - 2s_0s_{22} + 24s_8s_{14} + 2s_{21},\\
\dot{s}_{20} &= \frac{1}{2}(s_{22} + s_4s_{20} - c_2s_2s_{19}) + s_6s_{19} - s_7s_{16} - 8s_8s_{15},\\
\dot{s}_{21} &= 4c_1s_0 + 2s_{10}s_{13} - s_7s_{16},\\
\dot{s}_{22} &= \frac{1}{2}(s_0(c_3s_{19} - 16s_{24})) + 8\big(s_{10}s_{15} - 8s_{14}^2\big) + 4c_1,\\
\dot{s}_{23} &= \frac{1}{2}(s_{23}(s_4 + c_2s_0) - c_3(s_8(3c_3 + 40s_{10}) - 2s_3s_{16} - 4s_{19}(4s_2 + 3))) - 96s_{24}(1 + 2s_2)\\
&\quad + 8s_{16}(4s_{13} + 10s_{14} - c_2s_7) - 192c_1s_4 + 160s_{11}s_{19} - 96s_{24},\\
\dot{s}_{24} &= \frac{1}{2}(s_{10}s_{20} - s_{12}s_{19} - 2s_4s_{24}).
\end{aligned}
$$

This shows that the vector field $(\varphi_c)_*\mathcal{V}_1$ extends into a linear vector field $\overline{\mathcal{V}}_1$ on $\mathbb{P}^{24}$. ■

Thus, the item four of the complex Liouville theorem (Theorem (2.7)) is satisfied. We now show that the integral curves of $\overline{\mathcal{V}}_1$ that start at the three singular points go into the affine immediately

Proposition 3.10. *The flow Φ_t of the vector field $\overline{\mathcal{V}}_1$ on $\mathbb{P}^{24}$ coming from the points of $\mathcal{D}_c$ is sent into the affine part $\varphi_c(\mathbb{F}_c)$.*

Proof. Note that this is automatic for the points of $\varphi_c\big(\mathbf{\Gamma}_c^{(0)}\big)$, $\varphi_c\big(\mathbf{\Gamma}_c^{(1)}\big)$ and $\varphi_c\big(\mathbf{\Gamma}_c^{(2)}\big)$. We then just need to check for the singular points $Q_{\varepsilon_1\varepsilon_2}$, P_ε and T in $\mathbb{P}^{24}$.

Consider the 4 points $Q_{\varepsilon_1\varepsilon_2}$ intersections of $\mathcal{D}_c^{(0)}$ and $\mathcal{D}_c^{(2)}$. From (3.9) and (3.19) follow that the leading coefficient of $z_4\big(t;\mathbf{\Gamma}_c^{(0)}\big)$ has a pole for $\varsigma = 0$, that is maximal with the leading coefficient of $z_9\big(t;\mathbf{\Gamma}_c^{(0)}\big)$ and thus the function z_4 defines a map at these points. Let us show then that $\lim_{p\to\infty_{\epsilon_1\epsilon_2}} \frac{1}{z_4}\varphi_c^{(0)} \neq 0$.

Using (3.5), we get

$$z_4(t;m_0) = \frac{2a}{t} - 4d + (-e + 2ad - 2ac)t + \frac{1}{6}\big(ae - 4\big(2d^2 + da^2\big) + 4cd\big)t^2 + O\big(t^3\big).$$

The first terms of the inverse of this series are then given by

$$\frac{1}{z_4(t;m_0)} = \frac{t}{2a} + \frac{d}{a^2}t^2 + O\big(t^3\big). \tag{3.23}$$

Substituting $e = -\frac{4a^4-32a^2d-a^2c_2+64d^2+4dc_2+c_3}{8a}$, $c = \frac{a^2}{3} - \frac{4d}{3} - \frac{c_2}{12}$ in the second term and rewriting the coefficients according to the local parameter ς around $\infty_{\epsilon_1\epsilon_2}$ using (3.9), we obtain

$$\lim_{\varsigma\to 0}\frac{1}{z_4}(t,\varsigma) = \frac{1}{4}t^2 + O\big(t^3\big) \neq 0,$$

which prove that the integrals curves of $\overline{\mathcal{V}}_1$ which start from the points $Q_{\epsilon_1\epsilon_2}$ are immediately sent in the affine part $\varphi_c(\mathbb{F}_c)$.

For the point P_ϵ, intersection points of $\mathcal{D}_c^{(0)}$ and $\mathcal{D}_c^{(1)}$, the residue having the largest pole among the residues of $z_0\big(t;\boldsymbol{\Gamma}_c^{(1)}\big),\dots,z_{24}\big(t;\boldsymbol{\Gamma}_c^{(1)}\big)$ is $z_9\big(t;\boldsymbol{\Gamma}_c^{(1)}\big)$, the function z_{15} defines a local chart around this point. Consider the balances $x(t;m_1)$ 3.6 and rewriting a, b, c, d of $1/z_9\big(t;\boldsymbol{\Gamma}_c^{(1)}\big)$ depending on the local parameter ς around ∞_ϵ using $c = \frac{1}{12}a^2 - \frac{1}{24}c_2$, $e = \frac{c_1}{d^2}$ and (3.11), we obtain

$$\lim_{\varsigma\to 0}\frac{1}{z_9}(t,\varsigma) = \frac{1}{12}t^4 + O\big(t^6\big) \neq 0.$$

Consider the balances $x(t;m_0)$ (3.5) and the local parametrization (3.8), we also obtain

$$\lim_{\varsigma\to 0}\frac{1}{z_8}(t,\varsigma) = \frac{1}{12}t^4 + O\big(t^6\big) \neq 0.$$

Thus, the different limits found are different from zero, which also shows that the integral curve of $\overline{\mathcal{V}}_1$ which starts from the points P_ϵ are immediately sent in the affine part $\varphi_c(\mathbb{F}_c)$.

For the point T, intersection point of $\mathcal{D}_c^{(2)}$ and $\mathcal{D}_c^{(1)}$, the only non-zero coordinate corresponds to the function $z_{17}\big(t;\boldsymbol{\Gamma}_c^{(1)}\big)$, the function z_{17} defines a local map around this point. Consider the balances $x(t;m_1)$ (3.6) and $x(t;m_2)$ (3.7) by writing a, c, d, e of $1/z_{17}\big(t;\boldsymbol{\Gamma}_c^{(1)}\big)$ depending on the local parameter ς around ∞_ϵ and using $c = \frac{1}{12}a^2 - \frac{1}{24}c_2$, $e = \frac{c_1}{d^2}$ and (3.11), we get

$$\lim_{\varsigma\to 0}\frac{1}{z_{17}}(t,\varsigma) = \frac{1}{2304}t^7 + O\big(t^8\big) \neq 0.$$

Considering the balances $x(t;m_2)$ (3.7) and local parametrization (3.16), we also get

$$\lim_{\varsigma\to 0}\frac{1}{z_{17}}(t,\varsigma) = -\frac{1}{1152}t^7 + O\big(t^8\big) = -2\times\left(\frac{1}{2304}t^7\right) + O\big(t^8\big) \neq 0.$$

Thus, the different limits found are different from zero, which also shows that the integral curve of $\overline{\mathcal{V}}_1$ which starts from the points T are immediately sent in the affine part $\varphi_c(\mathbb{F}_c)$.

Thus, the flow of the vector field $\overline{\mathcal{V}}_1$ starting from each of the points of $\mathcal{D}_c^{(0)} \cup \mathcal{D}_c^{(1)} \cup \mathcal{D}_c^{(2)}$ goes into the affine part of $\varphi_c(\mathbb{F}_c)$.

In order to complete the proof of algebraic integrability, it is necessary to show that there exist no other divisors passing through the points $Q_{\epsilon_1\epsilon_2}$, P_ϵ and T.

For the four intersections points $Q_{\epsilon_1\epsilon_2}$ of $\mathcal{D}_c^{(0)}$ and $\mathcal{D}_c^{(2)}$, by rewriting the coefficients of the right-hand side of (3.23) with respect to the local parameter ς in the neighborhood of the points $\infty_{\epsilon_1\epsilon_2} \in \overline{\Gamma}_c^{(0)}$, we find

$$\frac{1}{z_4\big(t;\Gamma_c^{(0)}\big)} = \frac{1}{4}\big(2\varsigma t + t^2\big) + O\big(t^3, \varsigma t^2\big),$$

which shows that the multiplicity of $\frac{1}{z_4}$ at each of these points is equal to 2, which coincides with the sum of the orders of zero of $\frac{1}{z_4}$ on each of the divisors so there are no other divisors passing through the points $Q_{\epsilon_1\epsilon_2}$.

For the points P_ϵ and T of the divisor $\mathcal{D}_c$, obtained from the points at infinity ∞_ϵ, ∞_1 and ∞_2, let us check that the degree of $\mathcal{D}_c$ which is 3 is indeed equal to the degree of $\overline{\varphi_c(\mathbb{F}_c)} \setminus \varphi_c(\mathbb{F}_c)$ at these points. As the vector field $\overline{\mathcal{V}}_1$ is only tangent to one of the branches (transverse to the other) of $\mathcal{D}_c^1$ passing through these two points, we do the expansion along the non-tangent branch. As the function z_{17} defines a map at point T, we just need to substitute (3.11) into its inverse series,

$$\frac{1}{z_{17}\big(t;\mathbf{\Gamma}_c^{(1)}\big)} = -\frac{t}{d\big(-16d + 12ac + a^3\big)} + O\big(t^2\big) = -\frac{2t}{d\big(-32d - ac_2 + 4a^3\big)} + O\big(t^2\big),$$

which leads to

$$\frac{1}{z_{17}\big(t;\mathbf{\Gamma}_c^{(1)}\big)} = -\frac{1}{4\sqrt{c_1}}\varsigma t + O\big(t^3\big),$$

thus showing that there are no other divisors passing through T.

For points P_ϵ, we do this by calculating the first terms of the series $1/z_8\big(t;\mathbf{\Gamma}_c^{(0)}\big)$ using Laurent's solution $x(t;m_0)$ then we express the free parameters according to the local parameter ς in a neighborhood of (3.8). The resulting series in ς and t should start with monomials of degree 3 because the point P_ϵ has multiplicity 2 and 1 on the divisors $\mathcal{D}_c^{(0)}$ and $\mathcal{D}_c^{(1)}$, respectively, and the function z_8 has a simple pole on each of these divisors. We have

$$\frac{1}{z_8\big(t;\mathbf{\Gamma}_c^{(0)}\big)} = \frac{1}{e}t - \frac{at^2}{e} + O\big(t^3\big),$$

which leads to

$$\frac{1}{z_8\big(t;\mathbf{\Gamma}_c^{(0)}\big)} = 2\varsigma^2 t^2 + O\big(\varsigma^3, \varsigma t^3\big),$$

thus showing that there are no other passing divisors by P_ϵ. Furthermore this also shows that P_ϵ is an ordinary double point for the divisor $\mathcal{D}_c^{(0)}$. So, there are no other divisors in $\overline{\varphi_c(\mathbb{F}_c)} \setminus \varphi_c(\mathbb{F}_c)$ besides the divisors $\mathcal{D}_c^{(0)}$, $\mathcal{D}_c^{(1)}$ and $\mathcal{D}_c^{(2)}$ already found. ■

The Liouville complex theorem conditions being satisfied, it follows that for $c \in \Omega$, the projective variety $\overline{\varphi_c(\mathbb{F}_c)} = \varphi_c(\mathbb{F}_c) \cup \mathcal{D}_c$ is an abelian surface and the restrictions of vector fields $\overline{\mathcal{V}}_1$ and $\overline{\mathcal{V}}_2$ to these abelian surfaces are linear. Since $\overline{\varphi_c(\mathbb{F}_c)}$ contains a smooth curve of genus 2, it is the Jacobian of this curve. We have therefore proved the following theorem.

Theorem 3.11. *Let $(\mathcal{H}, \{\cdot,\cdot\}, \mathbf{F})$ be an integrable system describing the $a_4^{(2)}$ Toda lattice, where $\mathbf{F} = (F_1, F_2, F_3)$ and $\{\cdot,\cdot\}$ are given, respectively, by (3.3) and (3.2) with commuting vector fields (3.1).*

(*i*) *$(\mathcal{H}, \{\cdot,\cdot\}, \mathbf{F})$ is a weight homogeneous algebraical completely integrable system.*

(ii) For $c \in \Omega$, the fiber $\mathbb{F}_c$ of its momentum map is completed in an abelian surface $\mathbb{T}_c^2$ (the Jacobian of the hyperelliptic curve (of genus two) $\overline{\Gamma}_c^{(2)}$) by the addition of a singular divisor $\mathcal{D}_c$ composed of three irreducible components: $\mathcal{D}_c^{(0)}$ defined by

$$\begin{aligned}\Gamma_c^{(0)}\colon\ & 16d^2a^8 - \big(256d^3 + 8d^2c_2\big)a^6 + \big(1536d^2 + 96dc_2 + 8c_3 + c_2^2\big)d^2a^4 \\ & - \big(\big(8\big(8c_3 + 48dc_2 + c_2^2 + 512d^2\big)d + 2c_2c_3\big)d^2 + 64c_1\big)a^2 \\ & + \big(8d\big(c_2c_3 + 16dc_3 + 64d^2c_2 + 512d^3 + 2dc_2^2\big) + c_3^2\big)d^2 = 0,\end{aligned}$$

and $\mathcal{D}_c^{(1)}$ defined by

$$\Gamma_c^{(1)}\colon\ 256ad^3 - \big(\big(4a^2 - c_2\big)^2 - 16c_3\big)d^2 + 64c_1 = 0,$$

two singular curves of respective genus 3 and 4 and one smooth curve and $\mathcal{D}_c^{(2)}$ defined by

$$\Gamma_c^{(2)}\colon\ e^4a^4 - \big(8c_1 + c_2e^2\big)a^2e^2 - 64e^5 + 4e^2c_1c_2 + 4c_3e^4 + 16c_1^2 = 0$$

of genus 2 and isomorphic to $\overline{\Gamma}_c^{(2)}$. The curves intercept each other as indicated in Figure 1.

4 Geometry of the $a_4^{(2)}$ Toda lattice: holomorphic differentials forms

In this section, we show following an idea by Luc Haine (see [9]) how the holomorphic differentials are used to determine the tangency locus of the vector field $\mathcal{V}_1$ on the divisor $\mathcal{D}_c$, we compute the holomorphic differentials ω_1 and ω_2 on the three irreducible components of $\mathcal{D}_c$ that come from the differentials $\mathrm{d}t_1$ and $\mathrm{d}t_2$ on the abelian surface $\mathbb{T}_c^2$. We know that all irreducible components have multiplicity 1.

Let us calculate the holomorphic differential forms ω_1 and ω_2 on the divisor $\mathcal{D}_c$ which come from the differentials $\mathrm{d}t_1$ and $\mathrm{d}t_2$ on the abelian surface $\mathbf{T}_c^2$. Let $y_0 := z_1$ and $y := z_4$ be restricted to $\mathcal{D}_c^{(0)}$, the first coefficients of their Laurent series are given by

$$y_0^{(0)} = -2, \qquad y_0^{(1)} = 2c, \qquad y^{(0)} = 2a, \qquad y^{(1)} = -4d,$$

and using (3.5) and (3.4), we have

$$\mathcal{V}_2\left[\frac{1}{z_1}\right]_{|\mathcal{D}_c^{(0)}} = -2a^2 + 8d, \qquad \mathcal{V}_2\left[\frac{z_4}{z_1}\right]_{|\mathcal{D}_c^{(0)}} = -4ea - 24cd.$$

It then follows that

$$\begin{aligned}\delta &= \frac{1}{\big(y_0^{(0)}\big)^2}\begin{vmatrix} y_0^{(0)} & \mathcal{V}_2\big[\frac{1}{y_0}\big]_{|\mathcal{D}_c^{(0)}} \\ y_0^{(0)}y^{(1)} - y^{(0)}y_0^{(1)} & \mathcal{V}_2\big[\frac{z_4}{y_0}\big]_{|\mathcal{D}_c^{(0)}}\end{vmatrix} = \frac{1}{4}\begin{vmatrix} -2 & -2a^2 + 8d \\ 8d - 4ac & -4ea - 24cd\end{vmatrix} \\ &= 2ea + 12cd + 4a^2d - 2a^3c - 16d^2 + 8cad.\end{aligned}$$

The holomorphic differential forms $\mathrm{d}t_1$ and $\mathrm{d}t_2$ restricted to $\mathcal{D}_c^{(0)}$ are given by

$$\omega_1 = \frac{1}{\delta y_0^{(0)}}\mathrm{d}\left(\frac{y^{(0)}}{y_0^{(0)}}\right) = -\frac{\mathrm{d}a}{\delta} \qquad \text{and} \qquad \omega_2 = -\frac{1}{\delta}\mathcal{V}_2\left[\frac{1}{y_0}\right]_{|\mathcal{D}_c^{(0)}}\mathrm{d}\left(\frac{y^{(0)}}{y_0^{(0)}}\right) = -\frac{2a^2 - 8d}{\delta}\mathrm{d}a.$$

To determined differential forms on the divisor $\mathcal{D}_c^{(1)}$, consider the functions $y_0 := z_1$ and $y := z_3$, restricted to $\mathcal{D}_c^{(1)}$, the first coefficients of their Laurent series are given by

$$y_0^{(0)} = 2, \qquad y_0^{(1)} = a, \qquad y^{(0)} = -a, \qquad y^{(1)} = -3c + \frac{a^2}{4},$$

and using (3.6) and (3.4), we have

$$\mathcal{V}_2\left[\frac{1}{z_1}\right]_{|\mathcal{D}_c^{(1)}} = -6c - \frac{a^2}{2}, \qquad \mathcal{V}_2\left[\frac{z_3}{z_1}\right]_{|\mathcal{D}_c^{(1)}} = -2e + 4ad + 18c^2 - 3a^2c - \frac{3a^4}{8},$$

then

$$\delta = \frac{1}{\left(y_0^{(1)}\right)^2}\begin{vmatrix} y_0^{(0)} & \mathcal{V}_2\left[\frac{1}{y_0}\right]_{|\mathcal{D}_c^{(1)}} \\ y_0^{(0)}y^{(1)} - y^{(0)}y_0^{(1)} & \mathcal{V}_2\left[\frac{z_3}{y_0}\right]_{|\mathcal{D}_c^{(1)}} \end{vmatrix}$$
$$= \frac{1}{4}\begin{vmatrix} 2 & -6c - \frac{a^2}{2} \\ -6c + \frac{3a^2}{2} & -2e + 4ad + 18c^2 - 3a^2c - \frac{3a^4}{8} \end{vmatrix} = 2ad - e.$$

The holomorphic differential forms $\mathrm{d}t_1$ and $\mathrm{d}t_2$ restricted to $\mathcal{D}_c^{(1)}$ are given by

$$\omega_1 = \frac{1}{\delta y_0^{(0)}}\mathrm{d}\left(\frac{y^{(0)}}{y_0^{(0)}}\right) = -\frac{\mathrm{d}a}{4\delta} \qquad \text{and} \qquad \omega_2 = -\frac{1}{\delta}\mathcal{V}_2\left[\frac{1}{y_0}\right]_{|\mathcal{D}_c^{(1)}}\mathrm{d}\left(\frac{y^{(0)}}{y_0^{(0)}}\right) = -\frac{a^2 + 12c}{4\delta}\mathrm{d}a.$$

For calculating differential forms on the divisor $\mathcal{D}_c^{(2)}$, consider the functions $y_0 := z_2$ and $y := z_4$ restricted to $\mathcal{D}_c^{(2)}$, the first coefficients of their Laurent series are given by

$$y_0^{(0)} = -2, \qquad y_0^{(1)} = 2d, \qquad y^{(0)} = -2a, \qquad y^{(1)} = -4c,$$

and using (3.7) and (3.4), we have

$$\mathcal{V}_2\left[\frac{1}{z_2}\right]_{|\mathcal{D}_c^{(2)}} = 2c - \frac{a^2}{2}, \qquad \mathcal{V}_2\left[\frac{z_4}{z_2}\right]_{|\mathcal{D}_c^{(2)}} = 16e - 96cd.$$

It then follows that

$$\delta = \frac{1}{\left(y_0^{(1)}\right)^2}\begin{vmatrix} y_0^{(0)} & \mathcal{V}_2\left[\frac{1}{y_0}\right]_{|\mathcal{D}_c^{(2)}} \\ y_0^{(0)}y^{(1)} - y^{(0)}y_0^{(1)} & \mathcal{V}_2\left[\frac{z_3}{y_0}\right]_{|\mathcal{D}_c^{(2)}} \end{vmatrix} = \frac{1}{4}\begin{vmatrix} -2 & 2c - \frac{a^2}{2} \\ 8c + 4ad & 16e - 96cd \end{vmatrix}$$
$$= 48cd - 8e + ca^2 + \frac{a^3d}{2} - 4c^2 - 2acd.$$

The holomorphic differential forms $\mathrm{d}t_1$ and $\mathrm{d}t_2$ restricted to $\mathcal{D}_c^{(2)}$ are given by

$$\omega_1 = \frac{1}{\delta y_0^{(0)}}\mathrm{d}\left(\frac{y^{(0)}}{y_0^{(0)}}\right) = -\frac{\mathrm{d}a}{2\delta} \qquad \text{and} \qquad \omega_2 = -\frac{1}{\delta}\mathcal{V}_2\left[\frac{1}{y_0}\right]_{|\mathcal{D}_c^{(2)}}\mathrm{d}\left(\frac{y^{(0)}}{y_0^{(0)}}\right) = -\frac{4c - a^2}{2\delta}\mathrm{d}a.$$

The zero of differentials forms ω_1 and ω_2 gives the tangency points of fields vector $\mathcal{V}_1$ and $\mathcal{V}_2$ respectively.

Remark 4.1. In this paper, we use the Maple 13 application to develop and implement algorithms for determining Laurent series, various curves. We also use this application to determine the Painlevé divisors, the z_i functions, embedding to the projective space and give the different quadratic differential equations of the two chart Z_0 and Z_1.

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