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Asymptotic Expansions of Finite Hankel Transforms and the Surjectivity of Convolution Operators

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Abstract. A compactly supported distribution is called invertible in the sense of Ehrenpreis–Hörmander if the convolution with it induces a surjection from $\mathcal{C}^\infty(\mathbb{R}^n)$ to itself. We give sufficient conditions for radial functions to be invertible. Our analysis is based on the asymptotic expansions of finite Hankel transforms. The dominant term may be the contribution from the origin or from the boundary of the support of the function. For the proof, we propose a new method to calculate the asymptotic expansions of finite Hankel transforms of functions with singularities at a point other than the origin.

Key words: convolution; asymptotic expansion; Hankel transform; invertibility

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1 Introduction

It is known that

$$P(D)\colon\ \mathcal{C}^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty(\mathbb{R}^n) \qquad \text{and} \qquad P(D)\colon\ \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$$

are both surjective, where $P(D) \neq 0$ is an arbitrary linear partial differential operator with constant coefficients [6, 7, 14]. Since these mappings coincide with the convolution operator $P(D)\delta*$ on respective spaces, a natural question is to characterize compactly supported distributions that induce surjective convolution operators. Such a distribution is called invertible. Notice that $\delta(x-a)$ induces translation and hence is invertible. It was found that a compactly supported distribution is invertible if and only if its Fourier transform, an entire function, is slowly decreasing in a certain sense. This condition was found by Ehrenpreis [8] and was further studied by Hörmander [11]. A self-contained account can be found in [12]. Invertible distributions and slowly decreasing entire functions became and still are fundamental concepts in the research of convolution equations. Perturbation of invertible distributions are studied in [15], and a recent paper [5] investigates invertible distributions in an abstract setting. Convolution equations on symmetric spaces are discussed from the viewpoint of invertibility in [4].

Slow decrease is a technical estimate from below and is not easy to grasp. The present authors expect that the notions of slow decrease and of invertibility will become less mysterious if many examples or sufficient conditions are found. The delta function $\delta_{S(0,r)}$ supported on a sphere is invertible [13]. Its normal derivatives are invertible as well [16]. In [13, 16], the asymptotic behavior of Bessel functions were used. As is well known [18], the Fourier transform of a radial function can be written in terms of an integral involving a Bessel function. The Fourier transforms of $\delta_{S(0,r)}$ and its normal derivatives are limit cases.

In the present article, we prove the invertibility of some radial functions by using two methods of calculating asymptotic expansions of finite Hankel transforms $\int_0^1 \varphi(s)J_\nu(rs)\mathrm{d}s$. One is a result in Wong [20], in which singularities at $s=0$ determine the asymptotic behaviors. The other method, devised by the present authors, is useful to deal with singularities at $s=1$. As far as they know, this is the first result about the asymptotic behaviors of Hankel transforms of functions singular at a point other than $s=0$.

Notice that various additive formulas for supports and singular supports of convolutions and spaces of entire functions with a certain type of slow decrease are investigated in [3] and applied to study the surjectivity of convolution operators. Several examples are discussed in it and the characteristic function of an ellipsoid is studied by using a Bessel function. The tools developed in the present paper could be useful in that line of research.

2 Invertibility and slow decrease

In this article, we follow the convention

$$\hat{f}(\xi) = \left\langle f(x), \mathrm{e}^{-\mathrm{i}\xi\cdot x}\right\rangle = \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}\xi\cdot x} f(x)\mathrm{d}x, \qquad \xi\in\mathbb{R}^n$$

for the definition of the Fourier transform of a compactly supported distribution $f\in\mathcal{E}'(\mathbb{R}^n)$. When $f(x)\in L^1(\mathbb{R}^n)$ is a radial function, its Fourier transform can be written in terms of an integral involving a Bessel function.

Theorem 2.1 ([18, p. 155])**.** *Let $f_0(s)$, $s>0$, be a function of a single variable. The Fourier transform of $f(x)=f_0(|x|)$, $x\in\mathbb{R}^n$, is written in terms of a Hankel transform. More precisely,*

$$\hat{f}(\xi) = \frac{(2\pi)^{n/2}}{r^{n/2-1}}\int_0^\infty s^{n/2} f_0(s) J_{n/2-1}(rs)\mathrm{d}s, \qquad r=|\xi|, \qquad \xi\in\mathbb{R}^n.$$

Here $J_{n/2-1}(z)$ denotes the Bessel function of the first kind of order $n/2-1$.

Remark 2.2. Notice that $z^{-(n/2-1)}J_{n/2-1}(sz)$ is an *even* entire function in a single variable z for each $s>0$. If the support of $f_0(s)$ is bounded, then the Fourier transform of $f(x)=f_0(|x|)$ is an even entire function which is expressed by

$$\hat{f}(\zeta) = (2\pi)^{n/2}\int_0^\infty s^{n/2} f_0(s)\big(\sqrt{\zeta^2}\big)^{-(n/2-1)} J_{n/2-1}\big(s\sqrt{\zeta^2}\big)\mathrm{d}s,$$

where $\zeta=(\zeta_1,\dots,\zeta_n)\in\mathbb{C}^n$, $\zeta^2=\sum_{j=1}^n \zeta_j^2$.

We employ the notions of invertibility and slow decrease originated by Ehrenpreis [8] and refined by Hörmander [11].

Theorem 2.3 ([8, Theorem 2.2, Proposition 2.7], [11, Definition 3.1, Corollary 3.1], [12, Theorems 16.3.9 and 16.3.10])**.** *For $u\in\mathcal{E}'(\mathbb{R}^n)$, the following statements are equivalent.*

(*i*) *There is a constant $A>0$ such that we have*

$$\sup\big\{|\hat{u}(\zeta)|;\,\zeta\in\mathbb{C}^n,\,|\zeta-\xi|<A\log(2+|\xi|)\big\} > (A+|\xi|)^{-A}$$

for any $\xi\in\mathbb{R}^n$.

(*ii*) *There is a constant* $A > 0$ *such that we have*

$$\sup\big\{|\hat{u}(\eta)|;\, \eta \in \mathbb{R}^n,\, |\eta - \xi| < A\log(2+|\xi|)\big\} > (A+|\xi|)^{-A} \tag{2.1}$$

for any $\xi \in \mathbb{R}^n$.

(*iii*) *If* $w \in \mathcal{E}'(\mathbb{R}^n)$ *and* $\hat{w}/\hat{u}$ *is a holomorphic function, then* $\hat{w}/\hat{u}$ *is the Fourier transform of a distribution in* $\mathcal{E}'(\mathbb{R}^n)$.

(*iv*) *If* $v \in \mathcal{E}'(\mathbb{R}^n)$ *satisfies* $u * v \in \mathcal{C}^\infty(\mathbb{R}^n)$, *then* $v \in \mathcal{C}^\infty(\mathbb{R}^n)$.

(*v*) *The mapping* $u*\colon \mathcal{C}^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty(\mathbb{R}^n)$ *is surjective.*

(*vi*) *The mapping* $u*\colon \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ *is surjective.*

Remark 2.4. In the present paper, we only deal with surjectivity on $\mathbb{R}^n$. If one wants to study surjectivity on subsets of $\mathbb{R}^n$, one needs the notions of μ-convexity for supports or singular supports [12].

Definition 2.5 ([12, Definition 16.3.12])**.** An element $u(x)$ of $\mathcal{E}'(\mathbb{R}^n)$ is called *invertible* and its Fourier transform $\hat{u}(\zeta)$ is called *slowly decreasing* if the equivalent conditions in Theorem 2.3 are fulfilled. See Definition 2.8 below.

Remark 2.6. A finitely supported non-zero distribution is invertible [11, Theorem 4.4]. The most important example is $P(D)\delta(x)$, where $P(D) \neq 0$ is a linear partial differential operator with constant coefficients. It give rise to $P(D)\delta(x)*$, which is nothing but $P(D)$. So $P(D)\colon \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ and $P(D)\colon \mathcal{C}^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty(\mathbb{R}^n)$ are surjective.

Other classes of invertible distributions are discussed in [1]. Let u be a compactly supported measure with an atom, $v \in \mathcal{E}'(\mathbb{R}^n)$ have singular support disjoint from that of u and $P(D)$ be a non-zero linear partial differential operator with constant coefficients. Then $P(D)u + v$ is invertible.

Another useful fact in [1] is the following. Let $u \in \mathcal{E}'(\mathbb{R}^n)$, $f \in \mathcal{C}^\infty(\mathbb{R}^n)$. If f is real analytic in a neighborhood of $\operatorname{singsupp} u$ and fu is invertible, then u is invertible. In [13, 16], the spherical mean value operator and its variants are discussed from the view point of invertibility.

Proposition 2.7.

(*i*) *If* $u, v \in \mathcal{E}'(\mathbb{R}^n)$ *are invertible, then so is* $u * v$.

(*ii*) *Let* $\alpha \in \mathbb{R} \setminus \{0\}$. *If* $u(x) \in \mathcal{E}'(\mathbb{R}^n)$ *is invertible, so is* $u(\alpha x)$.

(*iii*) *Let* $a \in \mathbb{R}^n$. *If* $u(x) \in \mathcal{E}'(\mathbb{R}^n)$ *is invertible, so is* $u(x-a)$.

(*iv*) *If* $u(x) \in \mathcal{E}'(\mathbb{R}^n)$ *is invertible, then so is* $\sum_{j=1}^J P_j(D)u(x-a_j)$, *where* $P_j(D)$ *is a non-zero linear partial differential operator with constant coefficients and* $a_j \in \mathbb{R}^n$.

(*v*) *Let* $u \in \mathcal{E}'(\mathbb{R}^n)$ *and* $v \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. *Then,* $u + v$ *is invertible if and only if* u *is.*

(*vi*) *If* $u(x) \in \mathcal{E}'(\mathbb{R}^m)$ *and* $v(x') \in \mathcal{E}'(\mathbb{R}^n)$ *are invertible, then so is* $(u \otimes v)(x,x') = u(x)v(x') \in \mathcal{E}'(\mathbb{R}^{m+n})$.[1]

Proof. The proofs of (i)–(iv) are easy. Theorem 2.3 (ii) and the Paley–Wiener–Schwartz theorem imply (v). Finally, (vi) follows from Theorem 2.3 (ii). Indeed, if u and v satisfy (2.1) with a common constant A, then $u \otimes v$ satisfies (2.1) with $2A$ instead of A. ∎

[1]This can be proved in an alternative manner using general facts about the tensor product of nuclear Fréchet spaces.

Definition 2.8. In the present paper, we extend the terminology of Definition 2.5 slightly: an entire function $p(\zeta)$, not necessarily the Fourier transform of a compactly supported distribution, is called *slowly decreasing* if there is a constant $A > 0$ such that we have

$$\sup\big\{|p(\eta)|;\ \eta \in \mathbb{R}^n,\ |\eta - \xi| < A\log(2 + |\xi|)\big\} > (A + |\xi|)^{-A} \tag{2.2}$$

for any $\xi \in \mathbb{R}^n$.

Proposition 2.9. *Let $p(\zeta)$ be an entire function. There are constants $A, B > 0$ such that*

$$\sup\big\{|p(\eta)|;\ \eta \in \mathbb{R}^n,\ |\eta - \xi| < A\log(2 + |\xi|)\big\} > (A + |\xi|)^{-A} \tag{2.3}$$

for any $\xi \in \mathbb{R}^n$ satisfying $|\xi| \geq B$, if and only if $p(\zeta)$ is slowly decreasing in the sense of Definition 2.8.

Proof. The "if" part is trivial. We prove the "only if" part. If $|\xi_0| = B$, we set

$$S(\xi_0) = \big\{\eta \in \mathbb{R}^n;\ |\eta - \xi_0| < A\log(2 + B)\big\}.$$

By (2.3),

$$\sup\big\{|p(\eta)|;\ \eta \in S(\xi_0)\big\} > (A + B)^{-A}. \tag{2.4}$$

On the other hand, we set $S = \{\eta \in \mathbb{R}^n;\ |\eta| < B + A\log(2+B)\} \supset S(\xi_0)$. Here we may assume A is so large that $A\log(2 + B) > B$. Then

$$\bigcup_{|\xi_0|=B} S(\xi_0) = S. \tag{2.5}$$

By (2.4) and (2.5), we have

$$\sup\{|p(\eta)|;\ \eta \in S\} > (A + B)^{-A}. \tag{2.6}$$

Assume $|\xi| \leq B$ and set

$$S_\xi = \left\{\eta \in \mathbb{R}^n;\ |\eta - \xi| \leq \frac{2B + A\log(2 + B)}{\log 2}\log(2 + |\xi|)\right\}.$$

Since

$$S \subset \big\{\eta \in \mathbb{R}^n;\ |\eta - \xi| < 2B + A\log(2 + B)\big\} \subset S_\xi,$$

(2.6) implies $\sup\{|p(\eta)|;\ \eta \in S_\xi\} > (A + B)^{-A} > (A + B + |\xi|)^{-(A+B)}$. We set

$$\tilde{A} = \max\left\{\frac{2B + A\log(2 + B)}{\log 2},\ A + B\right\},$$

then (2.2) holds if we replace A with $\tilde{A}$. ■

Proposition 2.10. *Let $p(z)$ be an even entire function of a single variable. If $p(z)$ is slowly decreasing, then $p\big(\sqrt{\zeta^2}\big)$ is a slowly decreasing function in $\zeta \in \mathbb{C}^n$.*

Proof. Notice that $p\big(\sqrt{\zeta^2}\big)$ is well-defined since $p(z)$ is even. There are constants $A, B > 0$ such that we have

$$\sup\big\{|p(y)|;\, y \in \mathbb{R},\, |y - x| < A\log(2 + x)\big\} > (A + x)^{-A} \tag{2.7}$$

for any $x \in \mathbb{R}$ satisfying $x \geq B$. For any $\xi \in \mathbb{R}^n$, set $x = |\xi|$. The radial function $p(|\eta|)$, $\eta \in \mathbb{R}^n$, satisfies the condition in Proposition 2.9, since the combination of

$$\begin{aligned}\big\{p(|\eta|);\, \eta \in \mathbb{R}^n,\, |\eta - \xi| < A\log(2 + |\xi|)\big\} &= \big\{p(|\eta|);\, \eta \in \mathbb{R}^n,\, ||\eta| - |\xi|| < A\log(2 + |\xi|)\big\}\\ &= \{p(y);\, y \in \mathbb{R},\, |y - x| < A\log(2 + x)\}\end{aligned}$$

and (2.7) implies

$$\sup\big\{p(|\eta|);\, \eta \in \mathbb{R}^n,\, |\eta - \xi| < A\log(2 + |\xi|)\big\} > (A + x)^{-A} = (A + |\xi|)^{-A}$$

if $x = |\xi| \geq B$. ■

Proposition 2.11. *Let $p(z)$ be an even entire function of a single variable. Set $q(x) = x^\alpha p(x)$ for $x > 0$, $\alpha \geq 0$. Assume there is a sufficiently small constant $C > 0$ and sufficiently large constants $A, B > 0$ such that we have*

$$\sup\{|q(y)|;\, y > 0,\, |y - x| < B\} > Cx^{-A} \tag{2.8}$$

for any $x \geq B$. Then $p\big(\sqrt{\zeta^2}\big)$ is a slowly decreasing entire function in $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n$, where $\zeta^2 = \sum_{j=1}^n \zeta_j^2$.

Proof. The assertion is trivial when $\alpha = 0$ (see Proposition 2.10). We have only to prove the case $\alpha > 0$. By choosing a larger B if necessary, we may assume that

$$2^{-\alpha}C > B^{-\alpha}. \tag{2.9}$$

If $y > 0$, $|y - x| < B$, we have $y^\alpha < (x + B)^\alpha < (2x)^\alpha$ and

$$|p(y)| = |y^{-\alpha}q(y)| > (2x)^{-\alpha}|q(y)|. \tag{2.10}$$

By (2.8), (2.9) and (2.10), we have

$$\begin{aligned}\sup\{|p(y)|;\, y > 0,\, |y - x| < B\} &> (2x)^{-\alpha}Cx^{-A} > B^{-\alpha}x^{-(A+\alpha)}\\ &> x^{-(A+2\alpha)} > (A + 2\alpha + x)^{-(A+2\alpha)}.\end{aligned}$$

On the other hand, $B < A\log(2+x)$ holds if $B/\log 2 \leq A$. We see that $p(z)$ is slowly decreasing since (2.7) is valid if we adopt $\max\{A+2\alpha, B/\log 2\}$ as a new value of A. Apply Proposition 2.10 to complete the proof. ■

3 Wong's result

We review the main result of [20]. We incorporate the minor correction given in [22]. Our notation is different from that in [20]. In particular, we employ $s^{\mu+k}$ ($k = 0, 1, 2, \dots$) instead of $t^{s+\lambda-1}$ ($s = 0, 1, 2, \dots$). Let $\varphi(s)$ be a function in $s > 0$ and suppose that $\int_0^\infty \varphi(s)J_\nu(rs)\mathrm{d}s$ converges uniformly for all large values of r. We assume $\varphi(s)$ has the following three properties.

(Φ_1) $\varphi^{(m)}(s)$ is continuous in $s > 0$, where m is a nonnegative integer.

(Φ_2) $\varphi^{(j)}(s) \sim \sum_{k=0}^\infty c_k \frac{\mathrm{d}^j}{\mathrm{d}s^j} s^{\mu+k}$ ($s \to +0$; $j = 0, 1, 2, \dots, m$), where $c_0 \neq 0$, $\mathrm{Re}(\mu + \nu) > -1$, $m \geq \mathrm{Re}\,\mu + 1$.

(Φ_3) $\int_1^\infty s^{-1/2}\varphi(s)\mathrm{e}^{\mathrm{i}rs}\mathrm{d}s$, $\int_1^\infty s^{-1/2}s^{j-m-1/2}\varphi^{(j)}(s)\mathrm{e}^{\mathrm{i}rs}\mathrm{d}s$ ($1 \le j \le m$) converges uniformly for all large values of r.

Theorem 3.1 ([20, equations (3.6) and (4.2)], [22, p. 409])**.** *Assume that* (Φ_1)–(Φ_3) *hold, and let n be a positive integer satisfying $m - \operatorname{Re}\mu - 1 < n < m - \operatorname{Re}\mu + \frac{1}{2}$. Then*

$$\int_0^\infty \varphi(s)J_\nu(rs)\mathrm{d}s = \sum_{k=0}^{n-1} c_k \frac{\Gamma\left(\frac{1}{2}(\mu+k+\nu+1)\right)}{\Gamma\left(-\frac{1}{2}(\mu+k-\nu-1)\right)} \frac{2^{\mu+k}}{r^{\mu+k+1}} + o(r^{-m}), \qquad r \to \infty. \tag{3.1}$$

Remark 3.2. It is natural to assume $c_0 \neq 0$ as in (Φ_2) and we adopt it as a convention. See (4.3). If one wants to study the case $\varphi(s) \sim 0$, one has only to consider the difference of two functions with the same nontrivial expansion.

4 Asymptotic expansion and invertibility

First, we utilize Wong's results reviewed in the previous section to consider the contribution of the singularities at $s = 0$ to Hankel transforms. We consider an infinitely differentiable function $\varphi(s)$ and multiply it by a cut-off function. Therefore, m in (Φ_2) can be arbitrary and the conditions of uniform convergence are satisfied. The k-th coefficient in the right-hand side in (3.1) vanishes if and only if $c_k = 0$ or $\frac{1}{2}(\mu+k-\nu-1)$ is a nonnegative integer. Notice that $\frac{1}{2}(\mu+k+\nu+1)$ cannot be a pole of the Gamma function since $\operatorname{Re}(\mu+k+\nu+1) > k \ge 0$. This observation motivates us to introduce the set K in the proposition below.

Proposition 4.1. *Let $\mu, \nu \in \mathbb{C}$ and $\varphi(s)$ be an infinitely differentiable function in $(0,1)$. We make the following assumptions:*

$$\operatorname{Re}(\mu+\nu) > -1, \tag{4.1}$$

$$\varphi^{(j)}(s) \sim \sum_{k=0}^{\infty} c_k \frac{d^j}{ds^j} s^{\mu+k}, \qquad s \to +0; \quad j = 0,1,2,\dots, \tag{4.2}$$

$$c_0 \neq 0. \tag{4.3}$$

Moreover, let $\chi_0(s)$ be an infinitely differentiable function such that $\chi_0(s) = 1$ in $0 < s \le \varepsilon$ and $\chi_0(s) = 0$ in $1-\varepsilon \le s < 1$, where $0 < \varepsilon < 1/3$. We define the set $K = K(\mu,\nu,\{c_k\}_k)$ by

$$K = K(\mu,\nu,\{c_k\}_k) = \left\{ k \in \mathbb{N}_0;\, c_k \neq 0,\, \frac{1}{2}(\mu+k-\nu-1) \notin \mathbb{N}_0 \right\}, \tag{4.4}$$

where $\mathbb{N}_0$ is the set of nonnegative integers. Then, we have the following.

If $K \neq \varnothing$,

$$\int_0^1 \chi_0(s)\varphi(s)J_\nu(rs)\mathrm{d}s = c_{k_0} \frac{\Gamma\left(\frac{1}{2}(\mu+k_0+\nu+1)\right)2^{\mu+k_0}}{\Gamma\left(\frac{1}{2}(-\mu-k_0+\nu+1)\right)r^{\mu+k_0+1}} + o\big(r^{-\operatorname{Re}(\mu+k_0+1)}\big)$$

as $r \to \infty$, where $k_0 = \min K$.

If $K = \varnothing$,

$$\int_0^1 \chi_0(s)\varphi(s)J_\nu(rs)\mathrm{d}s = o\big(r^{-A}\big) \tag{4.5}$$

as $r \to \infty$, where A is an arbitrarily large real number.

Proposition 4.2. *Assume* $\mathrm{Re}(\nu) > -1$*. If* $\varphi(s)$ *is an infinitely differentiable function in* $(0,\infty)$*, its support is bounded and* $\varphi(s) = s^{\nu+1}\big(1-s^2\big)^{\alpha}$*,* $\alpha \in \mathbb{C}$ *near* $s = 0$*, then*

$$\int_0^{\infty} \varphi(s) J_\nu(rs)\mathrm{d}s = o\big(r^{-A}\big),$$

where A *is an arbitrarily large real number.*

Proof. The set K is empty in this case, since we have $\mu = \nu + 1$ and k is even if $c_k \neq 0$. ■

Next we consider the contribution of the singularities at $s = 1$ to finite Hankel transforms.

Proposition 4.3. *Let* N *be a nonnegative integer and* Λ *be a complex number with* $\mathrm{Re}\,\Lambda \geq N$*. Assume that* $\psi(t)$ *is an infinitely differentiable function in* $0 < t < 1$ *such that* $\psi(t) = 0$ *in* $0 < t < \varepsilon$ *and that* $\psi^{(k)}(t)$ *is integrable for* $0 \leq k \leq N$*. Set* $\phi(t) = (1-t)^{\Lambda}\psi(t)$*. Then* $\phi\big(s^2\big)$ *is an infinitely differentiable function in* $0 < s < 1$ *and if* $\nu \geq -N$*, we have*

$$\int_0^1 s^{\nu+1}\phi\big(s^2\big) J_\nu(rs)\mathrm{d}s = o\big(r^{-(N+1/2)}\big), \qquad \textit{as} \quad r \to \infty.$$

Proof. We have

$$\phi^{(k)}(t) = \sum_{j=0}^{k} \binom{k}{j} \Lambda(\Lambda-1)\cdots(\Lambda-j+1)(-1)^j(1-t)^{\Lambda-j}\psi^{(k-j)}(t).$$

Assume $N \geq 1$. We have $\phi^{(k)}(1) = 0$ if $k \leq N-1$. Set

$$I_k = \int_0^1 s^{\nu+k+1}\phi^{(k)}\big(s^2\big) J_{\nu+k}(rs)\mathrm{d}s, \qquad k = 0, 1, 2, \dots, N.$$

The recurrence relation $\big(z^{\nu+1}J_{\nu+1}(z)\big)' = z^{\nu+1}J_\nu(z)$ (see, for example, [17, formula (10.6.6)]) yields

$$\frac{\mathrm{d}}{\mathrm{d}s}\big\{s^{\nu+1}J_{\nu+1}(rs)\big\} = rs^{\nu+1}J_\nu(rs).$$

Combining this formula and integration by parts, we get

$$\begin{aligned} I_k &= \frac{1}{r}\int_0^1 \phi^{(k)}\big(s^2\big)\cdot rs^{\nu+k+1}J_{\nu+k}(rs)\mathrm{d}s \\ &= \frac{1}{r}\int_0^1 \phi^{(k)}\big(s^2\big)\frac{\mathrm{d}}{\mathrm{d}s}\big\{s^{\nu+k+1}J_{\nu+k+1}(rs)\big\}\mathrm{d}s = -\frac{2}{r}I_{k+1} \end{aligned} \tag{4.6}$$

if $k \leq N-1$. We obtain

$$I_0 = \left(-\frac{2}{r}\right)^N I_N \tag{4.7}$$

for $N \geq 1$. Notice that (4.7) holds for $N = 0$ as well.

On the other hand, it is well known that

$$J_\alpha(z) = \frac{2^{1/2}}{\pi^{1/2}}z^{-1/2}\cos\left(z - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) + O\big(z^{-3/2}\big) \tag{4.8}$$

as $\mathbb{R} \ni z \to \infty$ (see, for example, [17, formula (10.17.3)]). By (4.8) and the boundedness of $J_{\nu+N}(z)$ as $z \to +0$, there exists a bounded function $R(z)$ $(0 < z < \infty)$ such that

$$J_{\nu+N}(z) = \frac{2^{1/2}}{\pi^{1/2}} z^{-1/2} \cos\left(z - \frac{(\nu+N)\pi}{2} - \frac{\pi}{4}\right) + z^{-3/2} R(z).$$

We have

$$I_N = \frac{2^{1/2}}{\pi^{1/2}} r^{-1/2} \int_0^1 \phi^{(N)}\big(s^2\big) s^{\nu+N+1/2} \cos\left(rs - \frac{(\nu+N)\pi}{2} - \frac{\pi}{4}\right) \mathrm{d}s$$
$$+ r^{-3/2} \int_0^1 \phi^{(N)}\big(s^2\big) s^{\nu+N-1/2} R(rs) \mathrm{d}s.$$

The first term in the right hand side is of order $o\big(r^{-1/2}\big)$ as $r \to \infty$ by the Riemann–Lebesgue lemma. The second term is of order $O\big(r^{-3/2}\big)$ by the boundedness of R. We have shown $I_N = o\big(r^{-1/2}\big)$ and the combination of it and (4.7) gives $I_0 = o\big(r^{-(N+1/2)}\big)$. ■

Proposition 4.4. *Let N be a nonnegative integer and ν, $\lambda_0, \dots, \lambda_m$, Λ, $a_0, \dots, a_m$ be complex numbers. Assume $\operatorname{Re}\nu > -1$, $-1 < \operatorname{Re}\lambda_0 < \operatorname{Re}\lambda_1 < \cdots < \operatorname{Re}\lambda_m < \operatorname{Re}\Lambda$, $N \le \operatorname{Re}\Lambda$, $\operatorname{Re}\lambda_0 \le N-1$. Assume that the function $\phi(t)$ in $0 < t < 1$ satisfies*

$$\phi(t) = \sum_{k=0}^{m} a_k (1-t)^{\lambda_k} + (1-t)^{\Lambda} \psi(t), \qquad 1 - 2\varepsilon < t < 1,$$

where $a_0 \neq 0$ and $\psi(t)$ is an infinitely differentiable function in $0 < t < 1$ such that $\psi^{(k)}(t)$ is integrable in $1 - 2\varepsilon < t < 1$ for $0 \le k \le N$.

Let $\chi_1(t)$ be an infinitely differentiable function such that $\chi_1(t) = 0$ in $0 < t \le 1 - 2\varepsilon$ and $\chi_1(t) = 1$ in $1 - \varepsilon \le t < 1$. Set[2]

$$\tilde{\phi}(t) = \sum_{k=0}^{m} a_k (1-t)^{\lambda_k} + \chi_1(t)(1-t)^{\Lambda} \psi(t), \qquad 0 < t < 1.$$

Then as $r \to \infty$,

$$\int_0^1 s^{\nu+1} \tilde{\phi}\big(s^2\big) J_\nu(rs) \mathrm{d}s = a_0 \frac{2^{\lambda_0+1/2}}{\pi^{1/2}} \Gamma(\lambda_0+1) r^{-(\lambda_0+3/2)} \cos\left(r - \frac{\pi}{2}(\nu + \lambda_0 + 1) - \frac{\pi}{4}\right)$$
$$+ o\big(r^{-\operatorname{Re}(\lambda_0+3/2)}\big).$$

Proof. By [9, formula (26(33)a)] or [10, formula (6.567.1)], we have

$$\int_0^1 s^{\nu+1} \big(1 - s^2\big)^{\alpha} J_\nu(rs) \mathrm{d}s = 2^{\alpha} \Gamma(\alpha+1) r^{-(\alpha+1)} J_{\nu+\alpha+1}(r), \tag{4.9}$$

if $r > 0$, $\operatorname{Re}\nu > -1$, $\operatorname{Re}\alpha > -1$.[3] We employ this formula and Proposition 4.3 to obtain

$$\int_0^1 s^{\nu+1} \tilde{\phi}\big(s^2\big) J_\nu(rs) \mathrm{d}s = \sum_{k=0}^{m} a_k 2^{\lambda_k} \Gamma(\lambda_k+1) r^{-(\lambda_k+1)} J_{\nu+\lambda_k+1}(r) + o\big(r^{-(N+1/2)}\big).$$

We finish the proof by using (4.8). ■

[2]Notice that the cutoff is incomplete in the sense that $\chi_1(t)$ does not appear in the summation term.

[3]If we set $s = \sin\theta$, we get Sonine's first finite integral. Two methods of its evaluation are given in [19, formula (12.11) (1)]. There is another proof of (4.9). When α is a nonnegative integer, we can prove it following (4.6) in the proof of Proposition 4.3. To generalize it to $\operatorname{Re}\alpha > -1$, we divide the equality by $\Gamma(\alpha+1)$ and apply the Carlson's theorem [2, p. 110]. Boundedness is guaranteed by Poisson's integral representation of Bessel functions [17, formula (10.9.4)].

Now we give our main results.

Theorem 4.5. *Let $n \geq 2$ be an integer. Let $\varphi(s)$ be an infinitely differentiable function in the open interval $(0,1)$ and $\phi(t)$ $(0<t<1)$ be the function such that*

$$\varphi(s) = s^{n/2}\phi\big(s^2\big), \qquad 0<s<1.$$

Set $\nu = n/2-1$. If μ satisfies (4.1), $\varphi(s)$ satisfies (4.2), (4.3) and $\phi(t)$ satisfies the conditions of Proposition 4.4, then

$$f(x) = |x|^{-n/2}\varphi(|x|)\chi_{[0,1]}(|x|) = \phi\big(|x|^2\big)\chi_{[0,1]}(|x|), \qquad x \in \mathbb{R}^n \tag{4.10}$$

is an invertible distribution. Here $\chi_{[0,1]}(\cdot)$ is the indicator function of the interval $[0,1]$.

Proof. Set $\nu = \nu(n) = n/2-1$ and

$$\tilde{\varphi}(s) = \chi_0(s)\left\{\varphi(s) - s^{n/2}\sum_{k=0}^{m} a_k\big(1-s^2\big)^{\lambda_k}\right\}.$$

Notice that

$$\tilde{\varphi}^{(j)}(s) \sim \sum_{k=0}^{\infty} c_k \frac{\mathrm{d}^j}{\mathrm{d}s^j}s^{\mu+k} + \sum_{\ell=0}^{\infty} A_\ell \frac{\mathrm{d}^j}{\mathrm{d}s^j}s^{n/2+2\ell}, \qquad s \to +0$$

for some A_ℓ. The second sum, which corresponds to $s^{n/2}\sum_{k=0}^{m} a_k\big(1-s^2\big)^{\lambda_k}$, does not contribute to the finite Hankel transform because of Proposition 4.2.

Let $K_n = K(\mu, \nu(n), \{c_k\}_k)$ be the set defined by (4.4) with $\nu = \nu(n)$. If $K_n \neq \varnothing$, we have

$$H_0 := \int_0^1 \tilde{\varphi}(s)J_{\nu(n)}(rs)\mathrm{d}s = c_{k_0}\frac{\Gamma\left(\frac{1}{2}(\mu+k_0+\nu(n)+1)\right)2^{\mu+k_0}}{\Gamma\left(\frac{1}{2}(-\mu-k_0+\nu(n)+1)\right)r^{\mu+k_0+1}} + o\big(r^{-\operatorname{Re}(\mu+k_0+1)}\big),$$

and if $K_n = \varnothing$, H_0 decreases rapidly of arbitrary order by (4.5). On the other hand, for $\tilde{\phi}$ defined in Proposition 4.4, we have

$$\begin{aligned} H_1 &:= \int_0^1 s^{\nu(n)+1}\tilde{\phi}\big(s^2\big)J_{\nu(n)}(rs)\mathrm{d}s \\ &= a_0\frac{2^{\lambda_0+1/2}}{\pi^{1/2}}\Gamma(\lambda_0+1)r^{-(\lambda_0+3/2)}\cos\left(r - \frac{\pi}{2}(\nu(n)+\lambda_0+1) - \frac{\pi}{4}\right) + o\big(r^{-\operatorname{Re}(\lambda_0+3/2)}\big). \end{aligned}$$

We can prove that $H_0 + H_1$ satisfies (2.8) in the following way.

If $K_n = \varnothing$, H_1 is dominant. We assume $K_n \neq \varnothing$ from now on. If $\operatorname{Re}\mu + k_0 + 1 < \operatorname{Re}\lambda_0 + 3/2$, then H_0 is dominant. If $\operatorname{Re}\mu + k_0 + 1 > \operatorname{Re}\lambda_0 + 3/2$, then H_1 is dominant. If $\operatorname{Re}\mu + k_0 + 1 = \operatorname{Re}\lambda_0 + 3/2$, we can pick up those r's for which the cosine factor vanishes and the contribution of H_1 becomes irrelevant. In any case, $H_0 + H_1$ satisfies (2.8). By Theorem 2.1 and Proposition 2.11, $|x|^{-n/2}\tilde{\varphi}(|x|) + \tilde{\phi}\big(|x|^2\big)$ is invertible.

Notice that $\varphi(s) - \tilde{\varphi}(s) - s^{n/2}\tilde{\phi}\big(s^2\big)$ vanishes near $s = 0, 1$ and

$$f(x) - |x|^{-n/2}\tilde{\varphi}(|x|) - \tilde{\phi}\big(|x|^2\big) \in \mathcal{C}_0^\infty(\mathbb{R}^n).$$

By Proposition 2.7 (v), $f(x)$ is invertible. ∎

Example 4.6. If $\lambda > -n/2$, $\rho > 0$, then $f(x) = |x|^{\lambda - n/2}\bigl(1 - |x|^2\bigr)^{\rho-1}\chi_{[0,1]}(|x|)$ is an invertible distribution. This corresponds to the case of $\varphi(s) = s^\lambda\bigl(1-s^2\bigr)^{\rho-1}$ and $\phi(t) = t^{\lambda/2-n/4}(1-t)^{\rho-1}$.

The finite Hankel transform $\int_0^1 s^\lambda\bigl(1-s^2\bigr)^{\rho-1}J_\nu(rs)\mathrm{d}s$ can be written in terms of the generalized hypergeometric function ${}_2F_3$ by [10, formula (6.569)] and it is good enough to prove invertibility. The advantage of our method is that it is stable under small perturbations and works even if no closed form expression is available.

Remark 4.7. In (4.2), j can be arbitrarily large. This assumption can be relaxed in some cases. We give such an example. Assume

$$\varphi(s) \sim 0 = \sum_{k=0}^{\infty} 0 \cdot s^{\mu+k}$$

as $s \to +0$, where $-n/2 < \operatorname{Re}\mu \le -1$. We do not assume that term by term differentiation is possible:[4] m in (Φ_1) is 0. Moreover, we have removed the assumption $c_0 \neq 0$ and do not need the set K_n. We keep all the other assumptions of Theorem 4.5 unchanged. By (3.1) and Remark 3.2, we have $H_0 = o(1)$. If $\operatorname{Re}\lambda_0 \le -3/2$, H_1 is dominant and $f(x)$ defined by (4.10) is invertible.

Theorem 4.8. *Let $n \ge 2$ be an integer and N be a nonnegative integer. Let $\varphi(s)$ be an infinitely differentiable function in the open interval $(0,1)$ and $\phi(t)$ $(0 < t < 1)$ be the function such that $\varphi(s) = s^{n/2}\phi\bigl(s^2\bigr)$. Set $\nu = n/2 - 1$. Suppose that μ satisfies (4.1) and that $\varphi(s)$ satisfies (4.2), (4.3). Moreover, suppose $\operatorname{Re}\Lambda \ge N$ and*

$$\phi(t) = (1-t)^\Lambda \psi(t), \qquad 1 - 2\varepsilon < t < 1,$$

where $\psi(t)$ is an infinitely differentiable function in $0 < t < 1$ such that $\psi^{(k)}(t)$ is integrable in $1 - 2\varepsilon < t < 1$ for $0 \le k \le N$.

If $K_n = K(\mu, n/2-1, \{c_k\}_k) \neq \varnothing$ and $\operatorname{Re}(\mu + k_0 + 1/2) \le N$, then

$$f(x) = |x|^{-n/2}\varphi(|x|)\chi_{[0,1]}(|x|) = \phi\bigl(|x|^2\bigr)\chi_{[0,1]}(|x|), \qquad x \in \mathbb{R}^n$$

is an invertible distribution.

Proof. We have

$$H_0 = c_{k_0}\frac{\Gamma\left(\frac{1}{2}(\mu + k_0 + \nu(n) + 1)\right)2^{\mu+k_0}}{\Gamma\left(\frac{1}{2}(-\mu - k_0 + \nu(n) + 1)\right)r^{\mu+k_0+1}} + o\bigl(r^{-\operatorname{Re}(\mu+k_0+1)}\bigr), \qquad H_1 = o\bigl(r^{-(N+1/2)}\bigr)$$

by Propositions 4.1 and 4.3. Since $\operatorname{Re}(\mu + k_0 + 1) \le N + 1/2$, H_0 is dominant and H_1 is negligible. ∎

Theorem 4.9. *Let $n \ge 2$ be an integer. Let $\varphi(s)$ be an infinitely differentiable function in $s > 0$ such that $\varphi(s) = 0$ in $s \ge 1$. Set $\nu = n/2 - 1$. Suppose that μ satisfies (4.1) and that $\varphi(s)$ satisfies (4.2), (4.3). Then*

$$f(x) = |x|^{-n/2}\varphi(|x|)\chi_{[0,1]}(|x|), \qquad x \in \mathbb{R}^n$$

is an invertible distribution if and only if $K_n = K(\mu, n/2-1, \{c_k\}_k)$ is non-empty.

Proof. The quantity Λ in Theorem 4.8 can be an arbitrarily large integer and we have $H_1 = o\bigl(r^{-A}\bigr)$, where A is arbitrarily large. By Proposition 4.1, the invertibility of $f(x)$ is equivalent to the non-emptiness of K_n. ∎

[4] For example, $\varphi(s) = \mathrm{e}^{-1/s}\sin \mathrm{e}^{1/s}$.

Remark 4.10. We can find a lot of invertible distributions by combining Theorems 4.5, 4.8 and 4.9 with Remark 2.6 and Proposition 2.7.

Remark 4.11. We can formulate an invertibility theorem by using [21] instead of [20]. In [21], the function $\varphi(s)$ is allowed to have an asymptotic expansion involving powers of logarithms.

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