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Polynomial Solutions Modulo p^s of Differential KZ and Dynamical Equations

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Abstract. We construct polynomial solutions modulo p^s of the differential KZ and dynamical equations where p is an odd prime number.

Key words: differential KZ and dynamical equations; polynomial solutions modulo p^s; hypergeometric integrals

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1 Introduction

The KZ equations were introduced by Knizhnik and Zamolodchikov [7] to describe the differential equations for conformal blocks on the Riemann sphere. Different versions of the KZ equations appear in mathematical physics, algebraic geometry and the theory of special functions, see, for example, [3, 9]. One of the important properties of the KZ equations is their realization as suitable Gauss–Manin connections. This construction gives a presentation of solutions of the KZ equations by multidimensional hypergeometric integrals, see [1, 2, 13].

The fact that certain integrals of closed differential forms over cycles satisfy a linear differential equation follows by Stokes' theorem from a suitable cohomological relation, in which the result of the application of the corresponding differential operator to the integrand of an integral equals the differential of a differential form of one degree lower. Such cohomological relations for the KZ equations associated with arbitrary Kac–Moody algebras were developed in [14].

The KZ equations possess a bispectrality property – they have a compatible system of dynamical equations with respect to associated dynamical parameters, see [4, 5, 9, 11, 18].

Let p be an odd prime. In [15, 23], the differential KZ equations were considered modulo p^s, and polynomial solutions modulo p^s were constructed as analogs of the hypergeometric integrals. The construction was based on the fact that all cohomological relations described in [14] are defined over $\mathbb{Z}$ and can be reduced modulo p^s. Studying solutions modulo of p^s sheds light on solutions of the KZ equations both over the field of complex numbers and over p-adic fields, for example, see [16].

In this paper, we consider the joint system of the differential KZ and differential dynamical equations, the system introduced in [5], and construct polynomial solutions modulo p^s of the joint system as analogs of the corresponding hypergeometric integrals with an exponential term. For this purpose, one needs to represent the exponential function $e^{\lambda t}$ with an integer parameter λ by a polynomial in t modulo p^s. This can be done after replacing λ with $p\lambda$.

An interesting problem is to study the p-adic limit of the constructed polynomial solutions modulo p^s as $s \to \infty$, see examples of this limit for the differential KZ equations in [22, 23, 24].

The joint system of the KZ and dynamical equations has many versions: differential KZ equations and differential dynamical equations, differential KZ equations and difference dynamical equations, difference KZ equations and differential dynamical equations, difference KZ equations and difference dynamical equations, see, for example, [4, 9, 11, 18]. The polynomial solutions modulo p^s are constructed in this paper only for the original joint system of the differential KZ and differential dynamical equations, although there are examples of polynomial solutions modulo p^s in other cases, see [10, 12, 22] and also Appendix A.

In the remainder of the introduction we consider an example.

1.1 Solutions over $\mathbb{C}$

Consider the complex master function

$$\Phi(t,z,\lambda)=\mathrm{e}^{\lambda t}\prod_{i=1}^{2g+1}(t-z_i)^{-1/2}$$

and the tuple of integrals

$$I(z,\lambda)=(I_1(z,\lambda),\dots,I_{2g+1}(z,\lambda))=\int_\delta \Phi(t,z,\lambda)\left(\frac{1}{t-z_1},\dots,\frac{1}{t-z_{2g+1}}\right)\mathrm{d}t, \tag{1.1}$$

where δ is a 1-cycle.

Theorem 1.1. *The tuple $I(z,\lambda)$ satisfies the joint system of KZ and dynamical equations*

$$\frac{\partial I_j}{\partial z_i}=\frac12\frac{I_i-I_j}{z_i-z_j},\qquad i\neq j, \tag{1.2}$$

$$\frac{\partial I_i}{\partial z_i}=\lambda I_i-\frac12\sum_{j\neq i}\frac{I_i-I_j}{z_i-z_j}, \tag{1.3}$$

$$\frac{\partial I_i}{\partial \lambda}=z_iI_i+\frac{1}{2\lambda}\sum_{j=1}^{n}I_j. \tag{1.4}$$

The system of equations (1.2) and (1.3) is called the KZ equations of this example, the system of equations (1.4) is called the dynamical equation. The solutions $I(z,\lambda)$ are called the hypergeometric solutions.

Proof. The proof uses the following identities:

$$\frac{1}{(t-z_i)(t-z_j)}=\frac{1}{z_i-z_j}\left(\frac{1}{t-z_i}-\frac{1}{t-z_j}\right),$$

$$\frac{\partial}{\partial t}\Phi(t,z,\lambda)=\lambda\Phi(t,z,\lambda)-\frac12\Phi(t,z,\lambda)\sum_{j=1}^{2g+1}\frac{1}{t-z_j},$$

$$\frac{\partial}{\partial t}\frac{\Phi(t,z,\lambda)}{t-z_i}=\Phi(t,z,\lambda)\left(\frac{\lambda}{t-z_i}-\frac32\frac{1}{(t-z_i)^2}-\frac12\sum_{j\neq i}\frac{1}{(t-z_i)(t-z_j)}\right).$$

For $j\neq i$, we have

$$\frac{\partial I_j}{\partial z_i}=\int\Phi(t,z,\lambda)\frac{1/2}{(t-z_i)(t-z_j)}\mathrm{d}t=\frac12(I_i-I_j).$$

This proves the first equation. Then

$$\begin{aligned}\frac{\partial I_i}{\partial z_i} &= \int \Phi(t,z,\lambda)\frac{3/2}{(t-z_i)^2}\mathrm{d}t \\ &= -\int \frac{\partial}{\partial t}\frac{\Phi(t,z,\lambda)}{t-z_i}\mathrm{d}t + \int \Phi(t,z,\lambda)\bigg(\frac{\lambda}{t-z_i} - \frac{1}{2}\sum_{j\neq i}\frac{1}{(t-z_i)(t-z_j)}\bigg)\mathrm{d}t \\ &= \lambda I_i - \frac{1}{2}\sum_{j\neq i}\frac{I_i - I_j}{z_i - z_J}\end{aligned}$$

gives the second equation. We also have

$$\begin{aligned}\frac{\partial I_i}{\partial \lambda} &= \int \Phi(t,z,\lambda)\frac{t-z_i+z_i}{t-z_i}\mathrm{d}t = \int \Phi(t,z,\lambda)\mathrm{d}t + z_i\int \Phi(t,z,\lambda)\frac{1}{t-z_i}\mathrm{d}t \\ &= \frac{1}{\lambda}\int \frac{\partial}{\partial t}\Phi(t,z,\lambda)\mathrm{d}t + \frac{1}{2\lambda}\int \Phi(t,z,\lambda)\sum_{j=1}^{2g+1}\frac{1}{t-z_j}\mathrm{d}t + z_i\int \Phi(t,z,\lambda)\frac{1}{t-z_i}\mathrm{d}t \\ &= z_i I_i + \frac{1}{2\lambda}\sum_{j=1}^{2g+1} I_j. \qquad \blacksquare\end{aligned}$$

The complex vector space of (multi-valued) solutions of the joint system of KZ and dynamical equations (1.2)–(1.4) is $(2g+1)$-dimensional. Every solution of the joint system has the integral presentation (1.1) for a suitable cycle δ, see [8, Theorem 6.1] and an example in [8, Introduction].

1.2 Exponential function

We have

$$\mathrm{e}^{\lambda t} = \sum_{m=0}^{\infty}\lambda^{(m)}t^m, \qquad \lambda^{(m)} = \frac{\lambda^m}{m!}, \qquad \binom{m+n}{m}\lambda^{(m+n)} = \lambda^{(m)}\lambda^{(m)}, \qquad \frac{\mathrm{d}}{\mathrm{d}\lambda}\lambda^{(m)} = \lambda^{(m-1)}.$$

We set $\lambda^{(m)} = 0$ for $m<0$ by convention.

Let $f(t,z) = \sum_{m=0}^{\infty} b_m(z)t^m$, $b_m(z)\in\mathbb{Z}_p[z]$, where $\mathbb{Z}_p$ is the ring of p-adic integers and $z=(z_1,\dots,z_{2g+1})$ are parameters. Consider the decomposition

$$\mathrm{e}^{\lambda t}f(t,z) = \sum_{k=0}^{\infty} c_k(z,\lambda)t^k,$$

where each $c_k(\lambda,z)$ is a linear function in finitely many symbols $\lambda^{(m)}$, $m=0,1,\dots,$ whose coefficients lie in $\mathbb{Z}_p[z]$.

Lemma 1.2. *Let s, ℓ be positive integers. Then the coefficient of $t^{\ell p^s-1}$ in the series $\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathrm{e}^{\lambda t}f(t,z)\big)$ is divisible by p^s, that is, all coefficients of the corresponding linear function in symbols $\lambda^{(m)}$, $m\geq 0$, are divisible by p^s.*

Proof. It is enough to prove the lemma for $f(t)=t^a$. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\lambda t}t^a &= \sum_{m=0}^{\infty}\big(\lambda\cdot\lambda^{(m)}t^{m+a} + a\lambda^{(m)}t^{m+a-1}\big) = \sum_{m=0}^{\infty}\big((m+1)\lambda^{(m+1)}t^{m+a} + a\lambda^{(m)}t^{m+a-1}\big) \\ &= \sum_{k=0}^{\infty}\big((k-a+1)\lambda^{(k-a+1)}t^k + a\lambda^{(k-a+1)}t^k\big) = \sum_{k=0}^{\infty}(k+1)\lambda^{(k-a+1)}t^k. \qquad \blacksquare\end{aligned}$$

Lemma 1.3. *If $\lambda \in \mathbb{Z}_p$, then $p^k \lambda^{(k)} \in \mathbb{Z}_p$ for all $k \geq 0$. If s is a positive integer and $k > s\frac{p-1}{p-2}$, then $p^k \lambda^{(k)} \in p^s \mathbb{Z}_p$.*

Proof. The maximal power of p dividing $k!$ equals $\left[\frac{k}{p}\right] + \left[\frac{k}{p^2}\right] + \left[\frac{k}{p^3}\right] + \cdots$ which is not greater than $\frac{k}{p} + \frac{k}{p^2} + \frac{k}{p^3} + \cdots = \frac{k}{p-1}$. Hence the power of p dividing $\frac{p^k}{k!}$ is not less than $k - \frac{k}{p-1} = k\frac{p-2}{p-1} > 0$. Hence $p^k \lambda^{(k)} \in \mathbb{Z}_p$. We have $k\frac{p-2}{p-1} \geq s$ if $k \geq s\frac{p-1}{p-2}$. ■

Denote

$$d(s) = \left[s\frac{p-1}{p-2}\right] + 1, \qquad E_s(t) = \sum_{k=0}^{d(s)} \frac{t^k}{k!}.$$

Corollary 1.4. *If $\lambda \in \mathbb{Z}_p$, then $\mathrm{e}^{p\lambda t} \in \mathbb{Z}_p[[t]]$, $E_s(p\lambda t) \in \mathbb{Z}_p[t]$ and*

$$\begin{aligned}
&\mathrm{e}^{p\lambda t} \equiv E_s(p\lambda t) \pmod{p^s}, \\
&\frac{\partial}{\partial t} E_s(p\lambda t) \equiv p\lambda E_s(p\lambda t), \qquad \frac{\partial}{\partial \lambda} E_s(p\lambda t) \equiv pt E_s(p\lambda t) \pmod{p^s}, \\
&E_s(p\lambda(u+v)) \equiv E_s(p\lambda u) E_s(p\lambda v) \pmod{p^s}.
\end{aligned}$$

1.3 Remarks on p^r

Let $v_p(a)$ denote the p-adic evaluation of a.

Let r_1, r_2 be relatively prime positive integers. Denote $r = r_1/r_2$. Assume that $r > 1/(p-1)$. Then for a positive integer k, we have

$$v_p\big(p^{kr}/k!\big) = \left(kr - \left(\left[\frac{k}{p}\right] + \left[\frac{k}{p^2}\right] + \cdots\right)\right) > k\left(r - \frac{1}{p-1}\right) > 0.$$

Hence, if $\lambda \in \mathbb{Z}_p\big[p^{1/r_2}\big]$, then $p^{kr}\lambda^{(k)} \in \mathbb{Z}_p\big[p^{1/r_2}\big]$. Moreover, if s is a positive integer and $k > s\frac{p-1}{r(p-1)-1}$, then $p^{kr}\lambda^{(k)} \in p^s \mathbb{Z}_p\big[p^{1/r_2}\big]$.

Denote

$$d(r,s) = \left[s\frac{p-1}{r(p-1)-1}\right] + 1, \qquad E_{r,s}(t) = \sum_{k=0}^{d(r,s)} \frac{t^k}{k!}.$$

If $\lambda \in \mathbb{Z}_p\big[p^{1/r_2}\big]$, then $\mathrm{e}^{p^r\lambda t} \in \mathbb{Z}_p\big[p^{1/r_2}\big][[t]]$, $E_{r,s}(p^r\lambda t) \in \mathbb{Z}_p\big[p^{1/r_2}\big][t]$ and

$$\begin{aligned}
&\mathrm{e}^{p^r\lambda t} \equiv E_{r,s}(p^r\lambda t) \pmod{p^s}, \\
&\frac{\partial}{\partial t} E_{r,s}(p^r\lambda t) \equiv p^r\lambda E_{r,s}(p^r\lambda t), \\
&\frac{\partial}{\partial \lambda} E_{r,s}(p^r\lambda t) \equiv p^r t E_{r,s}(p^r\lambda t) \pmod{p^s}, \\
&E_{r,s}(p^r\lambda(u+v)) \equiv E_{r,s}(p^r\lambda u) E_{r,s}(p^r\lambda v) \pmod{p^s}.
\end{aligned}$$

If $r = 1/(p-1)$, then

$$v_p\big(p^{k/(p-1)}/k!\big) = \left(\frac{k}{p-1} - \left(\left[\frac{k}{p}\right] + \left[\frac{k}{p^2}\right] + \cdots\right)\right) > 0$$

and $\mathrm{e}^{p^{1/(p-1)}t} \in \mathbb{Z}_p\big[p^{1/(p-1)}\big][[t]]$ but $v_p\big(p^{k/(p-1)}/k!\big)$ does not grow as $k \to \infty$, and so we get an infinite series rather than a polynomial.

1.4 Reformulation of the equations

Change the variable $\lambda \mapsto p\lambda$. Then the KZ and dynamical equations take the form

$$\frac{\partial I_j}{\partial z_i} = \frac{1}{2}\frac{I_i - I_j}{z_i - z_j}, \qquad i \neq j, \tag{1.5}$$

$$\frac{\partial I_i}{\partial z_i} = p\lambda I_i - \frac{1}{2}\sum_{j\neq i}\frac{I_i - I_j}{z_i - z_j}, \tag{1.6}$$

$$\frac{\partial I_i}{\partial \lambda} = p z_i I_i + \frac{1}{2\lambda}\sum_{j=1}^{2g+1} I_j. \tag{1.7}$$

For any positive integer s, we construct below some vectors of polynomials

$$I(z,\lambda) = (I_1(z,\lambda), \dots, I_{2g+1}(z,\lambda))$$

with coefficients in $\mathbb{Z}_p$ which satisfy the KZ equations (1.5), (1.6) modulo p^s if $\lambda \in \mathbb{Z}_p$ and satisfy the dynamical equations (1.7) modulo p^s if $\lambda \in \mathbb{Z}_p^\times$.

1.5 Solutions modulo p^s

For a positive integer s, define

$$\Phi_s^o(t,z) = \prod_{i=1}^{2g+1}(t - z_i)^{(p^s-1)/2}, \qquad \Phi_s(t,z,\lambda) = E_s(p\lambda t)\Phi_s^o(t,z),$$

$$\Psi_s^o(t,z) = \Phi_s^o(t,z)\left(\frac{1}{t-z_1}, \dots, \frac{1}{t-z_{2g+1}}\right), \qquad \Psi_s(t,z,\lambda) = E_s(p\lambda t)\Psi_s^o(t,z).$$

Consider the Taylor expansions

$$\Psi_s^o(t,z) = \sum_{m=0}^{(2g+1)(p^s-1)/2-1} c_m^o(z) t^m, \qquad \Psi_s(t,z,\lambda) = \sum_{d=0}^{(2g+1)(p^s-1)/2-1+d(s)} c_d(z,\lambda) t^d,$$

where each $c_m^o(z)$ is a vector of polynomials in z with integer coefficients, and

$$c_d(z,\lambda) = \sum_{m=0}^{d} p^{d-m}\lambda^{(d-m)} c_m^o(z).$$

For any positive integer ℓ, denote

$$I^\ell(z,\lambda) = c_{\ell p^s - 1}(z,\lambda).$$

All coordinates of this vector are polynomials in z, λ with coefficients in $\mathbb{Z}_p$.

Theorem 1.5. *Let ℓ be a positive integer. If $\lambda \in \mathbb{Z}_p$, then $I^\ell(z,\lambda)$ is a solution modulo p^s of the KZ equations (1.5), (1.6). If $\lambda \in \mathbb{Z}_p^\times$, then $I^\ell(z,\lambda)$ is a solution modulo p^s of the dynamical equations (1.7).*

We call such solutions the p^s*-hypergeometric solutions* of the joint system of the KZ and dynamical equations.

Proof. The proof uses the following identities:

$$\frac{1}{(t-z_i)(t-z_j)} = \frac{1}{z_i - z_j}\left(\frac{1}{t-z_i} - \frac{1}{t-zj}\right),$$

$$\frac{\partial}{\partial t}\Phi_s(t,z,\lambda) \equiv p\lambda\Phi_s(t,z,\lambda) + \frac{p^s-1}{2}\Phi_s(t,z,\lambda)\sum_{j=1}^{2g+1}\frac{1}{t-z_j} \pmod{p^s},$$

$$\frac{\partial}{\partial t}\frac{\Phi_s(t,z,\lambda)}{t-z_i} \equiv \Phi_s(t,z,\lambda)\left(\frac{p\lambda}{t-z_i} + \frac{p^s-3}{2}\frac{1}{(t-z_i)^2} + \frac{p^s-1}{2}\sum_{j\neq i}\frac{1}{(t-z_i)(t-z_j)}\right)$$

modulo p^s. For $j \neq i$, we have

$$\begin{aligned}\frac{\partial}{\partial z_i}\Phi_s(t,z,\lambda)\frac{1}{t-z_j} &= \Phi(t,z,\lambda)\frac{1-p^s}{2}\frac{1}{(t-z_i)(t-z_j)}\\ &= \frac{1-p^s}{2}\Phi_s(t,z,\lambda)\left(\frac{1}{t-z_i} - \frac{1}{t-z_j}\right).\end{aligned}$$

Take the coefficient of $t^{\ell p^s-1}$ in both sides of the equation. As the result, we obtain modulo p^s,

$$\frac{\partial I_j^\ell}{\partial z_i}(z,\lambda) \equiv \frac{1}{2}\frac{I_i^\ell(z,\lambda) - I_j^\ell(z,\lambda)}{z_i - z_j}, \qquad i \neq j.$$

We have

$$\begin{aligned}\frac{\partial}{\partial z_i}\Phi_s(t,z,\lambda)\frac{1}{t-z_i} &= \Phi_s(t,z,\lambda)\frac{3-p^s}{2}\frac{1}{(t-z_i)^2}\\ &\equiv -\frac{\partial}{\partial t}\frac{\Phi_s(t,z,\lambda)}{t-z_i} + \Phi_s(t,z,\lambda)\left(\frac{p\lambda}{t-z_i} + \frac{p^s-1}{2}\sum_{j\neq i}\frac{1}{(t-z_i)(t-z_j)}\right).\end{aligned}$$

Take the coefficient of $t^{\ell p^s-1}$ in both sides of the equation. As the result, we obtain modulo p^s,

$$\frac{\partial I_i^\ell}{\partial z_i}(z,\lambda) \equiv p\lambda I_i^\ell(z,\lambda) - \frac{1}{2}\sum_{j\neq i}\frac{I_i^\ell(z,\lambda) - I_j^\ell(z,\lambda)}{z_i - z_J}.$$

Notice that $\frac{\partial}{\partial t}\frac{\Phi_s(t,z,\lambda)}{t-z_i}$ does not contribute to this result by Lemma 1.2.

We also have modulo p^s,

$$\begin{aligned}\frac{\partial}{\partial \lambda}\Phi_s(t,z,\lambda)\frac{1}{t-z_i} &\equiv p\Phi_s(t,z,\lambda)\frac{t-z_i+z_i}{t-z_i} = p\Phi_s(t,z,\lambda) + pz_i\Phi_s(t,z,\lambda)\frac{1}{t-z_i}\\ &\equiv \frac{1}{\lambda}\frac{\partial}{\partial t}\Phi_s(t,z,\lambda) + \frac{1}{\lambda}\frac{1-p^s}{2}\Phi_s(t,z,\lambda)\sum_{j=1}^{2g+1}\frac{1}{t-z_j} + pz_i\Phi_s(t,z,\lambda)\frac{1}{t-z_i}.\end{aligned}$$

Take the coefficient of $t^{\ell p^s-1}$ in both sides of the equation. As the result, we obtain modulo p^s,

$$\frac{\partial I_i^\ell}{\partial \lambda}(z,\lambda) \equiv pz_i I_i^\ell(z,\lambda) + \frac{1}{2\lambda}\sum_{j=1}^{2g+1} I_j^\ell(z,\lambda).$$

Notice that the coefficient of t^{p^s-1} in $\frac{1}{\lambda}\frac{\partial}{\partial t}\Phi_s(t,z,\lambda)$ is zero modulo p^s since λ is a unit in $\mathbb{Z}_p$ by assumption. The theorem is proved. ■

1.6 Properties of p^s-hypergeometric solutions

Lemma 1.6. *Assume that* $\lambda \in \mathbb{Z}_p^\times$ *and*

$$p^s + 2g - 1 > s\frac{2p-2}{p-2}. \tag{1.8}$$

Then a p^s*-hypergeometric solution* $I^\ell(z,\mu)$ *is zero unless* $\ell = 1, \dots, g$.

Proof. The degree of the polynomial $\Psi_s^o(t,z)$ with respect to t equals

$$(2g+1)\frac{p^s-1}{2} - 1 = (g+1)p^s - 1 - \frac{p^s+2g+1}{2}.$$

The degree of the polynomial $E_s(p\lambda t)$ with respect to t is not greater than $d(s) \le s\frac{p-1}{p-2} + 1$. Hence the degree of $\Psi_s(t,z,\mu)$ is not greater than

$$(g+1)p^s - 1 - \frac{p^s+2g+1}{2} + s\frac{p-1}{p-2} + 1 = (g+1)p^s - 1 - \frac{1}{2}\left(p^s + 2g - 1 - s\frac{2p-2}{p-2}\right).$$

If inequality (1.8) holds, then the polynomial $\Psi_s(t,z,\lambda)$ does not have monomials of degree $\ell p^s - 1$ for $\ell > g$. ■

For any p^s-hypergeometric solution $I^\ell(z,\lambda)$, consider its λ-independent term $I^\ell(z,0) = \big(I_1^\ell(z,0), \dots, I_{2g+1}^\ell(z,0)\big)$. This is a vector of polynomials in z with integer coefficients. It is a solution modulo p^s of the KZ equations (1.5) and (1.6) with $\lambda = 0$. We have

$$\sum_{j=1}^{2g+1} I_j^\ell(z,0) \equiv 0 \pmod{p^s}$$

since this sum is the coefficient of $t^{\ell p^s - 1}$ in

$$\sum_{j=1}^{2g+1} \frac{\Phi_s^o(t,z)}{t - z_j} = \frac{2}{p^s-1}\frac{\partial \Phi_s^o}{\partial t}(t,z).$$

The solution $I^\ell(z,\lambda)$ is a λ-deformation of the vector $I^\ell(z,0)$.

Theorem 1.7 ([22, Lemma 7.3])**.** *Assume that* $p^s > 2g+1$. *Consider the* λ*-independent terms* $I^1(z,0), \dots, I^g(z,0)$ *of the* p^s*-hypergeometric solutions* $I^1(z,\lambda), \dots, I^g(z,\lambda)$. *Project them to* $\mathbb{F}_p[z]^{2g+1}$. *Then the projections are linearly independent over the ring* $\mathbb{F}_p[z]$.

1.7 A generalization

Let r_1, r_2 be relatively prime positive integers. Denote $r = r_1/r_2$. Assume that $r > 1/(p-1)$. Change the variable $\lambda \to p^r\lambda$ in the KZ and dynamical equations (1.2), (1.3), (1.4). Then the equations take the form

$$\frac{\partial I_j}{\partial z_i} = \frac{1}{2}\frac{I_i - I_j}{z_i - z_j}, \qquad i \ne j, \tag{1.9}$$

$$\frac{\partial I_i}{\partial z_i} = p^r\lambda I_i - \frac{1}{2}\sum_{j \ne i}\frac{I_i - I_j}{z_i - z_j}, \tag{1.10}$$

$$\frac{\partial I_i}{\partial \lambda} = p^r z_i I_i + \frac{1}{2\lambda}\sum_{j=1}^{2g+1} I_j. \tag{1.11}$$

For a positive integer s, define

$$\Psi_{r,s}(t,z,\lambda) = E_{r,s}(p^r\lambda t)\prod_{i=1}^{2g+1}(t-z_i)^{(p^s-1)/2}\left(\frac{1}{t-z_1},\dots,\frac{1}{t-z_{2g+1}}\right).$$

Consider the Taylor expansion $\Psi_{r,s}(t,z,\lambda) = \sum_d c_d(z,\lambda)t^d$. For any positive integer ℓ, denote $I^\ell(z,\lambda) = c_{\ell p^s-1}(z,\lambda)$. All coordinates of this vector are polynomials in z,λ with coefficients in $\mathbb{Z}_p\big[p^{1/r_2}\big]$.

Theorem 1.8. *Let ℓ be a positive integer. If $\lambda \in \mathbb{Z}_p\big[p^{1/r_2}\big]$, then $I^\ell(z,\lambda)$ is a solution modulo p^s of the KZ equations (1.9), (1.10). If $\lambda \in \big(\mathbb{Z}_p\big[p^{1/r_2}\big]\big)^\times$, then $I^\ell(z,\lambda)$ is a solution modulo p^s of the dynamical equations (1.11).*

The proof is the same as the proof of Theorem 1.5. Theorem 1.5 is a special case of Theorem 1.8 for $r=1$.

Notice that for degree reasons, Theorem 1.8 gives for every s only finitely many solutions $I^\ell(z,\lambda)$.

If $r = 1/(p-1)$ and s is a positive integer, we may define

$$\Psi_{1/(p-1),s}(t,z,\lambda) = \mathrm{e}^{p^{1/(p-1)}\lambda t}\prod_{i=1}^{2g+1}(t-z_i)^{(p^s-1)/2}\left(\frac{1}{t-z_1},\dots,\frac{1}{t-z_{2g+1}}\right)$$

and then expand this vector into a power series in t: $\Psi_{1/(p-1),s}(t,z,\lambda) = \sum_d c_d(z,\lambda)t^d$. For any positive integer ℓ, denote $I^\ell(z,\lambda) = c_{\ell p^s-1}(z,\lambda)$. All coordinates of this vector are polynomials in z, λ with coefficients in $\mathbb{Z}_p\big[p^{1/(p-1)}\big]$.

Theorem 1.9. *Let ℓ be a positive integer. If $\lambda \in \mathbb{Z}_p\big[p^{1/(p-1)}\big]$, then $I^\ell(z,\lambda)$ is a solution modulo p^s of the KZ equations (1.9), (1.10) with $r = 1/(p-1)$ If $\lambda \in \big(\mathbb{Z}_p\big[p^{1/(p-1)}\big]\big)^\times$, then $I^\ell(z,\lambda)$ is a solution modulo p^s of the dynamical equations (1.11) with $r = 1/(p-1)$.*

Notice that this theorem gives infinitely many solutions $I^\ell(z,\lambda)$.

Remark 1.10. Another possibility to extend the construction of polynomial solutions is to replace the ring $\mathbb{Z}_p[p^r]$ by another p-adic ring, e.g., $\mathbb{Z}_p[\zeta]$, where ζ is a p^m-th root of 1.

1.8 Exposition of material

In Section 2, we describe the hypergeometric solutions of the joint system of the differential KZ and dynamical equations associated with $\mathfrak{sl}_2$ and explain their reduction to polynomial solutions modulo p^s. In Section 2.6, we briefly comment on how the results of Section 2 are extended to the joint system of the differential KZ and dynamical equations associated with arbitrary simple Lie algebras. In Appendix A, we consider an example and explain how to construct the polynomial solutions modulo p^s of qKZ difference equations.

2 The $\mathfrak{sl}_2$ differential KZ and dynamical equations

2.1 Equations

Let e, f, h be the standard basis of the complex Lie algebra $\mathfrak{sl}_2$ with relations $[e,f]=h$, $[h,e]=2e$, $[h,f]=-2f$. Denote

$$\Omega = e\otimes f + f\otimes e + \frac{1}{2}h\otimes h \in \mathfrak{sl}_2\otimes\mathfrak{sl}_2,$$

the Casimir element.

Given n, for any $x \in \mathfrak{sl}_2$, let $x^{(i)} \in U(\mathfrak{sl}_2)^{\otimes n}$ be the element equal to x in the i-th factor and to 1 in other factors. Similarly, for $1 \le i < j \le n$, let $\Omega^{(i,j)} \in U(\mathfrak{sl}_2)^{\otimes n}$ be the element equal to Ω in the i-th and j-th factors and to 1 in other factors.

Let $z_1, \dots, z_n \in \mathbb{C}$ be distinct and $\lambda \in \mathbb{C}^\times$. For $i = 1, \dots, n$, introduce the Gaudin Hamiltonians and the dynamical Hamiltonian by the formulas

$$H_i(z_1, \dots, z_n, \lambda) = \frac{\lambda}{2} h^{(i)} + \sum_{j \ne i} \frac{\Omega^{(i,j)}}{z_i - z_j} \in (U(\mathfrak{sl}_2))^{\otimes n},$$
$$D(z_1, \dots, z_n, \lambda) = \sum_{i=1}^{n} \frac{z_i}{2} h^{(i)} + \sum_{i,j=1}^{n} \frac{f^{(i)} e^{(j)}}{\lambda}.$$

Let $\otimes_{i=1}^n V_i$ be a tensor product of $\mathfrak{sl}_2$-modules and $\kappa \in \mathbb{C}^\times$. The system of differential equations on a $\otimes_{i=1}^n V_i$-valued function $I(z_1, \dots, z_n, \lambda)$,

$$\frac{\partial I}{\partial z_i} = \frac{1}{\kappa} H_i(z_1, \dots, z_n, \lambda) I, \qquad i = 1, \dots, n, \tag{2.1}$$
$$\frac{\partial I}{\partial \lambda} = \frac{1}{\kappa} D(z_1, \dots, z_n, \lambda) I, \tag{2.2}$$

is called the system of KZ and dynamical equations. The system depends on the parameter κ.

2.2 $\mathfrak{sl}_2$-modules

For a nonnegative integer i, denote by L_i the $(i+1)$-dimensional module with a basis $v_i, f v_i, \dots, f^i v_i$ and action

$$f \cdot f^k v_i = f^{k+1} v_i \qquad \text{for} \quad k = 0, \dots, i-1,$$
$$h \cdot f^k v_i = (i - 2k) f^k v_i \qquad \text{for} \quad k = 0, \dots, i,$$
$$e \cdot f^k v_i = k(i - k + 1) f^{k-1} v_i \qquad \text{for} \quad k = 1, \dots, i,$$

$f \cdot f^i v_i = 0$, $e \cdot v_i = 0$.

For $\vec{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n_{\ge 0}$, denote $|\vec{m}| = m_1 + \dots + m_n$ and $L^{\otimes \vec{m}} = L_{m_1} \otimes \dots \otimes L_{m_n}$. For $J = (j_1, \dots, j_n) \in \mathbb{Z}^n_{\ge 0}$, with $j_s \le m_s$ for $s = 1, \dots, n$, the vectors

$$f_J v := f^{j_1} v_{m_1} \otimes \dots \otimes f^{j_n} v_{m_n}$$

form a basis of $L^{\otimes \vec{m}}$. We have

$$f \cdot f_J v = \sum_{s=1}^{n} f_{J+1_s} v, \qquad h \cdot f_J v = (|m| - 2|J|) f_J v,$$
$$e \cdot f_J v = \sum_{s=1}^{n} j_s (m_s - j_s + 1) f_{J-1_s} v,$$

where $1_s = (0, \dots, 0, 1, 0, \dots, 0)$ with 1 staying at the s-th place.

For $w \in \mathbb{Z}$, introduce the weight subspace $L^{\otimes \vec{m}}[w] = \{v \in L^{\otimes \vec{m}} \mid h.v = wv\}$. We have the weight decomposition $L^{\otimes \vec{m}} = \oplus_{k=0}^{|m|} L^{\otimes \vec{m}}[|\vec{m}| - 2k]$. Denote

$$\mathcal{I}_k = \{J \in \mathbb{Z}^n_{\ge 0} \mid |J| = k,\ j_s \le m_s,\ s = 1, \dots, n\}.$$

The vectors $(f_J v)_{J \in \mathcal{I}_k}$ form a basis of $L^{\otimes \vec{m}}[|\vec{m}| - 2k]$.

2.3 Solutions over $\mathbb{C}$

Given $k, n \in \mathbb{Z}_{>0}$, $\vec{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n_{>0}$, $\kappa \in \mathbb{C}^\times$, denote $t = (t_1, \dots, t_k)$, $z = (z_1, \dots, z_n)$. Define the *master function*

$$\Phi(t, z, \lambda) := \Phi(t_1, \dots, t_k, z_1, \dots, z_n, \lambda) = \mathrm{e}^{\lambda \sum_{l=1}^n z_l/2\kappa - \lambda \sum_{i=1}^k t_i/\kappa} \\ \times \prod_{i<j} (z_i - z_j)^{m_i m_j/2\kappa} \prod_{1\le i \le j \le k} (t_i - t_j)^{2/\kappa} \prod_{l=1}^n \prod_{i=1}^k (t_i - z_l)^{-m_l/\kappa}.$$

For any function or differential form $F(t_1, \dots, t_k)$, denote

$$\mathrm{Sym}_t[F(t_1, \dots, t_k)] = \sum_{\sigma \in S_k} F(t_{\sigma_1}, \dots, t_{\sigma_k}),$$
$$\mathrm{Alt}_t[F(t_1, \dots, t_k)] = \sum_{\sigma \in S_k} (-1)^{|\sigma|} F(t_{\sigma_1}, \dots, t_{\sigma_k}).$$

For $J = (j_1, \dots, j_n) \in \mathcal{I}_k$, define the *weight function*

$$W_J(t, z) = \frac{1}{j_1! \dots j_n!} \mathrm{Sym}_t \left[\prod_{s=1}^n \prod_{i=1}^{j_s} \frac{1}{t_{j_1 + \dots + j_{s-1} + i} - z_s} \right].$$

For example,

$$W_{(1,0,\dots,0)} = \frac{1}{t_1 - z_1}, \qquad W_{(2,0,\dots,0)} = \frac{1}{t_1 - z_1} \frac{1}{t_2 - z_1},$$
$$W_{(1,1,0,\dots,0)} = \frac{1}{t_1 - z_1} \frac{1}{t_2 - z_2} + \frac{1}{t_2 - z_1} \frac{1}{t_1 - z_2}.$$

The function

$$W(t, z) = \sum_{J \in \mathcal{I}_k} W_J(t, z) f_J v$$

is the $L^{\otimes \vec{m}}[|\vec{m}| - 2k]$-valued *vector weight function.*

Consider the $L^{\otimes \vec{m}}[|\vec{m}| - 2k]$-valued function

$$I^{(\delta)}(z_1, \dots, z_n, \lambda) = \int_{\delta(z,\lambda)} \Phi(t, z, \lambda) W(t, z) \mathrm{d}t_1 \wedge \dots \wedge \mathrm{d}t_k, \tag{2.3}$$

where $\delta(z, \lambda)$ in $\{(z, \lambda)\} \times \mathbb{C}^k_t$ is a horizontal family of k-dimensional cycles of the twisted homology defined by the multivalued function $\Phi(t, z, \lambda)$, see, for example, [20, 21]. The cycles $\delta(z, \lambda)$ are multi-dimensional analogs of Pochhammer double loops.

Theorem 2.1 ([5, 14])**.** *The function $I^{(\delta)}(z, \lambda)$ is a solution of the KZ and dynamical equations* (2.1) *and* (2.2)*.*

The solutions in (2.3) are called the hypergeometric solutions.

The equations (1.2), (1.3), (1.4) in the Introduction and their solutions (1.1) are identified with equations (2.1), (2.2) for the weight subspace $L_1^{\otimes (2g+1)}[n-2]$ and their hypergeometric solutions (2.3) up to a gauge transformation.

In Section 2.4, we sketch the proof of Theorem 2.1 following [5, 14]. The intermediate statement in this proof will be used later when constructing solutions modulo p^s of the KZ and dynamical equations. The proof is based on the following cohomological relations.

2.4 Identities for differential forms

It is convenient to reformulate the definition of the hypergeometric integrals (2.3). Given $k,n\in\mathbb{Z}_{>0}$ and a multi-index $J=(j_1,\dots,j_n)$ with $|J|\le k$, denote

$$a_J=a_{J,1}\wedge a_{J,2}\wedge\cdots\wedge a_{J,|J|}:=\frac{\mathrm{d}(t_1-z_1)}{t_1-z_1}\wedge\cdots\wedge\frac{\mathrm{d}(t_{j_1}-z_1)}{t_{j_1}-z_1}\wedge\frac{\mathrm{d}(t_{j_1+1}-z_2)}{t_{j_1+1}-z_2}\wedge\cdots$$
$$\wedge\frac{\mathrm{d}(t_{j_1+\dots+j_{n-1}+1}-z_n)}{t_{j_1+\dots+j_{n-1}+1}-z_n}\wedge\cdots\wedge\frac{\mathrm{d}(t_{j_1+\dots+j_n}-z_n)}{t_{j_1+\dots+j_n}-z_n}.$$

Here $a_{J,\ell}$ is the ℓ-th factor of the product in the right-hand side.

Denote

$$b_J=b_{J,1}\wedge b_{J,2}\wedge\cdots\wedge b_{J,|J|}:=\frac{\mathrm{d}t_1}{t_1-z_1}\wedge\cdots\wedge\frac{\mathrm{d}t_{j_1}}{t_{j_1}-z_1}\wedge\frac{\mathrm{d}t_{j_1+1}}{t_{j_1+1}-z_2}\wedge\cdots$$
$$\wedge\frac{\mathrm{d}t_{j_1+\dots+j_{n-1}+1}}{t_{j_1+\dots+j_{n-1}+1}-z_n}\wedge\cdots\wedge\frac{\mathrm{d}t_{j_1+\dots+j_n}}{t_{j_1+\dots+j_n}-z_n},$$
$$c_J=\sum_{l=1}^{|J|}(-1)^{l+1}b_{J,1}\wedge b_{J,2}\wedge\cdots\wedge\widehat{b_{J,l}}\wedge\cdots\wedge b_{J,k}.$$

Define

$$\alpha_J=\frac{1}{j_1!\cdots j_n!}\mathrm{Alt}_{t_1,\dots,t_k}[a_J],\qquad \beta_J=\frac{1}{j_1!\cdots j_n!}\mathrm{Alt}_{t_1,\dots,t_k}[c_J].$$

Remark 2.2. Recall that we have k variables $t_1,t_2,\dots,t_k$. The differential form a_J is of degree $|J|=j_1+\dots+j_n\le k$ and depends on the variables $t_1,t_2,\dots,t_{j_1+\dots+j_n}$ only. While, the differential form α_J is of degree $j_1+\dots+j_n$ and depends on all the variables $t_1,t_2,\dots,t_k$.

If $|J|=k$, then for any fixed $z\in\mathbb{C}^n$, we have the identity

$$\alpha_J=W_J(t,z)\mathrm{d}t_1\wedge\cdots\wedge\mathrm{d}t_k$$

on $\{z\}\times\mathbb{C}^k$. Define

$$\alpha=\sum_{|J|=k}\alpha_J f_J v,\qquad \beta=\sum_{|J|=k}\beta_a J f_J v.$$

Example 2.3. For $k=n=2$, we have

$$\alpha_{(2,0)}=\frac{\mathrm{d}(t_1-z_1)}{t_1-z_1}\wedge\frac{\mathrm{d}(t_2-z_1)}{t_2-z_1},\qquad \beta_{(2,0)}=\frac{\mathrm{d}t_2}{t_2-z_1}-\frac{\mathrm{d}t_1}{t_1-z_1}.$$

The hypergeometric integrals (2.3) can be defined in terms of the differential forms α_J:

$$I^{(\delta)}(z_1,\dots,z_n,\lambda)=\int_{\delta(z,\lambda)}\Phi\alpha=\sum_{J\in\mathcal{I}_k}\left(\int_{\delta(z,\lambda)}\Phi\alpha_J\right)f_Jv.$$

Theorem 2.4 ([5, 15])**.**

(*i*) *We have the following algebraic identity for differential forms in t, z depending on parameter λ:*

$$\kappa\mathrm{d}_{t,z}\big(\Phi(t,z,\lambda)\alpha\big)=\sum_{l=1}^{n}H_i(z,\lambda)\mathrm{d}z_i\wedge\Phi(t,z,\lambda)\alpha,\tag{2.4}$$

where $\mathrm{d}_{t,z}$ denotes the differential with respect to variables t, z.

(*ii*) *For any fixed z, λ, we have the following algebraic identity for differential forms in t depending on parameters z, λ:*

$$\kappa\frac{\partial}{\partial\lambda}\big(\Phi(t,z,\lambda)\alpha\big)=D(t,z,\lambda)\Phi(t,z,\lambda)\alpha+\frac{1}{\lambda}\mathrm{d}_t\big(\Phi(t,z,\lambda)\beta\big), \tag{2.5}$$

where d_t denotes the differential with respect to variables t.

The assumptions in part (*ii*) mean that all differentials $\mathrm{d}z_1,\dots,\mathrm{d}z_n$ appearing in α must be put to zero to obtain identity (2.5).

Proof. Identity (2.4) follows from [15, Theorem 7.5.2″] and [5, Theorem 3.1]. Identity (2.5) follows from [5, Theorem 3.2]. ■

Integrating both sides of identities (2.4) and (2.5) over the cycle $\delta(z,\lambda)$ we conclude that the integral $\int_{\delta(z,\lambda)}\Phi(t,z,\lambda)\alpha$ satisfies the KZ and dynamical equations.

2.5 Solutions modulo p^s

Given $k,n\in\mathbb{Z}_{>0}$, $\vec m=(m_1,\dots,m_n)\in\mathbb{Z}^n_{>0}$, $\kappa\in\mathbb{Q}^\times$, let $p>2$ be a prime number such that p does not divide the numerator of κ. Change the variable $\lambda\to p\lambda$ in the KZ and dynamical equations (2.1), (2.2). Then the equations take the form

$$\frac{\partial I}{\partial z_i}=\frac{1}{\kappa}\bigg(p\frac{\lambda}{2}h^{(i)}+\sum_{j\neq i}\frac{\Omega^{(i,j)}}{z_i-z_j}\bigg)I,\qquad i=1,\dots,n, \tag{2.6}$$

$$\frac{\partial I}{\partial\lambda}=\frac{1}{\kappa}\bigg(p\sum_{i=1}^n\frac{z_i}{2}h^{(i)}+\sum_{i,j=1}^n\frac{f^{(i)}\mathrm{e}^{(j)}}{\lambda}\bigg)I. \tag{2.7}$$

Choose positive integers M_l for $l=1,\dots,n$, $M_{i,j}$ for $1\le i<j\le n$, and M^0, such that

$$M_s\equiv-\frac{m_s}{\kappa},\qquad M_{i,j}\equiv\frac{m_im_j}{2\kappa},\qquad M^0\equiv\frac{2}{\kappa}\qquad \pmod{p^s}.$$

Define the *master polynomial*

$$\Phi_s(t,z,\lambda)=\prod_{l=1}^n E_s\Big(p\frac{z_l\lambda}{2\kappa}\Big)\prod_{i=1}^n E_s\Big(-p\frac{t_i\lambda}{\kappa}\Big)\times$$
$$\times\prod_{1\le i<j\le n}(z_i-z_j)^{M_{i,j}}\prod_{1\le i\le j\le k}(t_i-t_j)^{M^0}\prod_{s=1}^n\prod_{i=1}^k(t_i-z_s)^{M_s}.$$

Consider the $L^{\otimes\vec m}[|\vec m|-2k]$-valued function

$$\Psi_s(t,z,\lambda)=\sum_{J\in\mathcal{I}_k}\Phi_s(t,z,\lambda)W_J(t,z)f_Jv.$$

This is a polynomial in t, z, λ with coefficients in $\mathbb{Z}_p$. Consider the Taylor expansion

$$\Psi_s(t,z,\lambda)=\sum_{\mathrm{d}_1,\dots,\mathrm{d}_k}c_{\mathrm{d}_1,\dots,\mathrm{d}_k}(z,\lambda)t_1^{\mathrm{d}_1}\dots t_k^{\mathrm{d}_k}.$$

For any vector $\ell=(\ell_1,\dots,\ell_k)$ with positive integer coordinates, denote

$$I^\ell(z,\lambda)=c_{\ell_1p^s-1,\dots,\ell_kp^s-1}(z,\lambda).$$

All coordinates of this vector are polynomials in z, λ with coefficients in $\mathbb{Z}_p$.

Theorem 2.5. *Let $\ell = (\ell_1, \dots, \ell_k)$ be a vector with positive integer coordinates. If $\lambda \in \mathbb{Z}_p$, then $I^\ell(z, \lambda)$ is a solution modulo p^s of the KZ equations (2.6). If $\lambda \in \mathbb{Z}_p^\times$, then $I^\ell(z, \lambda)$ is a solution modulo p^s of the dynamical equations (2.7).*

We call such solutions the p^s-*hypergeometric solutions* of the joint system of the KZ and dynamical equations.

Proof. The polynomial $\Psi_s(t, z, \lambda)$ satisfies identities (2.4) and (2.5) modulo p^s. Taking the coefficient of $t_1^{\ell_1 p^s - 1} \dots t_k^{\ell_k p^s - 1}$ in these identities kills the differentials with respect to t by Lemma 1.2. This proves the theorem. ■

Remark 2.6. We observed in Section 1.7 that Theorem 1.5 can be generalized to Theorem 1.8 by replacing p with p^r. Theorem 2.5 is generalized in the same way. We leave this exercise to readers.

2.6 Equations for other Lie algebras

The KZ and dynamical equations are defined for any simple Lie algebra $\mathfrak{g}$ or more generally for any Kac–Moody algebra, see, for example, [5, 14]. Similarly to what is done in Section 2.5, we can construct polynomial solutions modulo p^s of those KZ and dynamical equations.

The construction of the polynomial solutions modulo p^s in the $\mathfrak{sl}_2$ case is based on the algebraic identities for differential forms (2.4), (2.5). For an arbitrary Kac–Moody algebra, these algebraic identities were developed in [5, 14].

A Solutions modulo p^s of the qKZ equations

In Sections 1 and 2, we constructed solutions modulo p^s of the differential KZ and dynamical equations by first p^s-approximating the integrand of the hypergeometric solutions and then taking the coefficients of the monomials $t^{\ell p^s - 1}$ in the Taylor expansion of the approximated integrand. In this appendix, we show that the same idea can be applied to the qKZ difference equations, but instead of considering the Taylor expansion of the approximated integrand and then taking the coefficients of the monomials $t^{\ell p^s - 1}$ we first take the expansion of the p^s-approximated integrand into a sum of Pochhammer polynomials $[t]_m$ and then take the coefficients of the Pochhammer polynomials with indices $m = \ell p^s - 1$. See this approach in [10] for the qKZ equations with no exponential term. On the hypergeometric solutions of the qKZ difference equations see, for example, [17, 19].

In this paper, we consider only a baby example of the qKZ equations which illustrates these constructions.

A.1 Baby qKZ equation

Let $f(t)$ be a meromorphic function and $a \in \mathbb{C}$. The sum

$$\int_{(a)} f(t) \mathrm{d}_1 t := \sum_{n \in \mathbb{Z}} \mathrm{Res}_{t=a+n} f(t),$$

if defined, is called a Jackson integral. If $f(t) = g(t+1) - g(t)$ is a discrete differential, then the Jackson integral equals zero.

Consider the master function

$$\Phi(t, z, \lambda) = \mathrm{e}^{\lambda t} \frac{\Gamma(t-z)^2}{\Gamma\left(t - z + \frac{1}{2}\right) \Gamma\left(t - z - \frac{1}{2}\right)} \tag{A.1}$$

and the Jackson integral

$$I(z,\lambda) = \int_{(z)} \Phi(t,z,\lambda)\mathrm{d}t.$$

Lemma A.1. *The function $I(z,\lambda)$ satisfies the difference equation*

$$\mathrm{e}^{\lambda} I(z-1,\lambda) = I(z,\lambda). \tag{A.2}$$

Equation (A.2) is our baby qKZ equation and the function $I(z,\lambda)$ is its hypergeometric solution. More general qKZ equations and their hypergeometric solutions can be found, for example, in [3, 6, 17, 19].

Proof. We have

$$\Phi(t+1,z,\lambda) = \frac{\mathrm{e}^{\lambda}(t-z)^2}{\left(t-z+\frac{1}{2}\right)\left(t-z-\frac{1}{2}\right)}\Phi(t,z,\lambda),$$
$$\Phi(t,z-1,\lambda) = \frac{(t-z)^2}{\left(t-z+\frac{1}{2}\right)\left(t-z-\frac{1}{2}\right)}\Phi(t,z,\lambda),$$
$$\int_{(z)} \Phi(t+1,z,\lambda)d_1 t = \int_{(z)} \Phi(t,z,\lambda)d_1 t.$$

These relations imply (A.2). ■

A.2 Pochhammer polynomials

Let m be a positive integer. Define the Pochhammer polynomial

$$(t)_m = \prod_{i=1}^{m}(t-i+1).$$

We have

$$(t+1)_m = (t)_m\frac{t+1}{t-m+1}, \qquad (t+1)_m - (t)_m = m(t)_{m-1}. \tag{A.3}$$

A.3 Polynomial p^s-approximations

Let p be an odd prime. Let r_1, r_2 be relatively prime positive integers. Denote $r = r_1/r_2$. Assume that $r > 1/(p-1)$. Change the variable $\lambda \mapsto p^r\lambda$. Then equation (A.2) takes the form:

$$\mathrm{e}^{p^r\lambda} I(z-1,\lambda) = I(z,\lambda). \tag{A.4}$$

For any positive integer s, define the master polynomial

$$\Phi_{r,s}(t,z,\lambda) = E_{r,s}(p^r\lambda t)(t-z-1)_{\frac{p^s-1}{2}}(t-z-1)_{\frac{p^s+1}{2}}. \tag{A.5}$$

This is a polynomial in t, z, λ with coefficients in $\mathbb{Z}_p\left[p^{1/r_2}\right]$. Expand it into a sum of Pochhammer polynomials in the variable t,

$$\Phi_{r,s}(t,z,\lambda) = \sum_d c_d(z,\lambda)(t)_d.$$

Denote $I_{r,s}(z,\lambda) = c_{p^s-1}(z,\lambda)$.

Theorem A.2. *The polynomial $I_{r,s}(z,\lambda)$ is a solution modulo p^s of the qKZ difference equation* (A.4).

Proof. Let $F(t) = \sum_d a_d(t)_d$ be a polynomial. Denote $\operatorname{Coef}(F) = a_{p^s-1}$. We have

$$\Phi_{r,s}(t+1,z,\lambda) = \frac{E_{r,s}(p^r\lambda)(t-z)^2}{\left(t-z-\frac{p^s-1}{2}\right)\left(t-z-\frac{p^s+1}{2}\right)}\Phi_{r,s}(t,z,\lambda),$$
$$\Phi_{r,s}(t,z-1,\lambda) = \frac{(t-z)^2}{\left(t-z-\frac{p^s-1}{2}\right)\left(t-z-\frac{p^s+1}{2}\right)}\Phi_{r,s}(t,z,\lambda),$$
$$\operatorname{Coef}(\Phi_{r,s}(t,z,\lambda)) \equiv \operatorname{Coef}(\Phi_{r,s}(t+1,z,\lambda)) \pmod{p^s}, \tag{A.6}$$

where the congruence (A.6) follows from (A.3). Hence

$$E_{r,s}(p^r\lambda)I_{r,s}(z-1,\lambda) \equiv I_{r,s}(z,\lambda) \pmod{p^s},$$

and $I_{r,s}(z,\lambda)$ is a solution modulo p^s of the qKZ equation (A.4). ■

More general qKZ equations are given by multidimensional Jackson integrals whose integrand is a product of exponential factors and ratios of gamma functions like in (A.1). To construct polynomial solutions modulo p^s of the qKZ equations, we replace the exponential factors in the integrand by the product of the corresponding functions $E_{r,s}(t)$, replace the ratios of gamma functions by the corresponding Pochhammer polynomial as in (A.5), expand the result into Pochhammer polynomials and take the suitable coefficients of that expansion like in Theorem A.2.

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