

Spectral Asymmetry and Index Theory on Manifolds with Generalised Hyperbolic Cusps

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Abstract. We consider a complete Riemannian manifold, which consists of a compact interior and one or more φ -cusps: infinitely long ends of a type that includes cylindrical ends and hyperbolic cusps. Here φ is a function of the radial coordinate that describes the shape of such an end. Given an action by a compact Lie group on such a manifold, we obtain an equivariant index theorem for Dirac operators, under conditions on φ . These conditions hold in the cases of cylindrical ends and hyperbolic cusps. In the case of cylindrical ends, the cusp contribution equals the delocalised η -invariant, and the index theorem reduces to Donnelly's equivariant index theory on compact manifolds with boundary. In general, we find that the cusp contribution is zero if the spectrum of the relevant Dirac operator on a hypersurface is symmetric around zero.

Key words: equivariant index; Dirac operator; noncompact manifold; cusp

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1 Introduction

Index theory on noncompact manifolds

Index theory on noncompact, complete manifolds comes up naturally in several different contexts. Well-known results include:

- (1) the Gromov–Lawson relative index theorem [23] for differences of Dirac operators that are invertible, and equal, outside compact sets;
- (2) the Atiyah–Patodi–Singer (APS) index theorem [6, 21] on compact manifolds with boundary, where an approach is to attach a cylindrical end to the boundary to obtain a complete manifold without boundary; and
- (3) index theorems on noncompact locally symmetric spaces, and manifolds with cusps modelled on such spaces. Some important results include [13, 33, 34, 35, 39].

Other important areas, which are not considered in this paper, are index theory of Callias-type operators [1, 3, 16, 19, 20, 28], index theory where a group action is used to define an equivariant index of operators that are not Fredholm in the traditional sense, see, e.g., [5, 17], and index theory with values in the K -theory of C^* -algebras, see, e.g., [14].

In this paper and in [27], we work towards a common framework for studying the three types of index problems mentioned. In [27], we considered a complete Riemannian manifold M ,

a Clifford module $S \rightarrow M$ and a Dirac operator D on $\Gamma^\infty(S)$ that is “invertible at infinity” in the following sense. We assumed that there are a compact subset $Z \subset M$ and a $b > 0$ such that for all $s \in \Gamma_c^\infty(S)$ supported outside Z ,

$$\|Ds\|_{L^2} \geq b\|s\|_{L^2}.$$

Then D is Fredholm as an unbounded operator on $L^2(S)$ with a suitable domain [2, 23].

We assumed that M has a warped product structure outside Z (the φ -cusps as below, without assumptions on the function φ). Furthermore, we considered an action by a compact group G on M and S , commuting with D , and a group element $g \in G$. The main result in [27] is an expression for the value at g of the equivariant index of such an operator, as an Atiyah–Segal–Singer-type contribution from Z and a contribution from outside Z . This implies an equivariant version of the second index theorem mentioned at the start, and an equivariant version of the first for manifolds with the appropriate warped product form at infinity.

In this paper, we give an expression for the contribution from outside Z for manifolds with specified shapes outside Z , including cylindrical ends and hyperbolic cusps.

φ -cusps

More specifically, let M be a complete Riemannian manifold. Suppose a compact Lie group G acts isometrically on M , that $S \rightarrow M$ is a G -equivariant Clifford module, and that D is a G -equivariant Dirac operator on sections of S . Let $a > 0$, and let $\varphi \in C^\infty(a, \infty)$. We assume that there is a compact, G -invariant subset $Z \subset M$ with smooth boundary N , such that $C := M \setminus Z$ is G -equivariantly isometric to the product $N \times (a, \infty)$, with the Riemannian metric

$$e^{2\varphi}(B_N + dx^2),$$

where B_N is a G -invariant Riemannian metric on N , and x is the coordinate in (a, ∞) . Then we say that M has φ -cusps. (The results in this paper extend to cases where different functions φ are used on different connected components of N .)

A natural form of a Dirac operator on C is

$$e^{-\varphi}c_0\left(\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial x} + D_N + \frac{\dim(M) - 1}{2}\varphi'\right)$$

for a Dirac operator D_N on $S|_N$, where c_0 is the Clifford action for the product metric $B_N + dx^2$ on C . We assume $D|_C$ has this form, and, initially, that D_N is invertible. Then $D_N^2 \geq b^2$ for some $b > 0$.

We say that M has *weakly admissible φ -cusps* if

- (1) φ is bounded above, and
- (2) there is an $\alpha > 0$ such that $|\varphi'(x)| \leq b - \alpha$ for large enough x ,

and *strongly admissible φ -cusps* if

- (1) $\lim_{x \rightarrow \infty} \varphi(x) = -\infty$, and
- (2) $\lim_{x \rightarrow \infty} \varphi'(x) = 0$.

For example, if $\varphi(x) = -\mu \log(x)$ for $\mu \in \mathbb{R}$, then

- M is complete if and only if $\mu \leq 1$,
- M has weakly admissible φ -cusps if and only if $\mu \geq 0$, and
- M has strongly admissible φ -cusps if and only if $\mu > 0$.

Furthermore, M has finite volume if and only if $\mu > 1/\dim(M)$, but this is not directly relevant to us here.

Results

If M has weakly admissible φ -cusps, then our main result, Theorem 2.16 states that D is Fredholm, and the value of its equivariant index at $g \in G$ is

$$\text{index}_G(D)(g) = \int_{Z^g} \text{AS}_g(D) - \frac{1}{2} \eta_g^\varphi(D_N^+). \quad (1.1)$$

Here

- Z^g is the fixed point set of g in Z ,
- $\text{AS}_g(D)$ is the Atiyah–Segal–Singer integrand for D , and
- $\eta_g^\varphi(D_N^+)$ is the φ -cusp contribution associated to D_N^+ (the restriction of D_N to even-graded sections).

The cusp contribution $\eta_g^\varphi(D_N^+)$ equals

$$\eta_g^\varphi(D_N^+) = \lim_{a' \downarrow a} \int_0^\infty \sum_{\lambda \in \text{spec}(D_N^+)} \text{sgn}(\lambda) \text{tr}(g|_{\ker(D_N^+ - \lambda)}) F_\varphi(a', s, |\lambda|) ds \quad (1.2)$$

for a function F_φ depending on φ . (Here a is as in the definition of $C \cong N \times (a, \infty)$.) This function is expressed in terms of eigenfunctions of a Sturm–Liouville (or Schrödinger) operator on the half-line $(0, \infty)$, with Dirichlet boundary conditions at 0. See Definition 2.14 for details. If M has strongly admissible φ -cusps, then this operator has discrete spectrum. If M only has weakly admissible φ -cusps, then it may have a continuous spectral decomposition, and its spectral measure also appears in the expression for $\eta_g^\varphi(D_N^+)$.

There is also a version of (1.1) where the integral over Z^g is replaced by an integral over M^g , if this converges, and the limit in (1.2) is replaced by the limit $a' \rightarrow \infty$.

A possibly interesting feature of the cusp contribution $\eta_g^\varphi(D_N^+)$ is that it equals zero if the spectrum of D_N^+ has the equivariant symmetry property that

$$\text{tr}(g|_{\ker(D_N^+ - \lambda)}) = \text{tr}(g|_{\ker(D_N^+ + \lambda)}) \quad (1.3)$$

for all $\lambda \in \mathbb{R}$. If $g = e$, then this is exactly symmetry of the spectrum with respect to reflection in 0, including multiplicities. It is immediate from (1.2) that $\eta_g^\varphi(D_N^+) = 0$ if the spectrum of D_N^+ has this property. As noted in [6], the classical η -invariant also vanishes if D_N^+ has symmetric spectrum. So it seems that different ways of measuring spectral asymmetry are relevant to index theory on manifolds of the type we consider.

After we prove (1.1), we compute the function F_φ in (1.2), and hence the cusp contribution $\eta_g^\varphi(D_N^+)$ in the case where $\varphi(x) = 0$. Then $C = N \times (a, \infty)$ is a cylindrical end, and Proposition 4.4 states that $\eta_g^0(D_N^+)$ is the equivariant η -invariant [21] of D_N^+ . This computation is a spectral version of the geometric computation in [27, Section 5]. Then (1.1) becomes Donnelly’s equivariant version of the APS index theorem [21].

The eigenfunctions of the Sturm–Liouville operator involved in the expression (1.2) are known explicitly in several cases besides the cylinder case, such as $\varphi(x) = -\log(x)/2$ and the hyperbolic cusp case $\varphi(x) = -\log(x)$. Nevertheless, it seems to be a nontrivial problem to evaluate the cusp contribution (1.2) explicitly, even when these eigenfunctions are known. Concrete consequences and special cases of the main Theorem 2.16 are:

- (1) the Fredholm property of D ,
- (2) the fact that, for $\varphi = 0$, the cusp contribution $\eta_g^0(D_N^+)$ is the delocalised η -invariant of D_N^+ ,
- (3) the fact that, for general φ , the cusp contribution vanishes if the spectrum of D_N^+ has the symmetry property (1.3).

Related index theorems were obtained for manifolds with ends of the form $N \times (a, \infty)$ with metrics of the form $B_{N,x} + dx^2$, where now $B_{N,x}$ is a Riemannian metric on N depending on

$x \in (a, \infty)$. In many cases, this family of metrics on N has the form $B_{N,x} = \rho^2(x)B_N$, for a fixed Riemannian metric B_N on N and a function ρ on (a, ∞) . Results in this context include the ones in [8, 9, 10, 40, 42]. We discuss the relations between these results and (1.1) in Section 2.5.

2 Preliminaries and result

Throughout this paper, M is a p -dimensional Riemannian manifold with p even, and $S = S^+ \oplus S^- \rightarrow M$ is a $\mathbb{Z}/2$ -graded Hermitian vector bundle. We denote the Riemannian density on M by dm . We also assume that a compact Lie group G acts smoothly and isometrically on M , that S is a G -equivariant vector bundle and that the action on G preserves the metric and grading on S . We fix, once and for all, an element $g \in G$.

2.1 φ -cusps

Definition 2.1. The manifold M has (G -invariant) φ -cusps if there are

- a G -invariant compact subset $Z \subset M$ with smooth boundary N , and
- a number $a \geq 0$ and a function $\varphi \in C^\infty(a, \infty)$,

such that

- there is a G -equivariant isometry from $C := M \setminus Z$ onto the manifold $N \times (a, \infty)$ with the metric

$$B_\varphi := e^{2\varphi}(B_N + dx^2), \quad (2.1)$$

where B_N is the restriction of the Riemannian metric to N and x is the coordinate in (a, ∞) , and

- this isometry has a continuous extension to a map $\bar{C} \rightarrow N \times [a, \infty)$, which maps N onto $N \times \{a\}$.

The manifold M has *strongly admissible φ -cusps* or *strongly admissible cusps* if, in addition,

$$\lim_{x \rightarrow \infty} \varphi(x) = -\infty, \quad \lim_{x \rightarrow \infty} \varphi'(x) = 0. \quad (2.2)$$

Remark 2.2. There is no loss of generality in assuming that $a = 0$ in Definition 2.1. However, in examples, it may be that φ arises as the restriction to (a, ∞) of a function naturally defined on say $(0, \infty)$. Allowing nonzero a then means that we do not need to shift these functions over a to obtain a function on $(0, \infty)$. This does not matter for the results.

Remark 2.3. In Definition 2.1, the hypersurface N may be disconnected. Let N_1, \dots, N_k be its connected components. All results below generalise to the case where the Riemannian metric on $N \times (a, \infty)$ is of the form $e^{2\varphi_j}(B_N + dx^2)$ on $N_j \times (a, \infty)$, for $\varphi_j \in C^\infty(a, \infty)$ depending on j . This generalisation is straightforward, and we do not work out details here.

Lemma 2.4. *If M has φ -cusps, then it is complete if and only if*

$$\int_{a+1}^{\infty} e^{\varphi(x)} dx = \infty. \quad (2.3)$$

Proof. For any smooth, increasing $\gamma: [0, 1) \rightarrow [a+1, \infty)$ such that $\gamma(0) = a+1$, and any $n \in N$, consider the curve γ_n in C given by $\gamma_n(t) = (n, \gamma(t))$. Then the length of γ_n is

$$\text{sgn}(\gamma') \int_0^1 \gamma'(s) e^{\varphi(\gamma(s))} dt = \text{sgn}(\gamma') \int_{a+1}^{\gamma(1)} e^{\varphi(x)} dx.$$

Here $\gamma(1) := \lim_{t \rightarrow 1} \gamma(t)$ exists because γ is increasing. The lengths of all such curves with $\gamma(1) = \infty$ are infinite if and only if (2.3) holds. This implies that (2.3) is equivalent to the condition that any curve in C that goes to infinity has infinite length. Compare also the proof of Theorem 1 in [36]. ■

From now on, suppose that M is complete and has φ -cusps, and let Z , N , a , φ and B_N be as in Definition 2.1.

It will not directly be important for our results if M has finite or infinite volume. But because of the relevance of finite-volume manifolds, we note the following fact. Recall that p is the dimension of M .

Lemma 2.5. *The manifold M has finite volume if and only if*

$$\int_a^\infty e^{p\varphi(x)} dx < \infty.$$

Proof. The Riemannian density for B_φ is $\text{vol}_{B_\varphi} = e^{p\varphi} \text{vol}_{B_N} \otimes dx$. ■

Lemmas 2.4 and 2.5 are well known and our results do not logically depend on them. We chose to include their short proofs for the sake of completeness and to illustrate properties of φ -cusps.

Example 2.6. Suppose that M has φ -cusps, with $\varphi(x) = -\mu \log(x)$ for $\mu \in \mathbb{R}$. Then

- M has strongly admissible cusps if and only if $\mu > 0$,
- M is complete if and only if $\mu \leq 1$,
- M has finite volume if and only if $\mu > 1/p$.

All three conditions hold in the case $\mu = 1$ of hyperbolic cusps. In the case $\mu = 0$ of a cylindrical end, M has infinite volume and does not have strongly admissible cusps. But then M has weakly admissible cusps as in Definition 2.8 below, which is sufficient for our purposes.

2.2 Dirac operators on φ -cusps

From now on, we suppose that S is a G -equivariant Clifford module, which means that there is a G -equivariant vector bundle homomorphism $c: TM \rightarrow \text{End}(S)$, with values in the odd-graded endomorphisms, such that for all $v \in TM$,

$$c(v)^2 = -\|v\|^2 \text{Id}_S.$$

Let ∇ be a G -invariant connection on S that preserves the grading. Suppose that for all vector fields v and w on M ,

$$[\nabla_v, c(w)] = c(\nabla_v^{TM} w), \tag{2.4}$$

where ∇^{TM} is the Levi-Civita connection. Consider the Dirac operator

$$D: \Gamma^\infty(S) \xrightarrow{\nabla} \Gamma^\infty(S \otimes T^*M) \cong \Gamma^\infty(S \otimes TM) \xrightarrow{c} \Gamma^\infty(S). \tag{2.5}$$

It is odd with respect to the grading on S ; we denote its restrictions to even- and odd-graded sections by D^\pm , respectively.

Suppose that we have a G -equivariant vector bundle isomorphism $S|_C \cong S|_N \times (a, \infty) \rightarrow N \times (a, \infty)$.

We will assume that D has a natural form on $M \setminus Z$, (2.7) below. This assumption is motivated by a special case, Proposition 2.7, which we discuss now.

Let $B_0 := B_N + dx^2$ be the product metric on $N \times (a, \infty)$. Then

$$c_0 := e^{-\varphi} c|_C: T(N \times (a, \infty)) \rightarrow \text{End}(S|_C)$$

is a Clifford action for the metric B_0 . Let $\{e_1, \dots, e_p\}$ be a local orthonormal frame for TM with respect to B_0 , with $e_p = \frac{\partial}{\partial x}$, and $\{e_1, \dots, e_{p-1}\}$ a local orthonormal frame for TN with respect to B_N . (The objects that follow are defined globally by their expressions in terms of this frame, because they do not depend on the orthonormal frame.)

Because p is even, the operator

$$\gamma := (-i)^{p(p+1)/2} c_0(e_1) \cdots c_0(e_p)$$

defines a $\mathbb{Z}/2$ -grading on S , and $c_0(v)$ is odd for this grading for all vector fields v on C . We suppose that the given grading on S equals γ on $S|_C$. Let ∇^0 be a Clifford connection on $S|_C$ with respect to B_0 and c_0 (i.e., a Hermitian connection satisfying (2.4) for B_0 and c_0), and suppose that it preserves the grading γ , and that $\nabla_{\frac{\partial}{\partial x}} = \frac{\partial}{\partial x}$.

Let

$$D_N := -c_0(e_p) \sum_{j=1}^{p-1} c_0(e_j) \nabla_{e_j}^0.$$

This is a Dirac operator on $S|_N$ with respect to the Clifford multiplication

$$c_N(v) := -c_0(e_p) c_0(v)$$

for $v \in TN$.

Proposition 2.7. *There is a Clifford connection on $S|_C$, with respect to the Clifford action c and the metric (2.1), such that the resulting Dirac operator is*

$$e^{-\varphi} c_0(e_p) \left(\frac{\partial}{\partial x} + D_N + \frac{p-1}{2} \varphi' \right). \quad (2.6)$$

This fact follows from a standard expression for conformal transformations of Dirac operators. (See, e.g., the proof of Proposition 1.3 in [25] in the Spin^c case.) We summarise the arguments in Appendix A for the sake of completeness.

From now on, we make the two assumptions that

- (1) for some Dirac operator D_N on $S|_N$ that preserves the grading, and some grading-reversing, G -equivariant, isometric vector bundle endomorphism $\sigma: S|_N \rightarrow S|_N$ that anti-commutes with D_N ,

$$D|_{M \setminus Z} = e^{-\varphi} \sigma \left(\frac{\partial}{\partial x} + D_N + \frac{p-1}{2} \varphi' \right), \quad (2.7)$$

- (2) the Dirac operator D_N is invertible.

The first assumption is satisfied for a natural choice of Clifford connection on S , by Proposition 2.7. We indicate how to remove the second assumption in Section 5.

Because D_N is invertible, there is a $b > 0$ such that $D_N^2 \geq b^2$. For the purposes of our index theorem, the strong admissibility condition in Definition 2.1 may be weakened to the following.

Definition 2.8. The manifold M has *weakly admissible φ -cusps* or *weakly admissible cusps* (with respect to D_N) if φ is bounded above, and there are $a' \geq a$ and $\alpha > 0$ such that for all $x \in (a', \infty)$,

$$|\varphi'(x)| \leq b - \alpha. \quad (2.8)$$

Example 2.9. If M has φ -cusps with $\varphi(x) = -\mu \log(x)$ as in Example 2.6, then M has weakly admissible cusps if and only if $\mu \geq 0$. This includes the case $\mu = 0$ of a cylindrical end, relevant to the Atiyah–Patodi–Singer index theorem. For cusp metrics of this form, we have the implications

$$\text{finite volume} \Rightarrow \text{strongly admissible cusps} \Rightarrow \text{weakly admissible cusps}.$$

For general metrics of the form (2.1), only the second implication always holds.

Example 2.10. If φ is periodic, then φ -cusps are never strongly admissible. But if $\varphi(x) = \tilde{\varphi}(\sin(x))$ for some $\tilde{\varphi} \in C^\infty([-1, 1])$, then φ -cusps are weakly admissible if $|\tilde{\varphi}'| < b$.

2.3 Spectral theory for Sturm–Liouville operators

Let q be a real-valued, continuous function on the closed half-line $[0, \infty)$. A crucial role will be played by the spectral theory of Sturm–Liouville operators of the form

$$\Delta_q := -\frac{d^2}{dy^2} + q,$$

on $[0, \infty)$. We briefly review this theory here, and refer to [29, 41] for details.

For $\nu \in \mathbb{C}$, let $\theta_\nu \in C^\infty([0, \infty))$ be the unique solution of

$$\Delta_q \theta_\nu = \nu \theta_\nu,$$

such that

$$\theta_\nu(0) = 0, \quad \theta'_\nu(0) = 1. \quad (2.9)$$

The theory extends to more general boundary conditions, but we will only use the Dirichlet case. For $f \in C_c^\infty(0, \infty)$ and $\nu \in \mathbb{R}$, define the generalised Fourier transform

$$\mathcal{F}_q(f)(\nu) := \int_0^\infty f(y) \theta_\nu(y) dy.$$

For a function $\rho: \mathbb{R} \rightarrow \mathbb{C}$, let $L^2(\mathbb{R}, d\rho)$ be the space of square-integrable functions with respect to the measure $d\rho$, in the sense of Stieltjes integrals. Note that this measure may have singular points, if ρ is discontinuous.

Theorem 2.11. *There exists a unique increasing function $\rho: \mathbb{R} \rightarrow \mathbb{R}$ with the following properties:*

- (a) *The map $f \mapsto \mathcal{F}_q(f)$ extends to a unitary isomorphism from $L^2([0, \infty))$ onto $L^2(\mathbb{R}, d\rho)$.*
- (b) *For all continuous $f \in L^2([0, \infty))$ such that the integrands on the right-hand side are well-defined and the integral converges uniformly in y in compact intervals,*

$$f(y) = \int_{\mathbb{R}} \mathcal{F}_q(f)(\nu) \theta_\nu(y) d\rho(\nu). \quad (2.10)$$

See, for example, [29, Theorems 2.1.1 and 2.1.2].

The spectral measure $d\rho$ can be computed as follows.

Proposition 2.12. *For $\nu \in \mathbb{C}$, let $\theta_1(-, \nu)$ and $\theta_2(-, \nu)$ be the solutions of $\Delta_q \theta_j = \nu \theta_j$ on $[0, \infty)$, such that*

$$\theta_1(0, \nu) = 0, \quad \theta_1'(0, \nu) = 1, \quad \theta_2(0, \nu) = -1, \quad \theta_2'(0, \nu) = 0,$$

where the prime denotes the derivative with respect to the first variable. There is a function f on the upper half-plane in \mathbb{C} , with negative imaginary part, such that for all ν in the upper half-plane, $\theta_2(-, \nu) + f(\nu)\theta_1(-, \nu) \in L^2([0, \infty))$. And for all $\nu \in \mathbb{R}$,

$$\rho(\nu) = \frac{1}{\pi} \lim_{\delta \downarrow 0} \int_0^\nu -\operatorname{Im}(f(\nu' + i\delta)) d\nu'. \quad (2.11)$$

See [29, Theorem 2.4.1], or [41, Chapter 3], in particular Lemma 3.3 and the theorem on p. 60. The factor $\frac{1}{\pi}$ in (2.11) corresponds to the same factor in the inversion formula in the theorem on p. 60 of [41], which is not present in (2.10).

2.4 An index theorem for manifolds with admissible φ -cusps

We suppose, as before, that M is complete and has weakly admissible φ -cusps. Let a be as in Definition 2.1. Consider the function $\xi: (a, \infty) \rightarrow (0, \infty)$ defined by

$$\xi(x) := \int_a^x e^{\varphi(x')} dx'. \quad (2.12)$$

Then ξ is injective because its derivative is positive, and surjective by Lemma 2.4. For $\lambda \in \mathbb{R}$, define the functions $q_\lambda^\pm \in C^\infty(0, \infty)$ by

$$q_\lambda^\pm(y) := \lambda((\lambda \pm \varphi')e^{-2\varphi}) \circ \xi^{-1}. \quad (2.13)$$

(We only use q_λ^+ in the current section, but q_λ^- will also be used in Section 3.7.) In the case where $q = q_\lambda^\pm$, we write

$$\Delta_\lambda^\pm := \Delta_{q_\lambda^\pm}. \quad (2.14)$$

We then write $\rho^{\lambda, \pm}$ for the function ρ in Theorem 2.11, and $\theta_\nu^{\lambda, \pm}$ for the function θ_ν in Theorem 2.11.

Example 2.13. Suppose that $\varphi(x) = -\mu \log(x)$, for $\mu \in \mathbb{R}$. If $\mu = 1$, then

$$q_\lambda^\pm(y) = a^2 \lambda^2 e^{2y} \mp a \lambda e^y.$$

If $\mu \neq 1$, then

$$q_\lambda^\pm(y) = \lambda^2 \left((1 - \mu)y + a^{1-\mu} \right)^{\frac{2\mu}{1-\mu}} \mp \lambda \mu \left((1 - \mu)y + a^{1-\mu} \right)^{\frac{2\mu-1}{1-\mu}}.$$

For a finite-dimensional vector space V with a given representation by G , we write $\operatorname{tr}(g|_V)$ for the trace of the action by g on V .

Definition 2.14. The g -delocalised φ -cusp contribution associated to D_N^+ and a number $a' > a$ is

$$\begin{aligned} \eta_g^\varphi(D_N^+, a') &= 2e^{-p\varphi(a')} \int_0^\infty \sum_{\lambda \in \operatorname{spec}(D_N^+)} \operatorname{sgn}(\lambda) \operatorname{tr}(g|_{\ker(D_N^+ - \lambda)}) \\ &\quad \times \int_{\mathbb{R}} e^{-s\nu} \theta_\nu^{|\lambda|, +}(\xi(a')) \left((\theta_\nu^{|\lambda|, +})'(\xi(a')) + |\lambda| e^{-\varphi(a')} \theta_\nu^{|\lambda|, +}(\xi(a')) \right) d\rho^{|\lambda|, +}(\nu) ds, \end{aligned} \quad (2.15)$$

if the right-hand side converges.

We will see in Theorem 2.16 that (2.15) indeed converges in the situations we consider.

The notation η_g^φ and the factor 2 in the definition are motivated by the fact that η_e^0 is the usual η -invariant of [6], and, more generally, η_g^0 is the delocalised η -invariant [21]. See Proposition 4.4.

Definition 2.15. The spectrum of D_N^+ is *g-symmetric* if for all $\lambda \in \mathbb{R}$,

$$\mathrm{tr} \left(g|_{\ker(D_N^+ - \lambda)} \right) = \mathrm{tr} \left(g|_{\ker(D_N^+ + \lambda)} \right).$$

Note that if $g = e$, then the spectrum of D_N is *e-symmetric* precisely if it is symmetric around zero, including multiplicities.

For a G -equivariant, odd-graded, self-adjoint Fredholm operator F on $L^2(S)$, we denote its restrictions to even- and odd-graded sections by F^+ and F^- , respectively. Then

$$\mathrm{index}_G(F) := [\ker(F^+)] - [\ker(F^-)] \in R(G)$$

is the classical equivariant index of F^+ , in the representation ring $R(G)$ of G . We denote the value of its character at g by

$$\mathrm{index}_G(F)(g) = \mathrm{tr} \left(g|_{\ker(F^+)} \right) - \mathrm{tr} \left(g|_{\ker(F^-)} \right).$$

Let $\mathrm{AS}_g(D)$ be the Atiyah–Segal–Singer integrand associated to D , see, for example, [15, Theorem 6.16] or [7, Theorem 3.9]. It is a differential form of mixed degree on the fixed-point set M^g of g . The connected components of M^g may have different dimensions, and the integral of $\mathrm{AS}_g(D)$ over M^g is defined as the sum over these connected components of the integral of the component of the relevant degree.

Theorem 2.16 (index theorem on manifolds with admissible cusps). *Suppose that M has weakly admissible φ -cusps, that $D|_C = D_\varphi$ as in Proposition 2.7, and that D_N is invertible. Then D is Fredholm, the cusp contribution (2.15) converges for all $a' > a$, and*

$$\mathrm{index}_G(D)(g) = \int_{Z^g \cup (N^g \times (a, a'))} \mathrm{AS}_g(D) - \frac{1}{2} \eta_g^\varphi(D_N^+, a'). \quad (2.16)$$

Furthermore, $\lim_{a' \downarrow a} \eta_g^\varphi(D_N^+, a')$ converges, and

$$\mathrm{index}_G(D)(g) = \int_{Z^g} \mathrm{AS}_g(D) - \frac{1}{2} \lim_{a' \downarrow a} \eta_g^\varphi(D_N^+, a'). \quad (2.17)$$

If $\int_{M^g} \mathrm{AS}_g(D)$ converges, then $\lim_{a' \rightarrow \infty} \eta_g^\varphi(D_N^+, a')$ converges, and

$$\mathrm{index}_G(D)(g) = \int_{M^g} \mathrm{AS}_g(D) - \frac{1}{2} \lim_{a' \rightarrow \infty} \eta_g^\varphi(D_N^+, a'). \quad (2.18)$$

If the spectrum of D_N^+ is *g-symmetric*, then $\eta_g^\varphi(D_N^+, a') = 0$ for all $a' > a$. If M has strongly admissible φ -cusps, then $\Delta_{|\lambda|}^+$ has discrete spectrum for all λ , so the integral over ν in (2.15) becomes a sum.

The case (2.18) of Theorem 2.16 applies in some relevant special cases. The following fact follows from Proposition 4.4. This also follows from the fact that the \hat{A} -form on the cylinder $N \times (a, \infty)$ is zero.

Lemma 2.17. *In the setting of Theorem 2.16, if $\varphi = 0$, then (2.18) applies.*

The case (2.18) also applies if M^g is compact; this is equivalent to g having no fixed points on N .

Lemma 2.18. *In the setting of Theorem 2.16, if M^g is compact, then (2.18) applies.*

If M has strongly admissible cusps, so that $\Delta_{|\lambda|}^+$ has discrete spectrum, and we assume for simplicity that the eigenspaces are one-dimensional, then unitarity of $\mathcal{F}_{g,+}^+$ implies that the measure with respect to $d\rho_\nu^{|\lambda|,+}$ of every point $\nu \in \text{spec}(\Delta_{|\lambda|}^+)$ is $1/\|\theta_\nu^{|\lambda|,+}\|_{L^2}^2$. Then (2.15) becomes

$$\begin{aligned} \eta_g^\varphi(D_N^+, a') &= 2e^{-p\varphi(a')} \int_0^\infty \sum_{\lambda \in \text{spec}(D_N^+)} \text{sgn}(\lambda) \text{tr}(g|_{\ker(D_N^+ - \lambda)}) \\ &\quad \times \sum_{\nu \in \text{spec}(\Delta_{|\lambda|}^+)} \frac{e^{-s\nu}}{\|\theta_\nu^{|\lambda|,+}\|_{L^2}^2} \theta_\nu^{|\lambda|,+}(\xi(a')) ((\theta_\nu^{|\lambda|,+})'(\xi(a')) + |\lambda|e^{-\varphi(a')} \theta_\nu^{|\lambda|,+}(\xi(a'))) ds. \end{aligned}$$

It is generally not possible to normalise the eigenfunctions $\theta_\nu^{|\lambda|,+}$ so that their L^2 -norms are 1, because they should satisfy (2.9).

Remark 2.19. The expression (2.15) can be extended to $a' = a$, but then it equals zero because the functions $\theta_\nu^{|\lambda|,+}$ satisfy (2.9). So the limit $\lim_{a' \downarrow a} \eta_g^\varphi(D_N^+, a')$ on the right-hand side of (2.17) is generally different from the version of (2.15) with $a' = a$. See Remark 4.5 for an example, and also [27, Remarks 2.4 and 5.12].

Remark 2.20. As is the case for the classical η -invariant, the cusp contribution (2.15) measures (an equivariant version of) *spectral asymmetry*, in the sense that it is zero if the spectrum of D_N^+ is g -symmetric. It is intriguing that even in this more general setting, symmetry or asymmetry of the spectrum of D_N determines if a contribution ‘‘from infinity’’ is required in index theorems on manifolds of the type we consider. Spectral asymmetry of D_N^+ can be measured in different ways, and apparently, the cusp shape function φ determines what measure of spectral asymmetry is relevant for an index theorem on M .

2.5 Relations to other results

In some cases, the cusp metric (2.1) can be transformed to a metric of the form $\rho^2 B_N + du^2$, for a function ρ of a radial coordinate u . This helps to clarify the relations between Theorem 2.16 and the results in [8, 9, 10, 40].

Suppose that φ' has no zeroes. Let ρ be a positive, smooth function defined on (\tilde{a}, \tilde{b}) , with $\tilde{a} := \rho^{-1}(e^{\varphi(a)})$ and $\tilde{b} := \lim_{x \rightarrow \infty} \rho^{-1}(e^{\varphi(x)})$. Suppose that, on this interval,

$$\rho' = \varphi' \circ \varphi^{-1} \circ \log(\rho). \tag{2.19}$$

Then the map $(n, u) \mapsto (n, \varphi^{-1}(\log(\rho(u))))$ is an isometry from $N \times (\tilde{a}, \tilde{b})$, with Riemannian metric $\rho^2 B_N + du^2$, onto $N \times (a, \infty)$, with Riemannian metric $e^{2\varphi}(B_N + dx^2)$.

In the case where $\varphi(x) = -\mu \log(x)$, for $\mu \in [0, 1]$, the function

$$\rho(u) = (1 - \mu)^{\mu/(\mu-1)} u^{\mu/(\mu-1)}$$

is a solution of (2.19) if $\mu \neq 1$. If $\mu = 1$, then a solution is $\rho(u) = e^{-u}$, a well-known alternative form of hyperbolic metrics.

In his thesis [40], Stern computed the index of the signature operator on finite-volume manifolds with cusps $N \times (\tilde{a}, \infty)$ with metric $\rho^2 B_N + du^2$, under certain growth conditions on ρ and its derivatives. These conditions hold in the case where $\varphi(x) = -\mu \log(x)$, for $\mu \in [0, 1]$.

Baier [8] developed index theory on Spin-manifolds with cusps $N \times (\tilde{a}, \infty)$, with metrics of the form $\rho^2 B_N + du^2$. He considered indices of Spin-Dirac operators, and proved

- (1) vanishing of the index if there is a $c > 1$ such that $u^{-c}\rho(u)$ has a positive lower bound for large u ,
- (2) an index formula if there is a $c < 1$ such that $u^{-c}\rho(u)$ has an upper bound for large u ,
- (3) inequalities satisfied by the index in other cases.

In cylinder case $\mu = 0$ and the hyperbolic case $\mu = 1$, the second of Baier's results applies. For $\mu \in (0, 1)$, neither condition in the first two cases holds, so the inequalities for the index in the third case apply.

In cases where both Theorem 2.16 and Baier's index formula apply, Baier's formula has a simpler contribution from outside Z (the classical η -invariant), but a more complicated contribution from inside Z , involving the dimension of the kernel of $D^-|_Z$ with Dirichlet boundary conditions at N .

Ballmann and Brüning [9] considered two-dimensional manifolds of finite area, with cusps of a type that includes cusps $S^1 \times (\tilde{a}, \infty)$ with metrics of the form $\rho^2 B_N + du^2$ mentioned above. Then they obtained an explicit index formula for Dirac operators.

Ballmann, Brüning and Carron [10] studied manifolds with cusps of a different but related form. In their setting, the manifold's ends are diffeomorphic to $N \times (\tilde{a}, \infty)$ via the gradient flow of a function with certain properties. They obtain a general index theorem in this setting, and more concrete index theorems in the case where the metric on the ends is cuspidal. A metric of the form $\rho^2 B_N + du^2$, with ρ related to φ via (2.19), is cuspidal if $\varphi(x) = -\log(x)$, but not if $\varphi(x) = -\mu \log(x)$ for $\mu \in [0, 1)$.

Vaillant [42] obtained an index theorem for manifolds with fibred versions of hyperbolic cusps. In the case of hyperbolic cusps, he showed that the contribution from infinity equals zero. It is an interesting question to find conditions on φ that imply vanishing of the cusp contribution in general.

For some results on the spectrum of Dirac or Laplace operators on manifolds with hyperbolic cusps, see [11] for Spin-Dirac operators and [31] for the Hodge Laplacian. See [32] for the Hodge Laplacian on manifolds with a generalisation of hyperbolic cusps.

See [33] for finite-dimensionality of the kernels of Dirac operators on finite-volume hyperbolic locally symmetric spaces. The Fredholm property in Theorem 2.16 is a stronger version of this, under the condition that D_N is invertible. The latter condition can be removed as in Section 5.

In APS-type index theorems where the metric does not have a product structure near the boundary, the contribution from the boundary can usually be written as a local contribution involving a transgression form, and a spectral contribution, the usual η -invariant. See, e.g., [22, 24, 38], or [18] for an equivariant version. It is an interesting question if the cusp contribution in Theorem 2.16, defined in terms of the spectrum of the operator D_N^+ , can be decomposed in a similar way, into a local contribution from the manifold N and a possibly simpler spectral contribution.

3 Proof of Theorem 2.16

We first state an index theorem from [27] for Dirac operators that are invertible outside a compact set, see Theorem 3.1. Then we deduce Theorem 2.16 from this result.

3.1 Dirac operators that are invertible at infinity

We review the geometric setting and notation needed to formulate Theorem 2.2 in [27], this is Theorem 3.1 below. The text leading up to Theorem 3.1 is a slight reformulation of corresponding material from [27]. Compared to Theorem 2.16, some assumptions in the main result from [27], Theorem 3.1, are weaker, but there are additional assumptions on D , such as invertibility at

infinity. Part of the proof of Theorem 2.16 is to show that these additional assumptions hold in the setting of manifolds with admissible cusps.

We assume in this subsection and the next that M has φ -cusps, but not that these cusps are weakly or strongly admissible.

We say that D is *invertible at infinity* if there are a G -invariant compact subset $Z \subset M$ and a constant $b > 0$ such that for all $s \in \Gamma_c^\infty(S)$ supported in $M \setminus Z$,

$$\|Ds\|_{L^2} \geq b\|s\|_{L^2}. \quad (3.1)$$

We assume in this subsection and the next that D is invertible at infinity, and that the set Z may be taken as in Definition 2.1. (In [27], we took $a = 0$, now we allow general $a \geq 0$ for consistency with Theorem 2.16.)

For $k = 0, 1, 2, \dots$, let $W_D^k(S)$ be the completion of $\Gamma_c^\infty(S)$ in the inner product

$$(s_1, s_2)_{W_D^k} := \sum_{j=0}^k (D^j s_1, D^j s_2)_{L^2}.$$

Because D is invertible at infinity, it is Fredholm as an operator

$$D: W_D^1(S) \rightarrow L^2(S).$$

See [2, Theorem 2.1] or [23, Theorem 3.2].

Rather than the more specific form (2.7) of D on $C = M \setminus Z$, we assume that

$$D|_C = \sigma \left(f_1 \frac{\partial}{\partial x} + f_2 D_N + f_3 \right), \quad (3.2)$$

where

- $\sigma \in \text{End}(S|_N)^G$ interchanges $S^+|_N$ and $S^-|_N$,
- $f_1, f_2, f_3 \in C^\infty(a, a+2)$,
- D_N is a G -equivariant, invertible Dirac operator on $S|_N$ that preserves the grading.

Consider the vector bundle

$$S_C := S|_C \rightarrow C.$$

For $k \in \mathbb{N}$ at least 1, consider the Sobolev space

$$W_D^k(S_C) := \{s|_C; s \in W_D^k(S)\}.$$

(See [12] for other constructions of such Sobolev spaces on manifolds with boundary.) We denote the subspaces of even- and odd-graded sections by $W_D^k(S_C^\pm)$, respectively.

If $s \in W_D^{k+1}(S)$, then the restriction of $Ds \in W_D^k(S)$ to the interior of C is determined by the restriction of s to the interior of C . Since $k \geq 1$, the restriction of Ds to the interior of C has a unique extension to C . So $Ds|_C$ is determined by $s|_C$. In this way, D gives a well-defined, bounded operator from $W_D^{k+1}(S_C)$ to $W_D^k(S_C)$, which we denote by D_C .

Because N is compact, D_N has discrete spectrum. Let D_N^\pm be the restriction of D_N to sections of $S^\pm|_N$. Let $L^2(S^+|_N)_{>0}$ be the direct sum of these eigenspaces for positive eigenvalues (recall that 0 is not an eigenvalue). Consider the orthogonal projection

$$P^+: L^2(S^+|_N) \rightarrow L^2(S^+|_N)_{>0}. \quad (3.3)$$

(In [27], more general spectral projections are allowed, but this is the one relevant to the current setting.) We will also use the projection

$$P^- := \sigma_+ P^+ \sigma_+^{-1}: L^2(S^-|_N) \rightarrow \sigma_+ L^2(S^+|_N)_{>0},$$

where $\sigma_+ := \sigma|_{(S^+|_N)}$. In general, P^- is not necessarily a spectral projection for D_N^- , but in the setting that is relevant to us, we have $\sigma D_N^+ = -D_N^- \sigma$, so P^- is projection onto the *negative* eigenspaces of D_N^- ; see (3.12). We combine P^+ and P^- to an orthogonal projection

$$P := P^+ \oplus P^-: L^2(S|_N) \rightarrow L^2(S^+|_N)_{>0} \oplus \sigma_+ L^2(S^+|_N)_{>0}. \quad (3.4)$$

We will sometimes omit the superscripts \pm from P^\pm .

Consider the spaces

$$\begin{aligned} W_D^1(S_C^+; P) &:= \{s \in W_D^1(S_C^+); P^+(s|_N) = 0\}, \\ W_D^1(S_C^-; 1 - P) &:= \{s \in W_D^1(S_C^-); (1 - P^-)(s|_N) = 0\}, \\ W_D^2(S_C^+; P) &:= \{s \in W_D^2(S_C^+); P^+(s|_N) = 0, (1 - P^-)(D_C^+ s|_N) = 0\}, \\ W_D^2(S_C^-; 1 - P) &:= \{s \in W_D^2(S_C^-); (1 - P^-)(s|_N) = 0, P^+(D_C^- s|_N) = 0\}. \end{aligned} \quad (3.5)$$

Here we use the fact that there are well-defined, continuous restriction/extension maps

$$W_D^1(S_C) \rightarrow L^2(S|_N).$$

We assume that

- the operators

$$D_C^+: W_D^1(S_C^+; P) \rightarrow L^2(S_C^-) \quad (3.6)$$

and

$$D_C^-: W_D^1(S_C^-; 1 - P) \rightarrow L^2(S_C^+) \quad (3.7)$$

are invertible; and

- the operator $D_C^- D_C^+$ on $L^2(S_C^+)$, with domain $W_D^2(S_C^+; P)$, and the operator $D_C^+ D_C^-$ on $L^2(S_C^-)$, with domain $W_D^2(S_C^-; 1 - P)$, are self-adjoint.

3.2 An index theorem

Under the assumptions in Section 3.1, we state an index theorem using the following ingredients.

For $t > 0$, let $e_P^{-tD_C^- D_C^+}$ be the heat operator for the operator $D_C^- D_C^+$ on $L^2(S_C^+)$, with domain $W_D^2(S_C^+; P)$. By [27, Lemma 4.7], the operator $e_P^{-tD_C^- D_C^+} D_C^-$ has a smooth kernel λ_t^P . The *contribution from infinity* associated to D_C and $a' \in (a, a + 2)$ is

$$A_g(D_C, a') := -f_1(a') \int_0^\infty \int_N \text{tr}(g \lambda_s^P(g^{-1}n, a'; n, a')) \, dn \, ds, \quad (3.8)$$

defined whenever the integral in (3.8) converges.

The following result is a combination of Theorem 2.2 and Corollary 2.3 in [27].

Theorem 3.1 (index theorem for Dirac operators invertible at infinity). *For all $a' > a$, the quantity (3.8) converges, and*

$$\text{index}_G(D)(g) = \int_{Z^g \cup (N^g \times (a, a'])} \text{AS}_g(D) + A_g(D_C, a'). \quad (3.9)$$

Furthermore, the limit $\lim_{a' \downarrow a} A_g(D_C, a')$ converges, and

$$\text{index}_G(D)(g) = \int_{Z^g} \text{AS}_g(D) + \lim_{a' \downarrow a} A_g(D_C, a').$$

Theorem 2.16 follows from Theorem 3.1 because of the following three propositions.

Proposition 3.2. *In the setting of Theorem 2.16, the conditions of Theorem 3.1 hold.*

Proposition 3.3. *In the setting of Theorem 2.16, the cusp contribution (2.15) converges for all $a' > a$, and*

$$A_g(D_C, a') = -\frac{1}{2} \eta_g^\varphi(D_N^+, a').$$

If the spectrum of D_N is g -symmetric, then the right-hand side is zero.

Proposition 3.4. *If M has strongly admissible φ -cusps, then the operators $\Delta_{|\lambda|}^+$ have discrete spectrum for all $\lambda \in \text{spec}(D_N^+)$.*

Proof. It follows from the definition (2.13) of q_λ^+ that for all $\lambda \neq 0$,

$$\lim_{y \rightarrow \infty} q_{|\lambda|}^+(y) = \infty$$

if φ satisfies (2.2). It follows that $\Delta_{|\lambda|}^+$ has discrete spectrum, see [29, Theorem 1.3.1, Lemma 3.1.1 and equation (1.3)]. \blacksquare

In the rest of this section, we prove Propositions 3.2 (at the end of Section 3.6) and 3.3 (at the end of Section 3.7), and thus Theorem 2.16. (The case (2.18) follows immediately from the case (2.16).)

3.3 Transforming Dirac operators

We return to the setting of Section 2, where D is of the form in Proposition 2.7.

To compute (3.8), we use a Liouville-type transformation to relate D_C and D_C^2 to simpler operators. At the same time, this allows us to transform the Riemannian density of B_φ to a product density.

Define $\Phi \in C^\infty(a, \infty)$ by

$$\Phi(x) := e^{-\frac{p-1}{2}\varphi(x)}. \quad (3.10)$$

(Recall that $p = \dim(M)$.) Consider the vector bundle $\tilde{S}_C := S|_N \times (0, \infty) \rightarrow N \times (0, \infty)$. For a section s of \tilde{S}_C and $n \in N$ and $x \in (a, \infty)$, define

$$(Ts)(n, x) := \Phi(x)s(n, \xi(x))$$

with ξ as in (2.12).

Lemma 3.5. *The operator T defines a G -equivariant unitary isomorphism*

$$T: L^2(\tilde{S}_C, \text{vol}_{B_N} \otimes dy) \rightarrow L^2(S|_C, \text{vol}_{B_\varphi}).$$

Proof. We have $\text{vol}_{B_\varphi} = \text{vol}_{B_N} \otimes e^{p\varphi} dx$. And by a substitution $y = \xi(x)$, and the equality $\xi' = e^\varphi$, we have for all $f \in L^2((0, \infty), dy)$,

$$\|\Phi \cdot (f \circ \xi)\|_{L^2((a, \infty), e^{p\varphi} dx)}^2 = \int_a^\infty |f(\xi(x))|^2 e^{\varphi(x)} dx = \|f\|_{L^2((0, \infty), dy)}^2. \quad \blacksquare$$

Consider the function $h := e^{-\varphi \circ \xi^{-1}} \in C^\infty(0, \infty)$, and the Dirac operator

$$\tilde{D}_C := \sigma \left(\frac{\partial}{\partial y} + h D_N \right)$$

on $\Gamma^\infty(\tilde{S}_C)$. Here y is the coordinate in $(0, \infty)$. Let D_C be as in Section 3.1, but viewed as acting on smooth sections. It is given by (2.6).

Lemma 3.6. *The following diagram commutes:*

$$\begin{array}{ccc} \Gamma^\infty(S_C) & \xrightarrow{D_C} & \Gamma^\infty(S_C) \\ T \uparrow & & T \uparrow \\ \Gamma^\infty(\tilde{S}_C) & \xrightarrow{\tilde{D}_C} & \Gamma^\infty(\tilde{S}_C). \end{array}$$

Proof. This is a direct computation, based on (2.6) and (2.7). \blacksquare

It will be convenient to identify $S^-|_N \cong S^+|_N$ via σ in (2.7). Because signs and gradings are important in what follows, it is worth being explicit about details here. We write $\tau := \sigma|_{S^+|_N} \times 1: \tilde{S}_C^+ \rightarrow \tilde{S}_C^-$, and consider the isomorphism

$$1 \oplus \tau: \tilde{S}_C^+ \oplus \tilde{S}_C^+ \xrightarrow{\cong} \tilde{S}_C. \quad (3.11)$$

The operator D_N preserves the grading on $S|_N$; let D_N^\pm be its restrictions to even and odd-graded sections, respectively. Because D_N anticommutes with σ , we have

$$D_N^- \circ \tau = -\tau \circ D_N^+. \quad (3.12)$$

We will use the operators

$$\tilde{D}_C^\pm := \pm \frac{\partial}{\partial y} + h D_N^\pm: \Gamma^\infty(\tilde{S}_C^\pm) \rightarrow \Gamma^\infty(\tilde{S}_C^\pm).$$

Lemma 3.7. *Under the isomorphism (3.11), the operator \tilde{D}_C corresponds to the operator*

$$\begin{pmatrix} 0 & \tilde{D}_C^- \\ \tilde{D}_C^+ & 0 \end{pmatrix} \quad \text{on} \quad \Gamma^\infty(\tilde{S}_C^+ \oplus \tilde{S}_C^+).$$

Proof. Consider the vector bundle endomorphism

$$\Sigma := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}: \tilde{S}_C^+ \oplus \tilde{S}_C^+ \rightarrow \tilde{S}_C^+ \oplus \tilde{S}_C^+.$$

Then the diagram

$$\begin{array}{ccc} \tilde{S}_C & \xrightarrow{\sigma} & \tilde{S}_C \\ 1 \oplus \tau \uparrow & & 1 \oplus \tau \uparrow \\ \tilde{S}_C^+ \oplus \tilde{S}_C^+ & \xrightarrow{\Sigma} & \tilde{S}_C^+ \oplus \tilde{S}_C^+ \end{array}$$

commutes.

Because of (3.12) and the fact that the operator $\frac{\partial}{\partial y}$ commutes with τ ,

$$\begin{aligned} \left(\frac{\partial}{\partial y} + hD_N \right) \circ (1 \oplus \tau) &= \begin{pmatrix} \frac{\partial}{\partial y} + hD_N^+ & 0 \\ 0 & \frac{\partial}{\partial y} + hD_N^- \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial y} + hD_N^+ & 0 \\ 0 & \frac{\partial}{\partial y} - hD_N^+ \end{pmatrix}. \end{aligned}$$

We find that under the isomorphism (3.11), the operator \tilde{D}_C corresponds to

$$\Sigma \begin{pmatrix} \frac{\partial}{\partial y} + hD_N^+ & 0 \\ 0 & \frac{\partial}{\partial y} - hD_N^+ \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\partial}{\partial y} + hD_N^+ \\ \frac{\partial}{\partial y} + hD_N^+ & 0 \end{pmatrix}. \quad \blacksquare$$

For later use, we record expressions for the operators $\tilde{D}_C^\mp \tilde{D}_C^\pm$. Consider the Laplace-type operators

$$\Delta^\pm := -\frac{\partial^2}{\partial y^2} + e^{-2\varphi \circ \xi^{-1}} (D_N^+)^2 \pm (\varphi' \circ \xi^{-1}) e^{-2\varphi \circ \xi^{-1}} D_N^+ \quad \text{on } \Gamma^\infty(\tilde{S}_C).$$

Lemma 3.8. *We have $\tilde{D}_C^\mp \tilde{D}_C^\pm = \Delta^\pm$.*

Proof. By a direct computation,

$$\tilde{D}_C^\mp \tilde{D}_C^\pm = -\frac{\partial^2}{\partial y^2} + e^{-2\varphi \circ \xi^{-1}} (D_N^+)^2 \pm (\varphi \circ \xi^{-1})' e^{-\varphi \circ \xi^{-1}} D_N^+.$$

The right-hand side equals Δ^\pm , because

$$(\varphi \circ \xi^{-1})' = (\varphi' \circ \xi^{-1}) e^{-\varphi \circ \xi^{-1}}. \quad \blacksquare$$

3.4 APS-boundary conditions

Analogously to the Sobolev spaces defined in Section 3.2, we use the APS-type projection (3.3) to define the following Hilbert spaces. Here we identify $\tilde{S}_C^- \cong \tilde{S}_C^+$ via the map τ , and use the unitary isomorphism T from Lemma 3.5.

$$\begin{aligned} W_{\tilde{D}_C}^k(\tilde{S}_C^\pm) &:= T^{-1}(W_D^k(S_C^\pm)), \\ W_{\tilde{D}_C}^1(\tilde{S}_C^+; P) &:= \{s \in W_{\tilde{D}_C}^1(\tilde{S}_C^+); P^+(s|_N) = 0\}, \\ W_{\tilde{D}_C}^1(\tilde{S}_C^+; 1 - P) &:= \{s \in W_{\tilde{D}_C}^1(\tilde{S}_C^+); (1 - P^+)(s|_N) = 0\}, \\ W_{\tilde{D}_C}^2(\tilde{S}_C^+; P) &:= \{s \in W_{\tilde{D}_C}^2(\tilde{S}_C^+); P^+(s|_N) = 0, (1 - P^+)(\tilde{D}_C^+ s|_N) = 0\}, \\ W_{\tilde{D}_C}^2(\tilde{S}_C^+; 1 - P) &:= \{s \in W_{\tilde{D}_C}^2(\tilde{S}_C^+); (1 - P^+)(s|_N) = 0, P^+(\tilde{D}_C^- s|_N) = 0\}. \end{aligned}$$

(Analogously to (3.5), we use continuous restriction maps $W_{\tilde{D}_C}^1(\tilde{S}_C) \rightarrow L^2(S|_N)$.) By Lemma 3.6, we have the bounded operators

$$\tilde{D}_C^+: W_{\tilde{D}_C}^1(\tilde{S}_C^+; P) \rightarrow L^2(\tilde{S}_C^+), \quad (3.13)$$

$$\tilde{D}_C^-: W_{\tilde{D}_C}^1(\tilde{S}_C^+; 1 - P) \rightarrow L^2(\tilde{S}_C^+), \quad (3.14)$$

$$\tilde{D}_C^- \tilde{D}_C^+: W_{\tilde{D}_C}^2(\tilde{S}_C^+; P) \rightarrow L^2(\tilde{S}_C^+), \quad (3.15)$$

$$\tilde{D}_C^+ \tilde{D}_C^-: W_{\tilde{D}_C}^2(\tilde{S}_C^+; 1 - P) \rightarrow L^2(\tilde{S}_C^+). \quad (3.16)$$

The results in the previous subsection lead to the following conclusion, which shows that the conditions of Theorem 3.1 are equivalent to corresponding properties of the operator \tilde{D}_C in this setting.

Proposition 3.9.

- (a) The operator D is invertible at infinity in the sense of (3.1) if and only if there are $b_0 > 0$ and $u \geq a$ such that for all $s \in \Gamma_c^\infty(\tilde{S}_C)$ supported in (u, ∞) ,

$$\|\tilde{D}_C s\|_{L^2(\tilde{S}_C, \text{vol}_{B_N} \otimes dy)} \geq b_0 \|s\|_{L^2(\tilde{S}_C, \text{vol}_{B_N} \otimes dy)}.$$

- (b) The operator (3.6) is invertible if and only if the operator (3.13) is.
(c) The operator (3.7) is invertible if and only if the operator (3.14) is.
(d) The operator $D_C^- D_C^+$ on $L^2(S_C^+)$, with domain $W_D^2(S_C^+; P)$, is self-adjoint if and only if the operator $\tilde{D}_C^- \tilde{D}_C^+$ on $L^2(\tilde{S}_C^+)$, with domain $W_{\tilde{D}_C}^2(\tilde{S}_C^+; P)$ is.
(e) The operator the operator $D_C^+ D_C^-$ on $L^2(S_C^-)$, with domain $W_D^2(S_C^-; 1 - P)$ is self-adjoint if and only if the operator $\tilde{D}_C^+ \tilde{D}_C^-$ on $L^2(\tilde{S}_C^-)$, with domain $W_{\tilde{D}_C}^2(\tilde{S}_C^-; 1 - P)$ is.

Proof. Part (a) follows from Lemmas 3.5 and 3.6.

For any $s \in \Gamma^\infty(\tilde{S}_C)$, we have

$$(Ts)|_N = \Phi(a)s|_N,$$

because we view N as embedded into \bar{C} as $N \times \{a\}$ and into $N \times [0, \infty)$ as $N \times \{0\}$. Therefore, for any such section, $P(s|_N) = 0$ if and only if $P(Ts|_N) = 0$, and similarly for $1 - P$. Hence the operator T , together with (3.11), defines unitary isomorphisms

$$\begin{aligned} T: W_{\tilde{D}_C}^1(\tilde{S}_C^+; P) &\xrightarrow{\cong} W_D^1(S_C^+; P), \\ T \circ \tau: W_{\tilde{D}_C}^1(\tilde{S}_C^+; 1 - P) &\xrightarrow{\cong} W_D^1(S_C^-; 1 - P), \\ T: W_{\tilde{D}_C}^2(\tilde{S}_C^+; P) &\xrightarrow{\cong} W_D^2(S_C^+; P), \\ T \circ \tau: W_{\tilde{D}_C}^2(\tilde{S}_C^+; 1 - P) &\xrightarrow{\cong} W_D^2(S_C^-; 1 - P). \end{aligned} \tag{3.17}$$

In the second and fourth lines, we use the fact that by definition (3.4) of P , we have

$$P|_{L^2(S^-|_N)} = \tau_+ P^+ \tau_+^{-1},$$

so that the isomorphisms $T \circ \tau$ preserve the given boundary conditions.

Under the isomorphisms (3.17), the pairs of operators in parts (b)–(e) correspond to each other. Here we again used Lemmas 3.5 and 3.6, and also Lemma 3.7. \blacksquare

3.5 Lower bounds and invertibility

Two kinds of invertibility at infinity of D are assumed in Theorem 3.1: there is the condition (3.1) on D , and invertibility of (3.6) and (3.7). To verify these conditions in the context of manifolds with φ -cusps, we use Lemma 3.10 and Proposition 3.12 below.

The proof of the following lemma is the only place where we use the condition (2.8) in the definition of weakly admissible cusps.

Lemma 3.10. *Suppose that M has weakly admissible cusps, i.e., φ has the properties in Definition 2.8. Suppose that D_N is invertible. Then there are $u, b_0 > 0$ such that for all $s \in \Gamma_c^\infty(\tilde{S}_C)$ supported in $N \times (u, \infty)$,*

$$\|\tilde{D}_C s\|_{L^2} \geq b_0 \|s\|_{L^2}. \tag{3.18}$$

Proof. By Lemma 3.8,

$$\tilde{D}_C^2 \geq e^{-2\varphi \circ \xi^{-1}} (D_N^2 - |\varphi' \circ \xi^{-1}| |D_N|) = e^{-2\varphi \circ \xi^{-1}} |D_N| (|D_N| - |\varphi' \circ \xi^{-1}|). \quad (3.19)$$

Because D_N is invertible, there is a $b > 0$ such that $D_N^2 \geq b^2$. Because M has weakly admissible cusps, there is an upper bound β for φ , and there are $a' > 0$ and $\alpha > 0$ such that for all $x > a'$,

$$b - |\varphi'(x)| \geq \alpha.$$

Now $|D_N| \geq b$, so on $N \times (\xi(a'), \infty)$,

$$|D_N| - |\varphi' \circ \xi^{-1}| \geq \alpha, \quad \text{and} \quad e^{-2\varphi \circ \xi^{-1}} |D_N| \geq e^{-2\beta} b.$$

Hence, on $N \times (\xi(a'), \infty)$, the right-hand side of (3.19) is greater than or equal to

$$b_0^2 := \alpha b e^{-2\beta}.$$

Here we used that the various operators are self-adjoint and commute. ■

Remark 3.11. By a small adaptation of the proof of Lemma 3.10, we can show that if M has strongly admissible φ -cusps and D_N is invertible, then for all $b_0 > 0$, there is a $u > 0$ such that (3.18) holds for all $s \in \Gamma_c^\infty(\tilde{S}_C)$ supported in $N \times (u, \infty)$. By [4, Theorem SD] and Lemmas 3.5 and 3.6, this implies that D has discrete spectrum. This is an analogous result to Proposition 3.4.

Proposition 3.12. *If h has a positive lower bound, then the operators (3.13) and (3.14) are invertible.*

Lemma 3.13. *Let $\lambda \in \mathbb{R}$, $\zeta \in C_c^\infty(0, \infty)$ and $h \in C^\infty[0, \infty)$. For $u, v \geq 0$, define*

$$H_\lambda(u, v) := \exp\left(\lambda \int_u^v h(s) ds\right).$$

Define $f \in C^\infty[0, \infty)$ by

$$f(u) := \begin{cases} \int_0^u H_\lambda(u, v) \zeta(v) dv & \text{if } \lambda \geq 0, \\ -\int_u^\infty H_\lambda(u, v) \zeta(v) dv & \text{if } \lambda < 0. \end{cases} \quad (3.20)$$

Then

- (1) $f' + \lambda h f = \zeta$,
- (2) if $\zeta = \tilde{f}' + \lambda h \tilde{f}$ for some $f \in C_c^\infty(0, \infty)$, then $f = \tilde{f}$,
- (3) $f(0) = 0$ if $\lambda \geq 0$,
- (4) if $h \geq \varepsilon > 0$, then $f \in L^2(0, \infty)$, and $|\lambda| \|f\|_{L^2} \leq \frac{2}{\varepsilon} \|\zeta\|_{L^2}$.

Proof. The first two points follow from computations, the third point is immediate from the definition of f .

For the fourth point, note that because $h \geq \varepsilon$,

$$H_\lambda(u, v) \leq e^{\varepsilon \lambda (v-u)}, \quad (3.21)$$

if either

- $\lambda \geq 0$ and $v \leq u$, or
- $\lambda < 0$ and $v \geq u$.

For $\lambda' \in \mathbb{R}$, define

$$f^{\lambda'}(u) := \begin{cases} \int_0^u e^{\lambda'(v-u)} \zeta(v) \, dv & \text{if } \lambda' \geq 0, \\ -\int_u^\infty e^{\lambda'(v-u)} \zeta(v) \, dv & \text{if } \lambda' < 0. \end{cases}$$

It is shown in the proof of Proposition 2.5 in [6] that $f^{\lambda'} \in L^2(0, \infty)$, and $|\lambda'| \|f^{\lambda'}\|_{L^2} \leq 2\|\zeta\|_{L^2}$.

First suppose that $\lambda \geq 0$ and $\zeta \geq 0$. Then (3.21) implies that $|f| \leq |f^{\varepsilon\lambda}|$, so $f \in L^2(0, \infty)$, and

$$\varepsilon\lambda \|f\|_{L^2} \leq \varepsilon\lambda \|f^{\varepsilon\lambda}\|_{L^2} \leq 2\|\zeta\|_{L^2}.$$

If ζ also takes negative values, we decompose it as a difference of two nonnegative functions, and reach the same conclusion.

Now suppose that $\lambda < 0$ and $\zeta \leq 0$. Then again, (3.21) implies that $|f| \leq |f^{\varepsilon\lambda}|$, and

$$\varepsilon|\lambda| \|f\|_{L^2} \leq \varepsilon|\lambda| \|f^{\varepsilon\lambda}\|_{L^2} \leq 2\|\zeta\|_{L^2}.$$

This also extends to ζ with positive values by a decomposition of ζ into nonpositive functions. ■

For all $\lambda \in \text{spec}(D_N^+)$, let $\{\varphi_\lambda^1, \dots, \varphi_\lambda^{m_\lambda}\}$ be an orthonormal basis of $\ker(D_N^+ - \lambda)$. Then

$$\{\varphi_\lambda^j; \lambda \in \text{spec}(D_N^+), j = 1, \dots, m_\lambda\} \quad (3.22)$$

is a Hilbert basis of $L^2(S^+|_N)$ of eigensections of D_N^+ .

Proof of Proposition 3.12. We prove the claim for (3.13), the proof for (3.14) is similar.

Let $\zeta \in \Gamma_c^\infty(\tilde{S}_C^+)$. Write

$$\zeta = \sum_{\lambda \in \text{spec}(D_N^+)} \sum_{j=1}^{m_\lambda} \zeta_\lambda^j \otimes \varphi_\lambda^j, \quad (3.23)$$

where $\zeta_\lambda^j \in C_c^\infty(0, \infty)$. For every $\lambda \in \text{spec}(D_N^+)$ and j , define the function f_λ^j on $(0, \infty)$ as the function f in (3.20), with ζ replaced by ζ_λ^j . We claim that the series

$$\sum_{\lambda \in \text{spec}(D_N^+)} \sum_{j=1}^{m_\lambda} f_\lambda^j \otimes \varphi_\lambda^j \quad (3.24)$$

converges to an element $f \in W_{\tilde{D}_C}^1(\tilde{S}_C^+; P)$. We set $Q\zeta := f$ for any such ζ . Then the first point in Lemma 3.13 implies that $\tilde{D}_C^+ f = \zeta$. The second point in Lemma 3.13 implies that $Q\tilde{D}_C^+ \tilde{f} = \tilde{f}$ for any $\tilde{f} \in W_{\tilde{D}_C}^1(\tilde{S}_C^+; P)$. The third point in Lemma 3.13 implies that $P(f|_{N \times \{0\}}) = 0$.

To prove convergence of (3.24), we note that by the first point in Lemma 3.13,

$$\|f_\lambda^j \otimes \varphi_\lambda^j\|_{W_{\tilde{D}_C}^1}^2 = \|f_\lambda^j\|_{L^2}^2 + \|\zeta_\lambda^j\|_{L^2}^2.$$

Since $|\lambda|$ is bounded away from zero, the fourth point in Lemma 3.13 implies that

$$\|f_\lambda^j \otimes \varphi_\lambda^j\|_{W_{\tilde{D}_C}^1} \leq B \|\zeta_\lambda^j\|_{L^2} \quad (3.25)$$

for a constant $B > 0$ independent of λ . So convergence of (3.23) in $L^2(\tilde{S}_C^+)$ implies convergence of (3.24) in $W_{\tilde{D}_C}^1(\tilde{S}_C^+)$.

We have just seen that $\|Q\zeta\|_{W_{\tilde{D}_C}^1} \leq B\|\zeta\|_{L^2}$, thus Q extends continuously to an inverse of (3.13). \blacksquare

3.6 Adjoints

Proposition 3.14. *If h has a positive lower bound, then the two operators (3.13) and (3.14) are each other's adjoints.*

Proof. We claim that for all $s_P \in W_{\tilde{D}_C}^1(\tilde{S}_C^+; P)$ and $s_{1-P} \in W_{\tilde{D}_C}^1(\tilde{S}_C^+; 1-P)$,

$$(\tilde{D}_C^+ s_P, s_{1-P})_{L^2} = (s_P, \tilde{D}_C^- s_{1-P})_{L^2}. \quad (3.26)$$

Indeed, suppose that

$$s_P = s_P^N \otimes f_P, \quad s_{1-P} = s_{1-P}^N \otimes f_{1-P},$$

for $s_P^N, s_{1-P}^N \in L^2(S^+|_N)$ and $f_P, f_{1-P} \in L^2[0, \infty)$ such that $s_P \in W_{\tilde{D}_C}^1(\tilde{S}_C^+; P)$ and $s_{1-P} \in W_{\tilde{D}_C}^1(\tilde{S}_C^+; 1-P)$. Then by self-adjointness of D_N^+ and integration by parts, the left-hand side of (3.26) equals

$$(s_P^N, D_N^+ s_{1-P}^N)_{L^2(S^+|_N)} \left((f_P \overline{f_{1-P}})|_0^\infty + \int_0^\infty f_P(x) (-\overline{f'_{1-P}(x)} + h(x) \overline{f_{1-P}(x)}) dx \right). \quad (3.27)$$

If we further decompose the expressions with respect to eigenspaces of D_N^+ , and use that the components of f_P for positive eigenvalues equal zero at zero, and the components of f_{1-P} for negative eigenvalues equal zero at zero, then we find that the components for all eigenvalues of $f_P \overline{f_{1-P}}$ are zero at zero. So the term $(f_P \overline{f_{1-P}})|_0^\infty$ in (3.27) vanishes, and (3.27) equals the right-hand side of (3.26).

Now if $\sigma_P := \tilde{D}_C^+ s_P$ and $\sigma_{1-P} := \tilde{D}_C^- s_{1-P}$, then (3.26) becomes

$$(\sigma_P, (\tilde{D}_C^-)^{-1} \sigma_{1-P})_{L^2} = ((\tilde{D}_C^+)^{-1} \sigma_P, \sigma_{1-P})_{L^2}.$$

The sections σ_P and σ_{1-P} of this type are dense in $L^2(\tilde{S}_C^+)$. And the inverse operators $(\tilde{D}_C^+)^{-1}$ and $(\tilde{D}_C^-)^{-1}$ are bounded, so we find that

$$((\tilde{D}_C^+)^{-1})^* = (\tilde{D}_C^-)^{-1}.$$

This implies that $(\tilde{D}_C^+)^* = \tilde{D}_C^-$. \blacksquare

Proposition 3.15. *If h has a positive lower bound, then the operators (3.15) and (3.16) are self-adjoint.*

Proof. We prove the claim for (3.15), the proof for (3.16) is similar.

The operator (3.13) is invertible by Proposition 3.12. It maps the subspace

$$W_{\tilde{D}_C}^2(\tilde{S}_C^+; P) \subset W^1(\tilde{S}_C^+; P)$$

onto

$$W^1(\tilde{S}_C^+; 1 - P) \subset L^2(\tilde{S}_C^+).$$

So we obtain an invertible operator

$$\tilde{D}_C^+ : W_{\tilde{D}_C}^2(\tilde{S}_C^+; P) \rightarrow W^1(\tilde{S}_C^+; 1 - P).$$

So, again by Proposition 3.12, the composition

$$W_{\tilde{D}_C}^2(\tilde{S}_C^+; P) \xrightarrow{\tilde{D}_C^+} W^1(\tilde{S}_C^+; 1 - P) \xrightarrow{\tilde{D}_C^-} L^2(\tilde{S}_C^+) \quad (3.28)$$

is invertible, with bounded inverse. The adjoint of the bounded operator

$$(\tilde{D}_C^+)^{-1} : W_{\tilde{D}_C}^1(\tilde{S}_C^+; 1 - P) \rightarrow W_{\tilde{D}_C}^2(\tilde{S}_C^+; P)$$

is the restriction of the adjoint of

$$(\tilde{D}_C^+)^{-1} : L^2(\tilde{S}_C^+) \rightarrow W_{\tilde{D}_C}^1(\tilde{S}_C^+; P)$$

to $W_{\tilde{D}_C}^2(\tilde{S}_C^+; P)$. By Proposition 3.14, this is

$$(\tilde{D}_C^-)^{-1}|_{W_{\tilde{D}_C}^2(\tilde{S}_C^+; P)}.$$

Again applying Proposition 3.14, we find that the inverse of (3.28) has adjoint

$$W_{\tilde{D}_C}^2(\tilde{S}_C^+; P) \xrightarrow{(\tilde{D}_C^-)^{-1}} W_{\tilde{D}_C}^1(\tilde{S}_C^+; 1 - P) \xrightarrow{(\tilde{D}_C^+)^{-1}} L^2(\tilde{S}_C^+).$$

Hence, as maps from $W_{\tilde{D}_C}^2(\tilde{S}_C^+; P)$ to $L^2(\tilde{S}_C^+)$,

$$((\tilde{D}_C^- \tilde{D}_C^+)^{-1})^* = (\tilde{D}_C^- \tilde{D}_C^+)^{-1}.$$

This implies that $(\tilde{D}_C^- \tilde{D}_C^+)^* = \tilde{D}_C^- \tilde{D}_C^+$. ■

Remark 3.16. Proposition 3.15 can also be deduced from Proposition 3.14 via [37, Theorem X.25].

If M has weakly admissible φ -cusps, then φ has an upper bound, so $h = e^{-\varphi \circ \xi}$ has a positive lower bound. Therefore, Proposition 3.2 follows from Proposition 3.9, Lemma 3.10 and Propositions 3.12 and 3.15.

3.7 Cusp contributions

To prove Proposition 3.3, we express the contribution from infinity in Theorem 3.1 in terms of the operator \tilde{D}_C . Recall the definition of the function Φ in (3.10).

Lemma 3.17. *Suppose that the operator (3.15) is self-adjoint, and let $\tilde{\lambda}_s^P$ be the Schwartz kernel of $e_P^{-s\tilde{D}_C^- \tilde{D}_C^+} \tilde{D}_C^-$. Then for all $n, n' \in N$, $x, x' \in (a, \infty)$ and $s > 0$,*

$$\lambda_s^P(n, x; n', x') = e^{(1-p)\varphi(x')} \frac{\Phi(x)}{\Phi(x')} \tilde{\lambda}_s^P(n, \xi(x); n', \xi(x')).$$

Proof. The kernel λ_s^P is defined with respect to the Riemannian density $e^{p\varphi} dn dx$ on C , whereas $\tilde{\lambda}_s^P$ is defined with respect to the Riemannian density $dn dy$ on $N \times (0, \infty)$. Furthermore, Lemma 3.6 and the third isomorphism on (3.17) imply that

$$e_P^{-sD_C^- D_C^+} D_C^- = T \circ e_P^{-s\tilde{D}_C^- \tilde{D}_C^+} \tilde{D}_C^- \circ T^{-1}.$$

We find that for all $n \in N$ and $x' \in (a, \infty)$ and $s \in \Gamma_c^\infty(S_C)$,

$$\begin{aligned} & \int_N \int_a^\infty \lambda_s^P(n, x; n', x') s(n', x') e^{p\varphi(x')} dx' dn' \\ &= (e_P^{-sD_C^- D_C^+} D_C^- s)(n, x) = T(e_P^{-s\tilde{D}_C^- \tilde{D}_C^+} \tilde{D}_C^- T^{-1} s)(n, x) \\ &= \Phi(x) \int_N \int_0^\infty \tilde{\lambda}_s^P(n, \xi(x); n', y') \frac{1}{\Phi(\xi^{-1}(y'))} s(n', \xi^{-1}(y')) dn' dy'. \end{aligned}$$

By a substitution $x' = \xi^{-1}(y')$, the latter integral equals

$$\Phi(x) \int_N \int_a^\infty \tilde{\lambda}_s^P(n, \xi(x); n', \xi(x')) \frac{1}{\Phi(x')} s(n', x') e^{\varphi(x')} dn' dx'.$$

Here we used that $\xi' = e^\varphi$. ■

Recall the choice of the Hilbert basis (3.22) of $L^2(S^+|_N)$ of eigensections of D_N^+ . Let $\rho^{\lambda, \pm}$ and $\theta_\nu^{\lambda, \pm}$ be as ρ and θ_ν in Theorem 2.11, with $q = q_\lambda^\pm$ as in (2.13).

Lemma 3.18. *For all $s > 0$, the Schwartz kernel $\tilde{\lambda}_s^P$ in Lemma 3.17 equals*

$$\begin{aligned} & \sum_{\lambda > 0} \sum_{j=1}^{m_\lambda} \left(\int_{\mathbb{R}} e^{-s\nu} \theta_\nu^{\lambda, +} \otimes \left(\frac{d}{dy} + \lambda e^{-\varphi \circ \xi^{-1}} \right) \theta_\nu^{\lambda, +} d\rho^{\lambda, +}(\nu) \right) \otimes (\varphi_\lambda^j \otimes \varphi_\lambda^j) \\ &+ \sum_{\lambda < 0} \sum_{j=1}^{m_\lambda} \left(\int_{\mathbb{R}} e^{-s\nu} \left(-\frac{d}{dy} + \lambda e^{-\varphi \circ \xi^{-1}} \right) \theta_\nu^{\lambda, -} \otimes \theta_\nu^{\lambda, -} d\rho^{\lambda, -}(\nu) \right) \otimes (\varphi_\lambda^j \otimes \varphi_\lambda^j). \end{aligned} \quad (3.29)$$

Here we identify $S_N^+ \cong (S_N^+)^*$ using the metric, so we view $\varphi_\lambda^j \otimes \varphi_\lambda^j$ as a section of $S_N^+ \boxtimes (S_N^+)^*$.

Proof. We extend the projection (3.3) to a projection

$$P: L^2(\tilde{S}_C^+) \cong L^2(S^+|_N) \otimes L^2(0, \infty) \xrightarrow{P \otimes 1} L^2(S^+|_N)_{>0} \otimes L^2(0, \infty) \hookrightarrow L^2(\tilde{S}_C^+).$$

Then [27, Proposition 3.5] states that

$$e_P^{-s\tilde{D}_C^- \tilde{D}_C^+} \tilde{D}_C^- = e_F^{-s\tilde{D}_C^- \tilde{D}_C^+} \tilde{D}_C^- P + \tilde{D}_C^- e_F^{-s\tilde{D}_C^+ \tilde{D}_C^-} (1 - P). \quad (3.30)$$

For $s > 0$, let $e_F^{-s\tilde{D}_C^\mp \tilde{D}_C^\pm}$ be the heat operator for the Friedrichs extension of

$$\tilde{D}_C^\mp \tilde{D}_C^\pm: \Gamma_c^\infty(\tilde{S}_C^+) \rightarrow L^2(\tilde{S}_C^+).$$

Let $\kappa_s^{F, \pm}$ be its Schwartz kernel. By (2.14), (2.13) and Lemma 3.8, the restriction of $\tilde{D}_C^\mp \tilde{D}_C^\pm$ to $\ker(D_N^+ - \lambda) \otimes L^2(0, \infty)$ equals $\Delta_{q_\lambda^\pm}$. So by Theorem 2.11, the Schwartz kernel $\kappa_{\lambda, s}^{F, \pm}$ of the restriction of $e_F^{-s\tilde{D}_C^\mp \tilde{D}_C^\pm}$ to $\ker(D_N^+ - \lambda) \otimes L^2(0, \infty)$ is

$$\kappa_{\lambda, s}^{F, \pm} = \sum_{j=1}^{m_\lambda} \left(\int_{\mathbb{R}} e^{-s\nu} \theta_\nu^{\lambda, \pm} \otimes \theta_\nu^{\lambda, \pm} d\rho^{\lambda, \pm}(\nu) \right) \otimes (\varphi_\lambda^j \otimes \varphi_\lambda^j).$$

(Note that $\sum_{j=1}^{m_\lambda} \varphi_\lambda^j \otimes \varphi_\lambda^j$ is the identity operator on $\ker(D_N^+ - \lambda)$.) The Schwartz kernel of $\tilde{D}_C^- e_F^{-s\tilde{D}_C^+ \tilde{D}_C^-}$ equals \tilde{D}_C^- applied to the first entry of $\kappa_s^{F,-}$, so its restriction to $\ker(D_N^+ - \lambda) \otimes L^2([0, \infty))$ is

$$\sum_{j=1}^{m_\lambda} \left(\int_{\mathbb{R}} e^{-s\nu} \left(-\frac{d}{dy} + \lambda e^{-\varphi \circ \xi^{-1}} \right) \theta_\nu^{\lambda,-} \otimes \theta_\nu^{\lambda,-} d\rho^{\lambda,-}(\nu) \right) \otimes (\varphi_\lambda^j \otimes \varphi_\lambda^j). \quad (3.31)$$

The Schwartz kernel of $e_F^{-s\tilde{D}_C^- \tilde{D}_C^+} \tilde{D}_C^-$ equals the adjoint \tilde{D}_C^+ of \tilde{D}_C^- applied to the second entry of $\kappa_s^{F,+}$, so its restriction to $\ker(D_N^+ - \lambda) \otimes L^2(0, \infty)$ is

$$\sum_{j=1}^{m_\lambda} \left(\int_{\mathbb{R}} e^{-s\nu} \theta_\nu^{\lambda,+} \otimes \left(\frac{d}{dy} + \lambda e^{-\varphi \circ \xi^{-1}} \right) \theta_\nu^{\lambda,+} d\rho^{\lambda,+}(\nu) \right) \otimes (\varphi_\lambda^j \otimes \varphi_\lambda^j). \quad (3.32)$$

The claim follows from (3.30) and the expressions (3.31) and (3.32) for the relevant Schwartz kernels on eigenspaces of D_N^+ . \blacksquare

Lemma 3.19. *In the situation of Lemma 3.17, we have for all $a' > a$,*

$$\begin{aligned} \int_N \operatorname{tr}(g\lambda_s^P(g^{-1}n, a'; n, a')) dn &= e^{(1-p)\varphi(a')} \sum_{\lambda \in \operatorname{spec}(D_N^+)} \operatorname{sgn}(\lambda) \operatorname{tr}(g|_{\ker(D_N^+ - \lambda)}) \\ &\times \int_{\mathbb{R}} e^{-s\nu} \theta_\nu^{|\lambda|,+}(\xi(a')) \left(\left(\frac{d}{dy} + |\lambda| e^{-\varphi \circ \xi^{-1}} \right) \theta_\nu^{|\lambda|,+} \right) (\xi(a')) d\rho^{|\lambda|,+}(\nu). \end{aligned} \quad (3.33)$$

Proof. It follows directly from (2.13) that for all $\lambda \in \mathbb{R}$,

$$q_\lambda^+ = q_{-\lambda}^-.$$

This implies that, for all $\lambda \in \mathbb{R}$ and $\nu \in \mathbb{C}$, with notation as in (2.14) and below,

$$\Delta_\lambda^+ = \Delta_{-\lambda}^-, \quad \theta_\nu^{\lambda,+} = \theta_\nu^{-\lambda,-}, \quad \rho^{\lambda,+} = \rho^{-\lambda,-}.$$

The last two relations imply in particular that for all $\lambda < 0$,

$$\theta_\nu^{\lambda,-} = \theta_\nu^{|\lambda|,+}, \quad \rho^{\lambda,-} = \rho^{|\lambda|,+}, \quad -\frac{d}{dy} + \lambda e^{-\varphi \circ \xi^{-1}} = \operatorname{sgn}(\lambda) \left(\frac{d}{dy} + |\lambda| e^{-\varphi \circ \xi^{-1}} \right).$$

These equalities, together with Lemmas 3.17 and 3.18 imply that for all $n \in N$,

$$\begin{aligned} g\lambda_s^P(g^{-1}n, a'; n, a') &= e^{(1-p)\varphi(a')} \sum_{\lambda > 0} \sum_{j=1}^{m_\lambda} \operatorname{sgn}(\lambda) \left(\int_{\mathbb{R}} e^{-s\nu} \theta_\nu^{|\lambda|,+}(\xi(a')) \left(\frac{d}{dy} + |\lambda| e^{-\varphi \circ \xi^{-1}} \right) \right. \\ &\quad \left. \times \theta_\nu^{|\lambda|,+}(\xi(a')) d\rho^{\lambda,+}(\nu) \right) (g\varphi_\lambda^j(g^{-1}n) \otimes \varphi_\lambda^j(n)). \end{aligned}$$

This equality and

$$\sum_{j=1}^{m_\lambda} \int_N \operatorname{tr}(g\varphi_\lambda^j(g^{-1}n) \otimes \varphi_\lambda^j(n)) dn = \sum_{j=1}^{m_\lambda} (g \cdot \varphi_\lambda^j, \varphi_\lambda^j)_{L^2} = \operatorname{tr}(g|_{\ker(D_N^+ - \lambda)})$$

together imply (3.33). \blacksquare

Proof of Proposition 3.3. By Proposition 2.7, the function f_1 in (3.2) equals $e^{-\varphi}$ in our situation. So it follows from Lemma 3.19 that for all $a' > a$

$$\begin{aligned} A_g(D_C, a') &= -e^{-p\varphi(a')} \int_0^\infty \sum_{\lambda \in \text{spec}(D_N^+)} \text{sgn}(\lambda) \text{tr}(g|_{\ker(D_N^+ - \lambda)}) \\ &\quad \times \int_{\mathbb{R}} e^{-s\nu} \theta_\nu^{|\lambda|, +}(\xi(a')) \left(\left(\frac{d}{dy} + |\lambda| e^{-\varphi \circ \xi^{-1}} \right) \theta_\nu^{|\lambda|, +}(\xi(a')) \right) d\rho^{|\lambda|, +}(\nu) ds \\ &= -\frac{1}{2} \eta_g^\varphi(D_N^+, a'). \end{aligned} \tag{3.34}$$

In particular, because $A_g(D_C, a')$ converges by Theorem 3.1, so does $\eta_g^\varphi(D_N^+, a')$.

Vanishing of $\eta_g^\varphi(D_N^+, a')$ when D_N^+ has g -symmetric spectrum around zero follows directly from the definition (2.15): then the term corresponding to $\lambda \in \text{spec}(D_N)$ equals minus the term corresponding to $-\lambda$. \blacksquare

Proposition 3.4 was proved at the start of this section, and Proposition 3.2 was proved at the end of Section 3.6. So Propositions 3.2–3.4 are proved, and the proof of Theorem 2.16 is complete.

4 Cylinders

If $\varphi = 0$, then the metric (2.1) is the cylinder metric $B_N + dx^2$. We show that the cusp contribution $\eta_g^0(D_N^+, a')$ then equals Donnelly's g -delocalised version of the Atiyah–Patodi–Singer η -invariant, for all $a' > a$. The computation in this subsection is a spectral counterpart of the geometric computation in [27, Section 4].

We start by recalling [27, Proposition 5.1].

Proposition 4.1. *Let $(\lambda_j)_{j=1}^\infty$ and $(a_j)_{j=1}^\infty$ be sequences in \mathbb{R} such that $|\lambda_1| > 0$, and $|\lambda_j| \leq |\lambda_{j+1}|$ for all j , and such that there are $c_1, c_2, c_3, c_4 > 0$ such that for all j ,*

$$|\lambda_j| \geq c_1 j^{c_2}, \quad |a_j| \leq c_3 j^{c_4}.$$

Then for all $a' > 0$,

$$\int_0^\infty \sum_{j=1}^\infty \text{sgn}(\lambda_j) a_j \frac{e^{-\lambda_j^2 s} e^{-a'^2/s}}{\sqrt{s}} \left(\frac{a'}{s} - |\lambda_j| \right) ds = 0.$$

For a function $f \in L^1(\mathbb{R})$, we write

$$\check{f}(x) := \int_{\mathbb{R}} e^{ix\zeta} f(\zeta) d\zeta$$

for its inverse Fourier transform (up to a possible power of 2π).

Lemma 4.2. *Let $f \in L^1(\mathbb{R})$, and $\alpha, \beta \in \mathbb{R}$. If f is even, then*

$$\int_0^\infty \sin(\alpha\mu) \sin(\beta\mu) f(\mu) d\mu = \frac{1}{4} (-\check{f}(\alpha + \beta) + \check{f}(\alpha - \beta)).$$

If f is odd, then

$$\int_0^\infty \sin(\alpha\mu) \cos(\beta\mu) f(\mu) d\mu = \frac{1}{4i} (\check{f}(\alpha + \beta) + \check{f}(\alpha - \beta)).$$

Lemma 4.3. *If $\varphi = 0$, then the spectral measure $d\rho^{|\lambda|,+}$ in (2.15) equals*

$$d\rho^{|\lambda|,+}(\nu) = \begin{cases} \frac{1}{\pi} \sqrt{\nu - \lambda^2} d\nu & \text{if } \nu \geq \lambda^2, \\ 0 & \text{if } \nu < \lambda^2. \end{cases}$$

Proof. The proof is analogous to the computation in [41, Section 4.1] in the Neumann case.

With notation as in Proposition 2.12, we now have

$$\theta_1(x, \nu) = \frac{1}{\mu} \sin(\mu x), \quad \theta_2(x, \nu) = -\cos(\mu x),$$

with $\mu := \sqrt{\nu - \lambda^2}$. If ν has positive imaginary part, then we choose the square root with positive real and imaginary parts. Then $f(\nu) = -i\mu$ has negative imaginary part, and

$$\theta_2(x, \nu) + f(\nu)\theta_1(x, \nu) = -e^{i\mu x}$$

defines a function in $L^2([0, \infty))$ if $\text{Im}(\nu) > 0$. With our choice of square roots, we have for all $\nu \in \mathbb{R}$,

$$\lim_{\delta \downarrow 0} -\text{Im}(f(\nu + i\delta)) = \begin{cases} \mu & \text{if } \nu \geq \lambda^2, \\ 0 & \text{if } \nu < \lambda^2. \end{cases}$$

This implies the claim via Proposition 2.12. ■

The g -delocalised η -invariant of D_N^+ [21, 26, 30] is

$$\eta_g(D_N^+) = \frac{1}{\sqrt{\pi}} \int_0^\infty \text{Tr} (g D_N^+ e^{-s(D_N^+)^2}) \frac{1}{\sqrt{s}} ds.$$

If $g = e$, this equals the classical η -invariant of D_N^+ .

Proposition 4.4. *If $\varphi = 0$, then for all $a' > a$*

$$\eta_g^0(D_N^+, a') = \eta_g(D_N^+).$$

Proof. We apply the definition (2.15) of cusp contributions with $\varphi = 0$. We look for solutions of $-\theta'' + \lambda^2 = \nu\theta$ satisfying

$$\theta(0, \nu) = 0, \quad \theta'(0, \nu) = 1,$$

and find the eigenfunctions $\theta_\nu^{|\lambda|,+}(y) = \frac{1}{\sqrt{\nu - \lambda^2}} \sin(\sqrt{\nu - \lambda^2}y)$. Let $a' > a$, and set $a'' := a' - a = \xi(a')$. Then by Lemma 4.3, (2.15) becomes

$$\begin{aligned} \eta_g^0(D_N^+, a') &= \frac{2}{\pi} \int_0^\infty \sum_{\lambda \in \text{Spec}(D_N^+)} \text{sgn}(\lambda) \text{tr} (g|_{\ker(D_N^+ - \lambda)}) \int_{\lambda^2}^\infty e^{-s\nu} \frac{\sin(\sqrt{\nu - \lambda^2}a'')}{\sqrt{\nu - \lambda^2}} \\ &\quad \times \left(\cos(\sqrt{\nu - \lambda^2}a'') + |\lambda| \frac{\sin(\sqrt{\nu - \lambda^2}a'')}{\sqrt{\nu - \lambda^2}} \right) \sqrt{\nu - \lambda^2} d\nu ds. \end{aligned} \quad (4.1)$$

The change of variables $\mu = \sqrt{\nu - \lambda^2}$ and $d\nu = 2\sqrt{\nu - \lambda^2} d\mu$ reduces it to

$$\begin{aligned} \eta_g^0(D_N^+, a') &= \frac{4}{\pi} \int_0^\infty \sum_{\lambda \in \text{Spec}(D_N^+)} \text{sgn}(\lambda) \text{tr} (g|_{\ker(D_N^+ - \lambda)}) \\ &\quad \times \int_0^\infty e^{-s(\mu^2 + \lambda^2)} \sin(\mu a'') (\mu \cos(\mu a'') + |\lambda| \sin(\mu a'')) d\mu ds. \end{aligned}$$

Applying Lemma 4.2 and using $f(\mu) = e^{-s\mu^2}$, so $\check{f}(x) = \sqrt{\frac{\pi}{s}}e^{-\frac{x^2}{4s}}$ and $(\mu \mapsto \mu f(\mu))^\vee = \frac{1}{i}(\check{f})'$, we have

$$\begin{aligned} \int_0^\infty \sin^2(\mu a'') e^{-s\mu^2} d\mu &= \frac{\sqrt{\pi}}{4\sqrt{s}} (1 - e^{-\frac{a''^2}{s}}), \\ \int_0^\infty \sin(\mu a'') \cos(\mu a'') \mu e^{-s\mu^2} d\mu &= \frac{\sqrt{\pi} a''}{4s^{\frac{3}{2}}} e^{-\frac{a''^2}{s}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \eta_g^0(D_N^+, a') &= \frac{1}{\sqrt{\pi}} \int_0^\infty \sum_{\lambda \in \text{spec}(D_N^+)} \text{sgn}(\lambda) \text{tr}(g|_{\ker(D_N^+ - \lambda)}) e^{-s\lambda^2} \left(\frac{a''}{s^{\frac{3}{2}}} e^{-\frac{a''^2}{s}} + \frac{|\lambda|}{\sqrt{s}} - \frac{|\lambda|}{\sqrt{s}} e^{-\frac{a''^2}{s}} \right) ds \\ &= \frac{1}{\sqrt{\pi}} \int_0^\infty \sum_{\lambda \in \text{spec}(D_N^+)} \text{tr}(g|_{\ker(D_N^+ - \lambda)}) \frac{e^{-s\lambda^2} \lambda}{\sqrt{s}} ds \\ &\quad - \frac{1}{\sqrt{\pi}} \int_0^\infty \sum_{\lambda \in \text{spec}(D_N^+)} \text{sgn}(\lambda) \text{tr}(g|_{\ker(D_N^+ - \lambda)}) \frac{e^{-s\lambda^2} e^{-\frac{a''^2}{s}}}{\sqrt{s}} \left(\frac{a''}{s} - |\lambda| \right) ds. \end{aligned} \quad (4.2)$$

The first term equals $\eta_g(D_N^+)$. For the second term, we use Proposition 4.1, and take λ_j to be the j th eigenvalue of D_N^+ (ordered by absolute values), and $a_j := \text{tr}(g|_{\ker(D_N^+ - \lambda_j)})$. Then Weyl's law for D_N^+ shows that λ_j has the growth behaviour assumed in the proposition. And

$$|a_j| \leq \dim(\ker(D_N^+ - \lambda_j))$$

grows at most polynomially by Weyl's law. Hence Proposition 4.1 applies, and implies that the second term in (4.2) is zero. \blacksquare

Remark 4.5. By Proposition 4.4, the cusp contribution $\eta_g^0(D_N^+, a')$ is independent of $a' > a$ in this case. Furthermore, we see directly from (4.1) that a version of $\eta_g^0(D_N^+, a')$ with a' replaced by a equals zero. This illustrates the fact that the limit on the right-hand side of (2.17) does not equal the expression (2.15) with a' replaced by a .

As a consequence of Theorem 2.16 and Proposition 4.4, we obtain Donnelly's equivariant APS index theorem [21] for the index of $D|_Z$ with APS boundary conditions at N :

$$\text{index}_G^{\text{APS}}(D|_Z)(g) = \int_{Z^g} \text{AS}_g(D) - \frac{1}{2} \eta_g(D_N^+).$$

Indeed, if D_N^+ is invertible, then the left-hand side equals $\text{index}_G(D)(g)$. In general, one replaces Theorem 2.16 by Theorem 5.3 below.

5 Non-invertible D_N

In this section, we do not assume that D_N is invertible. Because N is compact, D_N^+ has discrete spectrum. Let $\varepsilon > 0$ be such that

$$\text{spec}(D_N^+) \cap (-2\varepsilon, 2\varepsilon) \subset \{0\}. \quad (5.1)$$

Let $w \in C^\infty(M)$ be a function such that for all $x \geq a$ and all $n \in N$,

$$w(n, x) = x. \quad (5.2)$$

Consider the operator

$$D^{\varepsilon w} := e^{-\varepsilon w} D e^{\varepsilon w}. \quad (5.3)$$

This operator equals (2.6) on C , with D_N^+ replaced by the invertible operator $D_N^+ + \varepsilon$. Therefore, much of the proof of Theorem 2.16 applies to $D^{\varepsilon w}$, apart from the fact that this operator and $D_N^+ + \varepsilon$ are not Dirac operators of the form (2.5). This affects the limits as $t \downarrow 0$ of heat operators associated to these operators.

In the proof of Theorem 3.1, given in [27], heat kernel asymptotics were used that may not apply to $D_N^+ + \varepsilon$. We therefore start from a version of this theorem where the hard cutoff between $Z^g \cup (N^g \times (a, a'])$ and the contribution from infinity $A_g(D_C, a')$ is replaced by a smooth cutoff function. Let $\psi \in C^\infty(M)$ be such that

$$\psi|_Z \equiv 1, \quad \psi|_{N \times [a+1, \infty)} \equiv 0. \quad (5.4)$$

For $t > 0$, define

$$A_g^t(D_C, \psi) := \int_0^\infty \int_N \int_a^\infty \text{tr} (g \lambda_s^P (g^{-1} n, x; n, x)) \psi'(x) f_1(x) dx dn ds,$$

and

$$\begin{aligned} \eta_g^{\varphi, t}(D_N^+, \psi) &= -2 \int_t^\infty \int_a^\infty \psi'(x) e^{-p\varphi(x)} \sum_{\lambda \in \text{spec}(D_N^+)} \text{sgn}(\lambda) \text{tr} (g|_{\ker(D_N^+ - \lambda)}) \\ &\quad \times \int_{\mathbb{R}} e^{-sv} \theta_\nu^{|\lambda|, +}(\xi(x)) ((\theta_\nu^{|\lambda|, +})'(\xi(x)) + |\lambda| e^{-\varphi(x)} \theta_\nu^{|\lambda|, +}(\xi(x))) d\rho^{|\lambda|, +}(\nu) dx ds. \end{aligned}$$

We use the following standard regularisation method.

Definition 5.1. For a function $f(t)$ that has an asymptotic expansion in t as $t \downarrow 0$, the *regularised limit* $\text{LIM}_{t \downarrow 0} f(t)$ is the coefficient of t^0 in such an asymptotic expansion.

The *regularised g -delocalised φ -cusp contribution* associated to $D_N^+ + \varepsilon$ and ψ is

$$\eta_g^{\varphi, \text{reg}}(D_N^+ + \varepsilon, \psi) := \text{LIM}_{t \downarrow 0} \eta_g^{\varphi, t}(D_N^+ + \varepsilon, \psi).$$

The condition (2.8) with b replaced by ε implies that $D^{\varepsilon w}$ is Fredholm, via Lemma 3.10. Theorem 4.14 in [27] then becomes

$$\text{index}_G(D^{\varepsilon w})(g) = \text{LIM}_{t \downarrow 0} (\text{Tr} (g \circ e^{-t \tilde{D}_{\varepsilon w}^- \tilde{D}_{\varepsilon w}^+} \psi) - \text{Tr} (g \circ e^{-t \tilde{D}_{\varepsilon w}^+ \tilde{D}_{\varepsilon w}^-} \psi) + A_g^t(D_C^{\varepsilon w}, \psi)). \quad (5.5)$$

Here $\tilde{D}_{\varepsilon w}$ is an extension of $D^{\varepsilon w}$ to a closed manifold containing $M \setminus (N \times (a+1, \infty))$, and we used the fact that the left-hand side is independent of t . The proof of (5.5) is a direct analogy of the proof of Theorem 4.14 in [27].

Example 5.2. Suppose that N is the circle, and $D_N^+ = i \frac{d}{d\theta}$. Then $D_N^+ + 1/2$ is invertible and has symmetric spectrum. So

$$A_e^t(D_C^{\varepsilon w}, \psi) = \eta_e^{\varphi, t}(D_N^+ + 1/2, \psi) = 0$$

for all t . Hence (5.5) becomes

$$\text{index}(D^{w/2}) = \text{LIM}_{t \downarrow 0} (\text{Tr} (e^{-t \tilde{D}_{w/2}^- \tilde{D}_{w/2}^+} \psi) - \text{Tr} (e^{-t \tilde{D}_{w/2}^+ \tilde{D}_{w/2}^-} \psi)).$$

Theorem 5.3. *Suppose that M has weakly admissible φ -cusps, where (2.8) holds on an interval (a', ∞) , with $a' > a$, and b replaced by ε . Then $D^{\varepsilon w}$ is Fredholm, and its index is independent of ε and w with the properties mentioned. And for $\psi \in C^\infty(M)$ satisfying (5.4),*

$$\text{index}_G(D^{\varepsilon w})(g) = \int_{M^g} \psi|_{M^g} \text{AS}_g(D) - \frac{1}{2} \lim_{\varepsilon \downarrow 0} \eta_g^{\varphi, \text{reg}}(D_N^+ + \varepsilon, \psi). \quad (5.6)$$

Proof. As noted above (5.5), the operator $D^{\varepsilon w}$ is Fredholm if M has weakly admissible φ -cusps with respect to the spectral gap 2ε of this operator. If $\varepsilon' > 0$ has the same property (5.1) as ε , then

$$D^{\varepsilon w} - D^{\varepsilon' w} = (\varepsilon' - \varepsilon)c(dw).$$

This is a bounded vector bundle endomorphism, so the linear path between $D^{\varepsilon w}$ and $D^{\varepsilon' w}$ is continuous. And all operators on this path are Fredholm, so $\text{index}(D^{\varepsilon w}) = \text{index}(D^{\varepsilon' w})$.

If $w' \in C^\infty(M)$ has the same property (5.2) as w , then

$$D^{\varepsilon w} - D^{\varepsilon w'} = -\varepsilon c(d(w - w')).$$

Because $w - w' = 0$ outside a compact set, $D^{\varepsilon w'}$ is a compact perturbation of $D^{\varepsilon w}$, when viewed as acting on the relevant Sobolev space. Hence $\text{index}(D^{\varepsilon w}) = \text{index}(D^{\varepsilon w'})$. We find that $\text{index}(D^{\varepsilon w})$ is independent of ε and w .

By the arguments leading up to (3.34), with integrals over s replaced by integrals from $t > 0$ to ∞ , we have

$$A_g^t(D_C^{\varepsilon w}, \psi) = -\frac{1}{2} \eta_g^{\varphi, t}(D_N^+ + \varepsilon, \psi).$$

Hence (5.5) becomes

$$\begin{aligned} \text{index}_G(D^{\varepsilon w})(g) &= \text{LIM}_{t \downarrow 0} (\text{Tr}(g \circ e^{-t\tilde{D}_{\varepsilon w}^- \tilde{D}_{\varepsilon w}^+} \psi) - \text{Tr}(g \circ e^{-t\tilde{D}_{\varepsilon w}^+ \tilde{D}_{\varepsilon w}^-} \psi)) \\ &\quad - \frac{1}{2} \eta_g^{\varphi, \text{reg}}(D_N^+ + \varepsilon, \psi). \end{aligned} \quad (5.7)$$

The coefficients of the heat operator $e^{-s\tilde{D}_{\varepsilon w}^2}$ are continuous in ε . And standard heat kernel asymptotics and localisation apply to $e^{-s\tilde{D}_0^2}$, the analogous operator with $\varepsilon = 0$. These imply that

$$\lim_{\varepsilon \downarrow 0} \text{LIM}_{t \downarrow 0} (\text{Tr}(g \circ e^{-t\tilde{D}_{\varepsilon w}^- \tilde{D}_{\varepsilon w}^+} \psi) - \text{Tr}(g \circ e^{-t\tilde{D}_{\varepsilon w}^+ \tilde{D}_{\varepsilon w}^-} \psi)) = \int_{M^g} \psi|_{M^g} \text{AS}_g(D).$$

Because the left-hand side of (5.7) is independent of ε , the claim follows. \blacksquare

Remark 5.4. The arguments of [27, Section 4.5] showing that ψ may be replaced by a step function involve an actual limit $t \downarrow 0$, not the regularised limit $\text{LIM}_{t \downarrow 0}$. For this reason, it is not immediately obvious to us if a version of Theorem 5.3 with ψ replaced by a step function is true.

Example 5.5. If φ is the zero function, then the left-hand side of (5.6) is the equivariant index of the restriction of D to $M \setminus C$, with Atiyah–Patodi–Singer boundary conditions at ∂C . Then for all suitable ψ , a slight modification of the proof of Proposition 4.4 shows that $\eta_g^{\varphi, \text{reg}}(D_N^+ + \varepsilon, \psi)$ is the regularised g -delocalised η -invariant of $D_N^+ + \varepsilon$. Hence

$$\lim_{\varepsilon \downarrow 0} \eta_g^{\varphi, \text{reg}}(D_N^+ + \varepsilon, \psi) = \text{tr}(g|_{\ker(D_N^+)}) + \eta_g^{\text{reg}}(D_N^+).$$

This fact is standard; see, for example, [26, Lemma 6.7].

A Conformal transformations of Dirac operators

Let M be a manifold of dimension p . Let B_0 be a Riemannian metric on M . Let $S \rightarrow M$ be a Clifford module for this metric, with Clifford action $c_0: TM \rightarrow \text{End}(S)$. Fix a Clifford connection ∇^0 on S preserving a Hermitian metric on S , and let $D_0 = c_0 \circ \nabla^0$ be the associated Dirac operator. Let $\varphi \in C^\infty(M)$, and consider the Riemannian metric $B_\varphi := e^{2\varphi} B_0$. We denote the gradient operator for B_0 by grad .

Proposition A.1. *There are a Clifford action c_φ by TM on S , with respect to B_φ , and a Clifford connection ∇^φ on S , with respect to c_φ and B_φ , such that the associated Dirac operator $D_\varphi = c_\varphi \circ \nabla^\varphi$ equals*

$$D_\varphi = e^{-\varphi} \left(D_0 + \frac{p-1}{2} c_0(\text{grad } \varphi) \right) = e^{-\frac{p+1}{2}\varphi} D_0 e^{\frac{p-1}{2}\varphi}. \quad (\text{A.1})$$

Remark A.2. The operator $c_0(\text{grad } \varphi)$ in (A.1) is fibrewise antisymmetric. But the operator D_φ is symmetric with respect to the L^2 -inner product defined with the Riemannian density associated to B_φ . This follows, for example, from Proposition A.1 and the usual argument why Dirac operators are symmetric.

We write $\mathfrak{X}(M)$ for the space of smooth vector fields on M . Let $\nabla^{TM,0}$ be the Levi-Civita connection for B_0 , and let $\nabla^{TM,\varphi}$ be the Levi-Civita connection for B_φ .

Lemma A.3. *For all $v, w \in \mathfrak{X}(M)$,*

$$\nabla_v^{TM,\varphi} w = \nabla_v^{TM,0} w + v(\varphi)w + w(\varphi)v - B_0(v, w) \text{grad } \varphi.$$

Proof. This is a computation based on the Koszul formulas for $\nabla^{TM,\varphi}$ and $\nabla^{TM,0}$. ■

Consider the Clifford action $c_\varphi := e^\varphi c_0$ with respect to B_φ . For $A \in \Omega^1(M; \text{End}(S))$, consider the connection $\nabla^A := \nabla^0 + A$ on S . For $v \in \mathfrak{X}(M)$, let $A_v \in \text{End}(S)$ be the pairing of A and v .

Lemma A.4. *The connection ∇^A is a Clifford connection for c_φ and B_φ if and only if for all $v, w \in \mathfrak{X}(M)$,*

$$[A_v, c_0(w)] = w(\varphi)c_0(v) - B_0(v, w)c_0(\text{grad } \varphi).$$

Proof. For all $v, w \in \mathfrak{X}(M)$,

$$[\nabla_v^A, c_\varphi(w)] = e^\varphi [\nabla_v, c_0(w)] + e^\varphi [A_v, c_0(w)] + e^\varphi v(\varphi)c_0(w). \quad (\text{A.2})$$

And by Lemma A.3,

$$c_\varphi(\nabla_v^{TM,\varphi} w) = e^\varphi c_0(\nabla_v^{TM,0} w) + e^\varphi v(\varphi)c_0(w) + e^\varphi w(\varphi)c_0(v) - e^\varphi B_0(v, w)c_0(\text{grad } \varphi). \quad (\text{A.3})$$

Because ∇^0 is a Clifford connection for c_0 and B_0 , (A.2) and (A.3) are equal if and only if

$$e^\varphi [A_v, c_0(w)] + e^\varphi v(\varphi)c_0(w) = e^\varphi v(\varphi)c_0(w) + e^\varphi w(\varphi)c_0(v) - e^\varphi B_0(v, w)c_0(\text{grad } \varphi). \quad \blacksquare$$

Lemma A.5. *For all $u, v, w \in \mathfrak{X}(M)$,*

$$[c_0(u)c_0(v), c_0(w)] = -2B_0(v, w)c_0(u) + 2B_0(u, w)c_0(v).$$

Proof. This is a straightforward computation, involving the equality

$$c_0(v_1)c_0(v_2) + c_0(v_2)c_0(v_1) = -2B_0(v_1, v_2)$$

for all $v_1, v_2 \in \mathfrak{X}(M)$. ■

Let $f \in C^\infty(M)$, and define $A^{\varphi,f} \in \Omega^1(M; \text{End}(S))$ by

$$A_v^{\varphi,f} := \frac{1}{2}c_0(\text{grad } \varphi)c_0(v) + fB_0(\text{grad } \varphi, v).$$

We write $\nabla^{\varphi,f} := \nabla^{A^{\varphi,f}}$.

Lemma A.6. *For all $f \in C^\infty(M)$, the connection $\nabla^{\varphi,f}$ is a Clifford connection for c_φ and B_φ .*

Proof. Lemma A.5 implies that $A^{\varphi,f}$ satisfies the condition in Lemma A.4. ■

Lemma A.7. *The connection $\nabla^{\varphi,f}$ preserves the metric on S if and only if $f|_{\text{supp}(\text{grad } \varphi)} = \frac{1}{2}$.*

Proof. Because ∇^0 preserves the metric on S , $\nabla^{\varphi,f}$ preserves the same metric if and only if $A_v^{\varphi,f}$ is anti-Hermitian for any vector field v . And because $c_0(w)$ is anti-Hermitian for any vector field w ,

$$(A_v^{\varphi,f})^* = -A^{\varphi,f} + (2f - 1)B_0(\text{grad } \varphi, v). \quad \blacksquare$$

Proof of Proposition A.1. Let c_φ and $\nabla^{\varphi, \frac{1}{2}}$ as defined above, where $f \equiv 1/2$. Then $\nabla^{\varphi, \frac{1}{2}}$ is a Clifford connection and preserves the metric by Lemmas A.6 and A.7.

Let $\{e_1, \dots, e_p\}$ be a local orthonormal frame for TM with respect to B_0 . Then the frame $\{e^{-\varphi}e_1, \dots, e^{-\varphi}e_p\}$ is a local orthonormal frame for TM with respect to B_φ . So

$$D_\varphi = \sum_{j=1}^p c_\varphi(e^{-\varphi}e_j) \nabla_{e^{-\varphi}e_j}^{\varphi, \frac{1}{2}} = e^{-\varphi} \sum_{j=1}^p c_0(e_j) \nabla_{e_j}^0 + e^{-\varphi} \sum_{j=1}^p c_0(e_j) A_{e_j}^{\varphi, \frac{1}{2}}.$$

The first term on the right-hand side equals $e^{-\varphi}D_0$, and the second term equals

$$\begin{aligned} & \frac{1}{2}e^{-\varphi} \sum_{j=1}^p c_0(e_j)c_0(\text{grad } \varphi)c_0(e_j) + \frac{1}{2}e^{-\varphi} \sum_{j=1}^p c_0(e_j)B_0(\text{grad } \varphi, e_j) \\ &= \frac{1}{2}e^{-\varphi} \sum_{j=1}^p (c_0(\text{grad } \varphi) - 2B_0(e_j, \text{grad } \varphi)c_0(e_j)) + \frac{1}{2}c_0(\text{grad } \varphi) \\ &= \frac{p-1}{2}e^{-\varphi}c_0(\text{grad } \varphi). \quad \blacksquare \end{aligned}$$

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