Quantum Toroidal Comodule Algebra of Type A_{n-1} and Integrals of Motion

Boris FEIGIN [ab]*, Michio JIMBO* [c] *and Evgeny MUKHIN* [d]

[a)] *National Research University Higher School of Economics,*
  20 Myasnitskaya Str., Moscow, 101000, Russia

[b)] *Landau Institute for Theoretical Physics, 1a Akademika Semenova Ave.,*
  Chernogolovka, 142432, Russia
  E-mail: *bfeigin@gmail.com*

[c)] *Department of Mathematics, Rikkyo University, Toshima-ku, Tokyo 171-8501, Japan*
  E-mail: *jimbomm@rikkyo.ac.jp*

[d)] *Department of Mathematics, Indiana University Purdue University Indianapolis,*
  402 N. Blackford St., LD 270, Indianapolis, IN 46202, USA
  E-mail: *emukhin@iupui.edu*

Abstract. We introduce an algebra $\mathcal{K}_n$ which has a structure of a left comodule over the quantum toroidal algebra of type A_{n-1}. Algebra $\mathcal{K}_n$ is a higher rank generalization of $\mathcal{K}_1$, which provides a uniform description of deformed W algebras associated with Lie (super)algebras of types BCD. We show that $\mathcal{K}_n$ possesses a family of commutative subalgebras.

Key words: quantum toroidal algebras; comodule; integrals of motion

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1 Introduction

Integrable systems originated in quantum toroidal algebras have been drawing a lot of attention in recent years, see [1, 4, 5, 8, 9, 10, 11, 14, 15, 16], to name a few. In this paper we continue the study launched in [7] concerning deformed W algebras and a quantum toroidal version of integrable models with reflections in the spirit of [21].

The work [7] is based on a new algebra $\mathcal{K}_1$ generated by current $E(z)$, Heisenberg half-currents $K^{\pm}(z)$, and a central element C. In various representations the current $E(z)$ recovers the fundamental current of known deformed W algebras of types BCD as well as their supersymmetric analogs. It was shown that $\mathcal{K}_1$ has a family of commuting elements $\{\mathbf{I}_N\}_{N=1}^{\infty}$ called integrals of motion. The element $\mathbf{I}_N$ is written as an N-fold integral of the product $\mathbf{E}(z_N)\cdots\mathbf{E}(z_1)$ with the explicit elliptic kernel of [19]. Here $\mathbf{E}(z)$ is the dressed current of the form $\mathbf{E}(z) = E(z)\mathbf{K}^{+}(z)^{-1}$, where $\mathbf{K}^{+}(z)$ is given in terms of $K^{+}(z)$. In the representations corresponding to deformed W algebras, the integrals of motion $\mathbf{I}_N$ are deformations of local integrals of motion in the W algebras, see [2].

In the present paper we generalize this construction by introducing algebras $\mathcal{K}_n$ for $n > 1$. The algebra $\mathcal{K}_n$ depends on parameters $q_1, q_2 = q^2$, and is generated by currents $E_1(z), \ldots, E_n(z)$, Heisenberg half-currents $K_1^{\pm}(z), \ldots, K_n^{\pm}(z)$, and a central element C. We show that it has the following key properties:

- Algebra $\mathcal{K}_n$ has essentially the same size as the Borel subalgebra of quantum toroidal $\mathfrak{gl}_n$ algebra $\mathcal{E}_n$; see Remark 3.1.

- For $m < n$, algebra $\mathcal{K}_m$ can be obtained as a subalgebra of $\mathcal{K}_n$ using the fusion procedure in the same way as $\mathcal{E}_m$ is found as a subalgebra of $\mathcal{E}_n$ in [3]; see Proposition 6.2.

- Algebra $\mathcal{K}_n$ is a left comodule over quantum toroidal algebra $\mathcal{E}_n$; see Theorem 3.4.

- Algebra $\mathcal{K}_n$ has two "boundary" modules $\mathbb{F}^D$ of level $C = q^{-1}$ and $\mathbb{F}^B$ of level $C = q^{1/2}$; see Lemmas 3.5 and 3.6.

- Algebra $\mathcal{K}_n$ has a family of commuting integrals of motion $\{\mathbf{I}_N\}_{N=1}^{\infty}$, such that $\mathbf{I}_N$ is an nN-fold integral of a current given by an ordered product of the form $\prod_{1 \leq i \leq n}^{\curvearrowright} \prod_{1 \leq a \leq N}^{\curvearrowright} \mathbf{E}_i(z_a)$ together with the kernel of [12]; see (4.9) and Theorem 4.2. Here the dressed currents have the form $\mathbf{E}_i(z) = \mathbf{K}_i^-(z)^{-1} E_i(z)$, where $\mathbf{K}_i^-(z)$ is given in terms of $K_i^-(z)$.[1]

Here we should make a disclaimer: the relations of $\mathcal{K}_n$ involve infinite sums, so that an appropriate completion is necessary. The same is true with the fusion procedure. Also the comodule map is defined at the level of a class of representations called admissible ones.

These properties ensure that $\mathcal{K}_n$ has a rich representation theory. At least, we have representations of the form $V \otimes W$, where W is one of the two boundary modules and V is any admissible $\mathcal{E}_n$ module. Here by an admissible module we mean a homogeneously graded module with the degree bounded from above; for example any tensor product of Fock modules (or, more generally, modules with highest weight with respect to the "rotated" set of generators) is admissible. Then, under a mild restriction on parameters q_1, q_2 and the level C, in any $\mathcal{K}_n$ module we have an integrable system given by action of $\{\mathbf{I}_N\}$. This is the principal result of this paper. For $n = 1$, this system is related to the system with reflections, see [7, Section 5.1], and the Bethe ansatz for it is studied in [16] in the setting of affine Yangians. Although the expressions of $\{\mathbf{I}_N\}$ are non-local, we shall sometimes refer to them as "local" integrals of motion, since in the case $n = 1$ they are deformations of local ones as already mentioned.

We do not completely understand the nature of algebras $\mathcal{K}_n$. The $\mathcal{K}_n$ algebras we discuss here are of A type. Similar algebras can be defined in the same way for arbitrary simply laced types, though the construction of integrals of motion is more challenging. At first glance, algebra $\mathcal{K}_n$ bears resemblance to $\imath$quantum groups; formula (3.16) for comodule structure is reminiscent of the Chevalley generators [13, formula (5.1)], and the defining relations (3.1)–(3.6) of $\mathcal{K}_n$ look similar to the Drinfeld realization given in [17, Theorem 5.1]. However, the details seem to be very different. For example, the relations in $\mathcal{K}_n$ are invariant under scaling $z \to cz$ but such homogeneity does not exist in those of [17]. The current appearing in the residues in the right-hand side of (5.5) in [17] is commutative, whereas our current $K_i^-(z)K_i^+(C^2 z)$ in (3.4) is not. The $\imath$quantum groups are coideals in the corresponding quantum groups, while we are not aware of inclusion of $\mathcal{K}_n$ to $\mathcal{E}_n$. We do not know if $\imath$quantum groups have interesting families of integrals of motion in general. Natural candidates which could be compared to $\mathcal{K}_n$ are the reflection algebras and, in particular, quantum twisted Yangians, see [18]. The quantum twisted Yangians are coideal subalgebras behind integrable systems with reflections. However, the exact relation to $\mathcal{K}_n$ algebras is unclear. The quantum twisted Yangians are given in terms of the R-matrix realization (as opposed to the Drinfeld-type realization of $\mathcal{K}_n$) and the formulas for the generators of the coideal are quadratic in generators of quantum group.

In quantum toroidal algebras $\mathcal{E}_n$, the integrals of motion originate from the standard construction of transfer matrices and depend on n arbitrary parameters. They commute for a simple reason following from the Yang–Baxter equation. Then a computation provides the expression

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[1]We change the convention slightly from [7].

for the integrals of motion in the form of integrals. The integrals of motion in the $\mathcal{K}_n$ algebra are given virtually by the same formula as in $\mathcal{E}_n$, except that one of the parameters (the elliptic nome) is fixed by the central element C. But we do not have any general reason for their commutativity; we simply check it directly.

It is known that in type A there is a natural duality of such models of $(\mathfrak{gl}_m, \mathfrak{gl}_n)$ type, see [6]. Under that duality, the algebras $\mathcal{E}_m$ and $\mathcal{E}_n$ act on the same space in such a way that the screening operators for one algebra are given in terms of generating currents of the other. Furthermore the integrals of motion of both algebras commute with each other. For $\mathcal{K}_1$ acting in a representation, we have a system of screening operators, see [7]. We expect that the algebras generated by the corresponding screening currents also possess a commutative family which should be called "non-local" integrals of motion, and that "local" and "non-local" integrals of motion commute.

Similarly, we expect that algebras $\mathcal{K}_n$ acting in representations commute with a family of screening operators. Moreover the algebra generated by the screening currents does not depend on n. For the case of the tensor product of ℓ Fock $\mathcal{E}_n$ modules with a boundary $\mathcal{K}_n$ module, we anticipate $\ell + 1$ screening currents each given as a sums of n vertex operators. The corresponding "non-local" integrals of motion should commute with the "local" integrals of motion $\{\mathbf{I}_N\}$ constructed in this paper. We plan to address this issue in future publications.

The text is organized as follows.

In Section 2 we introduce our convention about the quantum toroidal algebra $\mathcal{E}_n$ of type $\mathfrak{gl}_n$. In Section 3 we define algebra $\mathcal{K}_n$. Working with admissible representations, we rephrase the Serre relations as zero conditions (called wheel conditions) on matrix coefficients of products of generating currents. We show that on admissible representations, $\mathcal{K}_n$ may be viewed as a left comodule over $\mathcal{E}_n$. We use this fact to construct representations starting with elementary ones. In Section 4 we introduce integrals of motion. They are integrals of products of generating currents of $\mathcal{K}_n$ obtained by a dressing procedure. The kernel functions entering integrals of motion are precisely the same as those used for those of $\mathcal{E}_n$. We then prove that integrals of motion mutually commute by direct computation.

For simplicity of presentation, we treat the case $n \geq 3$ in the body of the text. We shall mention the modifications necessary for $n = 1, 2$ in Section 5.

In Section 6 we show that, for any $k = 1, \ldots, n-1$, algebra $\mathcal{K}_n = \mathcal{K}_n(q_1, q_2, q_3)$ with parameters q_1, q_2, q_3 contains mutually commuting subalgebras $\mathcal{K}_k(\bar{q}_1, \bar{q}_2, \bar{q}_3)$ and $\mathcal{K}_{n-k}(\bar{\bar{q}}_1, \bar{\bar{q}}_2, \bar{\bar{q}}_3)$ with appropriate parameters $\bar{q}_i$, $\bar{\bar{q}}_i$.

In Appendix A, we give a proof of the theta function identities used in the commutativity of integrals of motion.

Notation 1.1. Throughout the text we fix parameters $q^{1/2}, d^{1/2} \in \mathbb{C}^\times$ and define

$$q_1 = q^{-1}d, \qquad q_2 = q^2, \qquad q_3 = q^{-1}d^{-1},$$

so that $q_1 q_2 q_3 = 1$. We assume that $q_1^i q_2^j = 1$ for $i, j \in \mathbb{Z}$ implies $i = j = 0$.

For a positive integer N, we write $\delta_{i,k}^{(N)} = 1$ if $i \equiv k \bmod N$ and $\delta_{i,k}^{(N)} = 0$ otherwise. We use the standard symbols for infinite products

$$(z_1, \ldots, z_r; p)_\infty = \prod_{i=1}^{r} \prod_{k=0}^{\infty} \left(1 - z_i p^k\right), \qquad \Theta_p(z) = \left(z, pz^{-1}, p; p\right)_\infty.$$

We use also the formal series $\delta(z) = \sum_{k \in \mathbb{Z}} z^k$.

2   Preliminaries

In this section we fix our convention regarding the quantum toroidal algebra $\mathcal{E}_n$ of type $\mathfrak{gl}_n$.

2.1   Quantum toroidal algebra $\mathcal{E}_n$

Fix a positive integer $n \geq 3$. We define $g_{i,j}(z,w)$, $G_{i,j}(x)$ for $i, j \in \mathbb{Z}/n\mathbb{Z}$ as follows

$$
g_{i,i}(z,w) = z - q^2 w, \qquad G_{i,i}(x) = q^2 \frac{1 - q^{-2}x}{1 - q^2 x},
$$

$$
g_{i,i\pm1}(z,w) = z - q^{-1}d^{\pm1}w, \qquad G_{i,i\pm1}(x) = q^{-1}\frac{1 - qd^{\pm1}x}{1 - q^{-1}d^{\pm1}x},
$$

$$
g_{i,j}(z,w) = z - w, \qquad G_{i,j}(x) = 1 \qquad \text{if} \quad j \not\equiv i, i\pm1.
$$

Here $G_{i,j}(x)$ should be understood as power series expansion in x. As rational function we have

$$
G_{i,j}(w/z) = -\frac{g_{j,i}(w,z)}{d_{i,j}g_{i,j}(z,w)} = G_{j,i}(z/w)^{-1},
$$

where $d_{i,j} = d^{\pm1}$, $i \equiv j \pm 1$, and $d_{i,j} = 1$ otherwise.

Let P be a free $\mathbb{Z}$ module with basis $\{\varepsilon_i \mid i \in \mathbb{Z}/n\mathbb{Z}\}$, equipped with a symmetric bilinear form $(\, , \,) \colon P \times P \to \mathbb{Z}$ such that ε_i are orthonormal. We write $\bar{\alpha}_i = \varepsilon_{i-1} - \varepsilon_i$.

The quantum toroidal algebra $\mathcal{E}_n$ of type $\mathfrak{gl}_n$ is a unital associative algebra generated by

$$
e_{i,k}, \quad f_{i,k}, \quad h_{i,r}, \quad i \in \mathbb{Z}/n\mathbb{Z}, \ k \in \mathbb{Z}, \ r \in \mathbb{Z}\backslash\{0\}, \qquad q^h, \quad h \in P, \qquad C^{\pm1}.
$$

In terms of the generating series

$$
e_i(z) = \sum_{k\in\mathbb{Z}} e_{i,k} z^{-k}, \qquad f_i(z) = \sum_{k\in\mathbb{Z}} f_{i,k} z^{-k},
$$

$$
\psi_i^{\pm}(z) = \psi_{i,0}^{\pm} \exp\left(\pm(q - q^{-1})\sum_{r>0} h_{i,\pm r} z^{\mp r}\right), \qquad \psi_{i,0}^{\pm} = q^{\pm\bar{\alpha}_i},
$$

the defining relations read as follows

$$
q^h q^{h'} = q^{h+h'}, \quad h, h' \in P, \qquad q^0 = 1, \qquad C \text{ is central},
$$

$$
q^h e_i(z) q^{-h} = q^{(h,\bar{\alpha}_i)} e_i(z), \quad q^h f_i(z) q^{-h} = q^{-(h,\bar{\alpha}_i)} f_i(z), \quad q^h \psi_i^{\pm}(z) q^{-h} = \psi_i^{\pm}(z), \quad h \in P,
$$

$$
\psi_i^{\pm}(z)\psi_j^{\pm}(w) = \psi_j^{\pm}(w)\psi_i^{\pm}(z),
$$

$$
\psi_i^{+}(z)\psi_j^{-}(w) = \psi_j^{-}(w)\psi_i^{+}(z)G_{i,j}(Cw/z)^{-1}G_{i,j}(C^{-1}w/z),
$$

$$
\psi_i^{+}(z)e_j(w) = e_j(w)\psi_i^{+}(z)G_{i,j}(C^{-1}w/z), \qquad e_j(w)\psi_i^{-}(z) = \psi_i^{-}(z)e_j(w)G_{j,i}(z/w),
$$

$$
\psi_i^{+}(z)f_j(w) = f_j(w)\psi_i^{+}(z)G_{i,j}(w/z)^{-1}, \qquad f_j(w)\psi_i^{-}(z) = \psi_i^{-}(z)f_j(w)G_{j,i}(Cz/w)^{-1},
$$

$$
[e_i(z), f_j(w)] = \frac{\delta_{i,j}}{q - q^{-1}}\left(\delta(Cw/z)\psi_i^{+}(w) - \delta(Cz/w)\psi_i^{-}(z)\right),
$$

$$
[e_i(z), e_j(w)] = 0, \qquad [f_i(z), f_j(w)] = 0, \qquad i \not\equiv j, j \pm 1,
$$

$$
d_{i,j}g_{i,j}(z,w)e_i(z)e_j(w) + g_{j,i}(w,z)e_j(w)e_i(z) = 0,
$$

$$
d_{j,i}g_{j,i}(w,z)f_i(z)f_j(w) + g_{i,j}(z,w)f_j(w)f_i(z) = 0,
$$

$$
\mathop{\mathrm{Sym}}_{z_1,z_2}[e_i(z_1), [e_i(z_2), e_{i\pm1}(w)]_q]_{q^{-1}} = 0,
$$

$$
\mathop{\mathrm{Sym}}_{z_1,z_2}[f_i(z_1), [f_i(z_2), f_{i\pm1}(w)]_q]_{q^{-1}} = 0.
$$

Here $[A, B]_p = AB - pBA$, and the symbol Sym stands for symmetrization:

$$\operatorname*{Sym}_{x_1,\ldots,x_N} f(x_1,\ldots,x_N) = \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} f(x_{\sigma(1)},\ldots,x_{\sigma(N)}).$$

The relations above imply in particular

$$[h_{i,r}, h_{j,s}] = \delta_{r+s,0} \cdot a_{i,j}(r) \frac{C^r - C^{-r}}{q - q^{-1}}, \qquad r, s \neq 0,$$

$$a_{i,j}(r) = \frac{1}{r} \frac{q^r - q^{-r}}{q - q^{-1}} \left((q^r + q^{-r}) \delta_{i,j}^{(n)} - d^r \delta_{i,j-1}^{(n)} - d^{-r} \delta_{i,j+1}^{(n)} \right). \tag{2.1}$$

Algebra $\mathcal{E}_n$ has a $\mathbb{Z}$ grading which we call homogeneous grading, given by

$$\deg x_{i,r} = r, \quad x = e, f, h, \qquad \deg q^h = \deg C = 0.$$

2.2 Coproduct

Quite generally, let $\mathcal{A}$ be a $\mathbb{Z}$ graded algebra with a central element C. The completion of $\mathcal{A}$ in the positive direction is the algebra $\tilde{\mathcal{A}}$, linearly spanned by products of series of the form $\sum_{k=M}^{\infty} f_k g_k$, where $M \in \mathbb{Z}$, $f_k, g_k \in \mathcal{A}$ and $\deg g_k = k$. We call an $\mathcal{A}$ module V admissible if C is diagonalizable, and if V is $\mathbb{Z}$ graded with finite dimensional components of degree bounded from above: $V = \oplus_{k=-\infty}^{N} V_k$, where $V_k = \{v \in V \mid \deg v = k\}$, $\dim V_k < \infty$. The completion $\tilde{\mathcal{A}}$ acts on all admissible modules.

Let $\mathcal{E}_n \tilde{\otimes} \mathcal{E}_n$ be the tensor algebra $\mathcal{E}_n \otimes \mathcal{E}_n$ completed in the positive direction. We use the topological coproduct $\Delta \colon \mathcal{E}_n \to \mathcal{E}_n \tilde{\otimes} \mathcal{E}_n$

$$\Delta e_i(z) = e_i(C_2 z) \otimes \psi_i^-(z) + 1 \otimes e_i(z),$$
$$\Delta f_i(z) = f_i(z) \otimes 1 + \psi_i^+(z) \otimes f_i(C_1 z),$$
$$\Delta \psi_i^+(z) = \psi_i^+(z) \otimes \psi_i^+(C_1 z),$$
$$\Delta \psi_i^-(z) = \psi_i^-(C_2 z) \otimes \psi_i^-(z),$$
$$\Delta x = x \otimes x, \qquad x = q^h, C.$$

Here $C_1 = C \otimes 1$ and $C_2 = 1 \otimes C$.

Later on we shall use the elements in the completed algebra $\tilde{\mathcal{E}}_n$

$$\tilde{f}_i(z) = S^{-1}\big(f_i(z)\big) = -f_i\big(C^{-1}z\big)\psi_i^+\big(C^{-1}z\big)^{-1},$$

where S denotes the antipode. The $\tilde{f}_i(z)$'s satisfy the same quadratic relations and the Serre relations as do $e_i(z)$'s. In addition

$$d_{j,i} g_{j,i}(w, z) e_i(z) \tilde{f}_j(w) + g_{i,j}(z, w) \tilde{f}_j(w) e_i(z)$$
$$= -\delta_{i,j} \frac{g_{i,i}(w, z)}{q - q^{-1}} \big(\delta(w/z) - \delta\big(C^2 z/w\big) \psi_i^-(z) \psi_i^+(Cz)^{-1} \big). \tag{2.2}$$

2.3 Fock modules

It is well known [20] that $\mathcal{E}_n$ has admissible representations given in terms of vertex operators. Consider a vector space with basis $|\mathbf{m}\rangle$ labeled by n tuples of integers $\mathbf{m} = (m_0, m_1, \ldots, m_{n-1})$ $\in \mathbb{Z}^n$. We define linear operators $\mathrm{e}^{\pm \epsilon_i}$ and ∂_i, $0 \leq i \leq n-1$, by

$$\mathrm{e}^{\pm \epsilon_i} |\mathbf{m}\rangle = (-1)^{\sum_{s=0}^{i-1} m_s} |\mathbf{m} \pm \mathbf{1}_i\rangle, \qquad \partial_i |\mathbf{m}\rangle = m_i |\mathbf{m}\rangle,$$

with $\mathbf{1}_i = (\delta_{i,j})_{0 \le j \le n-1}$. We have $e^{\epsilon_i} e^{\epsilon_j} = -e^{\epsilon_j} e^{\epsilon_i}$ for $i \ne j$. We set

$$\mathbb{F} = \mathbb{C}[\{h_{i,-r}\}_{r>0, 0 \le i \le n-1}] \otimes \left(\bigoplus_{\mathbf{m} \in \mathbb{Z}^n} \mathbb{C}|\mathbf{m}\rangle \right).$$

We write operators $h_{i,\pm r} \otimes \mathrm{id}$, $\mathrm{id} \otimes e^{\pm \epsilon_i}$ and $\mathrm{id} \otimes \partial_i$ simply as $h_{i,\pm r}$, $e^{\pm \epsilon_i}$ and ∂_i. We extend the range of the suffix i by periodicity, e.g., $\partial_{i+n} = \partial_i$. Define a vertex operator which acts on $\mathbb{F}$

$$V_i(z) = e^{-\epsilon_i} e^{\epsilon_{i-1}} z^{\partial_{i-1}-\partial_i} d^{(\partial_{i-1}+\partial_i)/2} \exp\left(\sum_{r>0} \frac{q^{-r}}{[r]} h_{i,-r} z^r \right) \exp\left(-\sum_{r>0} \frac{1}{[r]} h_{i,r} z^{-r} \right).$$

For $u \in \mathbb{C}^\times$, the following assignment gives $\mathbb{F}$ a structure of an admissible $\mathcal{E}_n$ module denoted by $\mathbb{F}(u)$,

$$e_i(z) \mapsto u^{-\delta_{i,0}} z V_i(z), \qquad \tilde{f}_i(z) \mapsto -q^{-1} u^{\delta_{i,0}} z \, {:} V_i(z)^{-1} {:},$$

$$h_{i,r} \mapsto h_{i,r}, \quad r \ne 0, \qquad q^{\varepsilon_i} \mapsto q^{\partial_i}, \qquad C \mapsto q.$$

Here we use the usual normal ordering symbol to bring $e^{\pm \epsilon_i}$, $h_{i,-r}$, $r > 0$, to the left and ∂_i, $h_{i,r}$, $r > 0$, to the right, except that the order between $e^{\pm \epsilon_i}$'s is kept unchanged. The $\mathcal{E}_n$ module $\mathbb{F}(u)$ is a direct sum of irreducible submodules, see [6, Lemma 3.2].

3   Comodule algebra $\mathcal{K}_n$

In this section we introduce an algebra $\mathcal{K}_n$, and show that it has the structure of an $\mathcal{E}_n$ comodule in the sense to be made precise below. Algebra $\mathcal{K}_1$ was first introduced and studied in [7] in order to give a uniform description of deformed W algebras associated with simple Lie (super)algebras of types BCD.

3.1   Algebra $\mathcal{K}_n$

We introduce an algebra $\mathcal{K}_n$ through generating series

$$E_i(z) = \sum_{k \in \mathbb{Z}} E_{i,k} z^{-k}, \qquad K_i^\pm(z) = K_{i,0}^\pm \exp\left(\pm(q - q^{-1}) \sum_{\pm r > 0} H_{i,r} z^{-r} \right), \qquad i \in \mathbb{Z}/n\mathbb{Z},$$

and an invertible central element C. The defining relations read as follows

$$K_i^\pm(z) K_j^\pm(w) = K_j^\pm(w) K_i^\pm(z), \tag{3.1}$$

$$K_i^+(z) K_j^-(w) = K_j^-(w) K_i^+(z) G_{i,j}(w/z) G_{i,j}(C^2 w/z), \tag{3.2}$$

$$K_i^+(z) E_j(w) = E_j(w) K_i^+(z) G_{i,j}(w/z), \quad E_j(w) K_i^-(z) = K_i^-(z) E_j(w) G_{j,i}(z/w), \tag{3.3}$$

$$d_{i,j} g_{i,j}(z, w) E_i(z) E_j(w) + g_{j,i}(w, z) E_j(w) E_i(z) \tag{3.4}$$

$$= \frac{\delta_{i,j}}{q - q^{-1}} \left(g_{i,i}(z, w) \delta(C^2 z/w) K_i^-(z) K_i^+(C^2 z) + g_{i,i}(w, z) \delta(C^2 w/z) K_i^-(w) K_i^+(C^2 w) \right),$$

$$E_i(z) E_j(w) = E_j(w) E_i(z), \qquad i \not\equiv j, j \pm 1. \tag{3.5}$$

In addition, we impose the Serre relations

$$\operatorname*{Sym}_{z_1, z_2} [E_i(z_1), [E_i(z_2), E_{i\pm 1}(w)]_q]_{q^{-1}} = -\operatorname*{Sym}_{z_1, z_2} \left\{ \left(\frac{1}{1 - q^{-1} d^{\mp 1} z_1/w} + \frac{q^{-1} d^{\pm 1} w/z_2}{1 - q^{-1} d^{\pm 1} w/z_2} \right) \right.$$

$$\left. \times \delta(C^2 z_1/z_2) K_i^-(z_1) E_{i\pm 1}(w) K_i^+(z_2) \right\}. \tag{3.6}$$

In the right-hand side, the rational functions of the form $1/(1 - az_1/w)$ or $1/(1 - bw/z_2)$ stand for their expansions in non-negative powers of z_1/w or w/z_2, respectively.

In the right-hand sides of (3.4) and (3.6), the Fourier coefficients of $K_i^-(z)K_i^+(C^2z)$ are infinite series. Therefore the relations require some justification. We proceed as follows. We define a $\mathbb{Z}$ grading of generators by

$$\deg E_{i,k} = k, \qquad \deg H_{i,\pm r} = \pm r, \qquad \deg K_{i,0}^\pm = \deg C = 0.$$

We call it the homogeneous grading. Consider the free algebra $\mathcal{A}$ generated by $E_{i,k}$, $H_{i,\pm r}$, $K_{i,0}^\pm$, $C^{\pm 1}$. Let $\widetilde{\mathcal{A}}$ be the completion of $\mathcal{A}$ with respect to the grading in the positive direction. Then we define $\mathcal{K}_n$ to be the graded quotient algebra of $\widetilde{\mathcal{A}}$ under the relations (3.1)–(3.6).

The quadratic relations (3.4) imply

$$d_{i,j}\hat{g}_{i,j}(z,w)E_i(z)E_j(w) + \hat{g}_{j,i}(w,z)E_j(w)E_i(z) = 0, \tag{3.7}$$

where

$$\hat{g}_{i,j}(z,w) = \begin{cases} (z - C^2w)(z - C^{-2}w)(z - q^2w), & i = j, \\ g_{i,j}(z,w), & i \neq j. \end{cases}$$

The factor $(z - C^2w)(z - C^{-2}w)$ for $i = j$ is chosen to kill the delta functions in the right-hand side of (3.4).

Remark 3.1. Algebra $\mathcal{K}_n$ has a filtration defined by $\mathrm{pdeg}\, E_i(z) = 1$, $\mathrm{pdeg}\, K_i^\pm(z) = 0$ for all i, which we call the principal filtration. In the associated graded algebra, the subalgebra generated by $\{E_i(z)\}$ is isomorphic to the subalgebra of the quantum toroidal algebra $\mathcal{E}_n$ generated by $\{e_i(z)\}$.

The relations for $K_i^\pm(z)$ are slightly different from those of $\psi_i^\pm(z)$ in $\mathcal{E}_n$, since the right-hand side of (3.2) contains a product of $G_{i,j}$'s as opposed to the ratio. In particular, while the elements $\psi_{i,0}^\pm \in \mathcal{E}_n$ are inverse to each other, the elements $K_{i,0}^\pm \in \mathcal{K}_n$ are not mutually commutative,

$$K_{i,0}^+K_{j,0}^- = q^{2(\bar{\alpha}_i,\bar{\alpha}_j)}K_{j,0}^-K_{i,0}^+.$$

We have the relations

$$[H_{i,r}, H_{i,s}] = -\delta_{r+s,0} \cdot a_{i,j}(r)\frac{1 + C^{2r}}{q - q^{-1}}, \qquad r > 0,$$

which are to be compared with (2.1). In the limit $q \to 1$ with $C = q^c$, algebra $\mathcal{E}_n$ reduces to the enveloping algebra of a Lie algebra of matrix valued difference operators, see [3, Section 3.7]. In contrast, algebra $\mathcal{K}_n$ does not seem to have a reasonable limit due to the presence of the sum $1 + C^{2r}$.

3.2 Wheel conditions

All representations of $\mathcal{E}_n$ or $\mathcal{K}_n$ considered in this paper are admissible representations. In this setting, we rewrite the formal series relations of generating currents in the language of matrix coefficients.

Let $V = \oplus_{k=-\infty}^{k_0} V_k$ be a graded vector space with degrees bounded from above and $\dim V_k < \infty$. We denote by $V^* = \oplus_{k=-\infty}^{k_0} V_k^*$ the restricted dual space. Suppose that we are given a set of formal series $\{E_i(z) = \sum_{k \in \mathbb{Z}} E_{i,k}z^{-k} \mid i \in \mathbb{Z}/n\mathbb{Z}\}$, such that $E_{i,k} \in \mathrm{End}\, V$, $E_{i,k}V_l \subset V_{k+l}$,

satisfying the relations (3.5) and (3.7) with some $C \in \mathbb{C}^\times$. We assume that the zeros of $\hat{g}_{i,i}(z, w)$ are distinct:

$$C^2 \neq \pm 1, q^{\pm 2}.$$

It follows from (3.5) and (3.7) that all matrix coefficients

$$\langle w, E_{i_1}(z_1) \cdots E_{i_N}(z_N) v \rangle, \qquad v \in V, \quad w \in V^*,$$

converge in the region $|z_1| \gg \cdots \gg |z_N|$ to rational functions which have at most simple poles at $\hat{g}_{i_r, i_s}(z_r, z_s) = 0$, where $r < s$ and $i_r \equiv i_s, i_s \pm 1$.

Quite generally, let $A_1(z)$, $A_2(z)$ be operator-valued formal series and $r_{i,j}(x)$ be rational functions. We shall say that an exchange relation

$$r_{1,2}(z_2/z_1) A_1(z_1) A_2(z_2) = r_{2,1}(z_1/z_2) A_2(z_2) A_1(z_1)$$

holds as rational functions, if an arbitrary matrix coefficient of each side converges (in regions $|z_1| \gg |z_2|$ and $|z_2| \gg |z_1|$ respectively) to the same rational function. Similarly we shall use the term "exchange relation as meromorphic functions".

Equations (3.5) and (3.7) imply that we have relations as rational functions

$$\lambda_{i,j}^0(w/z) E_i(z) E_j(w) = \lambda_{j,i}^0(z/w) E_j(w) E_i(z), \tag{3.8}$$

where we set

$$\lambda_{i,i}^0(x) = \frac{1 - C^2 x}{1 - x} \frac{1 - C^{-2} x}{1 - x} \frac{1 - q^2 x}{1 - x},$$

$$\lambda_{i,i\pm 1}^0(x) = d^{\mp 1/2} \frac{1 - q^{-1} d^{\pm 1} x}{1 - x},$$

and $\lambda_{i,j}^0(x) = 1$ for $j \not\equiv i, i \pm 1$.

Set

$$E_{i_1, \dots, i_N}(z_1, \dots, z_N) = \prod_{1 \leq r < s \leq N} \lambda_{i_r, i_s}^0(z_s/z_r) E_{i_1}(z_1) \cdots E_{i_N}(z_N). \tag{3.9}$$

It follows from (3.8) that

$$E_{i_{\sigma(1)}, \dots, i_{\sigma(N)}}(z_{\sigma(1)}, \dots, z_{\sigma(N)}) = E_{i_1, \dots, i_N}(z_1, \dots, z_N), \qquad \sigma \in \mathfrak{S}_N. \tag{3.10}$$

In particular (3.9) is symmetric with respect to z_r and z_s when they have the same "color" $i_r = i_s$. Moreover all matrix coefficients of (3.9) have the form

$$\frac{P(z_1, \dots, z_N)}{\prod_{r<s}(z_r - z_s)^{N_{r,s}}}, \qquad P(z_1, \dots, z_N) \in \mathbb{C}[z_1^{\pm 1}, \dots, z_N^{\pm 1}], \tag{3.11}$$

where $N_{r,s} = 2$ if $i_r = i_s$, $N_{r,s} = 1$ if $i_r \equiv i_s \pm 1$, and $N_{r,s} = 0$ otherwise. Note that, when $i_r = i_s$, third order poles are absent due to the symmetry in variables of the same color.

The following is easy to see.

Proposition 3.2. *Under the conditions* (3.5) *and* (3.7), *the condition* (3.4) *holds if and only if*

$$E_{i,i}(z, C^2 z) = \frac{1}{q - q^{-1}} \frac{1 + C^2}{1 - C^2} \frac{1 - q^2 C^2}{1 - C^2} K_i^-(z) K_i^+(C^2 z). \tag{3.12}$$

We have also

Proposition 3.3. *Under the conditions* (3.5) *and* (3.7), *the Serre relations* (3.6) *are satisfied if and only if the following zero conditions hold:*

$$E_{i,i,i\pm1}(z, q^2 z, qd^{\mp1}z) = 0, \tag{3.13}$$

$$E_{i,i,i\pm1}(z, C^2 z, C^2 q^{-1}d^{\mp1}z) = 0. \tag{3.14}$$

If it is the case then we have also

$$E_{i,i,i\pm1}(z, C^{-2}z, C^{-2}qd^{\mp1}z) = 0. \tag{3.15}$$

We call (3.13)–(3.15) "wheel conditions".

Proof. The left-hand side and the right-hand side of the Serre relations (3.6) read respectively as

$$\mathrm{LHS} = \underset{z_1,z_2}{\mathrm{Sym}}\big(E_i(z_1)E_i(z_2)E_{i+1}(w) - (q+q^{-1})E_i(z_1)E_{i+1}(w)E_i(z_2) + E_{i+1}(w)E_i(z_1)E_i(z_2)\big),$$

$$\mathrm{RHS} = -\underset{z_1,z_2}{\mathrm{Sym}}\left(-\frac{q_3^{-1}w/z_1}{1 - q_3^{-1}w/z_1} + \frac{q_1 w/z_2}{1 - q_1 w/z_2} + \delta(q_3 z_1/w)\right)\delta(C^2 z_1/z_2)K_i^-(z_1)$$
$$\times E_{i+1}(w)K_i^+(z_2).$$

The case involving $E_i(z_1)$, $E_i(z_2)$ and $E_{i-1}(w)$ can be treated by interchanging q_3 with q_1.

Using the definition (3.9), LHS can be rewritten as

$$\mathrm{LHS} = E_{i,i,i+1}(z_1, z_2, w)\left(\frac{1}{\lambda_{i,i}^0\left(\frac{z_2}{z_1}\right)\lambda_{i,i+1}^0\left(\frac{w}{z_1}\right)\lambda_{i,i+1}^0\left(\frac{w}{z_2}\right)} - \frac{q+q^{-1}}{\lambda_{i,i}^0\left(\frac{z_2}{z_1}\right)\lambda_{i,i+1}^0\left(\frac{w}{z_1}\right)\underline{\lambda_{i+1,i}^0\left(\frac{z_2}{w}\right)}}\right.$$
$$+ \frac{1}{\lambda_{i,i}^0\left(\frac{z_2}{z_1}\right)\underline{\lambda_{i+1,i}^0\left(\frac{z_1}{w}\right)}\lambda_{i+1,i}^0\left(\frac{z_2}{w}\right)} + \frac{1}{\lambda_{i,i}^0\left(\frac{z_1}{z_2}\right)\lambda_{i,i+1}^0\left(\frac{w}{z_1}\right)\lambda_{i,i+1}^0\left(\frac{w}{z_2}\right)}$$
$$\left. - \frac{q+q^{-1}}{\lambda_{i,i}^0\left(\frac{z_1}{z_2}\right)\underline{\lambda_{i+1,i}^0\left(\frac{z_1}{w}\right)}\lambda_{i,i+1}^0\left(\frac{w}{z_2}\right)} + \frac{1}{\lambda_{i,i}^0\left(\frac{z_1}{z_2}\right)\lambda_{i+1,i}^0\left(\frac{z_1}{w}\right)\lambda_{i+1,i}^0\left(\frac{z_2}{w}\right)}\right).$$

The matrix coefficients of $E_{i,i,i+1}(z_1, z_2, w)$ are Laurent polynomials up to poles on $z_1 = z_2$ or $z_i = w$, which are cancelled by the zeros of $1/\lambda_{i,j}^0$.

This is a sum of six formal series, each expanded in a different region. In order to bring them to expansions in a common region, we use the identity $1/(1-z) = \delta(z) - z^{-1}/(1-z^{-1})$, where $1/(1-z)$ stands for its expansion in non-negative powers of z, while $z^{-1}/(1-z^{-1})$ means the one in negative powers of z. In the underlined factors, we substitute

$$\frac{1}{\lambda_{i+1,i}^0\left(\frac{z}{w}\right)} = (\text{expansion in } |z| \gg |w|) + d^{-1/2}(1 - q_3^{-1})\delta(q_3 z/w),$$

$$\frac{1}{\lambda_{i,i}^0\left(\frac{z_1}{z_2}\right)} = (\text{expansion in } |z_1| \gg |z_2|) - \frac{q^{-2}(1-q^2)^3}{(1 - C^{-2}q^2)(1 - C^2 q^2)}\delta(q^2 z_1/z_2)$$
$$+ \frac{1 - C^2}{1 + C^2}\frac{1 - C^2}{1 - q^2 C^2}\delta(C^2 z_2/z_1) + \frac{1 - C^{-2}}{1 + C^{-2}}\frac{1 - C^{-2}}{1 - q^2 C^{-2}}\delta(C^{-2}z_2/z_1)$$

and bring all terms into the sum of their expansions in $|z_1| \gg |z_2| \gg |w|$ and additional delta functions. We then compare the coefficients of each product of delta functions.

Up to the symmetry $z_1 \leftrightarrow z_2$, there are the following cases to consider:

(1) 1,   (2) $\delta(q_3 z_1/w)$,   (3) $\delta(q^2 z_1/z_2)$,

(4) $\delta\big(C^2 z_2/z_1\big),$ (5) $\delta(q_3 z_1/w)\delta\big(q^2 z_1/z_2\big),$ (6) $\delta(q_3 z_1/w)\delta\big(C^2 z_2/z_1\big),$

(7) $\delta(q_3 z_1/w)\delta\big(C^{-2} z_2/z_1\big),$ (8) $\delta(q_3 z_1/w)\delta(q_3 z_2/w).$

In cases (1), (2) and (3), the coefficients vanish due to the identities of rational functions

$$
0 = \operatorname*{Sym}_{z_1,z_2}\left\{ \frac{1}{\lambda^0_{i,i}\big(\frac{z_2}{z_1}\big)\lambda^0_{i,i+1}\big(\frac{w}{z_1}\big)\lambda^0_{i,i+1}\big(\frac{w}{z_2}\big)} - \frac{q+q^{-1}}{\lambda^0_{i,i}\big(\frac{z_2}{z_1}\big)\lambda^0_{i,i+1}\big(\frac{w}{z_1}\big)\lambda^0_{i+1,i}\big(\frac{z_2}{w}\big)} \right.
$$
$$
\left. + \frac{1}{\lambda^0_{i,i}\big(\frac{z_2}{z_1}\big)\lambda^0_{i+1,i}\big(\frac{z_1}{w}\big)\lambda^0_{i+1,i}\big(\frac{z_2}{w}\big)} \right\},
$$
$$
0 = \frac{1}{\lambda^0_{i,i}(z_2/z_1)}\frac{1}{\lambda^0_{i+1,i}(z_2/w)} - \big(q+q^{-1}\big)\frac{1}{\lambda^0_{i,i}(z_1/z_2)}\frac{1}{\lambda^0_{i,i+1}(w/z_2)} + \frac{1}{\lambda^0_{i,i}(z_1/z_2)}\frac{1}{\lambda^0_{i+1,i}(z_2/w)},
$$
$$
0 = \frac{1}{\lambda^0_{i,i+1}(w/z_1)}\frac{1}{\lambda^0_{i,i+1}(w/z_2)} - \big(q+q^{-1}\big)\frac{1}{\lambda^0_{i+1,i}(z_1/w)}\frac{1}{\lambda^0_{i,i+1}(w/z_2)}
$$
$$
+ \frac{1}{\lambda^0_{i+1,i}(z_1/w)}\frac{1}{\lambda^0_{i+1,i}(z_2/w)}.
$$

In case (8) the coefficient vanishes on $z_1 = z_2$ due to the triple zero of $\lambda^0_{i,i}(z_1/z_2)^{-1}$ there.

In case (4), the relevant terms are the 3 terms containing $\lambda^0_{i,i}(z_1/z_2)^{-1}$. Using the relation (3.12), we find that the coefficient of $\delta\big(C^2 z_2/z_1\big)$ is

$$
\frac{1-C^2}{1+C^2}\frac{1-C^2}{1-q^2 C^2}E_{i,i}\big(C^2 z_2, z_2\big)E_{i+1}(w)\lambda^0_{i,i+1}(w/z_1)\lambda^0_{i,i+1}(w/z_2)
$$
$$
\times \left(\frac{1}{\lambda^0_{i,i+1}(w/z_1)}\frac{1}{\lambda^0_{i,i+1}(w/z_2)} - \big(q+q^{-1}\big)\frac{1}{\lambda^0_{i+1,i}(z_1/w)}\frac{1}{\lambda^0_{i,i+1}(w/z_2)} \right.
$$
$$
\left. + \frac{1}{\lambda^0_{i+1,i}(z_1/w)}\frac{1}{\lambda^0_{i+1,i}(z_2/w)} \right)
$$
$$
= -\frac{d(w/z_1 - q^2 w/z_2)}{\big(1-q_3^{-1}w/z_1\big)\big(1-q_3^{-1}w/z_2\big)}K_i^-(z_2)K_i^+\big(C^2 z_2\big)E_{i+1}(w),
$$

which coincides with the corresponding term in RHS.

Likewise, in case (7) the coefficient of $\delta\big(C^2 z_2/z_1\big)\delta\big(q_3^{-1}w/z_1\big)$ comes from

$$
E_{i,i,i+1}(z_1, z_2, w)\frac{1}{\lambda^0_{i,i}(z_1/z_2)}\frac{1}{\lambda^0_{i+1,i}(z_2/w)}\big(-\big(q+q^{-1}\big)\lambda^0_{i,i+1}(w/z_2) + \lambda^0_{i+1,i}(z_2/w)\big).
$$

Computing similarly we find that it matches with the corresponding term in RHS.

It remains to consider (5) and (6). The coefficients of the product of delta functions are proportional (with non-zero multipliers) to the left-hand side of (3.13) and (3.14), respectively. Since there are no terms that come from RHS, we have shown that the Serre relations are equivalent to (3.13) and (3.14).

Instead of rewriting terms into expansions in $|z_1| \gg |z_2| \gg |w|$, one can equally well proceed to the opposite region $|z_1| \ll |z_2| \ll |w|$. Computing in the same way we obtain (3.15) in place of (3.14).   ∎

3.3   Comodule structure

In this section we show that, when we restrict to admissible modules, $\mathcal{K}_n$ may be viewed as a left comodule of $\mathcal{E}_n$. To be precise we prove the following.[2]

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[2]Formula for Δ is slightly changed from the one used in [7]. The $\mathcal{E}_1$ comodule structure discussed there should be understood at the level of admissible representations.

Theorem 3.4. *Let V be an admissible $\mathcal{E}_n$ module and W an admissible $\mathcal{K}_n$ module, such that C acts as scalar C_1, C_2 respectively. We set $C_{12} = C_1 C_2$ and assume that*

$$C_2^2 \neq \pm 1, q^{\pm 2}, \qquad C_{12}^2 \neq \pm 1, q^{\pm 2}.$$

Then $V \otimes W$ is given the structure of an admissible $\mathcal{K}_n$ module by the action of the following currents:

$$\Delta E_i(z) = e_i(C_2 z) \otimes K_i^-(z) + 1 \otimes E_i(z) + \tilde{f}_i\big(C_2^{-1} z\big) \otimes K_i^+(z), \tag{3.16}$$

$$\Delta K_i^+(z) = \psi_i^+ \big(C_1^{-1} C_2^{-1} z\big)^{-1} \otimes K_i^+(z), \tag{3.17}$$

$$\Delta K_i^-(z) = \psi_i^- (C_2 z) \otimes K_i^-(z), \tag{3.18}$$

$$\Delta C = C_{12}. \tag{3.19}$$

Moreover the following "coassociativity" and "counit property" hold:

$$(V_1 \otimes V_2) \otimes W = V_1 \otimes (V_2 \otimes W),$$
$$\mathbb{C} \otimes W = W.$$

Here V_1, V_2 are admissible $\mathcal{E}_n$ modules and $\mathbb{C}$ denotes the trivial $\mathcal{E}_n$ module.

Proof. On each vector of an admissible module, the currents of $\mathcal{E}_n$ or $\mathcal{K}_n$ comprise only a finite number of negative powers in z. Hence each Fourier coefficient of (3.16)–(3.18) is a well-defined operator on $V \otimes W$.

It is easy to verify the relations (3.1)–(3.3) and (3.5) by direct calculation. With the aid of (2.2), one can also check (3.4) for $i \neq j$.

Let us check (3.4) for $i = j$. We shall write $A_1(z, w) \equiv A_2(z, w)$ if each matrix coefficient of $A_1(z, w) - A_2(z, w)$ is a Laurent polynomial. Then we have on V

$$g_{i,j}(z, w) e_i(z) e_j(w) \equiv 0, \qquad g_{i,j}(z, w) \tilde{f}_i(z) \tilde{f}_j(w) \equiv 0, \tag{3.20}$$

$$e_i(z) \tilde{f}_j(w) \equiv \frac{\delta_{i,j}}{q - q^{-1}} \left(-\frac{1}{1 - w/z} + \frac{1}{1 - C^{-2} w/z} \psi_i^-(z) \psi_i^+ \big(C^{-1} w\big)^{-1} \right), \tag{3.21}$$

and on W

$$g_{i,j}(z, w) E_i(z) E_j(w)$$
$$\equiv \frac{\delta_{i,j}}{q - q^{-1}} \left(\frac{z - q^2 w}{1 - C^{-2} w/z} K_i^-(z) K_i^+(w) + \frac{w - q^2 z}{1 - C^2 w/z} K_i^-(w) K_i^+(z) \right). \tag{3.22}$$

Using these we obtain

$$\big(q - q^{-1}\big)\big(z - q^2 w\big) \Delta E_i(z) \Delta E_i(w)$$
$$\equiv \big(q - q^{-1}\big)\big(z - q^2 w\big) \big(e_i(C_2 z) \tilde{f}_i\big(C_2^{-1} w\big) \otimes K_i^-(z) K_i^+(w)$$
$$\quad + \tilde{f}_i\big(C_2^{-1} z\big) e_i(C_2 w) \otimes K_i^+(z) K_i^-(w) + 1 \otimes E_i(z) E_i(w)\big)$$
$$\equiv -\left(\frac{z - q^2 w}{1 - C_2^{-2} w/z} - \frac{z - q^2 w}{1 - C_{12}^{-2} w/z} \psi_i^-(C_2 z) \psi_i^+ \big(C_{12}^{-1} w\big)^{-1} \right) \otimes K_i^-(z) K_i^+(w)$$
$$\quad + \left(\frac{w - q^2 z}{1 - C_2^{-2} z/w} - \frac{w - q^2 z}{1 - C_{12}^{-2} z/w} \psi_i^-(C_2 w) \psi_i^+ \big(C_{12}^{-1} z\big)^{-1} \right) \otimes K_i^-(w) K_i^+(z)$$
$$\quad + \frac{z - q^2 w}{1 - C_2^{-2} w/z} 1 \otimes K_i^-(z) K_i^+(w) + \frac{w - q^2 z}{1 - C_2^2 w/z} 1 \otimes K_i^-(w) K_i^+(z)$$

$$\equiv \frac{z - q^2 w}{1 - C_{12}^{-2} w/z} \psi_i^-(C_2 z) \psi_i^+ \big(C_{12}^{-1} w\big)^{-1} \otimes K_i^-(z) K_i^+(w)$$

$$- \frac{w - q^2 z}{1 - C_{12}^{-2} z/w} \psi_i^-(C_2 w) \psi_i^+ \big(C_{12}^{-1} z\big)^{-1} \otimes K_i^-(w) K_i^+(z).$$

Interchanging z with w and summing these, we arrive at (3.4) for $i = j$.

Next let us verify the Serre relations (3.6). By Proposition 3.3, it suffices to show that the current $\Delta E_{i,i,i\pm1}(z_1, z_1, w)$ satisfies the wheel conditions (3.13)–(3.14). It is comprised of 27 terms which we group together according to the principal grading in the first component, defined by $\mathrm{pdeg}\, e_i(z) = 1$, $\mathrm{pdeg}\, f_i(z) = -1$, $\mathrm{pdeg}\, \psi_i^\pm(z) = \mathrm{pdeg}\, C = 0$. For the terms of principal degrees ±3 and ±2, the wheel conditions are easily checked.

Let us verify the case of principal degree 1. There are 6 terms coming from $\Delta E_{i,i,i\pm1}(z_1, z_2, w)$:

$$A_1 = \phi\, e_i(C_2 z_1) e_i(C_2 z_2) \tilde{f}_{i\pm1}\big(C_2^{-1} w\big) \otimes K_i^-(z_1) K_i^-(z_2) K_{i\pm1}^+(w),$$

$$A_2 = \phi\, e_i(C_2 z_1) e_{i\pm1}(C_2 w) \tilde{f}_i\big(C_2^{-1} z_2\big) \otimes K_i^-(z_1) K_{i\pm1}^-(w) K_i^+(z_2) G_{i,i\pm1}(w/z_2),$$

$$A_3 = \phi\, e_i(C_2 z_2) e_{i\pm1}(C_2 w) \tilde{f}_i\big(C_2^{-1} z_1\big) \otimes K_i^-(z_2) K_{i\pm1}^-(w) K_i^+(z_1) G_{i,i}(z_2/z_1) G_{i,i\pm1}(w/z_1),$$

$$A_4 = \phi\, e_i(C_2 z_1) \otimes K_i^-(z_1) E_i(z_2) E_{i\pm1}(w),$$

$$A_5 = \phi\, e_i(C_2 z_2) \otimes K_i^-(z_2) E_i(z_1) E_{i\pm1}(w) G_{i,i}(z_2/z_1),$$

$$A_6 = \phi\, e_{i\pm1}(C_2 w) \otimes K_{i\pm1}^-(w) E_i(z_1) E_i(z_2) G_{i,i\pm1}(w/z_1) G_{i,i\pm1}(w/z_2),$$

where

$$\phi = \frac{z_1 - C_{12}^2 z_2}{z_1 - z_2} \frac{z_1 - C_{12}^{-2} z_2}{z_1 - z_2} \frac{z_1 - q^2 z_2}{z_1 - z_2} \frac{z_1 - q^{-1} d^{\pm1} w}{z_1 - w} \frac{z_2 - q^{-1} d^{\pm1} w}{z_1 - w}.$$

From (3.20)–(3.22), we see that the sum $\sum_{i=1}^6 A_i$ has no poles other than $z_1 = z_2$, $z_i = w$. In fact, the only possible poles at $z_2 = C_2^2 z_1$ (resp. $z_2 = C_2^{-2} z_1$) arise from A_2 and A_6 (resp., A_3 with A_6), and they cancel by virtue of (3.21) and (3.22).

Let us check (3.13). Under the specialization $z_2 = q^2 z_1$, $w = q d^{\mp1} z_1 = q^{-1} d^{\mp1} z_2$, each term has a zero due to the vanishing factors

$$\big(z_1 - q^2 z_2\big) G_{i,i}(z_2/z_1), \qquad z_1 - q^{-1} d^{\pm1} w, \qquad \big(z_2 - q^{-1} d^{\pm1} w\big) G_{i,i\pm1}(w/z_2).$$

To check (3.14), set $z_2 = C_{12}^2 z_1$. All A_i's vanish at $z_2 = C_{12}^2 z_1$ with the exception of A_2. Since the latter has the factor $\big(z_2 - q^{-1} d^{\pm1} w\big) G_{i,i\pm1}(w/z_2)$, it also vanishes by setting further $w = q^{-1} d^{\mp1} z_2$.

In the case of principal degree 0, there are 7 terms:

$$B_0 = \phi\, 1 \otimes E_i(z_1) E_i(z_2) E_{i\pm1}(w),$$

$$B_1 = \phi\, e_i(C_2 z_1) \tilde{f}_i\big(C_2^{-1} z_2\big) \otimes K_i^-(z_1) E_{i\pm1}(w) K_i^+(z_2) G_{i,i\pm1}(w/z_2),$$

$$B_2 = \phi\, e_i(C_2 z_2) \tilde{f}_i\big(C_2^{-1} z_1\big) \otimes K_i^-(z_2) E_{i\pm1}(w) K_i^+(z_1) G_{i,i}(z_2/z_1) G_{i,i\pm1}(w/z_1),$$

$$B_3 = \phi\, e_i(C_2 z_1) \tilde{f}_{i\pm1}\big(C_2^{-1} w\big) \otimes K_i^-(z_1) E_i(z_2) K_{i\pm1}^+(w),$$

$$B_4 = \phi\, e_i(C_2 z_2) \tilde{f}_{i\pm1}\big(C_2^{-1} w\big) \otimes K_i^-(z_2) E_i(z_1) K_{i\pm1}^+(w) G_{i,i}(z_2/z_1),$$

$$B_5 = \phi\, e_{i\pm1}(C_2 w) \tilde{f}_i\big(C_2^{-1} z_2\big) \otimes K_{i\pm1}^-(w) E_i(z_1) K_i^+(z_2) G_{i,i\pm1}(w/z_1) G_{i,i\pm1}(w/z_2),$$

$$B_6 = \phi\, e_{i\pm1}(C_2 w) \tilde{f}_i\big(C_2^{-1} z_1\big) \otimes K_{i\pm1}^-(w) E_i(z_2) K_i^+(z_1) G_{i,i}(z_2/z_1) G_{i,i\pm1}(w/z_1) G_{i,i\pm1}(w/z_2).$$

Again possible poles at $z_2 = C_2^{\pm2} z_1$ are cancelled among B_0, B_1, B_2 and the sum $\sum_{i=0}^6 B_i$ has no poles other than $z_1 = z_2$, $z_i = w$.

Consider $z_2 = q^2 z_1$, $w = qd^{\mp 1}z_1 = q^{-1}d^{\mp 1}z_2$. By the assumption on C_2^2, B_0 vanishes. The rest of the terms are zero for the same reason as in principal degree 1. Next let $z_2 = C_{12}^2 z_1$. The only term which may survive is B_1. Due to the factor $\left(z_2 - q^{-1}d^{\pm 1}w\right)G_{i,i+1}(w/z_2)$ it also vanishes by setting further $w = q^{-1}d^{\mp 1}z_2$.

The case of principal degree -1 being similar to principal degree 1, we omit the details. Finally the coassociativity and counit property can be readily checked by using (3.16)–(3.19). ∎

3.4 Representations

Theorem 3.4 allows us to construct a large family of admissible representations of $\mathcal{K}_n$ starting from simple ones.

Algebra $\mathcal{K}_n$ has a Heisenberg subalgebra $\mathcal{H}_n$ generated by $H_{i,r}$, $i \in \mathbb{Z}/n\mathbb{Z}$, $r \neq 0$, $K_{i,0}^\pm$, and $C^{\pm 1}$. Let $\mathbb{F}^\mathsf{D}$ be an irreducible representation of $\mathcal{H}_n$ on which C acts as q^{-1}.

Lemma 3.5. *With the following action, $\mathbb{F}^\mathsf{D}$ is an admissible $\mathcal{K}_n$ module:*

$$E_i(z) \mapsto 0, \qquad K_i^\pm(z) \mapsto K_i^\pm(z), \qquad C \mapsto q^{-1}.$$

Proof. The defining relations are obviously satisfied except (3.4). To see the latter it suffices to note that $g_{i,i}(z, C^2 z) = 0$ when $C = q^{-1}$. ∎

In the next case we need to adjoin $\left(K_{i,0}^\pm\right)^{1/2}$ to $\mathcal{H}_n$ and $\mathcal{K}_n$. We define $\tilde{K}_i^\pm(z)$ by

$$\tilde{K}_i^\pm(z)\tilde{K}_i^\pm(qz) = K_i^\pm(z).$$

Let $\mathbb{F}^\mathsf{B}$ be an irreducible representation of $\mathcal{H}_n$ on which C acts as $q^{1/2}$.

Lemma 3.6. *With the following action, $\mathbb{F}^\mathsf{B}$ is an admissible $\mathcal{K}_n$ module:*

$$E_i(z) \mapsto k\tilde{K}_i^-(z)\tilde{K}_i^+(qz), \qquad K_i^\pm(z) \mapsto K_i^\pm(z), \qquad C \mapsto q^{1/2},$$

where $k = 1/(q^{1/2} - q^{-1/2})$.

Proof. We can write $G_{i,j}(x) = \tilde{G}_{i,j}(x)\tilde{G}_{i,j}(q^{-1}x)$, where

$$\tilde{G}_{i,i}(x) = q\frac{1 - q^{-1}x}{1 - x}\frac{1 - qx}{1 - q^2 x}, \qquad \tilde{G}_{i,i\pm 1}(x) = q^{-1/2}\frac{1 - qd^{\pm 1}x}{1 - d^{\pm 1}x},$$

and $\tilde{G}_{i,j}(x) = 1$ otherwise. The assertion can be checked by noting

$$\tilde{K}_i^+(z)\tilde{K}_j^-(w) = \tilde{K}_j^-(w)\tilde{K}_i^+(z)\tilde{G}_{i,j}(w/z).$$

We use also

$$E_{i,i}(z, w) \mapsto qk^2 \tilde{K}_i^-(z)\tilde{K}_i^-(w)\tilde{K}_i^+(qz)\tilde{K}_i^+(qw)\frac{1 - q^{-2}w/z}{1 - w/z}\frac{1 - q^2 w/z}{1 - w/z}$$

and Proposition 3.2 to check the coefficients of delta functions in (3.4), and use Proposition 3.3 along with

$$E_{i,i,i+1}(z_1, z_2, w) \mapsto k^3 \tilde{K}_i^-(z_1)\tilde{K}_i^-(z_2)\tilde{K}_{i\pm 1}^-(w)\tilde{K}_i^+(qz_1)\tilde{K}_i^+(qz_2)\tilde{K}_{i\pm 1}^+(qw)$$

$$\times \frac{1 - q^{-2}z_2/z_1}{1 - z_2/z_1}\frac{1 - q^2 z_2/z_1}{1 - z_2/z_1}\frac{1 - d^{\pm 1}w/z_1}{1 - w/z_1}\frac{1 - d^{\pm 1}w/z_2}{1 - w/z_2}$$

for the Serre relations (3.6). ∎

Together with Theorem 3.4, the above lemmas imply

Corollary 3.7. *Let V be an admissible $\mathcal{E}_n$ module, where C acts as scalar C_1 with $q^{-2}C_1^2 \neq \pm 1, q^{\pm 2}$. Then $V \otimes \mathbb{F}^D$ is an admissible $\mathcal{K}_n$ module with the action*

$$\Delta E_i(z) = e_i(q^{-1}z) \otimes K_i^-(z) + \tilde{f}_i(qz) \otimes K_i^+(z),$$
$$\Delta K_i^+(z) = \psi_i^+\big(qC_1^{-1}z\big)^{-1} \otimes K_i^+(z), \qquad \Delta K_i^-(z) = \psi_i^-\big(q^{-1}z\big) \otimes K_i^-(z),$$
$$\Delta C = C_1 q^{-1}.$$

Proof. Since $C_2^2 = q^{-2}$, Theorem 3.4 does not literally apply. Nevertheless, in the proof there, the only place where this matters is the term B_0 in degree 0. Since this term is identically zero in $\mathbb{F}^D$, the same proof works. ∎

Corollary 3.8. *Let V be an admissible $\mathcal{E}_n$ module, where C acts as scalar C_1 with $qC_1^2 \neq \pm 1, q^{\pm 2}$. Then $V \otimes \mathbb{F}^B$ is an admissible $\mathcal{K}_n$ module with the action*

$$\Delta E_i(z) = e_i(q^{1/2}z) \otimes K_i^-(z) + 1 \otimes k\tilde{K}_i^-(z)\tilde{K}_i^+(qz) + \tilde{f}_i(q^{-1/2}z) \otimes K_i^+(z),$$
$$\Delta K_i^+(z) = \psi_i^+\big(q^{-1/2}C_1^{-1}z\big)^{-1} \otimes K_i^+(z), \qquad \Delta K_i^-(z) = \psi_i^-\big(q^{1/2}z\big) \otimes K_i^-(z),$$
$$\Delta C = C_1 q^{1/2},$$

where $\tilde{K}_i^\pm(z)$ and k are as in Lemma 3.6.

Example 3.9. Consider the $\mathcal{K}_n$ module

$$\mathbb{F}(u_1) \otimes \cdots \otimes \mathbb{F}(u_\ell) \otimes \mathbb{F}^D, \qquad \ell \geq 3.$$

The central element acts as $C^2 = q^{2\ell-2}$. The current $E_i(z)$ is represented as a sum of 2ℓ vertex operators:

$$z^{-1}E_i(z) = -\sum_{k=1}^{\ell} q^{k-\ell}u_k\Lambda_{i,k}(z) + \sum_{k=1}^{\ell} q^{-k+\ell-1}u_k^{-1}\Lambda_{i,\bar{k}}(z), \qquad (3.23)$$

where

$$\Lambda_{i,k}(z) = \overbrace{1 \otimes \cdots \otimes 1}^{k-1} \otimes :V_i\big(q^{k-\ell+1}z\big)^{-1}: \otimes \psi_i^+\big(q^{k-\ell+1}z\big)^{-1} \otimes \cdots \otimes \psi_i^+(z)^{-1} \otimes K_i^+(z),$$

$$\Lambda_{i,\bar{k}}(z) = \overbrace{1 \otimes \cdots \otimes 1}^{k-1} \otimes V_i\big(q^{-k+\ell-1}z\big) \otimes \psi_i^-\big(q^{-k+\ell-2}z\big) \otimes \cdots \otimes \psi_i^-\big(q^{-1}z\big) \otimes K_i^-(z).$$

In the right-hand side, $\psi_i^\pm(z)$ or $K_i^\pm(z)$ stand for their action on $\mathbb{F}(u_i)$ or $\mathbb{F}^D$, respectively.

In what follows we introduce an ordering $1 \prec \cdots \prec \ell \prec \bar{\ell} \prec \cdots \prec \bar{1}$ to the set $\{1, \ldots, \ell, \bar{\ell}, \ldots, \bar{1}\}$. We use letters $a, b, \ldots$ for elements of the latter, with the convention that $\bar{\bar{a}} = a$. Each $\Lambda_{i,a}(z)$ is a product of oscillator part $\Lambda_{i,a}^{\mathrm{osc}}(z)$ and the zero mode part. The contractions of the former are given by the following table.

Table 1. Contractions $\overset{\frown}{\Lambda_{i,a}^{\mathrm{osc}}(z)\Lambda_{j,b}^{\mathrm{osc}}(w)}$ with $x = w/z$

	$\begin{array}{c}a \prec b\\b \neq \bar{a}\end{array}$	$a = b$	$\begin{array}{c}a \succ b\\b \neq \bar{a}\end{array}$	$\begin{array}{c}a \prec b\\b = \bar{a}\end{array}$	$\begin{array}{c}a \succ b\\b = \bar{a}\end{array}$
$i \equiv j$	$\dfrac{1-q_2^{-1}x}{1-q_2 x}$	$(1-x)(1-q_2^{-1}x)$	1	$\dfrac{\dfrac{1-q_2^{-1}x}{1-q_2 x}\times 1}{(1! -C^2 q_2^{1-a}x)(1-C^2 q_2^{2-a}x)}$	$\dfrac{1}{(1-C^{-2}q_2^b x)(1-C^{-2}q_2^{b-1}x)}$
$i+1 \equiv j$	$\dfrac{1-q_3^{-1}x}{1-q_1 x}$	$\dfrac{1}{1-q_1 x}$	1	$\dfrac{1-q_3^{-1}x}{1-q_1 x}\,(1-C^2 q_2^{1-a}q_1 x)$	$1-C^{-2}q_2^{b-1}q_3^{-1}x$

The contractions in the case $i - 1 \equiv j$ can be obtained by switching q_1 and q_3.

Using this table one can check directly the wheel conditions, namely that $E_{i,i,i+1}(z_1, z_2, w)$ represented by

$$\frac{1 - C^2 z_2/z_1}{1 - z_2/z_1} \frac{1 - C^{-2} z_2/z_1}{1 - z_2/z_1} \frac{1 - q_2 z_2/z_1}{1 - z_2/z_1} \frac{1 - q_1 w/z_1}{1 - w/z_1} \frac{1 - q_1 w/z_2}{1 - w/z_2} \Lambda_{i,a}(z_1) \Lambda_{i,b}(z_2) \Lambda_{i+1,c}(w)$$

vanishes for all $a, b, c \in \{1, \ldots, \ell, \bar{\ell}, \ldots, \bar{1}\}$ under the following specializations:

(i) $(z_1, z_2, w) = (z, q_2 z, q_1^{-1} z)$,

(ii) $(z_1, z_2, w) = (z, C^2 z, C^2 q_3 z)$,

(iii) $(z_1, z_2, w) = (z, C^{-2} z, C^{-2} q_1^{-1} z)$.

Case (i). This follows from

$$\overbrace{\Lambda_{i,a}(z_1) \Lambda_{i,b}(z_2)}\big|_{z_2 = q_2 z_1} = 0 \qquad \text{if} \quad a \preceq b,$$

$$\overbrace{\Lambda_{i,b}(z_2) \Lambda_{i+1,c}(w)}\big|_{w = q_3 z_2} = 0 \qquad \text{if} \quad b \prec c,$$

$$(1 - q_1 w/z_1) \overbrace{\Lambda_{i,a}(z_1) \Lambda_{i+1,c}(w)}\big|_{w = q_1^{-1} z_1} = 0 \qquad \text{if} \quad a \succ c.$$

Case (ii). This follows from

$$(1 - C^{-2} z_2/z_1) \overbrace{\Lambda_{i,a}(z_1) \Lambda_{i,b}(z_2)}\big|_{z_2 = C^2 z_1} = 0 \qquad \text{if} \quad (a, b) \neq (\bar{1}, 1),$$

$$\overbrace{\Lambda_{i,1}(z_2) \Lambda_{i+1,c}(w)}\big|_{w = q_3 z_2} = 0 \qquad \text{if} \quad c \neq 1,$$

$$\overbrace{\Lambda_{i,\bar{1}}(z_1) \Lambda_{i+1,1}(w)}\big|_{w = C^2 q_3 z_1} = 0.$$

Case (iii). This follows from

$$(1 - C^2 z_2/z_1) \overbrace{\Lambda_{i,a}(z_1) \Lambda_{i,b}(z_2)}\big|_{z_2 = C^{-2} z_1} = 0 \qquad \text{if} \quad (a, b) \neq (1, \bar{1}),$$

$$(1 - q_1 w/z_2) \overbrace{\Lambda_{i,\bar{1}}(z_2) \Lambda_{i+1,c}(w)}\big|_{w = q_1^{-1} z_2} = 0 \qquad \text{if} \quad c \neq \bar{1},$$

$$\overbrace{\Lambda_{i,1}(z_1) \Lambda_{i+1,\bar{1}}(w)}\big|_{w = C^{-2} q_1^{-1} z_1} = 0.$$

Similarly for $E_{i,i,i-1}(z_1, z_2, w)$.

Remark 3.10. As it turns out, the currents (3.23) commute with a system of screening operators derived from quantum toroidal algebra of type D. We hope to address this point elsewhere.

4 Integrals of motion

We continue to work with admissible representations of $\mathcal{K}_n$ on which C acts as a scalar. Our goal in this section is to introduce a family of commuting operators acting on each such representation (with mild restrictions on parameters), which we call integrals of motion (IMs). Loosely speaking, we may think of them as generators of a commutative subalgebra of $\mathcal{K}_n$.

The IMs will depend on q, d, C as well as additional arbitrary parameters

$$\mu_1, \ldots, \mu_n \in \mathbb{C}, \qquad \sum_{i=1}^{n} \mu_i = 0. \tag{4.1}$$

In addition, they depend also on an "elliptic" parameter p. However, our experience with the algebra $\mathcal{K}_1$ tells that the IMs exist only when p and C are related in a special way. Guided by the result in [7], we shall impose the relation

$$C^2 = pq^2$$

and make the following assumption:

$$|p| < 1, \qquad \left|qd^{\pm 1}\right| < 1, \qquad \left|pq^{-2}\right| < 1, \qquad \left|pq^{-1}d^{\pm 1}\right| < 1. \tag{4.2}$$

In particular $|q^2| < 1$. We use parameters $\tau \in \mathbb{C}$ with $\operatorname{Im}\tau > 0$ and $\beta, \gamma \in \mathbb{C}$ given by

$$p = \mathrm{e}^{-2\pi\mathrm{i}/\tau}, \qquad q^2 = p^\gamma, \qquad d = p^\beta. \tag{4.3}$$

We use also $\theta(u) = p^{u^2/2 - u/2}\Theta_p(p^u)$, which satisfies

$$\theta(u + 1) = -\theta(u), \qquad \theta(u + \tau) = -\mathrm{e}^{-2\pi\mathrm{i}u - \pi\mathrm{i}\tau}\theta(u). \tag{4.4}$$

4.1 Dressed currents

A basic constituent of IMs is the dressed currents, obtained by modifying the currents $E_i(z)$ of $\mathcal{K}_n$.

In the following we define operators $H_{i,0}$ by $(K_{i,0}^-)^{-1}K_{i,0}^+ = q^{2H_{i,0}}$. The dressed currents of $\mathcal{K}_n$ are defined by

$$\mathbf{E}_i(z) = \mathbf{K}_i^-(z)^{-1}E_i(z)z^{-\gamma H_{i,0} - \gamma + \mu_i},$$
$$\mathbf{K}_i^-(z) = \exp\left(-(q - q^{-1})\sum_{r>0}\frac{H_{i,-r}}{1 - p^r}z^r\right),$$

where γ and μ_i are as in (4.3) and (4.1).

The dressed currents have well-defined matrix coefficients on admissible representations. They satisfy the exchange relations as meromorphic functions

$$\lambda_{i,j}(z_2/z_1)\mathbf{E}_i(z_1)\mathbf{E}_j(z_2) = \lambda_{j,i}(z_1/z_2)\mathbf{E}_j(z_2)\mathbf{E}_i(z_1),$$

with

$$\lambda_{i,i}(x) = x^{-\gamma}\frac{1 - C^2 x}{1 - x}\frac{1 - C^{-2}x}{1 - x}\frac{1 - q^2 x}{1 - x}\frac{\left(q^{-2}x; p\right)_\infty}{\left(q^2 x; p\right)_\infty},$$
$$\lambda_{i,i\pm 1}(x) = x^{\gamma/2}d^{\mp 1/2}\frac{1 - q^{-1}d^{\pm 1}x}{1 - x}\frac{\left(qd^{\pm 1}x; p\right)_\infty}{\left(q^{-1}d^{\pm 1}x; p\right)_\infty},$$

and $\lambda_{i,j}(x) = 1$ in all other cases. Up to a power of x, $\lambda_{i,j}(x)$ specializes to $\lambda_{i,j}^0(x)$ at $p = 0$. Each matrix coefficient of products of dressed currents is a meromorphic function (up to an overall power function). We have "elliptic" exchange relations as meromorphic functions

$$\mathbf{E}_i(z)\mathbf{E}_i(w) = \mathbf{E}_i(w)\mathbf{E}_i(z)\frac{\theta(v - u + \gamma)}{\theta(v - u - \gamma)},$$
$$\mathbf{E}_i(z)\mathbf{E}_{i\pm 1}(w) = \mathbf{E}_{i\pm 1}(w)\mathbf{E}_i(z)q^{\pm 2\beta}\frac{\theta(v - u \pm \beta - \gamma/2)}{\theta(v - u \pm \beta + \gamma/2)},$$

where $z = p^u$, $w = p^v$.

As before we define

$$\mathbf{E}_{i_1,\ldots,i_N}(z_1,\ldots,z_N) = \prod_{1\leq r<s\leq N} \lambda_{i_r,i_s}(z_s/z_r)\mathbf{E}_{i_1}(z_1)\cdots\mathbf{E}_{i_N}(z_N). \tag{4.5}$$

Then it has the same symmetry as (3.10), and we have

$$\mathbf{E}_{i_1,\ldots,i_N}(z_1,\ldots,z_N) = \prod_{r<s} z_s^{-\gamma(\bar\alpha_{i_r},\bar\alpha_{i_s})} \prod_{s=1}^{N} \mathbf{K}_{i_s}^{-}(z_s)^{-1} E_{i_1,\ldots,i_N}(z_1,\ldots,z_N) \prod_{s=1}^{N} z_s^{-\gamma H_{i_s,0}-\gamma+\mu_{i_s}}.$$

The product $\prod_{s=1}^{N} \mathbf{K}_{i_s}^{-}(z_s)^{-1}$ contains only $\{H_{i,-r}\}$ with $r > 0$. Its contribution to any matrix coefficient of $\mathbf{E}_{i_1,\ldots,i_N}(z_1,\ldots,z_N)$ is a Laurent polynomial in z_s's for degree reasons. Hence the equality above shows that, up to an overall power of z_i's, each matrix coefficient of operator (4.5) has the same functional form (3.11) as that of $E_{i_1,\ldots,i_N}(z_1,\ldots,z_N)$. In particular the wheel conditions (3.13)–(3.15) are valid also for (4.5).

4.2 Integrals of motion

Given parameters (4.1), let $X_n = X_n(\mu_1,\ldots,\mu_n)$ denote the space of entire functions $\vartheta(u_1,\ldots,u_n)$ satisfying

$$\vartheta(u_1,\ldots,u_i+1,\ldots,u_n) = \vartheta(u_1,\ldots,u_i,\ldots,u_n), \tag{4.6}$$

$$\vartheta(u_1,\ldots,u_i+\tau,\ldots,u_n) = e^{-2\pi i(2u_i-u_{i-1}-u_{i+1}-\mu_i+\tau)} \vartheta(u_1,\ldots,u_i,\ldots,u_n), \tag{4.7}$$

where $u_{i+n} = u_i$. We have $\dim X_n = n$.

For $M \geq 1$ and $\vartheta \in X_n$, we introduce the kernel functions

$$h_M(\mathbf{u}_1,\ldots,\mathbf{u}_n;\vartheta)$$
$$= \frac{\vartheta(\bar u_1,\ldots,\bar u_n)\prod_{i=1}^{n}\prod_{1\leq a<b\leq M}\theta(u_{i,a}-u_{i,b})\theta(u_{i,a}-u_{i,b}-\gamma)}{\prod_{i=1}^{n-1}\prod_{1\leq a,b\leq M}\theta\left(u_{i,a}-u_{i+1,b}-\beta-\frac{\gamma}{2}\right)\prod_{1\leq a,b\leq M}\theta\left(u_{n,a}-u_{1,b}-\beta+\frac{\gamma}{2}\right)}, \tag{4.8}$$

where

$$\mathbf{u}_i = (u_{i,1},\ldots,u_{i,M}), \qquad \bar u_i = \sum_{a=1}^{M} u_{i,a}.$$

We define the integrals of motion $\mathbf{G}_M(\vartheta)$ of $\mathcal{K}_n$ by

$$\mathbf{G}_M(\vartheta) = \int\cdots\int \overset{\frown}{\prod_{1\leq i\leq n}} \overset{\frown}{\prod_{1\leq a\leq M}} \mathbf{E}_i(x_{i,a}) \cdot h_M(\mathbf{u}_1,\ldots,\mathbf{u}_n;\vartheta) \prod_{i=1}^{n}\prod_{a=1}^{M} \frac{dx_{i,a}}{x_{i,a}}. \tag{4.9}$$

Here $x_{i,a} = p^{u_{i,a}}$, and the symbol $\overset{\frown}{\prod}_{1\leq i\leq N}$ stands for ordered product

$$\overset{\frown}{\prod_{1\leq i\leq N}} A_i = A_N A_{N-1}\cdots A_1.$$

Even though individual factors contain fractional powers, the integrand of (4.9) comprises only integer powers with respect to each $x_{i,a}$. The integrals are taken over a common circle $|x_{i,a}| = R$. The result is independent of the choice of $R > 0$.

Formally, (4.9) looks identical to that of the integrals of motion of type A [5, 8, 12]. In particular the kernel functions (4.8) are the same as those used there. The difference is hidden in the pole structure of products of dressed currents.

For later use we rewrite the integrand in terms of the current in (4.5),

$$\mathbf{E}^{(M)}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \mathbf{E}_{1,\ldots,1,\ldots,n,\ldots,n}(\mathbf{x}_1, \ldots, \mathbf{x}_n),$$

where $\mathbf{x}_i = (x_{i,1}, \ldots, x_{i,M})$. In the right-hand side, each index i occurs M times.

Proposition 4.1. *Set* $x = p^u$ *and*

$$\xi(u) = p^{u^2/2}(1-x)\big(x, p^2 q^2 x, p; p\big)_\infty,$$

$$\eta(u) = p^{u^2/2-(\beta+1/2)u} \frac{d^{-1/2}}{1-x^{-1}} \big(qd^{-1}x, qdx^{-1}, p; p\big)_\infty.$$

Then the IM's can be expressed as

$$\mathbf{G}_M(\vartheta) = c_M \int \cdots \int \mathbf{E}^{(M)}(\mathbf{x}_1, \ldots, \mathbf{x}_n)\, k_M(\mathbf{u}_1, \ldots, \mathbf{u}_n; \vartheta) \prod_{i=1}^{n} \prod_{a=1}^{M} \frac{\mathrm{d}x_{i,a}}{x_{i,a}}, \qquad (4.10)$$

where c_M *is a constant, and*

$$k_M(\mathbf{u}_1, \ldots, \mathbf{u}_n; \vartheta) = \vartheta(\bar{u}_1, \ldots, \bar{u}_n) \frac{\prod_{i=1}^{n} \prod_{1 \le a \ne b \le M} \xi(u_{i,a} - u_{i,b})}{\prod_{i=1}^{n} \prod_{1 \le a,b \le M} \eta(u_{i,a} - u_{i+1,b})},$$

$$x_{i,a} = p^{u_{i,a}}, \qquad \bar{u}_i = \sum_{a=1}^{M} u_{i,a}.$$

Here and after, we set $u_{n+1,a} = u_{1,a}$.

Proof. This follows from the relations

$$\frac{\lambda_{i,i}(p^{-u})}{\lambda_{i,i}(p^u)} = \frac{\theta(u+\gamma)}{\theta(u-\gamma)}, \qquad \frac{\lambda_{i,i\mp1}(p^{-u})}{\lambda_{i,i\pm1}(p^u)} = q^{\pm2\beta} \frac{\theta(u \pm \beta - \gamma/2)}{\theta(u \pm \beta + \gamma/2)},$$

$$\lambda_{i,i}(p^u)\xi(u)\xi(-u) = -p^{-\gamma^2/2-3\gamma/2-1}\theta(u)\theta(u-\gamma),$$

$$\lambda_{i,i\pm1}\big(p^{\mp u}\big)\theta(u \pm \gamma/2 - \beta) = p^{(\gamma/2\mp\beta)^2/2-(\gamma/2-\beta)/2}\, \eta(u). \qquad \blacksquare$$

4.3   Commutativity

We are now in a position to state the commutativity of IMs.

Theorem 4.2. *For all* $M, N \ge 1$ *and* $\vartheta_1, \vartheta_2 \in X_n$, *the integrals of motion mutually commute:*

$$[\mathbf{G}_M(\vartheta_1), \mathbf{G}_N(\vartheta_2)] = 0.$$

Theorem 4.2 is the main result of this paper. The rest of this section is devoted to its proof.

Proof. We start by examining the product of operators, dropping irrelevant constants

$$\mathbf{G}_M(\vartheta_1) = \int \cdots \int_{\mathcal{C}_1} \mathbf{E}^{(M)}(\mathbf{x}_1, \ldots, \mathbf{x}_n)\, k_M(\mathbf{u}_1, \ldots, \mathbf{u}_n; \vartheta_1) \prod_{i=1}^{n} \prod_{a=1}^{M} \frac{\mathrm{d}x_{i,a}}{x_{i,a}},$$

$$\mathbf{G}_N(\vartheta_2) = \int \cdots \int_{\mathcal{C}_R} \mathbf{E}^{(N)}(\mathbf{y}_1, \ldots, \mathbf{y}_n)\, k_N(\mathbf{v}_1, \ldots, \mathbf{v}_n; \vartheta_2) \prod_{i=1}^{n} \prod_{b=1}^{N} \frac{\mathrm{d}y_{i,b}}{y_{i,b}}.$$

Here $x_{i,a} = p^{u_{i,a}}$, $y_{i,b} = p^{v_{i,b}}$. We choose the contours to be the unit circle $\mathcal{C}_1 \colon |x_{i,a}| = 1$ for $\mathbf{G}_M(\vartheta_1)$, and a circle $\mathcal{C}_R \colon |y_{i,b}| = R$ of radius R for $\mathbf{G}_N(\vartheta_2)$.

Consider first the product

$$\mathbf{G}_N(\vartheta_2)\mathbf{G}_M(\vartheta_1) = \int \cdots \int_{\mathcal{C}_R} \int \cdots \int_{\mathcal{C}_1} \mathbf{E}^{(N)}(\mathbf{y}_1, \ldots, \mathbf{y}_n)\, \mathbf{E}^{(M)}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$$

$$\times\, k_N(\mathbf{v}_1, \ldots, \mathbf{v}_n; \vartheta_2)\, k_M(\mathbf{u}_1, \ldots, \mathbf{u}_n; \vartheta_1) \prod_{i=1}^{n}\prod_{b=1}^{N} \frac{\mathrm{d}y_{i,b}}{y_{i,b}} \prod_{i=1}^{n}\prod_{a=1}^{M} \frac{\mathrm{d}x_{i,a}}{x_{i,a}}. \quad (4.11)$$

We choose $R \gg 1$ to ensure that the product of operators in the integrand converges absolutely. Introducing the combined variables $\mathbf{z}_i = (z_{i,1}, \ldots, z_{i,M+N})$, where $z_{i,a} = x_{i,a}$, $1 \leq a \leq M$, $z_{i,a+M} = y_{i,a}$, and $u_{i,M+a} = v_{i,a}$, $1 \leq a \leq N$, and substituting

$$\mathbf{E}^{(N)}(\mathbf{y}_1, \ldots, \mathbf{y}_n)\mathbf{E}^{(M)}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$$

$$= \mathbf{E}^{(M+N)}(\mathbf{z}_1, \ldots, \mathbf{z}_n) \prod_{i=1}^{n} \prod_{\substack{1 \leq a \leq M \\ 1 \leq b \leq N}} \lambda_{i,i}(x_{i,a}/y_{i,b})^{-1} \prod_{i=1}^{n} \prod_{\substack{1 \leq a \leq M \\ 1 \leq b \leq N}} \lambda_{i+1,i}(x_{i,a}/y_{i+1,b})^{-1}$$

$$\times \prod_{i=1}^{n} \prod_{\substack{1 \leq a \leq M \\ 1 \leq b \leq N}} \lambda_{i-1,i}(x_{i,a}/y_{i-1,b})^{-1},$$

we find that the integrand of (4.11) can be written as a product of two factors ST, with

$$S(\mathbf{z}_1, \ldots, \mathbf{z}_n)$$

$$= \mathbf{E}^{(M+N)}(\mathbf{z}_1, \ldots, \mathbf{z}_n) \prod_{i=1}^{n} \prod_{1 \leq a \neq b \leq M+N} \xi(u_{i,a} - u_{i,b}) \prod_{i=1}^{n} \prod_{1 \leq a,b \leq M+N} \eta(u_{i,a} - u_{i+1,b})^{-1},$$

$$T(\mathbf{u}_1, \ldots, \mathbf{u}_n | \mathbf{v}_1, \ldots, \mathbf{v}_n)$$

$$= \vartheta_1(\overline{u}_1, \ldots, \overline{u}_n)\vartheta_2(\overline{v}_1, \ldots, \overline{v}_n)$$

$$\times \prod_{i=1}^{n} \prod_{\substack{1 \leq a \leq M \\ 1 \leq b \leq N}} \frac{\theta(u_{i,a} - v_{i+1,b} - \gamma/2 - \beta)\theta(v_{i,b} - u_{i+1,a} + \gamma/2 - \beta)}{\theta(u_{i,a} - v_{i,b})\theta(u_{i,a} - v_{i,b} - \gamma)}. \quad (4.12)$$

Starting from $R \gg 1$, we bring the contours $\mathcal{C}_R$ for $y_{i,b}$ to the unit circle. For that purpose we need to locate the position of poles between the two circles $\mathcal{C}_R$ and $\mathcal{C}_1$. The poles between y's and x's come from ξ, η and θ:

$$\prod_{i=1}^{n} \prod_{\substack{1 \leq a \leq M \\ 1 \leq b \leq N}} (1 - x_{i,a}/y_{i,b})(1 - y_{i,b}/x_{i,a}) \left(\overline{x_{i,a}/y_{i,b}}, p^2q^2x_{i,a}/y_{i,b}, \overline{y_{i,b}/x_{i,a}}^{\,1-y_{i,b}/x_{i,a}}, p^2q^2y_{i,b}/x_{i,a}; p\right)_{\infty}$$

$$\times \prod_{i=1}^{n} \prod_{\substack{1 \leq a \leq M \\ 1 \leq b \leq N}} \frac{(1 - y_{i+1,b}/x_{i,a})(1 - x_{i+1,a}/y_{i,b})}{\left(qd^{-1}x_{i,a}/y_{i+1,b}, \overline{qdy_{i+1,b}/x_{i,a}}, \overline{qd^{-1}y_{i,b}/x_{i+1,a}}, qdx_{i+1,a}/y_{i,b}; p\right)_{\infty}}$$

$$\times \prod_{i=1}^{n} \prod_{\substack{1 \leq a \leq M \\ 1 \leq b \leq N}} \frac{\left(pq^{-1}d^{-1}x_{i,a}/y_{i+1,b}, \overline{qdy_{i+1,b}/x_{i,a}}, \overline{qd^{-1}y_{i,b}/x_{i+1,a}}, pq^{-1}dx_{i+1,a}/y_{i,b}; p\right)_{\infty}}{\left(\overline{x_{i,a}/y_{i,b}}, py_{i,b}/x_{i,a}, q^{-2}x_{i,a}/y_{i,b}, pq^2y_{i,b}/x_{i,a}; p\right)_{\infty}}.$$

Under our assumption on parameters (4.2), we see that the only poles between $\mathcal{C}_R$ and $\mathcal{C}_1$ are

$$y_{i,b} = q_2^{-1}x_{i,a}, \, p^{-1}q_2^{-1}x_{i,a}, \qquad 1 \leq i \leq n, \quad 1 \leq a \leq M, \quad 1 \leq b \leq N. \quad (4.13)$$

Including the poles between y's which come from η, poles of the integrand with respect to each $y_{i,b}$ are as follows (see the figure below):

$$
\begin{aligned}
y_{i,b} &= p^{-k}q_1 y_{i+1,c}, & p^{-k}q_3 y_{i-1,c}, & \qquad k \geq 0, \\
y_{i,b} &= p^k q_3^{-1} y_{i+1,c}, & p^k q_1^{-1} y_{i-1,c}, & \qquad k \geq 0, \\
y_{i,b} &= p^k q_2^{-1} x_{i,a}, & k \geq -1, & \\
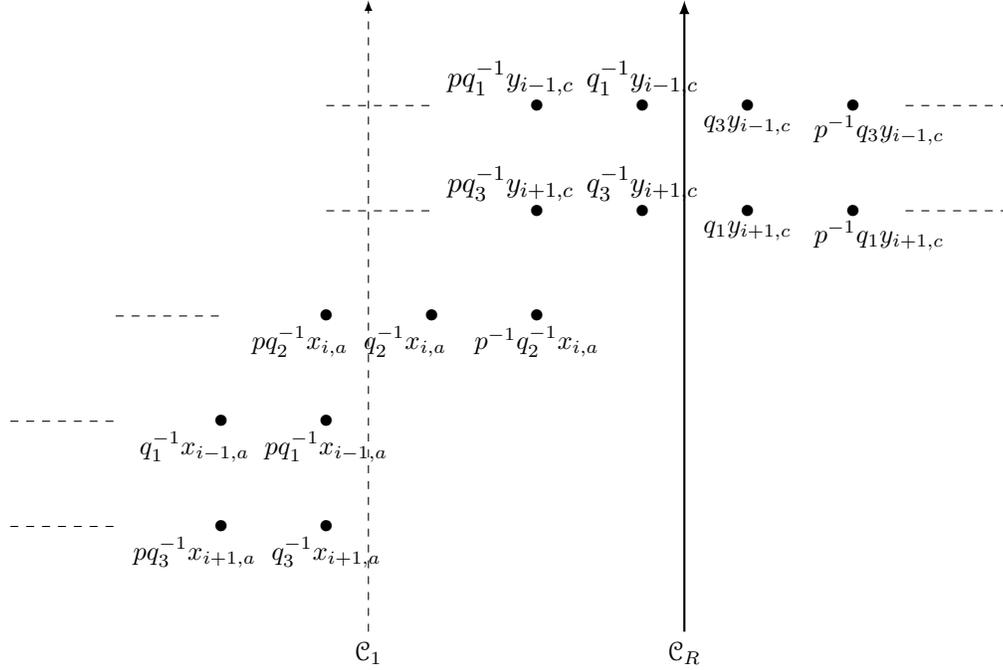y_{i,b} &= p^k q_1^{-1} x_{i-1,a}, & p^k q_3^{-1} x_{i+1,a}, & \qquad k \geq 0.
\end{aligned}
$$

Figure 1. The contour in the $y_{i,b}$ plane.

We move the contour to the unit circle, picking up residues of poles (4.13) along the way. At first glance, taking residues in one variable $y_{i,b} = p^{-1}q_2^{-1}x_{i,a}$ or $y_{i,b} = q_2^{-1}x_{i,a}$ seems to produce new poles in other variables,

$$
\begin{aligned}
y_{i+1,c} &= q_2^{-1}q_1^{-1}x_{i,a}, & p^{-1}q_2^{-1}q_1^{-1}x_{i,a}, \\
y_{i-1,c} &= q_2^{-1}q_3^{-1}x_{i,a}, & p^{-1}q_2^{-1}q_3^{-1}x_{i,a}.
\end{aligned}
$$

An important point is that these poles are cancelled by the zeros of $\mathbf{E}^{(M+N)}(\mathbf{z}_1, \ldots, \mathbf{z}_n)$ due to the wheel conditions (3.13)–(3.15).

For the product in the opposite order

$$
\mathbf{G}_M(\vartheta_1)\mathbf{G}_N(\vartheta_2) = \int \cdots \int_{\mathcal{C}_1} \int \cdots \int_{\mathcal{C}_R} \mathbf{E}^{(M)}(\mathbf{x}_1, \ldots, \mathbf{x}_n)\,\mathbf{E}^{(N)}(\mathbf{y}_1, \ldots, \mathbf{y}_n)
$$

$$
\times k_M(\mathbf{u}_1, \ldots, \mathbf{u}_n; \vartheta_1)k_N(\mathbf{v}_1, \ldots, \mathbf{v}_n; \vartheta_2) \prod_{i=1}^{n}\prod_{a=1}^{M} \frac{\mathrm{d}x_{i,a}}{x_{i,a}} \prod_{i=1}^{n}\prod_{b=1}^{N} \frac{\mathrm{d}y_{i,b}}{y_{i,b}}, \quad (4.14)
$$

we choose $R \ll 1$ for convergence. The integrand can be obtained from the one for (4.11) by switching the roles of $\mathbf{x}$, M and $\mathbf{y}$, N. It amounts to changing (4.12) to

$$T'(\mathbf{u}_1, \ldots, \mathbf{u}_n | \mathbf{v}_1, \ldots, \mathbf{v}_n) = \vartheta_1(\overline{u}_1, \ldots, \overline{u}_n)\vartheta_2(\overline{v}_1, \ldots, \overline{v}_n)$$
$$\times \prod_{i=1}^{n} \prod_{\substack{1 \le a \le M \\ 1 \le b \le N}} \frac{\theta(v_{i,b} - u_{i+1,a} - \gamma/2 - \beta)\theta(u_{i,a} - v_{i+1,b} + \gamma/2 - \beta)}{\theta(v_{i,b} - u_{i,a})\theta(v_{i,b} - u_{i,a} - \gamma)}.$$

Similarly as above, we start from $R \ll 1$ and bring $\mathcal{C}_R$ to the unit circle. The relevant poles in between are

$$y_{i,b} = q_2 x_{i,a}, \, p q_2 x_{i,a}, \qquad 1 \le i \le n, \quad 1 \le a \le M, \quad 1 \le b \le N.$$

Moving the contour we pick up residues of these poles. Again the wheel conditions ensure that no new poles arise for the remaining variables.

In order to prove the commutativity, we must show that, after moving contours to the unit circle, (4.11) and (4.14) give the same result.

First, let us compare terms that arise without taking residues. Since all integrals are taken on the unit circle, we can symmetrize the integrand. Taking into account the symmetry of $S(\mathbf{z}_1, \ldots, \mathbf{z}_n)$, we see that the equality of integrals reduces to the identity of theta functions

$$\text{Sym } T(\mathbf{u}_1, \ldots, \mathbf{u}_n | \mathbf{v}_1, \ldots, \mathbf{v}_n) = \text{Sym } T'(\mathbf{u}_1, \ldots, \mathbf{u}_n | \mathbf{v}_1, \ldots, \mathbf{v}_n), \tag{4.15}$$

where Sym stands for symmetrization in each group of variables $\{u_{i,a}\}_{a=1}^{M} \cup \{v_{i,b}\}_{b=1}^{N}$, $1 \le i \le n$. This identity was stated in [12]. Since their proof contains some gaps, we prove it in Appendix; see Theorem A.1 and Remark A.6 below.

In general, we compare terms obtained by taking residues with respect to some group of variables. First consider terms coming from (4.11). In view of symmetry and the zeros of $S(\mathbf{z}_1, \ldots, \mathbf{z}_n)$ at $y_{i,a} = y_{i,b}$, it is enough to consider the case

$$y_{i,a} = p^{-1} q_2^{-1} x_{i,a}, \qquad 1 \le a \le k_i, \tag{4.16}$$
$$y_{i,b} = q_2^{-1} x_{i,b}, \qquad k_i + 1 \le b \le l_i, \tag{4.17}$$

with some $0 \le k_i \le l_i \le \min(M, N)$, $1 \le i \le n$. We compare it with the corresponding residue coming from (4.14)

$$y_{i,a} = p q_2 x_{i,a}, \qquad 1 \le a \le k_i, \tag{4.18}$$
$$y_{i,b} = q_2 x_{i,b}, \qquad k_i + 1 \le b \le l_i, \tag{4.19}$$

for $1 \le i \le n$. We further rename variables $x_{i,a}$ to $p^{-1} q_2^{-1} x_{i,a}$ for (4.18) and to $q_2^{-1} x_{i,a}$ for (4.19). The contours for them change to $\left| p^{-1} q_2^{-1} x_{i,a} \right| = 1$ and $\left| q_2^{-1} x_{i,a} \right| = 1$, respectively. We shift them back to the unit circle $|x_{i,a}| = 1$, noting that the wheel conditions ensure there are no poles which hinder the shift.

The factor $S(\mathbf{z}_1, \ldots, \mathbf{z}_n)$ has no poles at (4.16) and (4.17) and specializes to

$$S\big(\ldots, z_{i,a}, \ldots, z_{i,b}, \ldots, p^{-1} q_2^{-1} z_{i,a}, \ldots, q_2^{-1} z_{i,b}, \ldots\big). \tag{4.20}$$

Similarly, at (4.18) and (4.19) it specializes to

$$S\big(\ldots, z_{i,a}, \ldots, z_{i,b}, \ldots, p q_2 z_{i,a}, \ldots, q_2 z_{i,b}, \ldots\big). \tag{4.21}$$

After the renaming, (4.21) is brought to (4.20) because $S(\mathbf{z}_1, \ldots, \mathbf{z}_n)$ is symmetric in $\{z_{i,a}\}_{1 \le a \le M+N}$ for each i.

It remains to compare the residues of the functions T, T'. Since they are periodic with period 1, their residues at $v_{i,a} = u_{i,a} - \gamma - 1$ are the same as those at $v_{i,a} = u_{i,a} - \gamma$. Hence we are to show the equality

$$\mathrm{Sym}^* \operatorname*{res}_{\substack{v_{i,a}=u_{i,a}-\gamma, \\ 1\le a\le l_i,\, 1\le i\le n}} T(\mathbf{u}_1,\ldots,\mathbf{u}_n|\mathbf{v}_1,\ldots,\mathbf{v}_n)$$

$$= \mathrm{Sym}^* \left\{ (-1)^{\sum_{i=1}^n l_i} \operatorname*{res}_{\substack{v_{i,a}=u_{i,a}+\gamma, \\ 1\le a\le l_i,\, 1\le i\le n}} T'(\mathbf{u}_1,\ldots,\mathbf{u}_n|\mathbf{v}_1,\ldots,\mathbf{v}_n) \right\} \Bigg|_{\substack{u_{i,a}\to u_{i,a}-\gamma, \\ 1\le a\le l_i,\, 1\le i\le n}},$$

where Sym^* stands for the symmetrization with respect to the remaining variables $\{u_{i,a}\}_{a=l_i+1}^M \cup \{v_{i,b}\}_{b=l_i+1}^N$, $1 \le i \le n$. Note that moving the contours $\mathcal{C}_R$ from $R < 1$ to $\mathcal{C}_1$ we obtain a sign factor.

To check the equality, we start from the residues of the identity (4.15)

$$\operatorname*{res}_{\substack{v_{i,a}=u_{i,a}-\gamma, \\ 1\le a\le l_i,\, 1\le i\le n}} \mathrm{Sym}\, T(\mathbf{u}_1,\ldots,\mathbf{u}_n|\mathbf{v}_1,\ldots,\mathbf{v}_n) = \operatorname*{res}_{\substack{v_{i,a}=u_{i,a}-\gamma, \\ 1\le a\le l_i,\, 1\le i\le n}} \mathrm{Sym}\, T'(\mathbf{u}_1,\ldots,\mathbf{u}_n|\mathbf{v}_1,\ldots,\mathbf{v}_n). \quad (4.22)$$

Symmetrization Sym amounts to replacing variables $\mathbf{u}_i$, $\mathbf{v}_i$ as

$$\mathbf{u}_i \to \{u_{i,a}\}_{a\in I_i} \cup \{v_{i,b}\}_{b\in J_i'}, \qquad \mathbf{v}_i \to \{u_{i,a}\}_{a\in I_i'} \cup \{v_{i,b}\}_{b\in J_i},$$

and sum over partitions of indices

$$I_i \sqcup I_i' = \{1,\ldots,M\}, \qquad J_i \sqcup J_i' = \{1,\ldots,N\}, \qquad |I_i'| = |J_i'|.$$

In order to have non-zero residues at $v_{i,a} = u_{i,a} - \gamma$ $(1 \le a \le l_i)$ in the left-hand side of (4.22), I_i and J_i must contain $\{1,\ldots,l_i\}$. Therefore the left-hand side of (4.22) reduces to

$$\mathrm{Sym}^* \operatorname*{res}_{\substack{v_{i,a}=u_{i,a}-\gamma, \\ 1\le a\le l_i,\, 1\le i\le n}} T(\mathbf{u}_1,\ldots,\mathbf{u}_n|\mathbf{v}_1,\ldots,\mathbf{v}_n).$$

Similarly, in the right-hand side of (4.22), non-zero residues appear only when J_i' and I_i' contain $\{1,\ldots,l_i\}$. For each i we have

$$\operatorname*{res}_{\substack{v_{i,a}=u_{i,a}-\gamma, \\ 1\le a\le l_i}} T'(\ldots,v_{i,1},\ldots,v_{i,l_i},\ldots|\ldots,u_{i,1},\ldots,u_{i,l_i},\ldots)$$

$$= (-1)^{l_i} \operatorname*{res}_{\substack{u_{i,a}'=v_{i,a}'+\gamma, \\ 1\le a\le l_i}} T'(\ldots,v_{i,1}',\ldots,v_{i,l_i}',\ldots|\ldots,u_{i,1}',\ldots,u_{i,l_i}',\ldots)\Big|_{\substack{v_{i,a}'\to u_{i,a}-\gamma. \\ 1\le a\le l_i}}$$

Altogether we can rewrite the right-hand side of (4.22) as

$$\mathrm{Sym}^* \left\{ (-1)^{\sum_{i=1}^n l_i} \operatorname*{res}_{\substack{v_{i,a}=u_{i,a}+\gamma, \\ 1\le a\le l_i,\, 1\le i\le n}} T'(\mathbf{u}_1,\ldots,\mathbf{u}_n|\mathbf{v}_1,\ldots,\mathbf{v}_n) \right\} \Bigg|_{\substack{u_{i,a}\to u_{i,a}-\gamma. \\ 1\le a\le l_i,\, 1\le i\le n}}.$$

Proof of Theorem 4.2 is now complete.   ■

5   The cases $n = 1, 2$

All results discussed so far are valid also in the case $n = 1, 2$ with suitable modifications. In this section we indicate the necessary changes.

5.1   Case $n = 2$

Dealing with algebras $\mathcal{E}_n$ and $\mathcal{K}_n$ for $n = 2$, there are two points to be taken care of.

First, the structure functions are changed. We keep $g_{i,i}(z, w)$ unchanged but replace $g_{i,j}(z, w)$ for $i \neq j$ with $g_{i,j}^{n \geq 3}(z, w) g_{j,i}^{n \geq 3}(z, w)$. Similar changes are due for $G_{i,j}(x)$, $\lambda_{i,j}^0(x)$ and $\lambda_{i,j}(x)$. Namely we use

$$g_{i,1-i}(z, w) = (z - q_1 w)(z - q_3 w),$$

$$G_{i,1-i}(x) = q^{-2} \frac{\left(1 - q_1^{-1} x\right)\left(1 - q_3^{-1} x\right)}{(1 - q_1 x)(1 - q_3 x)},$$

$$\lambda_{i,1-i}^0(x) = \frac{1 - q_1 x}{1 - x} \frac{1 - q_3 x}{1 - x},$$

$$\lambda_{i,1-i}(x) = x^\gamma \frac{1 - q_1 x}{1 - x} \frac{1 - q_3 x}{1 - x} \frac{\left(q_1^{-1} x, q_3^{-1} x; p\right)_\infty}{(q_1 x, q_3 x; p)_\infty}.$$

We set also $d_{i,i} = 1$, $d_{i,1-i} = -1$.

Second, the Serre relations are modified.

The defining relations of $\mathcal{E}_2$ are the same as those for $n \geq 3$ except for the Serre relations

$$\underset{z_1, z_2, z_3}{\mathrm{Sym}} [e_i(z_1), [e_i(z_2), [e_i(z_3), e_{1-i}(w)]_{q^2}]]_{q^{-2}} = 0, \tag{5.1}$$

$$\underset{z_1, z_2, z_3}{\mathrm{Sym}} [f_i(z_1), [f_i(z_2), [f_i(z_3), f_{1-i}(w)]_{q^2}]]_{q^{-2}} = 0. \tag{5.2}$$

The defining relations of $\mathcal{K}_2$ are (3.1)–(3.4), together with the Serre relations

$$\underset{z_1, z_2}{\mathrm{Sym}} \big(q_1(z_1 - q_3 w)(z_2 - q_3 w) E_i(z_1) E_i(z_2) E_{1-i}(w) - \big(1 + q_2^{-1}\big)(z_1 - q_3 w)(q_1 z_2 - w)$$

$$\times E_i(z_1) E_{1-i}(w) E_i(z_2) + q_3(q_1 z_1 - w)(q_1 z_2 - w) E_{1-i}(w) E_i(z_1) E_i(z_2) \big)$$

$$= \underset{z_1, z_2}{\mathrm{Sym}} \bigg\{ q^{-3} w \bigg((z_2 - d^{-2} z_1) \frac{1 - q_3^{-1} z_1/w}{1 - q_3 z_1/w} + (z_2 - d^2 z_1) \frac{1 - q_1^{-1} w/z_2}{1 - q_1 w/z_2} - z_2 + q^2 z_1 \bigg)$$

$$\times \delta\big(C^2 z_1/z_2\big) K_i^-(z_1) E_{1-i}(w) K_i^+(z_2) \bigg\}, \tag{5.3}$$

and the same relation with q_1 and q_3 interchanged.

At first glance, the quartic Serre relations (5.1) and (5.2) for $\mathcal{E}_2$ and the cubic Serre relations (5.3) for $\mathcal{K}_2$ look very different. Actually the former are equivalent (under the rest of the relations) to cubic relations which are the left-hand sides of (5.3) and their "f-version". See [3, Lemma 2.1] and remark after that.

With the above changes, formula for comodule structure (Theorem 3.4), formulas for integrals of motion (4.9) and (4.10) and their commutativity (Theorem 4.2) remain valid.

5.2   Case $n = 1$

Algebra $\mathcal{K}_1$ has been discussed in [7]. For reader's reference we mention the necessary changes. We drop suffixes i, j from structure functions and set

$$g(z, w) = \prod_{s=1}^{3} (z - q_s w),$$

$$G(x) = \prod_{s=1}^{3} \frac{1 - q_s^{-1} x}{1 - q_s x},$$

$$\lambda^0(x) = \frac{1 - C^2 x}{1 - x} \frac{1 - C^{-2} x}{1 - x} \prod_{s=1}^{3} \frac{1 - q_s x}{1 - x},$$

$$\lambda(x) = \frac{1 - C^2 x}{1 - x} \frac{1 - C^{-2} x}{1 - x} \prod_{s=1}^{3} \frac{1 - q_s x}{1 - x} \frac{(q_s^{-1} x; p)_\infty}{(q_s x; p)_\infty}.$$

We change also the normalization of generators of $\mathcal{E}_1$ slightly, so that

$$\psi^\pm(z) = \exp\left(\sum_{\pm r > 0} \kappa_r h_{\pm r} z^{\mp r} \right),$$

$$[e(z), f(w)] = \frac{1}{\kappa_1} \left(\delta(Cw/z)\psi^+(w) - \delta(Cz/w)\psi^-(z) \right),$$

where $\kappa_r = \prod_{s=1}^{3} (1 - q_s^r)$.

The Serre relations for $\mathcal{E}_1$ are

$$\operatorname*{Sym}_{z_1, z_2, z_3} z_2 z_3^{-1} [e(z_1), [e(z_2), e(z_3)]] = 0,$$

$$\operatorname*{Sym}_{z_1, z_2, z_3} z_2 z_3^{-1} [f(z_1), [f(z_2), f(z_3)]] = 0.$$

The Serre relation for $\mathcal{K}_1$ reads

$$\operatorname*{Sym}_{z_1, z_2, z_3} \frac{z_2}{z_3} [E(z_1), [E(z_2), E(z_3)]]$$

$$= \operatorname*{Sym}_{z_1, z_2, z_3} \left\{ X(z_1, z_2, z_3) \frac{1}{q - q^{-1}} \delta(C^2 z_1 / z_3) K^-(z_1) E(z_2) K^+(z_3) \right\},$$

where

$$X(z_1, z_2, z_3) = \frac{(z_1 + z_2)(z_3^2 - z_1 z_2)}{z_1 z_2 z_3} G(z_2/z_3) + \frac{(z_2 + z_3)(z_1^2 - z_2 z_3)}{z_1 z_2 z_3} G(z_1/z_2)$$

$$+ \frac{(z_3 + z_1)(z_2^2 - z_3 z_1)}{z_1 z_2 z_3}.$$

The space of theta functions X_1 is defined to be that of constant functions $\mathbb{C}$. Replacing $h_M(\mathbf{u}_1, \ldots, \mathbf{u}_n; \vartheta)$ by

$$h_M(\mathbf{u}) = \prod_{1 \le a < b \le M} \frac{\theta(u_a - u_b)\theta(u_a - u_b - \gamma)}{\theta(u_a - u_b - \beta - \gamma/2)\theta(u_a - u_b + \beta - \gamma/2)},$$

we define integrals of motion by (4.9). Then the commutativity Theorem 4.2 holds true.

6 Fusion

It is known [3] that the completed quantum toroidal algebra $\tilde{\mathcal{E}}_n$ with parameters q_1, q_2, q_3 contains various subalgebras isomorphic to $\tilde{\mathcal{E}}_k$, with $1 \le k \le n - 1$ and suitable parameters $\bar{q}_1$, $\bar{q}_2$, $\bar{q}_3$. In this section we discuss an analogue of this construction for $\mathcal{K}_n$. Throughout this section we take $n \ge 2$, and fix an admissible $\mathcal{K}_n$ module V, where C acts as a scalar such that $C^2 \ne \pm 1, q^{\pm 2}$.

Consider the following set of operators $\left\{ \bar{E}_i(z), \bar{K}_i^\pm(z) \right\}_{0 \le i \le n-2}$ acting on V:

$$\bar{E}_0(z) = \bar{c}\, E_{n-1,0}(q_1 z, z), \qquad \bar{K}_0^\pm(z) = K_{n-1}^\pm(q_1 z) K_0^\pm(z), \tag{6.1}$$

$$\bar{E}_i(z) = E_i\big(q_1^{\frac{i}{n-1}} z\big), \qquad \bar{K}_i^\pm(z) = K_i^\pm\big(q_1^{\frac{i}{n-1}} z\big), \qquad 1 \le i \le n-2. \tag{6.2}$$

Here $\bar{c} = (qq_1)^{1/2}\big(1 - q_1^{-1}\big)$ for $n \ge 3$ and $\bar{c} = \big(1 - q_1^{-1}\big)^2$ for $n = 2$.

Proposition 6.1. *The operators* (6.1) *and* (6.2) *together with the same C give an action of $\mathcal{K}_{n-1}$ on V with parameters*

$$\bar{q}_1 = q_1 q_1^{\frac{1}{n-1}}, \qquad \bar{q}_2 = q_2, \qquad \bar{q}_3 = q_3 q_1^{-\frac{1}{n-1}}.$$

Proof. Denote by $\bar{G}_{i,j}(x)$, $0 \leq i, j \leq n-2$ the functions $G_{i,j}(x)$ for $\mathcal{K}_{n-1}$ with the parameters given above. Then we have

$$\bar{G}_{i,i}(x) = G_{i,i}(x)^2 G_{i+1,i}(q_1 x) G_{i,i+1}(q_1^{-1} x), \qquad n \geq 2,$$

and

$$\bar{G}_{i,i\pm 1}(x) = \begin{cases} G_{i,i\pm 1}(q_1^{\pm \frac{1}{n-1}} x), & n \geq 4, \\ G_{i,i\pm 1}(q_1^{\pm \frac{1}{n-1}} x) G_{i\pm 1,i}(q_1^{\mp \frac{1}{n-1}} x), & n = 3. \end{cases}$$

Using these one can verify the relations (3.1)–(3.4) as rational functions.

For simplicity we assume $n \geq 3$ below. The case $n = 2$ can be treated with minor modifications. Let us check the condition (3.12) for $\bar{E}_0(z)$. We have

$$\bar{E}_0(z_1)\bar{E}_0(z_2) = E_{n-1,n-1,0,0}(q_1 z_1, q_1 z_2, z_1, z_2)\bar{c}\bar{d}(1 - q_1^{-1})^2 \frac{(1 - z_2/z_1)^2}{(1 - C^2 z_2/z_1)^2} \frac{(1 - z_2/z_1)^2}{(1 - C^{-2} z_2/z_1)^2}$$

$$\times \frac{(1 - z_2/z_1)^2}{(1 - q^2 z_2/z_1)^2} \frac{(1 - q_1^{-1} z_2/z_1)(1 - q_1 z_2/z_1)}{(1 - q^{-2} z_2/z_1)(1 - z_2/z_1)}.$$

The wheel conditions (3.13)–(3.15) ensure that the apparent double poles in the right-hand side are actually all simple. To compute the residue at $z_2 = C^2 z_1$, it suffices to consider

$$\frac{1}{1 - C^{-2} z_2/z_1} E_{n-1,n-1,0,0}(q_1 z_1, q_1 z_2, z_1, z_2) \Big|_{z_2 = C^2 z_1}$$

$$= w \frac{\partial}{\partial w} E_{n-1,n-1,0,0}(q_1 z_1, q_1 z_2, w, z_2) \Big|_{\substack{w = z_1 \\ z_2 = C^2 z_1}} + E_{n-1,n-1,0,0}(q_1 w, C^2 q_1 z_1, z_1, C^2 z_1) \Big|_{\substack{w = z_1 \\ z_2 = C^2 z_1}}.$$

The second term in the right-hand side vanishes due to the wheel condition. Substituting

$$E_{n-1,n-1,0,0}(q_1 z_1, q_1 z_2, w, z_2) = K_{n-1}^-(q_1 z_1) E_{0,0}(w, q_1 z_1) K_{n-1}^+(C^2 q_1 z_1)$$

$$\times \frac{1 - w/z_1}{1 - q_1^{-1} w/z_1} \frac{1 - q^2 C^{-2} w/z_1}{1 - q_1^{-1} C^{-2} w/z_1} \frac{-d^{-2} q^{-1}}{1 - q_1^{-1}} \frac{1 + C^2}{1 - C^2} \frac{1 - q^2 C^2}{1 - q_1^{-1} C^2},$$

we arrive at

$$\lambda_{0,0}^0(z_2/z_1)\bar{E}_0(z_1)\bar{E}_0(z_2)\Big|_{z_2 = C^2 z_1} = \frac{1}{q - q^{-1}} \frac{1 + C^2}{1 - C^2} \frac{1 - q^2 C^2}{1 - C^2} \bar{K}_0^-(z_1)\bar{K}_0^+(C^2 z_1).$$

The wheel conditions for $\bar{E}_i(z)$ can be verified similarly. As an example, consider $\bar{E}_{0,0,1}(z_1, z_2, w)$. Up to an irrelevant factor it can be written as

$$\bar{E}_{0,0,1}(z_1, z_2, w) \propto \frac{1}{\lambda_{0,0}^0(z_2/z_1)} E_{n-1,n-1,0,0,1}(q_1 z_1, q_1 z_2, z_1, z_2, q_1^{\frac{1}{n-1}} w). \qquad (6.3)$$

Since

$$E_{n-1,n-1,0}(q_1 z, q_1 q_2 z, z) = 0 \qquad \text{and} \qquad E_{0,0,1}(z, q_2 z, q_2 q_3 z) = 0,$$

$E_{n-1,n-1,0,0,1}(q_1 z_1, q_1 z_2, z_1, z_2, q_1^{-1} z_1)$ is divisible by $(z_2 - q_2 z_1)^2$. Therefore (6.3) vanishes at $z_2 = q_2 z_1$, $w = \bar{q}_1^{-1} z_1$. Similarly, the condition $E_{n-1,n-1,0}(q_1 z, q_1 C^2 z, z) = 0$ and $E_{0,0,1}(z, C^2 z, C^2 q_3 z) = 0$ imply that (6.3) vanishes at $z_2 = C^2 z_1$, $w = C^2 \bar{q}_3 z_1$. ∎

Interchanging the roles of q_1 and q_3, one can equally well consider action of $\mathcal{K}_{n-1}$ with parameters

$$\bar{\bar{q}}_1 = q_1 q_3^{-\frac{1}{n-1}}, \qquad \bar{\bar{q}}_2 = q_2, \qquad \bar{\bar{q}}_3 = q_3 q_3^{\frac{1}{n-1}}.$$

Note that in either case the values of q, C are unchanged.

Iterating this construction we obtain in general

Proposition 6.2. *Let $1 \le k \le n-1$. Then the following hold with an appropriate choice of constants $\bar{c}_{n,k}$, $\bar{\bar{c}}_{n,n-k}$:*

(*i*) *The operators*

$$\bar{E}_0(z) = \bar{c}_{n,k} E_{k,k+1,\ldots,n-1,0}\big(q_1^{n-k}z, q_1^{n-k-1}z, \ldots, q_1 z, z\big),$$
$$\bar{K}_0^{\pm}(z) = K_k^{\pm}\big(q_1^{n-k}z\big) K_{k+1}^{\pm}\big(q_1^{n-k-1}z\big) \cdots K_{n-1}^{\pm}(q_1 z) K_0^{\pm}(z),$$
$$\bar{E}_i(z) = E_i\big(q_1^{\frac{n-k}{k}i}z\big), \qquad \bar{K}_i^{\pm}(z) = K_i^{\pm}\big(q_1^{\frac{n-k}{k}i}z\big), \qquad 1 \le i \le k-1,$$

together with the same C give an action of $\mathcal{K}_k$ on V with parameters

$$\bar{q}_1 = q_1 q_1^{\frac{n-k}{k}}, \qquad \bar{q}_2 = q_2, \qquad \bar{q}_3 = q_3 q_1^{-\frac{n-k}{k}}.$$

(*ii*) *The operators*

$$\bar{\bar{E}}_0(z) = \bar{\bar{c}}_{n,n-k} E_{k,k-1,\ldots,1,0}\big(q_3^k z, q_3^{k-1}z, \ldots, q_3 z, z\big),$$
$$\bar{\bar{K}}_0^{\pm}(z) = K_k^{\pm}\big(q_3^k z\big) K_{k-1}^{\pm}\big(q_3^{k-1}z\big) \cdots K_1^{\pm}(q_3 z) K_0^{\pm}(z),$$
$$\bar{\bar{E}}_i(z) = E_{i+k}\big(q_3^{\frac{k}{n-k}i}z\big), \qquad \bar{\bar{K}}_i^{\pm}(z) = K_{i+k}^{\pm}\big(q_3^{\frac{k}{n-k}i}z\big), \qquad 1 \le i \le n-k-1,$$

together with the same C give an action of $\mathcal{K}_{n-k}$ on V with parameters

$$\bar{\bar{q}}_1 = q_1 q_3^{-\frac{k}{n-k}}, \qquad \bar{\bar{q}}_2 = q_2, \qquad \bar{\bar{q}}_3 = q_3 q_3^{\frac{k}{n-k}}.$$

(*iii*) *The actions* (*i*) *and* (*ii*) *mutually commute.*

Proof. We need only to prove the commutativity (*iii*). For simplicity of presentation we assume $n \ge 3$. Evidently $\{\bar{E}_i(z), \bar{K}_i^{\pm}(z)\}_{1 \le i \le k-1}$ commute with $\{\bar{\bar{E}}_i(z), \bar{\bar{K}}_i^{\pm}(z)\}_{1 \le i \le n-k-1}$.

To show that $\bar{\bar{K}}_i^+(z)$, $1 \le i \le n-k-1$, commutes with $\bar{E}_0(w)$, it suffices to note that

$$K_j^+(z) E_{k,k+1,\ldots,n-1,0}\big(q_1^{n-k}w, \ldots, w\big) = E_{k,k+1,\ldots,n-1,0}\big(q_1^{n-k}w, \ldots, w\big) K_j^+(z)$$
$$\times G_{j,j-1}\big(q_1^{n-j+1}w/z\big) G_{j,j}\big(q_1^{n-j}w/z\big)$$
$$\times G_{j,j+1}\big(q_1^{n-j-1}w/z\big), \qquad k+1 \le j \le n-1,$$

and apply the identity

$$G_{i,i-1}(q_1 x) G_{i,i}(x) G_{i,i+1}\big(q_1^{-1}x\big) = 1.$$

Commutativity of $\{\bar{\bar{E}}_i(z), \bar{\bar{K}}_i^{\pm}(z)\}_{1 \le i \le n-k-1}$ with $\{\bar{E}_0(w), \bar{K}_0^{\pm}(w)\}$ (for $\bar{\bar{E}}_i(z)$ with $\bar{E}_0(w)$ as rational functions) can be shown similarly.

We have further

$$\bar{\bar{K}}_0(z) \bar{E}_0(w) = \bar{E}_0(w) \bar{\bar{K}}_0(z) G_{k-1,k}\big(q_1^n q_2^k q_3 w/z\big) G_{k,k}\big(q_1^n q_2^k w/z\big) G_{k,k+1}\big(q_1^n q_2^k q_1^{-1}w/z\big)$$
$$\times G_{1,0}\big(q_3^{-1}w/z\big) G_{0,0}(w/z) G_{0,n-1}(q_1 w/z),$$

so that we can apply

$$G_{i-1,i}(q_3 x)G_{i,i}(x)G_{i,i+1}(q_1^{-1}x) = 1.$$

By the same token $\{\bar{\bar{E}}_0(z), \bar{K}_0^\pm(z)\}$ commute with $\{\bar{E}_0(w), \bar{K}_0^\pm(w)\}$ (for $\bar{\bar{E}}_0(z)$ with $\bar{E}_0(w)$ as rational functions).

To finish the proof, it remains to examine the singularities of $\bar{\bar{E}}_i(z)\bar{E}_0(w)$. For $1 \le i \le n-k-1$, we use

$$E_j(z)E_{j-1,j,j+1}(q_1 w, w, q_1^{-1} w) = E_{j-1,j,j,j+1}(q_1 w, z, w, q_1^{-1} w)\frac{1-q_1 w/z}{1-q_2^{-1}w/z}\frac{1-w/z}{1-C^2 w/z}$$
$$\times \frac{1-w/z}{1-q^2 w/z}\frac{1-w/z}{1-C^{-2}w/z}\frac{1-q_1^{-1}w/z}{1-w/z},$$

and observe that all poles are cancelled due to the wheel conditions. Finally for $i = 0$ we have

$$E_{k,k-1,\dots,1,0}(q_3^k z, \dots, z)E_{k,k+1,\dots,n-1,0}(q_1^{n-k}w, \dots, w)$$
$$= E_{0,0,1,\dots,k-1,k,k,k+1,\dots,n-1}(z, w, q_3 z, \dots, q_3^{k-1}z, q_3 z, q_1^{n-k}w, \dots, q_1 w)$$
$$\times \left(\lambda_{k-1,k}^0(q_1^n q_2^k q_3 w/z)\lambda_{k,k}^0(q_1^n q_2^k w/z)\lambda_{k,k+1}^0(q_1^n q_2^k q_1^{-1}w/z)\right)^{-1}$$
$$\times \left(\lambda_{0,n-1}^0(q_1 w/z)\lambda_{0,0}^0(w/z)\lambda_{1,0}^0(q_3^{-1}w/z)\right)^{-1},$$

so the cancellation works similarly as above. ∎

A   Theta function identities

As in Section 4.2 we use the space of theta functions $X_n(\mu_1, \dots, \mu_n)$ given by (4.6) and (4.7) for $n \ge 2$. For $n = 1$, we define $X_1 = \mathbb{C}$ to be the space of constant functions. In this section we prove the following identity.

Theorem A.1. *The following identities hold for all $n, M, N \ge 1$, $\alpha, \gamma \in \mathbb{C}$ and $\vartheta_1, \vartheta_2 \in X_n$:*

$$\sum_{I_1, J_1} \cdots \sum_{I_n, J_n} \vartheta_1(\overline{u}_{1,I_1}, \dots, \overline{u}_{n,I_n})\vartheta_2(\overline{u}_{1,J_1}, \dots, \overline{u}_{n,J_n})$$
$$\times \prod_{i=1}^n \frac{\prod_{\substack{a \in I_i \\ b \in J_{i+1}}} \theta(u_{i,a} - u_{i+1,b} - \alpha)\prod_{\substack{b \in J_i \\ a \in I_{i+1}}} \theta(u_{i,b} - u_{i+1,a} - \alpha + \gamma)}{\prod_{\substack{a \in I_i \\ b \in J_i}} \theta(u_{i,a} - u_{i,b})\theta(u_{i,a} - u_{i,b} - \gamma)}$$
$$= \sum_{I_1, J_1} \cdots \sum_{I_n, J_n} \vartheta_1(\overline{u}_{1,I_1}, \dots, \overline{u}_{n,I_n})\vartheta_2(\overline{u}_{1,J_1}, \dots, \overline{u}_{n,J_n})$$
$$\times \prod_{i=1}^n \frac{\prod_{\substack{b \in J_i \\ a \in I_{i+1}}} \theta(u_{i,b} - u_{i+1,a} - \alpha)\prod_{\substack{a \in I_i \\ b \in J_{i+1}}} \theta(u_{i,a} - u_{i+1,b} - \alpha + \gamma)}{\prod_{\substack{a \in I_i \\ b \in J_i}} \theta(u_{i,b} - u_{i,a})\theta(u_{i,b} - u_{i,a} - \gamma)}. \tag{A.1}$$

Here we set $\overline{u}_{i,I} = \sum_{a \in I} u_{i,a}$ for a subset I of $\{1, \dots, M+N\}$, and the sum is taken over all partitions

$$I_i \sqcup J_i = \{1, \dots, M+N\} \qquad \text{satisfying} \quad |I_i| = M, \quad |J_i| = N, \quad i = 1, \dots, n.$$

For the proof of theorem, we prepare several lemmas.

Lemma A.2. *If $\vartheta \in X_n(\mu_1, \dots, \mu_n)$, then*

$$\vartheta(u_1 + u, \dots, u_n + u) = \vartheta(u_1, \dots, u_n), \qquad u \in \mathbb{C}.$$

Proof. Let $\phi(u) = \vartheta(u_1 + u, \ldots, u_n + u)$. Then $\phi(u)$ is entire and satisfies $\phi(u + 1) = \phi(u)$, $\phi(u + \tau) = \phi(u)$. Hence it is a constant: $\phi(u) = \phi(0)$. ∎

Lemma A.3. *Let $\phi(v_1, \ldots, v_L)$ be an entire function with quasi-periodicity property*

$$\phi(v_1, \ldots, v_i + 1, \ldots, v_L) = \phi(v_1, \ldots, v_i, \ldots, v_L),$$

$$\phi(v_1, \ldots, v_i + \tau, \ldots, v_L) = e^{-2\pi i m \sum_{i=1}^{L} v_i + k} \phi(v_1, \ldots, v_i, \ldots, v_L)$$

with some $m \in \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{C}$. Then it is a function of $\sum_{i=1}^{L} v_i$.

Proof. Fix $i \neq j$, and set $\varphi(u) = \phi(v_1, \ldots, v_i + u, \ldots, v_j - u, \ldots, v_L)$. Then $\varphi(u)$ is an entire function and satisfies $\varphi(u + 1) = \varphi(u)$, $\varphi(u + \tau) = \varphi(u)$. Hence it is a constant: $\varphi(u) = \varphi(0)$. Since i, j, u are arbitrary, this proves the lemma. ∎

Denote both sides of the identity (A.1) by $\mathrm{LHS}_{n,M,N}$ and $\mathrm{RHS}_{n,M,N}$, respectively, and set $\Phi_{n,M,N} = \mathrm{LHS}_{n,M,N} - \mathrm{RHS}_{n,M,N}$. Where necessary we exhibit also their dependence on the variables $\mathbf{u}_i = (u_{i,1}, \ldots, u_{i,M+N})$ and the choice of $\vartheta_1, \vartheta_2 \in X_n$. We set $\Phi_{n,M,N} = 0$ if $M = 0$ or $N = 0$.

Lemma A.4. *For $n \geq 2$, $\Phi_{n,M,N}$ has the quasi-periodicity in each variable $u_{i,a}$,*

$$\Phi_{n,M,N}(\ldots, u_{i,a} + 1, \ldots) = \Phi_{n,M,N}(\ldots, u_{i,a}, \ldots),$$

$$\Phi_{n,M,N}(\ldots, u_{i,a} + \tau, \ldots) = e^{-2\pi i(2\bar{u}_i - \bar{u}_{i-1} - \bar{u}_{i+1} - \mu_i + \tau)} \Phi_{n,M,N}(\ldots, u_{i,a}, \ldots).$$

For $n = 1$, $\Phi_{1,M,N}$ has periods 1 and τ.

Modulo $\mathbb{Z} \oplus \mathbb{Z}\tau$, the only poles of $\Phi_{n,M,N}$ are $u_{i,a} = u_{i,b} + \gamma$ with some a, b, i.

Proof. The quasi-periodicity follows from (4.4), (4.6) and (4.7). Since $\Phi_{n,M,N}$ is symmetric in each group of variables $\{u_{i,a}\}_{a=1,\ldots,M+N}$, it cannot have a simple pole at $u_{i,a} = u_{i,b}$. ∎

Lemma A.5. *We have*

$$\mathop{\mathrm{res}}_{u_{1,1}=u_{1,2}+\gamma} \cdots \mathop{\mathrm{res}}_{u_{n,1}=u_{n,2}+\gamma} \Phi_{n,M,N}(\mathbf{u}_1, \ldots, \mathbf{u}_n | \vartheta_1, \vartheta_2)$$

$$= A^n \frac{\prod_{i=1}^{n} \prod_{a=2}^{M+N} \theta(u_{i,a} - u_{i+1,2} - \alpha)\theta(u_{i,2} - u_{i+1,a} - \alpha + \gamma)}{\prod_{i=1}^{n} \prod_{a=3}^{M+N} \theta(u_{i,a} - u_{i,2})\theta(u_{i,a} - u_{i,2} - \gamma)}$$

$$\times \Phi_{n,M-1,N-1}(\mathbf{u}'_1, \ldots, \mathbf{u}'_n | \vartheta'_1, \vartheta'_2),$$

where

$$\mathbf{u}'_i = (u_{i,3}, \ldots, u_{i,M+N}),$$
$$\vartheta'_s(v_1, \ldots, v_n) = \vartheta_s(v_1 + u_{1,2}, \ldots, v_n + u_{n,2}), \qquad s = 1, 2,$$

and $A = \theta(\gamma)^{-1} \mathrm{res}_{u=0} \theta(u)^{-1}$.

Proof. In the sum $\mathrm{LHS}_{n,M,N}$, the iterated residue is non-zero only for terms such that $1 \in I_i$ and $2 \in J_i$ for all $i = 1, \ldots, n$. Write $I_i = \{1\} \sqcup \bar{I}_i$ and $J_i = \{2\} \sqcup \bar{J}_i$. Similarly, for $\mathrm{RHS}_{n,M,N}$ one must have $2 \in I_i$ and $1 \in J_i$ for $i = 1, \ldots, n$. Write $I_i = \{2\} \sqcup \bar{I}_i$ and $J_i = \{1\} \sqcup \bar{J}_i$ in that case.

In either case, the functions ϑ_s become

$$\vartheta_1(u_{1,2} + \bar{u}_{1,\bar{I}_1}, \ldots, u_{n,2} + \bar{u}_{n,\bar{I}_n})\vartheta_2(u_{1,2} + \bar{u}_{1,\bar{J}_1}, \ldots, u_{n,2} + \bar{u}_{n,\bar{J}_n})$$

due to Lemma A.2.

The rest is a simple computation. ∎

Remark A.6. Lemma A.5 is proved in [12]. However the rest of the argument necessary for completing the proof of Theorem A.1 is missing there.

Lemma A.7. *Fix $i \in \{1, \ldots, n\}$. Then the following hold:*

$$\Phi_{n,M,N}(\mathbf{u}_1, \ldots, \mathbf{u}_n | \vartheta_1, \vartheta_2)\big|_{\substack{u_{i+1,a}=u_{i,a}-\alpha \\ 1 \le a \le M+N}}$$
$$= (-1)^{MN} \Phi_{n-1,M,N}(\mathbf{u}_1, \ldots, \mathbf{u}_i, \mathbf{u}_{i+2}, \ldots, \mathbf{u}_n | \vartheta_1'', \vartheta_2''),$$
$$\Phi_{n,M,N}(\mathbf{u}_1, \ldots, \mathbf{u}_n | \vartheta_1, \vartheta_2)\big|_{\substack{u_{i+1,a}=u_{i,a}-\alpha+\gamma \\ 1 \le a \le M+N}}$$
$$= (-1)^{MN} \Phi_{n-1,M,N}(\mathbf{u}_1, \ldots, \mathbf{u}_i, \mathbf{u}_{i+2}, \ldots, \mathbf{u}_n | \vartheta_1''', \vartheta_2'''),$$

where

$$\vartheta_1''(v_1, \ldots, v_{n-1}) = \vartheta_1(v_1, \ldots, v_i, v_i - M\alpha, v_{i+1}, \ldots, v_{n-1}),$$
$$\vartheta_2''(v_1, \ldots, v_{n-1}) = \vartheta_2(v_1, \ldots, v_i, v_i - N\alpha, v_{i+1}, \ldots, v_{n-1}),$$

and ϑ_s''' is given by replacing α by $\alpha - \gamma$ in ϑ_s''.

Proof. Since the calculations are similar, we consider the first case. Under this specialization, terms in $\mathrm{LHS}_{n,M,N}$ survive only when $I_i \cap J_{i+1} = \varnothing$, which implies $I_i = I_{i+1}$ and $J_i = J_{i+1}$. Then the factor $\prod_{\substack{a \in I_i \\ b \in J_{i+1}}} \theta(u_{i,a} - u_{i+1,b} - \alpha) \prod_{\substack{a \in I_{i+1} \\ b \in J_i}} \theta(u_{i,b} - u_{i+1,a} - \alpha + \gamma)$ from the numerator cancels $\prod_{\substack{a \in I_{i+1} \\ b \in J_{i+1}}} \theta(u_{i+1,a} - u_{i+1,b})\theta(u_{i+1,a} - u_{i+1,b} - \gamma)$ from the denominator, thereby reducing n to $n-1$. For $\mathrm{RHS}_{n,M,N}$ the situation is entirely similar. ∎

Lemma A.8 ([19]). *The identity* (A.1) *holds for $n = 1$.*

Proof. For $n = 1$ we may assume $\vartheta_1 = \vartheta_2 = 1$, so the identity takes the form

$$\sum_{I,J} \prod_{\substack{a \in I \\ b \in J}} \frac{\theta(u_a - u_b - \alpha)\theta(u_b - u_a - \alpha + \gamma)}{\theta(u_a - u_b)\theta(u_a - u_b - \gamma)} = \sum_{I,J} \prod_{\substack{a \in I \\ b \in J}} \frac{\theta(u_b - u_a - \alpha)\theta(u_a - u_b - \alpha + \gamma)}{\theta(u_b - u_a)\theta(u_b - u_a - \gamma)}.$$

Lemma A.5 together with the symmetry shows that $\Phi_{1,M,N}$ has no pole at $u_a = u_b + \gamma$. Hence $\Phi_{1,M,N}$ is an elliptic function without poles, therefore it is a constant. Upon specializing $u_a = a\alpha$, $a = 1, \ldots, M + N$, we obtain

$$\mathrm{LHS}_{1,M,N} = (-1)^{MN} \frac{\prod_{a=1}^M \theta((a-1)\alpha + \gamma)/\theta(a\alpha) \prod_{a=1}^N \theta((a-1)\alpha + \gamma)/\theta(a\alpha)}{\prod_{a=1}^{M+N} \theta((a-1)\alpha + \gamma)/\theta(a\alpha)}$$
$$= \mathrm{RHS}_{1,M,N},$$

thereby proving $\Phi_{1,M,N} = 0$. ∎

Proof of Theorem A.1. We use induction on n. The base of induction is proved in Lemma A.8. Assume that theorem is true for $n-1$.

For $i = 1, \ldots, n$ we set

$$Z_i = \operatorname*{res}_{u_{i,2}=u_{i,1}-\gamma} \operatorname*{res}_{u_{i+1,2}=u_{i+1,1}-\gamma} \cdots \operatorname*{res}_{u_{n,2}=u_{n,1}-\gamma} \Phi_{n,M,N}(\mathbf{u}_1, \ldots, \mathbf{u}_n, \alpha, \gamma | \vartheta_1, \vartheta_2).$$

We show $Z_i = 0$ by induction on i. By Lemma A.5 and the induction hypothesis $\Phi_{n-1,M,N} = 0$, we have $Z_1 = 0$. Suppose $Z_i = \operatorname{res}_{u_{i,2}=u_{i,1}-\gamma} Z_{i+1} = 0$. Then Z_{i+1} has no poles in $\mathbf{u}_i$.

Since $u_{i+1,2} = u_{i+1,1} - \gamma$, non-zero terms in Z_{i+1} are such that $1 \in I_{i+1}$, $2 \in J_{i+1}$ in LHS. Then in the numerator we have factors $\theta(u_{i,a} - u_{i+1,2} - \alpha)$, $a \in I_i$, and $\theta(u_{i,b} - u_{i+1,1} - \alpha + \gamma)$, $b \in J_i$. Therefore each term has a factor

$$\prod_{a=1}^{M+N} \theta(u_{i,a} - u_{i+1,2} - \alpha).$$

By the same argument, RHS, and hence Z_{i+1}, is shown to have the same factor. If $Z_{i+1} \neq 0$, this is a contradiction because the quasi-periodicity property of Z_{i+1} implies that it is a function of $\bar{u}_i = \sum_{a=1}^{M+N} u_{i,a}$, see Lemma A.3. Hence we have $Z_{i+1} = 0$.

It follows that $\Phi_{n,M,N}$ has no poles at $u_{n,a} = u_{n,b} - \gamma$. Therefore, if non-zero, it is a function of the sum $\bar{u}_n = \sum_{a=1}^{M+N} u_{n,a}$. On the other hand, by Lemma A.7 and the induction hypothesis, $\Phi_{n,M,N}$ has the following zeros with respect to $\bar{u}_n$:

$$\bar{u}_n = \bar{u}_1 + (M+N)\alpha, \qquad \bar{u}_n = \bar{u}_1 + (M+N)(\alpha - \gamma),$$
$$\bar{u}_n = \bar{u}_{n-1} - (M+N)\alpha, \qquad \bar{u}_n = \bar{u}_{n-1} - (M+N)(\alpha - \gamma).$$

From the quasi-periodicity, if $\Phi_{n,M,N} \neq 0$, then it can have only two zeros modulo $\mathbb{Z} \oplus \mathbb{Z}\tau$, which contradicts to the above zeros.

This completes the proof of Theorem A.1. ∎

Acknowledgments

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