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Quasi-Polynomials and the Singular $[Q, R] = 0$ Theorem

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Abstract. In this short note we revisit the 'shift-desingularization' version of the $[Q, R] = 0$ theorem for possibly singular symplectic quotients. We take as starting point an elegant proof due to Szenes–Vergne of the quasi-polynomial behavior of the multiplicity as a function of the tensor power of the prequantum line bundle. We use the Berline–Vergne index formula and the stationary phase expansion to compute the quasi-polynomial, adapting an early approach of Meinrenken.

Key words: symplectic geometry; Hamiltonian G-spaces; symplectic reduction; geometric quantization; quasi-polynomials; stationary phase

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1 Introduction

Let (M, ω) be a compact connected symplectic manifold equipped with an action of a compact connected Lie group G by symplectomorphisms. Suppose that the action of G is Hamiltonian, meaning that there is a G-equivariant map, the moment map,

$$\mu_{\mathfrak{g}}\colon\ M \to \mathfrak{g}^*,$$

where $\mathfrak{g}^*$ is the dual of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$, satisfying the moment map condition

$$\iota(X_M)\omega = -\mathrm{d}\langle \mu_{\mathfrak{g}}, X\rangle, \qquad X \in \mathfrak{g}. \tag{1.1}$$

Let $\big(L, \nabla^L\big)$ be a G-equivariant prequantum line bundle with connection on M, i.e., L is a G-equivariant Hermitian line bundle with compatible connection ∇^L, $\big(\nabla^L\big)^2 = -2\pi\mathrm{i}\omega$ and the derivative of the G-action on L satisfies Kostant's condition

$$\mathcal{L}^L_X - \nabla^L_{X_M} = 2\pi\mathrm{i}\langle \mu_{\mathfrak{g}}, X\rangle.$$

Choose a compatible almost complex structure J on M, i.e., $\omega(Jw, Jv) = \omega(w, v)$ and $\omega(w, Jv)$ $=: g(w, v)$ is a Riemannian metric. Let D_L denote the Dolbeault–Dirac operator twisted by $\big(L, \nabla^L\big)$, an elliptic differential operator acting on sections of the spinor bundle $\wedge T^*_{0,1}M \otimes L$. The kernel of D_L carries an action of G, and the G-equivariant index is defined to be the difference $\mathrm{index}_G(D_L) := \ker\big(D_L^{\mathrm{even}}\big) - \ker\big(D_L^{\mathrm{odd}}\big)$ of the kernel of D_L on even/odd degree forms, regarded as an element of the representation ring $R(G)$.

The quantization-commutes-with-reduction theorem ($[Q, R] = 0$ theorem) describes the multiplicity of the trivial representation in $\mathrm{index}_G(D_L)$ in terms of the symplectic quotient $M^{\mathrm{red}} := \mu_{\mathfrak{g}}^{-1}(0)/G$. When 0 is a regular value of $\mu_{\mathfrak{g}}$, M^{red} is an orbifold and the theorem states that $\mathrm{index}_G(D)^G$ equals the index of the twisted Dolbeault–Dirac operator $D^{\mathrm{red}}_{L^{\mathrm{red}}}$ on M^{red}. The theorem was first conjectured by Guillemin–Sternberg [3], and the general case (M, G both compact, 0 a regular value) was first proved by Meinrenken [8]. Different proofs of the $[Q, R] = 0$ theorem were given by Tian–Zhang [15] and Paradan [11]. The theorem has since been extended in various directions.

There are versions of the $[Q,R]=0$ theorem when 0 is not necessarily a regular value, due to Meinrenken–Sjamaar [10]; below we will give a precise statement of one of these results, involving a partial *shift desingularization*, i.e., $\mathrm{index}_G(D_L)^G$ is related to the index on the symplectic quotient at a nearby weakly regular value. At the same time, we introduce some notation that will be of use later on.

Fix a maximal torus T with Lie algebra $\mathfrak{t}$. Let $\Lambda \subset \mathfrak{t}^*$ be the (real) weight lattice. Given $\lambda \in \Lambda$, the corresponding character $T \to U(1)$ is written $t \mapsto t^\lambda = \mathrm{e}^{2\pi\mathrm{i}\langle\lambda,X\rangle}$ where $t=\mathrm{e}^X$, $X \in \mathfrak{t}$. Let $\mathcal{R} \subset \Lambda$ be the set of roots. We also fix a closed positive Weyl chamber $\mathfrak{t}_+$, which determines a set of positive (resp. negative) roots $\mathcal{R}_\pm$. For each relatively open face $\sigma \subset \mathfrak{t}^*_+$, the stabilizer G_ξ of points $\xi \in \sigma$ under the coadjoint action, does not depend on ξ, and will be denoted G_σ. If $\sigma_1 \subset \overline{\sigma}_2$ then $G_{\sigma_1} \supset G_{\sigma_2}$. Note also that G_σ is connected and contains the maximal torus T. The Lie algebra $\mathfrak{g}_\sigma$ decomposes into its semi-simple and central parts $\mathfrak{g}_\sigma = [\mathfrak{g}_\sigma,\mathfrak{g}_\sigma] \oplus \mathfrak{z}_\sigma$. The subspace $\mathfrak{z}^*_\sigma \subset \mathfrak{t}^*$ is defined to be the annihilator of $[\mathfrak{g}_\sigma,\mathfrak{g}_\sigma]$, or equivalently the fixed point set of the coadjoint G_σ action. The face σ is an open subset of $\mathfrak{z}^*_\sigma$.

Let $\Delta = \mu_\mathfrak{g}(M) \cap \mathfrak{t}^*_+$ be the moment polytope. A well-known theorem in symplectic geometry states that there is a unique face $\sigma \subset \mathfrak{t}^*_+$ of minimal dimension such that $\Delta \subset \overline{\sigma}$ (briefly, this is a consequence of (1.1), which implies that $\mathrm{d}\mu_\mathfrak{g}$ has constant rank on the top dimensional G-orbit type stratum, and the complement of the latter has codimension at least 2); σ is called the *principal face* or *principal wall*. The corresponding symplectic cross-section, called the *principal cross-section*, $Y = \mu_\mathfrak{g}^{-1}(\sigma)$ is a Hamiltonian G_σ-space. Moreover the semi-simple part $[G_\sigma, G_\sigma]$ of G_σ acts trivially on Y. For further details, see for example [5] and references therein.

Let $I \subset \mathfrak{z}^*_\sigma$ be the smallest affine subspace containing Δ. Let $\mathfrak{t}_I \subset \mathfrak{t}$ be the annihilator of the subspace parallel to I, and let $T_I = \exp(\mathfrak{t}_I) \subset T$ be the corresponding subtorus. By equation (1.1), $\mathfrak{t}_I$ is the generic infinitesimal stabilizer of Y. In particular T_I acts trivially, hence the quotient torus T/T_I acts on Y. The moment map $\mu_\mathfrak{g}$ may have no non-trivial regular values. But the restriction

$$\mu_\mathfrak{g}|_Y \colon\ Y \to I$$

viewed as a map with codomain I, always has non-trivial regular values, and we will refer to these as *weakly-regular values*. If ξ is a weakly-regular value, then the reduced space $M_\xi = \mu_\mathfrak{g}^{-1}(\xi)/G_\sigma$ is an orbifold. Let $L_\xi = L|_{\mu_\mathfrak{g}^{-1}(\xi)}/G_\sigma$ be the corresponding (orbifold) line bundle over M_ξ.

Theorem 1.1 ([10], see also [11, 13])**.** *Let $(M,\omega,\mu_\mathfrak{g})$ be a compact connected Hamiltonian G-space with moment polytope Δ. If $0 \notin \Delta$ then $\mathrm{index}_G(D_L)^G = 0$. Otherwise for every weakly-regular value $\xi \in \Delta$ sufficiently close to 0, $\mathrm{index}_G(D_L)^G$ equals the index of the Dolbeault–Dirac operator $D^{\mathrm{red}}_{L_\xi}$ on the reduced space M_ξ.*

We will now describe the main result of this article and its relation to Theorem 1.1. Consider tensor powers L^k, $k \in \mathbb{Z}_{>0}$ of the prequantum line bundle. For a dominant weight λ, let $\chi_\lambda \in R(G)$ denote the character of the irreducible representation of G with highest weight λ. We define the *multiplicity function* $m_G(k,\lambda)$ by the expression

$$\mathrm{index}_G(D_{L^k}) = \sum_{\lambda \in \Lambda \cap \mathfrak{t}^*_+} m_G(k,\lambda)\chi_\lambda. \tag{1.2}$$

An important theme in the work of Szenes–Vergne [14] and also in our approach, is that the function $m_G(k,\lambda)$ has more coherent behavior than its restriction to any fixed value of k.

The statement of the result requires some further background on orbifolds, for which we refer the reader to, for example, [2, Appendix A], [8, Section 2]. A small warning is that we will not require the action of isotropy groups in orbifold charts to be effective (this is in agreement with the references [2, 8] mentioned above). One advantage of permitting this, is that for a locally free action of a compact Lie group K on a manifold P, the corresponding orbifold P/K has orbifold charts given automatically by the slice theorem, with the isotropy groups being simply the isotropy groups for the action of K on P.

In fact all the orbifolds that we will encounter arise naturally as such quotients P/K, and one could avoid mentioning orbifolds altogether by working instead with suitable K-basic structures on P. An example is the description of characteristic forms for orbifold vector bundles, which can be defined in terms of orbifold charts for P/K, or alternatively in terms of K-basic differential forms on P. In brief, the latter approach goes as follows. One can take the complex $(\Omega_{\mathrm{bas}}(P), \mathrm{d})$ of K-basic differential forms on P as a working definition of the de Rham complex of P/K (if K acts freely then P/K is a manifold and pullback of forms from P/K to P is an isomorphism of complexes $(\Omega(P/K), \mathrm{d}) \simeq (\Omega_{\mathrm{bas}}(P), \mathrm{d})$). A K-equivariant vector bundle $E \to P$ determines an orbifold vector bundle E/K over P/K. Let θ be a connection on P with curvature F_θ. The choice of connection determines a *Cartan map* (cf. [9]) from closed K-equivariant forms $\alpha(X)$ on P to closed K-basic forms: $\alpha(X) \mapsto \mathrm{Car}_\theta(\alpha) := \Pi_{\mathrm{hor}}\alpha(F_\theta)$, where Π_{hor} is the projection onto the horizontal part relative to the connection. The Cartan map induces an isomorphism from the K-equivariant cohomology of P to the cohomology of the complex of basic differential forms on P. If $\alpha(X)$ is a K-equivariant characteristic form (constructed via the K-equivariant analogue of the usual Chern–Weil construction cf. [1, 9]), then one may take $\mathrm{Car}_\theta(\alpha) \in \Omega_{\mathrm{bas}}(P)$ as the definition of the corresponding characteristic form for E/K.

Let $\xi \in \Delta$ be a weakly-regular value. By the moment map equation (1.1), the action of $K = T/T_I$ on the level set

$$P = \mu_{\mathfrak{g}}^{-1}(\xi)$$

is locally free. The set S_P of elements $g \in T/T_I$ such that $P^g \neq \varnothing$ is finite. For each $g \in S_P$, we obtain an orbifold

$$\Sigma_g = P^g/(T/T_I), \qquad \Sigma = \bigsqcup_{g \in S_P} \Sigma_g.$$

Note that $\Sigma_1 = P/(T/T_I) = M_\xi$ identifies with the reduced space itself, and more generally Σ_g identifies with a symplectic quotient of Y^g. For each $g \in S_P$ there is an immersion $\Sigma_g \hookrightarrow \Sigma$ induced by $P^g \hookrightarrow P$. Let $\nu_{\Sigma_g,\Sigma}$ denote the (orbifold) normal bundle (the quotient $\nu_{P^g,P}/(T/T_I)$), which inherits a complex structure from the almost complex structures on Y, Y^g. Define the characteristic form

$$\mathcal{D}^g_{\mathbb{C}}(\nu_{\Sigma_g,\Sigma}) = \det_{\mathbb{C}}\big(1 - g_\nu^{-1} \mathrm{e}^{-\frac{\mathrm{i}}{2\pi} F_\nu}\big),$$

where g_ν denotes the action of g on the normal bundle (defined in terms of an orbifold chart, or in terms of $\nu_{P^g,P}$), and F_ν denotes the curvature. Taking the quotient of $L|_{P^g}$ we obtain (orbifold) line bundles

$$L_{\Sigma_g} = (L|_{P^g})/(T/T_I), \qquad L_\Sigma = \bigsqcup_{g \in S_P} L_{\Sigma_g}.$$

There is a locally constant function

$$g_L \colon\ \Sigma_g \to U(1)$$

giving the phase of the action of g on L_{Σ_g} (or equivalently on $L|_{P^g}$). Let $d \colon \Sigma \to \mathbb{Z}$ be the locally constant function giving the size of a generic isotropy group for Σ (or equivalently the number of elements in the generic stabilizer for the T/T_I action on $\sqcup P^g$).

Let θ be a connection for the locally free $K = T/T_I$-action on $\sqcup_{g \in S_P} P^g$. The curvature F_θ is horizontal and $\mathfrak{t}/\mathfrak{t}_I$-valued, hence for any $\lambda \in (\mathfrak{t}/\mathfrak{t}_I)^* = I$, the form $\langle \lambda, F_\theta \rangle$ is K-basic, hence descends to Σ. With the preparations above, we can state the main result of this note.

Theorem 1.2. *If $0 \notin \Delta$ then $m(k,0) = 0$ for all $k \geq 1$. If $0 \in \Delta$ then there is a closed polytope $\mathfrak{p} \subset \Delta$ of the same dimension as Δ and containing the origin such that the following is true. Let $C_\mathfrak{p}$ denote the cone*

$$C_\mathfrak{p} = \{(t, t\tau) \,|\, t \in (0,\infty), \tau \in \mathfrak{p}\} \subset \mathbb{R} \times \mathfrak{t}^*.$$

Fix a weakly regular value $\xi \in \Delta$ sufficiently close to 0 as in Theorem 1.1. Let $P = \mu_\mathfrak{g}^{-1}(\xi)$ and define Σ, L_Σ, etc. as above. Then for all $(k,\lambda) \in (\mathbb{Z}_{>0} \times \Lambda) \cap C_\mathfrak{p}$,

$$m_G(k,\lambda) = \sum_{g \in S_P} g^{-\lambda} \int_{\Sigma_g} \frac{1}{d} \frac{g_L^k \mathrm{Ch}(L_\Sigma)^k \mathrm{Td}(\Sigma)}{\mathcal{D}_\mathbb{C}^g(\nu_{\Sigma_g,\Sigma})} \mathrm{e}^{\langle \lambda, F_\theta \rangle}. \tag{1.3}$$

Of course this result is also originally due to Meinrenken–Sjamaar [10]. Theorem 1.1 follows immediately from Theorem 1.2 by applying Kawasaki's index theorem for orbifolds to $\mathrm{index}\big(D^{\mathrm{red}}_{L_\xi}\big)$ and comparing with the evaluation of (1.3) at $(k,\lambda) = (1,0)$.

Let us give a brief summary of our approach to deriving Theorem 1.2. Recall that a function f on a lattice Γ in a real vector space V is said to be *quasi-polynomial* if there is a sublattice Γ' with Γ/Γ' finite and f restricts to a polynomial function on each coset of Γ'. More generally, one says f is quasi-polynomial on a subset $\Gamma_0 \subset \Gamma$ if $f \upharpoonright \Gamma_0 = q \upharpoonright \Gamma_0$ for some quasi-polynomial q. A fundamental fact, originally derived from Theorem 1.1 by Meinrenken–Sjamaar [10], is that m_G is quasi-polynomial on the subset $C_\mathfrak{p} \cap (\mathbb{Z}_{>0} \times \Lambda)$. Our first goal, in Section 2, is to give an independent proof of this fact, taking as a starting point a formula for m_G due to Szenes–Vergne [14] (inspired by work of Paradan [11]), which they obtained by a combinatorial rearrangement of the fixed-point formula for the index.

Then in Section 3 we adapt an idea of Meinrenken [7] to compute the quasi-polynomial $m_G \upharpoonright C_\mathfrak{p}$ using the Berline–Vergne index formula and the principle of stationary phase. The output of the stationary phase formula is an asymptotic expansion for $m_G(k, k\xi)$ in powers of k (allowing coefficients that are periodic in k). As one knows in advance that $m_G(k,k\xi)$ is quasi-polynomial in k, one concludes that the expansion is exact, yielding Theorem 1.2.

The article of Meinrenken–Sjamaar [10] contains, besides Theorem 1.1, a wealth of detailed information about singular reduction and $[Q,R] = 0$. Our goal in this short note is much more modest. We also do not make a great claim of originality, and in particular the debt to [14] and [7] will be apparent. Part of our motivation stems from the hope that the article of Szenes–Vergne [14], in combination with this note, will provide a more elementary treatment of the $[Q,R] = 0$ theorem than was previously available.

2 Quasi-polynomials and the multiplicity function

The goal of this section is Theorem 2.2 on the quasi-polynomial behavior of the multiplicity function, which we prove using results of Szenes–Vergne [14] reviewed below.

The quotient $\mathfrak{g}/\mathfrak{t}$ can be identified with the unique $\mathrm{Ad}(T)$-invariant complement to $\mathfrak{t}$ in $\mathfrak{g}$. Let $\mathfrak{t} \subset \mathfrak{h} \subset \mathfrak{g}$ be a T-invariant subspace. We may similarly identify $\mathfrak{h}/\mathfrak{t}$ and $\mathfrak{g}/\mathfrak{h}$ with subspaces of $\mathfrak{g}$. The choice of positive roots $\mathcal{R}_+$ determines a complex structure on $\mathfrak{g}/\mathfrak{t}$, whose $+\mathrm{i}$-eigenspace is identified with the direct sum of the positive root spaces:

$$(\mathfrak{g}/\mathfrak{t})^{1,0} \simeq \bigoplus_{\alpha \in \mathcal{R}_+} \mathfrak{g}_\alpha.$$

We obtain similar complex structures on $\mathfrak{g}/\mathfrak{h}$, $\mathfrak{h}/\mathfrak{t}$, whose $+$i-eigenspaces are direct sums of positive roots spaces. We will write $\det_{\mathbb{C}}^{\mathfrak{g}/\mathfrak{t}}(a)$ (resp. $\det_{\mathbb{C}}^{\mathfrak{g}/\mathfrak{h}}(a)$, $\det_{\mathbb{C}}^{\mathfrak{h}/\mathfrak{t}}(a)$) for the determinant of a complex linear endomorphism a of $\mathfrak{g}/\mathfrak{t}$ (resp. $\mathfrak{g}/\mathfrak{h}$, $\mathfrak{h}/\mathfrak{t}$). An example is the endomorphism Ad_t, $t \in T$ (resp. ad_X, $X \in \mathfrak{t}$); in this case we will simply write $\det_{\mathbb{C}}^{\mathfrak{g}/\mathfrak{t}}(t)$ instead of $\det_{\mathbb{C}}^{\mathfrak{g}/\mathfrak{t}}(\mathrm{Ad}_t)$ (resp. $\det_{\mathbb{C}}^{\mathfrak{g}/\mathfrak{t}}(X)$ instead of $\det_{\mathbb{C}}^{\mathfrak{g}/\mathfrak{t}}(\mathrm{ad}_X)$), the action of T (resp. $\mathfrak{t}$) on $\mathfrak{g}/\mathfrak{t}$ being understood. Then for example if $t = \mathrm{e}^X \in T$,

$$\det_{\mathbb{C}}^{\mathfrak{g}/\mathfrak{t}}(1 - t^{-1}) = \prod_{\alpha \in \mathcal{R}_+} (1 - t^{-\alpha}) = \prod_{\alpha \in \mathcal{R}_+} (1 - \mathrm{e}^{-2\pi \mathrm{i}\langle \alpha, X\rangle}),$$
$$\det_{\mathbb{C}}^{\mathfrak{g}/\mathfrak{t}}(-X) = \prod_{\alpha \in \mathcal{R}_+} -2\pi\mathrm{i}\langle \alpha, X\rangle.$$

For $\lambda \in \Lambda \cap \mathfrak{t}^*_+$, the Weyl character formula says that for $t \in T$,

$$\chi_\lambda(t) \cdot \det_{\mathbb{C}}^{\mathfrak{g}/\mathfrak{t}}(1 - t^{-1}) = \sum_{w \in W} (-1)^{l(w)} t^{w(\lambda+\rho)-\rho}, \tag{2.1}$$

where W is the Weyl group, $l(w)$ is the length of the element $w \in W$, and ρ is the half sum of the positive roots. The right-hand-side is an element of $R(T)$ with multiplicity function m_λ obtained by Fourier transform. Note that

- m_λ is anti-symmetric under the *ρ-shifted action of the Weyl group*:

$$m_\lambda(w(\mu + \rho) - \rho) = (-1)^{l(w)} m_\lambda(\mu).$$

- The support of $m_\lambda|_{\Lambda \cap \mathfrak{t}^*_+}$ is $\{\lambda\}$, where it takes the value 1.

Conversely these two properties determine m_λ: it is the unique W-anti-symmetric function on Λ extending the multiplicity function of χ_λ. Applying these observations to the multiplicity function m_G defined in (1.2), we make the following definition.

Definition 2.1. Let $m(k, -)\colon \Lambda \to \mathbb{Z}$ be the unique ρ-shifted W-anti-symmetric function such that $m(k, \lambda) = m_G(k, \lambda)$ for all $\lambda \in \Lambda \cap \mathfrak{t}^*_+$. The corresponding character $Q(k, -)\colon T \to \mathbb{C}$ is defined as the inverse Fourier transform:

$$Q(k, t) = \sum_{\lambda \in \Lambda} m(k, \lambda) t^\lambda.$$

Using the Weyl character formula (2.1) and the definition of m_G, it is easy to verify that

$$Q(k, t) = \sum_{\lambda \in \Lambda} m(k, \lambda) t^\lambda = \mathrm{index}_T(D_{L^k})(t) \cdot \det_{\mathbb{C}}^{\mathfrak{g}/\mathfrak{t}}(1 - t^{-1}).$$

We define

$$\mu = \mathrm{pr}_{\mathfrak{t}^*} \circ \mu_{\mathfrak{g}}$$

to be the composition of the moment map $\mu_{\mathfrak{g}}$ with the projection to $\mathfrak{t}^*$. Then μ is a moment map for the action of T on M. Suppose $t \in T$ is sufficiently generic, so that $M^t = M^T$. The Atiyah–Bott–Segal formula for the index yields

$$Q(k, t) = \sum_{F \subset M^T} t^{k\mu_F} \int_F \frac{\mathrm{e}^{k\omega}\mathrm{Td}(F)}{\mathcal{D}^t_{\mathbb{C}}(\nu_F)} \det_{\mathbb{C}}^{\mathfrak{g}/\mathfrak{t}}(1 - t^{-1}), \tag{2.2}$$

where the sum is over connected components F of M^T, and μ_F denotes the constant value of the moment map μ on F. The multiplicity m is obtained by Fourier transform of (2.2).

Key to the approach in [14] is a different expression for $m(k,\lambda)$ that we briefly describe here. The formula depends on the choice of an invariant inner product on $\mathfrak{g}$, as well as a generic point γ contained in $\mathfrak{t}^*_+$ and sufficiently close to 0 (see [14, Section 4.1] for the meaning of 'generic' here). Using the inner product we identify $\mathfrak{t} \simeq \mathfrak{t}^*$. We need some additional notation:

- Let $\mathrm{Comp}_T(M)$ denote the set of connected components of M^H, as H ranges over all (connected) sub-tori of T.
- For $C \in \mathrm{Comp}_T(M)$, let $\mathfrak{t}_C \subset \mathfrak{t}$ be its generic infinitesimal stabilizer. Let A_C be the smallest affine subspace containing the image $\mu(C)$. In particular A_M is the smallest affine subspace containing $\mu(M)$. Note that A_C is a translate of the annihilator of $\mathfrak{t}_C$.
- Let $\gamma_C \in A_C$ be the orthogonal projection of γ onto A_C, and let $\tau_C = \gamma_C - \gamma$.

The Szenes–Vergne–Paradan formula [14, equation (39)] (see also [14, Proposition 41, Theorem 48]) is a sum of contributions:

$$m = \sum_C m_C, \tag{2.3}$$

where C ranges over components $C \in \mathrm{Comp}_T(M)$ such that $\gamma_C \in \mu_{\mathfrak{g}}(C)$. Szenes–Vergne derive this formula directly from (2.2) using an interesting combinatorial rearrangement, the main ingredient of which is a decomposition formula for Kostant-type partition functions. The formula is inspired by, and closely related to, the work of Paradan [11]. The fact that only a subset of the components in $\mathrm{Comp}_T(M)$ contribute is non-trivial and quite important for $[Q,R]=0$. The proof given by Szenes–Vergne involves studying the asymptotic behavior of the m_C's using the Berline–Vergne formula and the principle of stationary phase. It goes back to results of Paradan [11], who proved a closely related result using transversally elliptic symbols and K-theoretic methods. Note that Szenes–Vergne assume for simplicity that M^T consists of isolated fixed points, but it is not difficult to handle the general case with the same methods; see for example [6, Section 7] for some indications of how this can be done.

For the proof of Theorem 2.2 we do not need the precise definition of the terms m_C in (2.3), but we will need the following two crucial properties:

1. The function m_C restricts to a quasi-polynomial on each Λ-translate of the set $(\mathbb{Z}\times\Lambda)\cap\boldsymbol{A}_C$, where
$$\boldsymbol{A}_C = \{(t, t\tau) \,|\, t \in \mathbb{R}_{>0},\, \tau \in A_C\} \subset \mathbb{R} \times \mathfrak{t}^*.$$
2. Let $\mathrm{wt}(\nu_C)$ denote the list of complex weights (for the compatible almost complex structure J) for the $\mathfrak{t}_C$ action on the normal bundle ν_C. If $\lambda \in \Lambda$ is in the support of $m_C(k,-)$ then λ satisfies the inequality
$$\langle \tau_C, \lambda\rangle \geq k\langle \tau_C, \gamma_C\rangle + \langle \tau_C, \sigma_C\rangle, \qquad \sigma_C := \sum_{\substack{\delta\in\mathrm{wt}(\nu_C)\\ \langle\tau_C,\delta\rangle>0}} \delta - \sum_{\substack{\alpha\in\mathcal{R}_+\\ \langle\tau_C,\alpha\rangle>0}} \alpha. \tag{2.4}$$
See the proof of [14, Theorem 49]. Note that, except for the special case $\tau_C = 0$, (2.4) defines a half-space in $\mathfrak{t}^*$.

We will refer to these two properties as 'property (a)', 'property (b)' in the proof of the next result. Theorem 2.2 is a strengthening of [14, Theorem 49] (which says that the function $k \mapsto m(k,0)$ is quasi-polynomial), and our arguments are based on their elegant approach.

Theorem 2.2 ([10], see also [11, 12, 13])**.** *If $0 \notin \Delta$ then $m(k, 0) = 0$ for all $k \geq 1$. If $0 \in \Delta$ then there is a closed polytope $\mathfrak{p} \subset \Delta$ of the same dimension as Δ and containing the origin such that $m(k, \lambda)$ is quasi-polynomial on the set of integral points $(\mathbb{Z} \times \Lambda) \cap C_{\mathfrak{p}}$ contained in the cone*

$$C_{\mathfrak{p}} = \{(t, t\tau) \,|\, t \in (0, \infty),\, \tau \in \mathfrak{p}\} \subset \mathbb{R} \times \mathfrak{t}^*.$$

Proof. The strategy is based on choosing a suitable $\gamma \in \mathfrak{t}^*_+$ and then analyzing the supports of the contributions m_C to m in the corresponding Szenes–Vergne–Paradan formula (2.3) using property (b). The contribution m_C appears in (2.3) only if $\gamma_C \in \mu_{\mathfrak{g}}(M) \cap \mathfrak{t}^* = W \cdot \Delta \subset W \cdot I$ (recall by definition I is the smallest affine subspace containing Δ). Because γ is chosen generically, the only $C \in \mathrm{Comp}_T(M)$ which may contribute to (2.3) are those such that the affine subspace A_C is entirely contained in I or one of its Weyl reflections, and throughout the proof we assume this is the case.

Suppose $0 \in \Delta$. We argue that by a suitable choice of γ, one can arrange that *for all but one* of the contributions, (i) $\langle \tau_C, \gamma_C \rangle \geq 0$ with equality if and only if $0 \in A_C$, (ii) $\langle \tau_C, \gamma_C \rangle > \langle \tau_C, \gamma_I \rangle$, where γ_I is the orthogonal projection of γ onto I, and (iii) $\langle \tau_C, \sigma_C \rangle > 0$. The one special contribution is denoted m_{C_I} below and corresponds to the subspace $A_{C_I} = I$. By property (b), (i) and (iii) imply that for $C \neq C_I$, the support of $m_C(k, -)$ lies outside kH_C where H_C is the half-space

$$H_C = \{\xi \,|\, \langle \tau_C, \xi \rangle \leq \langle \tau_C, \gamma_C \rangle\}.$$

Let $\mathfrak{p}$ be the intersection of I with all of the half-spaces H_C for $C \neq C_I$. By (ii), the relative interior of $\mathfrak{p}$, viewed as a polytope in I, contains the point γ_I, hence in particular is non-empty. By construction $m \restriction C_{\mathfrak{p}} = m_{C_I} \restriction C_{\mathfrak{p}}$. Then property (a) implies that m_{C_I} is quasi-polynomial on $C_{\mathfrak{p}}$, hence the result.

We claim that one can ensure (i) holds for all C by choosing $\gamma \in \mathfrak{t}^*_+$ sufficiently close to 0. Indeed let A^0_C be the subspace parallel to A_C, and let $a_C \in A_C$ be the nearest point in A_C to 0. Then $\gamma_C - a_C \in A^0_C$ while τ_C, a_C are both orthogonal to A^0_C, hence $\langle \tau_C, \gamma_C - a_C \rangle = 0 = \langle a_C, \gamma_C - a_C \rangle$. These imply $\langle \tau_C, \gamma_C \rangle = \|a_C\|^2 - \langle a_C, \gamma \rangle$. If $0 \in A_C$ then $a_C = 0$ and this vanishes. Otherwise we can ensure $\langle \tau_C, \gamma_C \rangle > 0$ by choosing $\|\gamma\| < \|a_C\|$. Since only finitely many C occur, we can choose γ such that this holds for all C with $0 \notin A_C$. We now turn to verifying (ii), (iii), and also handle the case $0 \notin \Delta$ along the way.

Suppose $\gamma_C \in \mu_{\mathfrak{g}}(C)$, so that m_C indeed appears in (2.3). If $\alpha \in \mathcal{R}_+$ and $\langle \tau_C, \alpha \rangle > 0$, then since $\gamma \in \mathfrak{t}^*_+$ it follows that $\langle \gamma_C, \alpha \rangle > 0$. It is a consequence of the cross-section theorem (cf. [5]) that $\alpha|_{\mathfrak{t}_C}$ appears in the list of weights $\mathrm{wt}(\nu_C)$. Hence

$$\sigma_C = \sum_{\substack{\delta \in \mathrm{wt}(\nu_C) - \mathcal{R}^{\tau_C}_+ \\ \langle \tau_C, \delta \rangle > 0}} \delta, \tag{2.5}$$

where $\mathcal{R}^{\tau_C}_+$ denotes the set of positive roots α such that $\langle \tau_C, \alpha \rangle > 0$, and $\mathrm{wt}(\nu_C) - \mathcal{R}^{\tau_C}_+$ denotes the list of weights on ν_C with one copy of $\alpha|_{\mathfrak{t}_C}$ removed for each $\alpha \in \mathcal{R}_+$ satisfying $\langle \tau_C, \alpha \rangle > 0$. Hence

$$\langle \tau_C, \sigma_C \rangle \geq 0 \tag{2.6}$$

and the inequality is strict if at least one weight δ contributes in (2.5).

If $0 \notin \Delta$ then, choosing γ sufficiently close to 0, we can ensure that for each C such that $0 \in A_C$ we have $\gamma_C \notin \mu_{\mathfrak{g}}(M)$ (a fortiori $\gamma_C \notin \mu_{\mathfrak{g}}(C)$), hence m_C does not appear in (2.3) at all. On the other hand, by (i), (2.6) and property (b), if $0 \notin A_C$ then $m_C(k, 0) = 0$ for all $k \geq 1$. We conclude that if $0 \notin \Delta$ then $m(k, 0) = 0$ for all $k \geq 1$.

We turn to the case $0 \in \Delta \subset I$. In this case we may choose γ such that it is simultaneously close to 0 and arbitrarily close to γ_I, the orthogonal projection of γ onto I. Since $\tau_C = \gamma_C - \gamma$, $\langle \tau_C, \gamma \rangle \leq \langle \tau_C, \gamma_C \rangle$ with equality if and only if $\gamma_C = \gamma$. By taking γ sufficiently close to I, one can ensure that $\langle \tau_C, \gamma_I \rangle \leq \langle \tau_C, \gamma_C \rangle$ with equality if and only if $\gamma_C = \gamma_I$.

We first consider contributions from components $C \in \mathrm{Comp}_T(M)$ such that $\gamma_C \notin \mathfrak{t}^*_+$. In this case there exists a negative root $\alpha \in \mathcal{R}_-$ such that $\langle \gamma_C, \alpha \rangle > 0$. It follows from the cross-section theorem that $\alpha|_{\mathfrak{t}_C} \in \mathrm{wt}(\nu_C)$. Since $\gamma \in \mathfrak{t}^*_+$, $\langle \gamma, \alpha \rangle \leq 0$ and so

$$\langle \tau_C, \alpha \rangle = \langle \gamma_C, \alpha \rangle - \langle \gamma, \alpha \rangle > 0.$$

As $\alpha \notin \mathcal{R}_+$, we see that $\delta = \alpha$ indeed contributes in (2.5), hence $\langle \tau_C, \sigma_C \rangle > 0$. Moreover since $\gamma_C \notin \mathfrak{t}^*_+$, $\gamma_C \neq \gamma_I$, hence $\langle \tau_C, \gamma_I \rangle < \langle \tau_C, \gamma_C \rangle$. This establishes (ii), (iii) for this case.

We are left to consider contributions from $C \in \mathrm{Comp}_T(M)$ such that $\gamma_C \in \Delta = \mu_{\mathfrak{g}}(M) \cap \mathfrak{t}^*_+$. Let $\Delta_{\mathrm{reg}} \subset \Delta$ be the relatively open dense subset of weakly regular values. The connected components of Δ_{reg} are relatively open polytopes inside the subspace I. Choose a connected component $\mathfrak{a} \subset \Delta_{\mathrm{reg}}$ containing 0 in its closure. We may choose $\gamma \in \mathfrak{t}^*_+$ such that the orthogonal projection γ_I onto I lies in $\mathfrak{a}$. The fibre $\mu_{\mathfrak{g}}^{-1}(\gamma_I)$ is connected and contained in M^{T_I}, hence there is a unique connected component $C_I \subset M^{T_I}$ containing $\mu_{\mathfrak{g}}^{-1}(\gamma_I)$. Then $A_{C_I} = I$ and by property (a), m_{C_I} is quasi-polynomial on the set of integral points in $\boldsymbol{A}_C = \{(t, t\tau) \,|\, t > 0,\ \tau \in I\} \supset C_{\mathfrak{p}}$.

The final situation to consider consists of the contributions from $C \in \mathrm{Comp}_T(M)$ such that $\gamma_C \in \Delta \setminus \Delta_{\mathrm{reg}}$. In particular $\gamma_C \neq \gamma_I$ hence

$$\langle \tau_C, \gamma_I \rangle < \langle \tau_C, \gamma_C \rangle \tag{2.7}$$

establishing (ii) for this case. Let σ be the face of $\mathfrak{t}^*_+$ containing γ_C. The subset

$$U = G_\sigma \cdot \bigcup_{\overline{\tau} \supset \sigma} \tau,$$

where the union is taken over relatively open faces of $\mathfrak{t}^*_+$ whose closure contains σ, is a slice for the coadjoint G_σ-action. Let $Y = \mu_{\mathfrak{g}}^{-1}(U)$ be the corresponding symplectic cross-section, cf. [5, Remark 3.7, Theorem 3.8]. Consider the function $f = \langle \tau_C, \mu \rangle|_Y \colon Y \to \mathbb{R}$, for which $C \cap Y \subset Y^{\tau_C} = \mathrm{Crit}(f)$ is a critical submanifold. Note that $f|_{C \cap Y} = \langle \tau_C, \gamma_C \rangle$. A result from symplectic geometry says that in a suitable tubular neighborhood of $C \cap Y$, the function f takes the form

$$f(z_1, \ldots, z_n) = \langle \tau_C, \gamma_C \rangle - \pi \sum_j |z_j|^2 \langle \tau_C, \delta_j \rangle, \tag{2.8}$$

where $\delta_j \in \mathrm{wt}(\nu_{C \cap Y, Y})$, $\langle \tau_C, \delta_j \rangle \neq 0$, z_j is a vector in the subbundle of $\nu_{C \cap Y, Y}$ where $\mathfrak{t}_C$ acts with weight δ_j, and $|z_j|$ denotes its norm with respect to a suitable Hermitian structure.

Let S be the line segment with endpoints γ_I and γ_C. By convexity $S \subset \Delta$. The inverse image $\mu_{\mathfrak{g}}^{-1}(S) \subset Y$ is connected since $\mu_{\mathfrak{g}}$ has connected fibres. By (2.7), along the line segment S, f varies between its absolute minimum $\langle \tau_C, \gamma_I \rangle$ on the fibre $\mu_{\mathfrak{g}}^{-1}(\gamma_I)$ and its absolute maximum $\langle \tau_C, \gamma_C \rangle$ on the fibre $\mu_{\mathfrak{g}}^{-1}(\gamma_C)$. By connectedness of $\mu_{\mathfrak{g}}^{-1}(S)$ and equation (2.8), there must exist a δ_j such that $\langle \tau_C, \delta_j \rangle > 0$.

By the cross-section theorem $\nu_{Y,M}|_{C \cap Y} \simeq (C \cap Y) \times \mathfrak{g}^{\perp}_{\gamma_C}$, where the orthogonal complement $\mathfrak{g}^{\perp}_{\gamma_C}$ is embedded in $TM|_{C \cap Y}$ as the orbit directions. The weights $\mathcal{R}^{\tau_C}_+$ which are removed in (2.5) can be identified with the weights of the $\mathfrak{t}_C$-action on $\nu_{Y,M}|_{C \cap Y}$. With this understanding we have $\mathrm{wt}(\nu_{C \cap Y, Y}) \subset \mathrm{wt}(\nu_C) - \mathcal{R}^{\tau_C}_+$. Thus δ_j indeed contributes to (2.5), establishing (iii) for this case. This completes the proof. ∎

Corollary 2.3. *Suppose $0 \in \Delta$ and let $\mathfrak{p} \subset \Delta$ be as in Theorem 2.2. If $\xi \in \mathfrak{p}$ is rational and $n_\xi \in \mathbb{Z}_{>0}$ is the least positive integer such that $n_\xi \xi \in \Lambda$, then the function*

$$f_\xi\colon\ n_\xi \cdot \mathbb{Z}_{>0} \to \mathbb{Z}, \qquad f_\xi(k) = m(k, k\xi)$$

is quasi-polynomial. Moreover $m \restriction C_\mathfrak{p}$ is the unique quasi-polynomial function such that $m(k, k\xi) = f_\xi(k)$ for all rational, weakly regular values ξ in the relative interior of $\mathfrak{p}$.

Remark 2.4. A suitable finite collection of the functions f_ξ already fully determines $m \restriction C_\mathfrak{p}$.

3 Stationary phase calculation

Assume $0 \in \Delta$ and let $\mathfrak{p} \subset \Delta$ be as in Theorem 2.2, so that $m \restriction C_\mathfrak{p}$ is quasi-polynomial. By Corollary 2.3, $m \restriction C_\mathfrak{p}$ is completely determined by the collection of quasi-polynomial functions $f_\xi(k) = m(k, k\xi)$, for ξ ranging over rational, weakly regular values of $\mu_\mathfrak{g}$ lying in the relative interior of $\mathfrak{p}$. In this section we use the Berline–Vergne index formula and the stationary phase expansion to compute the functions f_ξ, and hence also $m \restriction C_\mathfrak{p}$. The end result will be the formula (1.3) in Theorem 1.2.

Let $t \in T$. By the Berline–Vergne formula, for $X \in \mathfrak{t}$ sufficiently small one has $Q\big(k, te^X\big) = Q_t(k, X)$ where

$$Q_t(k, X) := \int_{M^t} \frac{t_L^k \mathrm{e}^{k(\omega + 2\pi\mathrm{i}\langle \mu, X\rangle)} \mathrm{Td}\big(M^t, \frac{2\pi}{\mathrm{i}} X\big)}{\mathcal{D}^t_\mathbb{C}\big(\nu_{M^t,M}, \frac{2\pi}{\mathrm{i}} X\big)} \det{}^{\mathfrak{g}/\mathfrak{t}}_\mathbb{C}\big(1 - t^{-1}\mathrm{e}^{-X}\big), \tag{3.1}$$

and $\mathrm{Td}\big(M^t, \frac{2\pi}{\mathrm{i}} X\big)$, $\mathcal{D}^t_\mathbb{C}\big(\nu_{M^t,M}, \frac{2\pi}{\mathrm{i}} X\big)$ denote equivariant extensions of the usual Chern-Weil forms, closed with respect to the differential $\mathrm{d} + 2\pi\mathrm{i}\iota(X_M)$, obtained by replacing curvatures with equivariant curvatures (evaluated at $\frac{2\pi}{\mathrm{i}} X$) in the usual formulas (cf. [1] for details, although note that we are using the topologist's convention for characteristic classes).

Let B_r denote the ball of radius $r > 0$ around the origin in $\mathfrak{g}/\mathfrak{t}$. Let $\mu_{\mathfrak{g}/\mathfrak{t}}$ denote the composition of $\mu_\mathfrak{g}$ with the quotient map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{t}$. Let $\mathfrak{g}^t \subset \mathfrak{g}$ denote the fixed-point set of Ad_t. Then B_r^t is a neighborhood of 0 in $\mathfrak{g}^t/\mathfrak{t}$. Recall $\mathfrak{g}^t/\mathfrak{t}$, $\mathfrak{g}/\mathfrak{g}^t$ are equipped with complex structures such that their $+\mathrm{i}$-eigenspaces are identified with sums of positive root spaces. Equip $\mathfrak{g}^t/\mathfrak{t}$ with the orientation induced by the complex structure, and let $\tau_{\mathfrak{g}^t/\mathfrak{t}}(X)$ be a T-equivariant Thom form with support contained in B_r^t, closed for the differential $\mathrm{d} - \iota(X_M)$. Consider the T-equivariant differential form on $\mathfrak{g}^t/\mathfrak{t}$ (closed for the differential $\mathrm{d} + 2\pi\mathrm{i}\iota(X_{\mathfrak{g}^t/\mathfrak{t}})$) given by

$$\mathrm{Ch}^t\big(\mathsf{b}, \tfrac{2\pi}{\mathrm{i}} X\big) = \det{}^{\mathfrak{g}/\mathfrak{g}^t}_\mathbb{C}\big(1 - t^{-1}\mathrm{e}^{-X}\big) \det{}^{\mathfrak{g}^t/\mathfrak{t}}_\mathbb{C}\left(\frac{1 - \mathrm{e}^{-X}}{X}\right) \tau_{\mathfrak{g}^t/\mathfrak{t}}\big(\tfrac{2\pi}{\mathrm{i}} X\big),$$

The map $\mu_{\mathfrak{g}/\mathfrak{t}}$ restricts to a map $M^t \to \mathfrak{g}^t/\mathfrak{t}$, which we use to pull back the form $\mathrm{Ch}^t\big(\mathsf{b}, \frac{2\pi}{\mathrm{i}} X\big)$.

Lemma 3.1.

$$Q_t(k, X) = \int_{\mu^{-1}_{\mathfrak{g}/\mathfrak{t}}(B_r^t)} \frac{t_L^k \mathrm{e}^{k(\omega + 2\pi\mathrm{i}\langle \mu, X\rangle)} \mathrm{Td}\big(M^t, \frac{2\pi}{\mathrm{i}} X\big)}{\mathcal{D}^t_\mathbb{C}\big(\nu_{M^t,M}, \frac{2\pi}{\mathrm{i}} X\big)} \mathrm{Ch}^t\big(\mathsf{b}, \tfrac{2\pi}{\mathrm{i}} X\big). \tag{3.2}$$

Proof. The pullback of $\tau_{\mathfrak{g}^t/\mathfrak{t}}(X)$ to $0 \in \mathfrak{g}^t/\mathfrak{t}$ is the equivariant Euler class, which (since 0 is just a point) is the function

$$\prod_{\alpha \in \mathcal{R}_+^{\mathfrak{g}^t}} -\langle \alpha, X\rangle = \det{}^{\mathfrak{g}^t/\mathfrak{t}}_\mathbb{C}\big(\tfrac{\mathrm{i}}{2\pi} X\big),$$

where $\mathcal{R}_+^{\mathfrak{g}^t} \subset \mathcal{R}_+$ is a set of positive roots for $\mathfrak{g}^t$. Note also that t acts trivially on $\mathfrak{g}^t/\mathfrak{t}$, since $\mathfrak{g}^t$ is the fixed point subspace under the adjoint action. It follows that the pullback of $\mathrm{Ch}^t\big(\mathsf{b}, \frac{2\pi}{\mathrm{i}}X\big)$ to $0 \in \mathfrak{g}^t/\mathfrak{t}$ is the function $\det_{\mathbb{C}}^{\mathfrak{g}/\mathfrak{t}}\big(1 - t^{-1}\mathrm{e}^{-X}\big)$. Since pullback to $\{0\} = (\mathfrak{g}^t/\mathfrak{t})^T$ is injective on equivariant cohomology classes, $\mathrm{Ch}^t\big(\mathsf{b}, \frac{2\pi}{\mathrm{i}}X\big)$, $\det_{\mathbb{C}}^{\mathfrak{g}/\mathfrak{t}}\big(1 - t^{-1}\mathrm{e}^{-X}\big)$ determine the same class in T-equivariant cohomology of $\mathfrak{g}^t/\mathfrak{t}$. As M is compact, we may make this replacement in (3.1) without changing the value of the integral. ■

Remark 3.2. The reason for the notation is that $\mathrm{Ch}^t\big(\mathsf{b}, \frac{2\pi}{\mathrm{i}}X\big)$ is a representative for the t-twisted Chern character of a Bott element $\mathsf{b} \in K_T^0(\mathfrak{g}/\mathfrak{t})$, which generates the latter as an $R(T) = K_T^0(\mathrm{pt})$-module. To be more precise, b is the generator whose pullback to $0 \in \mathfrak{g}/\mathfrak{t}$ is $[\wedge^{\mathrm{ev}}\mathfrak{n}_-] - [\wedge^{\mathrm{odd}}\mathfrak{n}_-] \in K_T^0(\mathrm{pt})$, $\mathfrak{n}_-$ being the direct sum of the negative root spaces.

Since T is compact, there exists a finite set $S \subset T$ and an open cover $\{U_t \,|\, t \in S\}$ of T where U_t is a small open ball around t in T such that $Q\big(k, t\mathrm{e}^X\big) = Q_t(k, X)$ for $t\mathrm{e}^X \in U_t$. Let σ_t, $t \in S$ be bump functions on $\mathfrak{t}$ such that $\{\hat{t}_*\sigma_t \,|\, t \in S\}$ is a partition of unity subordinate to the cover, where $\hat{t}$ is the map

$$\hat{t}\colon\ \mathfrak{t} \to T, \qquad X \mapsto t\mathrm{e}^X,$$

which we may assume restricts to a diffeomorphism of a small ball around $0 \in \mathfrak{t}$ onto U_t. By equations (3.1) and (3.2)

$$Q = \sum_{t \in S} \hat{t}_*(\sigma_t Q_t).$$

The multiplicity function m is the Fourier transform of Q:

$$m(k, \lambda) = \sum_{t \in S} \int_{\mathfrak{t}} \sigma_t(X)\big(t\mathrm{e}^X\big)^{-\lambda} Q_t(k, X).$$

To do the stationary phase calculation (for $k \to \infty$) following the approach outlined at the beginning of this section, we now set $\lambda = k\xi$ where $\xi \in (\Lambda \otimes \mathbb{Q}) \cap \mathfrak{p}$ is a rational, weakly regular value of $\mu_{\mathfrak{g}}$ contained in the relative interior of $\mathfrak{p} \subset \Delta$ as in Corollary 2.3, $k \in n_\xi \mathbb{Z}_{>0}$ and n_ξ is the least positive integer such that $n_\xi \xi \in \Lambda$. Thus

$$m(k, k\xi) = \sum_t t^{-k\xi} \int_{\mathfrak{t}} \mathrm{d}X \sigma_t(X) \int_{\mu_{\mathfrak{g}/\mathfrak{t}}^{-1}(B_r^t)} \frac{t_L^k \mathrm{Td}\big(M^t, \frac{2\pi}{\mathrm{i}}X\big)}{\mathcal{D}_{\mathbb{C}}^t\big(\nu_{M^t,M}, \frac{2\pi}{\mathrm{i}}X\big)} \mathrm{Ch}^t\big(\mathsf{b}, \tfrac{2\pi}{\mathrm{i}}X\big) \mathrm{e}^{k(\omega + 2\pi\mathrm{i}\langle \mu - \xi, X\rangle)}. \quad (3.3)$$

Let $f(m, X) = \langle \mu(m) - \xi, X\rangle$ viewed as a real-valued function on $\mu_{\mathfrak{g}/\mathfrak{t}}^{-1}\big(B_r^t\big) \times \mathfrak{t}$. According to the principle of stationary phase, we can include a bump function supported in a small neighborhood of the critical set of f in the integrand of (3.3), and the error will be $o(k^{-\infty})$. The derivative

$$\mathrm{d}_{(m,X_0)} f = \langle \mathrm{d}_m \mu, X_0\rangle + \langle \mu(m) - \xi, \mathrm{d}_{X_0} X\rangle$$

and in particular $\mathrm{Crit}(f) \subset \mu^{-1}(\xi) \times \mathfrak{t}$. Let χ be the pullback by μ of a bump function in $\mathfrak{t}^*$ supported in a small neighborhood of ξ. Thus

$$m(k, k\xi) \sim \sum_t t^{-k\xi} \int_{\mathfrak{t}} \mathrm{d}X \sigma_t(X) \times \int_{\mu_{\mathfrak{g}/\mathfrak{t}}^{-1}(B_r^t)} \chi \frac{t_L^k \mathrm{Td}\big(M^t, \frac{2\pi}{\mathrm{i}}X\big)}{\mathcal{D}_{\mathbb{C}}^t\big(\nu_{M^t,M}, \frac{2\pi}{\mathrm{i}}X\big)} \mathrm{Ch}^t\big(\mathsf{b}, \tfrac{2\pi}{\mathrm{i}}X\big) \mathrm{e}^{k(\omega + 2\pi\mathrm{i}\langle \mu - \xi, X\rangle)}, \quad (3.4)$$

where $\sim$ denotes equality modulo an $o(k^{-\infty})$ error.

Let $Y=\mu_{\mathfrak{g}}^{-1}(\sigma)$ be the cross-section for the principal face. By the cross-section theorem, a neighborhood N of Y in M is G_σ-equivariantly diffeomorphic to

$$Y\times\mathfrak{g}/\mathfrak{g}_\sigma,$$

where $\mathfrak{g}/\mathfrak{g}_\sigma\simeq\mathfrak{g}_\sigma^\perp$ is embedded in the orbit directions. Since $\mu^{-1}(\xi)\cap\mu_{\mathfrak{g}}^{-1}(\mathfrak{t}^*)=\mu_{\mathfrak{g}}^{-1}(\xi)\subset Y$, by taking r and $\operatorname{supp}(\chi)$ sufficiently small, we can assume that $\operatorname{supp}(\chi)\cap\mu_{\mathfrak{g}/\mathfrak{t}}^{-1}(B_r)$ is contained in a small neighborhood of $\mu_{\mathfrak{g}}^{-1}(\xi)$ where the local model $Y\times\mathfrak{g}/\mathfrak{g}_\sigma$ is valid, and so we may replace $\mu_{\mathfrak{g}/\mathfrak{t}}^{-1}(B_r^t)$ with N^t in equation (3.4). In the next lemma we use the Thom form to integrate over the $(\mathfrak{g}/\mathfrak{g}_\sigma)^t$ directions.

Lemma 3.3.

$$\begin{aligned}m(k,k\xi)\sim{}&\sum_{t\in S}t^{-k\xi}\int_{\mathfrak{t}}\mathrm{d}X\sigma_t(X)\\&\times\int_{Y^t}\chi\frac{t_L^k\operatorname{Td}\big(Y^t,\frac{2\pi}{\mathrm{i}}X\big)}{\mathcal{D}_{\mathbb{C}}^t\big(\nu_{Y^t,Y},\frac{2\pi}{\mathrm{i}}X\big)}\mathrm{e}^{k(\omega+2\pi\mathrm{i}\langle\mu-\xi,X\rangle)}\det\nolimits_{\mathbb{C}}^{\mathfrak{g}_\sigma/\mathfrak{t}}\big(1-t^{-1}\mathrm{e}^{-X}\big).\end{aligned}\tag{3.5}$$

Proof. The neighborhood N^t of Y^t in M^t is T-equivariantly diffeomorphic to

$$Y^t\times(\mathfrak{g}/\mathfrak{g}_\sigma)^t=Y^t\times\mathfrak{g}^t/\mathfrak{g}_\sigma^t,$$

where $\mathfrak{g}_\sigma^t=(\mathfrak{g}_\sigma)^t$ is the subspace of $\mathfrak{g}_\sigma$ fixed by t. Moreover the almost complex structure on N^t is homotopic to a product almost complex structure, where Y^t is equipped with an almost complex structure compatible with the symplectic form in the cross-section, and $\mathfrak{g}^t/\mathfrak{g}_\sigma^t$ is equipped with the almost complex structure whose $+\mathrm{i}$-eigenspace is identified with a sum of positive root spaces. Let

$$p\colon\ N^t\to Y^t$$

denote the projection. For the normal bundle

$$\nu_{M^t,M}|_{N^t}\simeq p^*\nu_{Y^t,Y}\oplus(\mathfrak{g}/\mathfrak{g}_\sigma)/(\mathfrak{g}/\mathfrak{g}_\sigma)^t=p^*\nu_{Y^t,Y}\oplus\mathfrak{g}/\big(\mathfrak{g}^t+\mathfrak{g}_\sigma\big),$$

and again the almost complex structure is homotopic to a product one, using a compatible almost complex structure on the symplectic vector bundle $\nu_{Y^t,Y}$, and an almost complex structure on $\mathfrak{g}/\big(\mathfrak{g}^t+\mathfrak{g}_\sigma\big)$ whose $+\mathrm{i}$-eigenspace is identified with a sum of positive root spaces. Using the identifications above we obtain, up to equivariantly exact forms:

$$\begin{aligned}\operatorname{Td}\big(M^t,\tfrac{2\pi}{\mathrm{i}}X\big)|_{N^t}&=\operatorname{Td}\big(Y^t,\tfrac{2\pi}{\mathrm{i}}X\big)\det\nolimits_{\mathbb{C}}^{\mathfrak{g}^t/\mathfrak{g}_\sigma^t}\left(\frac{X}{1-\mathrm{e}^{-X}}\right),\\\mathcal{D}_{\mathbb{C}}^t\big(\nu_{M^t,M},\tfrac{2\pi}{\mathrm{i}}X\big)|_{N^t}&=\mathcal{D}_{\mathbb{C}}^t\big(\nu_{Y,Y^t},\tfrac{2\pi}{\mathrm{i}}X\big)\det\nolimits_{\mathbb{C}}^{\mathfrak{g}/(\mathfrak{g}^t+\mathfrak{g}_\sigma)}\big(1-t^{-1}\mathrm{e}^{-X}\big).\end{aligned}\tag{3.6}$$

Since $Y^t\subset\mu_{\mathfrak{g}}^{-1}(\mathfrak{t}^*)$, the pullback of the equivariant Thom form $\tau_{\mathfrak{g}^t/\mathfrak{t}}(X)$ to Y^t is just the function

$$\det\nolimits_{\mathbb{C}}^{\mathfrak{g}^t/\mathfrak{t}}\big(\tfrac{\mathrm{i}}{2\pi}X\big)=\det\nolimits_{\mathbb{C}}^{\mathfrak{g}^t/\mathfrak{g}_\sigma^t}\big(\tfrac{\mathrm{i}}{2\pi}X\big)\det\nolimits_{\mathbb{C}}^{\mathfrak{g}_\sigma^t/\mathfrak{t}}\big(\tfrac{\mathrm{i}}{2\pi}X\big).$$

We recognize $\det_{\mathbb{C}}^{\mathfrak{g}^t/\mathfrak{g}_\sigma^t}\big(\frac{\mathrm{i}}{2\pi}X\big)$ as the equivariant Euler class of the trivial bundle $Y^t\times\mathfrak{g}^t/\mathfrak{g}_\sigma^t$. Thus up to an equivariantly exact form, we have

$$\tau_{\mathfrak{g}^t/\mathfrak{t}}(X)=\tau_p(X)\det\nolimits_{\mathbb{C}}^{\mathfrak{g}_\sigma^t/\mathfrak{t}}\big(\tfrac{\mathrm{i}}{2\pi}X\big),\tag{3.7}$$

where $\tau_p(X)$ is an equivariant Thom form for the vector bundle $p\colon N^t=Y^t\times\mathfrak{g}^t/\mathfrak{g}_\sigma^t\to Y^t$.

We next want to make the replacements (3.6), (3.7) in equation (3.4), and then use the Thom form $\tau_p(X)$ to integrate over the fibres of $p\colon N^t \to Y^t$. In the integral over N^t in (3.4), the integrand has compact support and all terms in the integrand are equivariantly closed except for the bump function χ. By Stokes' theorem, replacing a form by a cohomologous form in the integrand leads to an error term containing $\mathrm{d}\chi$; but $\mathrm{d}\chi$ vanishes near $\mu^{-1}(\xi)$, so the principle of stationary phase implies the error will be $o(k^{-\infty})$. Let $\iota_{Y^t}\colon Y^t \hookrightarrow N^t$ denote the inclusion. Similarly the formula $p_*(\tau_p(X)\alpha(X)) = \iota^*_{Y^t}\alpha(X)$ applies when $\alpha(X)$ is equivariantly closed. But writing $\chi = 1-(1-\chi)$, the principle of stationary phase again shows that we can make this replacement up to an $o(k^{-\infty})$ error term.

After making these replacements and integrating over the fibre, the form $\tau_p\big(\frac{2\pi}{\mathrm{i}}X\big)$ disappears. There are various Lie theoretic factors left over:

$$\frac{\det_{\mathbb{C}}^{\mathfrak{g}/\mathfrak{g}^t}\big(1-t^{-1}\mathrm{e}^{-X}\big)}{\det_{\mathbb{C}}^{\mathfrak{g}/(\mathfrak{g}^t+\mathfrak{g}_\sigma)}\big(1-t^{-1}\mathrm{e}^{-X}\big)}\det_{\mathbb{C}}^{\mathfrak{g}^t/\mathfrak{t}}\left(\frac{1-\mathrm{e}^{-X}}{X}\right)\det_{\mathbb{C}}^{\mathfrak{g}^t_\sigma/\mathfrak{t}}(X)\det_{\mathbb{C}}^{\mathfrak{g}^t/\mathfrak{g}^t_\sigma}\left(\frac{X}{1-\mathrm{e}^{-X}}\right),$$

which simplify to $\det_{\mathbb{C}}^{\mathfrak{g}_\sigma/\mathfrak{t}}\big(1-t^{-1}\mathrm{e}^{-X}\big)$ (one uses that t acts trivially on $\mathfrak{g}^t_\sigma/\mathfrak{t}$ and that $\big(\mathfrak{g}^t+\mathfrak{g}_\sigma\big)/\mathfrak{g}^t \simeq \mathfrak{g}_\sigma/\mathfrak{g}^t_\sigma$). ■

Choose a complementary subtorus T'_I so that $T \simeq T_I \times T'_I$. The quotient map $T \to T/T_I$ induces an isomorphism of groups $T'_I \xrightarrow{\sim} T/T_I$. By adding additional points if necessary, we may assume the finite subset $S \subset T$ is a product $S_I \times S'_I$, where $S_I \subset T_I$, $S'_I \subset T'_I$ and that the image of S'_I in T/T_I contains the set S_P from the introduction. Thus we will write elements of S as products hg with $h \in S_I \subset T_I$ and $g \in S'_I \subset T'_I$. We may assume the bump function σ_t is a product $\sigma_h \cdot \sigma_g$, where σ_h (resp. σ_g) is a bump function on $\mathfrak{t}_I$ (resp. $\mathfrak{t}'_I$), satisfying

$$\sum_{h\in S_I}\hat{h}_*\sigma_h = 1, \qquad \sum_{g\in S'_I}\hat{g}_*\sigma_g = 1. \tag{3.8}$$

The next lemma gives a further simplification of (3.5).

Lemma 3.4.

$$m(k,k\xi) \sim \sum_{g\in S'_I} g^{-k\xi}\int_{\mathfrak{t}'_I}\mathrm{d}X\sigma_g(X)\int_{Y^g}\chi\frac{g_L^k\mathrm{Td}\big(Y^g,\frac{2\pi}{\mathrm{i}}X\big)}{\mathcal{D}^g_{\mathbb{C}}\big(\nu_{Y^g,Y},\frac{2\pi}{\mathrm{i}}X\big)}\mathrm{e}^{k(\omega+2\pi\mathrm{i}\langle\mu-\xi,X\rangle)}. \tag{3.9}$$

Proof. As T_I acts trivially on Y and $\mu(Y) \subset I$, the characteristic forms in (3.5) only depend on the component of X (resp. t) in $\mathfrak{t}'_I$ (resp. T'_I). Likewise as $\xi \in (\Lambda\otimes\mathbb{Q})\cap I$, $t^{-k\xi}$ only depends on the component g of t in T'_I. This means the following expression can be split off from (3.5) and evaluated separately:

$$\sum_{h\in S_I}\int_{\mathfrak{t}_I}\mathrm{d}X\sigma_h(X)\det_{\mathbb{C}}^{\mathfrak{g}_\sigma/\mathfrak{t}}\big(1-h^{-1}g^{-1}\mathrm{e}^{-X}\big). \tag{3.10}$$

The determinant is given by a product:

$$\prod_{\alpha\in\mathcal{R}_+^{\mathfrak{g}_\sigma}}\big(1-h^{-\alpha}g^{-\alpha}\mathrm{e}^{-2\pi\mathrm{i}\langle\alpha,X\rangle}\big).$$

When the product over $\mathcal{R}_+^{\mathfrak{g}_\sigma}$ is expanded, we obtain an alternating sum of terms of the form $h^{-\zeta}g^{-\zeta}\mathrm{e}^{-2\pi\mathrm{i}\langle\zeta,X\rangle}$, where ζ is a sum of a subset of $\mathcal{R}_+^{\mathfrak{g}_\sigma}$. The elements of $\mathcal{R}_+^{\mathfrak{g}_\sigma}$ lie in $\mathrm{ann}(\mathfrak{z}_\sigma)$, the annihilator of $\mathfrak{z}_\sigma$ in $\mathfrak{t}^*$. Since $\mathfrak{t}^* = \mathfrak{z}^*_\sigma \oplus \mathrm{ann}(\mathfrak{z}_\sigma)$ and $I \subset \mathfrak{z}^*_\sigma$, it follows that either $\zeta = 0$ or else $\zeta \notin I$.

We claim that if $\zeta \neq 0$, then the corresponding contribution to (3.10) is 0. Indeed taking the Fourier transform of the first equation in (3.8), we find that for any $[\zeta] \in \Lambda/(\Lambda \cap I)$, the weight lattice of T_I, we have

$$\sum_{h \in S_I} h^{-[\zeta]} \int_{\mathfrak{t}_I} \sigma_h(X) \mathrm{e}^{-2\pi \mathrm{i}\langle [\zeta], X\rangle} \mathrm{d}X = \delta_0([\zeta]),$$

where δ_0 is the function on $\Lambda/(\Lambda \cap I)$ equal to 1 at 0 and 0 otherwise, obtained by Fourier transform of the constant function 1 on T_I. Thus for $\zeta \in \Lambda$,

$$\sum_{h \in S_I} h^{-\zeta} \int_{\mathfrak{t}_I} \sigma_h(X) \mathrm{e}^{-2\pi \mathrm{i}\langle \zeta, X\rangle} \mathrm{d}X = \delta_{\Lambda \cap I}(\zeta),$$

where $\delta_{\Lambda \cap I}$ is the function on Λ equal to 1 on $\Lambda \cap I$ and 0 otherwise. In particular if $\zeta \notin I$ we see that the corresponding contribution in (3.10) vanishes.

On the other hand, using equation (3.8), the contribution from $\zeta = 0$ to (3.10) is

$$\sum_{h \in S_I} \int_{\mathfrak{t}_I} \mathrm{d}X \sigma_h(X) = 1.$$

This yields the expression on the right-hand-side of (3.9). ∎

We can now complete the proof of Theorem 1.2. The fibre $P = \mu^{-1}(\xi) \subset Y$ is smooth, and the quotient $\Sigma_e := M_\xi = P/G_\sigma = P/T_I'$ is an orbifold (T_I' acts locally freely on P). By the coisotropic embedding theorem, a neighborhood of P in Y is T-equivariantly symplectomorphic to

$$P \times B_I \subset P \times I,$$

where B_I is a small ball around ξ in the subspace $I \subset \mathfrak{t}^*$, the moment map μ is projection to the second factor, and the symplectic form

$$\omega|_{P \times B_I} = \omega_\xi + \mathrm{d}\langle \eta - \xi, \theta\rangle = \omega_\xi + \langle \mathrm{d}\eta, \theta\rangle + \langle \eta - \xi, F_\theta\rangle,$$

where ω_ξ is the pullback of the symplectic form on the reduced space M_ξ, η is the variable in B_I, $\theta \in \Omega^1(P, \mathfrak{t}^*)^T$ is a connection on P with curvature $F_\theta = \mathrm{d}\theta$, and here as well as below we have omitted pullbacks from the notation. A neighborhood of P^g in Y^g is T-equivariantly symplectomorphic to

$$P^g \times B_I^g = P^g \times B_I,$$

and T_I' acts locally freely on P^g, with the quotient $\Sigma_g = P^g/T_I'$ being an orbifold. On the same neighborhood we have

$$\begin{aligned} &\mathrm{Td}\big(Y^g, \tfrac{2\pi}{\mathrm{i}}X\big) = \mathrm{pr}_1^* \mathrm{Td}\big(P^g, \tfrac{2\pi}{\mathrm{i}}X\big), \\ &\nu_{Y^g,Y} = \mathrm{pr}_1^* \nu_{P^g,P} \quad \Rightarrow \quad \mathcal{D}_{\mathbb{C}}^g\big(\nu_{Y^g,Y}, \tfrac{2\pi}{\mathrm{i}}X\big) = \mathrm{pr}_1^* \mathcal{D}_{\mathbb{C}}^g\big(\nu_{P^g,P}, \tfrac{2\pi}{\mathrm{i}}X\big). \end{aligned}$$

Below we will omit pr_1^* from the notation.

Take the bump function χ to have its support contained in the neighborhood of P where the above local normal forms are valid. We may then integrate over I instead of B_I, since χ vanishes outside of $P \times B_I$ by assumption. On $\mathrm{supp}(\chi)$,

$$\mathrm{e}^{k(\omega + 2\pi \mathrm{i}\langle \mu - \xi, X\rangle)} = \mathrm{e}^{k(\omega_\xi + \langle \mathrm{d}\eta, \theta\rangle + \langle \eta - \xi, F_\theta\rangle + 2\pi \mathrm{i}\langle \eta - \xi, X\rangle)}.$$

Only the top degree part of $\mathrm{e}^{k\langle \mathrm{d}\eta,\theta\rangle}$ contributes to the integral over I; this top degree part is $(-1)^{n(n-1)/2}k^n \mathrm{d}\eta\cdot\Theta$, where $n=\dim(I)$, $\mathrm{d}\eta=\Pi \mathrm{d}\eta^a$, $\Theta=\Pi\theta_a$ in terms of coordinates on I. The sign $(-1)^{n(n-1)/2}$ relates the symplectic and product orientations for $P^g\times I$, so will be absorbed when we use Fubini's theorem to write the integral over $P^g\times I$ as an iterated integral. Let $\overline{\chi}(\eta)=\chi(\eta+\xi)$, a bump function on I supported near 0. Making these substitutions, as well as a change of variables $\eta\rightsquigarrow\eta+\xi$ in the integral over I, the asymptotic expression (3.9) for $m(k,k\xi)$ simplifies to

$$k^n\sum_g g^{-k\xi}\int_{\mathfrak{t}'_I\times I}\mathrm{d}X\,\mathrm{d}\eta\,\mathrm{e}^{2\pi\mathrm{i}k\langle\eta,X\rangle}\sigma_g(X)\overline{\chi}(\eta)\int_{P^g}\Theta\frac{g_L^k\mathrm{Td}\big(P^g,\frac{2\pi}{\mathrm{i}}X\big)}{\mathcal{D}^g_{\mathbb{C}}\big(\nu_{P^g,P},\frac{2\pi}{\mathrm{i}}X\big)}\mathrm{e}^{k(\omega_\xi+\langle\eta,F_\theta\rangle)}.$$

We need the following special case of the stationary phase expansion.

Proposition 3.5 (stationary phase expansion, cf. [4, Lemma 7.7.3])**.** *Let $u(X,\eta)$ be a Schwartz function. We have the following asymptotic expansion in k:*

$$\int_{\mathfrak{t}'_I\times(\mathfrak{t}'_I)^*}\mathrm{d}X\,\mathrm{d}\eta\,\mathrm{e}^{2\pi\mathrm{i}k\langle\eta,X\rangle}u(X,\eta)\sim\frac{1}{k^n}\sum_{j=0}^{\infty}\frac{1}{j!}\left(\sum_a\frac{\mathrm{i}}{2\pi k}\frac{\partial}{\partial\eta_a}\frac{\partial}{\partial X^a}\right)^j u(0,0).$$

Remark 3.6. To obtain the expression here from the expression appearing in *loc. cit.*, one sets $x=(X,\eta)\in\mathbb{R}^{2n}$ and $A(X,\eta)=(\eta,X)$. Note also that in Hörmander's notation $D=-\mathrm{i}(\mathrm{d}/\mathrm{d}x)$.

We apply this to the smooth compactly supported function

$$u(X,\eta)=\sigma_g(X)\overline{\chi}(\eta)\int_{P^g}\Theta\frac{g_L^k\mathrm{Td}\big(P^g,\frac{2\pi}{\mathrm{i}}X\big)}{\mathcal{D}^g_{\mathbb{C}}\big(\nu_{P^g,P},\frac{2\pi}{\mathrm{i}}X\big)}\mathrm{e}^{k(\omega_\xi+\langle\eta,F_\theta\rangle)}.$$

Although this function depends on k, the dependence is quasi-polynomial, and so the expansion still applies. Since $\sigma_g(X)$, $\overline{\chi}(\eta)$ equal 1 in a neighborhood of 0, they have no effect on the expansion. The η derivatives $k^{-1}\partial_{\eta_a}$ operate only on the factor $\mathrm{e}^{k\langle\eta,F_\theta\rangle}$. The combined effect of the operator $\Sigma_a\,(\mathrm{i}/2\pi k)\partial_{\eta_a}\partial_{X^a}$ is to replace X with $(\mathrm{i}/2\pi)F_\theta$, yielding the asymptotic expansion

$$m(k,k\xi)\sim\sum_g g^{-k\xi}\int_{P^g}\Theta\frac{g_L^k\mathrm{Td}\big(P^g,F_\theta\big)}{\mathcal{D}^g_{\mathbb{C}}(\nu_{P^g,P},F_\theta)}\mathrm{e}^{k\omega_\xi}.\tag{3.11}$$

(By substituting F_θ for X in $\mathrm{Td}\big(P^g,X\big)$, $\mathcal{D}^g_{\mathbb{C}}(\nu_{P^g,P},X)^{-1}$, we mean to take the Taylor expansion around $X=0$ and substitute the differential form F_θ.) At this stage we see that the contribution of $g\in S'_I$ vanishes unless $P^g\neq\varnothing$, so that $S'_I=S_P$ (S_P is as in Theorem 1.2). As the characteristic forms $\mathrm{Td}\big(P^g,F_\theta\big)$, $\mathcal{D}^g_{\mathbb{C}}(\nu_{P^g,P},F_\theta)$ appear multiplied by the form Θ, which has top degree in the T'_I orbit directions, we can replace these characteristic forms with their horizontal parts. Substituting F_θ for X and taking the horizontal part is the definition of the Cartan map Car_θ for the locally free action of T'_I on the space P^g, hence the result is the pullback along the map $P^g\to\Sigma_g=P^g/T'_I$ of the form

$$\frac{\mathrm{Td}(\Sigma_g)}{\mathcal{D}^g_{\mathbb{C}}(\nu_{\Sigma,\Sigma^g})}.$$

(See our remarks in the introduction regarding characteristic forms for orbifolds.) Similarly the 1^{st} Chern form $c_1(L_\Sigma)$ is obtained by applying the Cartan map to the equivariant symplectic form $\omega_{\mathfrak{t}}(X)=\omega-\langle\mu,X\rangle$, and results in $c_1(L_\Sigma)=\omega_\xi-\langle\xi,F_\theta\rangle$. Hence $\mathrm{Ch}(L_\Sigma)=\mathrm{e}^{c_1(L_\Sigma)}=\mathrm{e}^{\omega_\xi-\langle\xi,F_\theta\rangle}$. The integral over the fibres of $P^g\to\Sigma_g$ then gives $1/d$, where $d\colon\Sigma=\sqcup\Sigma_g\to\mathbb{Z}$ is the locally constant function giving the size of the generic stabilizer for the $T'_I \simeq T/T_I$ action on $\sqcup P^g \to \Sigma$. Equation (3.11) becomes

$$m(k, k\xi) \sim \sum_{g \in S_P} g^{-k\xi} \int_{\Sigma_g} \frac{1}{d} \frac{g_L^k \mathrm{Ch}(L_\Sigma)^k \mathrm{Td}(\Sigma)}{\mathcal{D}_{\mathbb{C}}^g(\nu_{\Sigma_g,\Sigma})} \mathrm{e}^{k\langle \xi, F_\theta\rangle}. \tag{3.12}$$

By Corollary 2.3, $m(k, k\xi)$ is a quasi-polynomial function of k, hence the asymptotic expansion must be exact, or in other words, '$\sim$' in equation (3.12) can be replaced with '$=$'. Thus setting $\lambda = k\xi$ we have

$$m(k, \lambda) = \sum_{g \in S_P} g^{-\lambda} \int_{\Sigma_g} \frac{1}{d} \frac{g_L^k \mathrm{Ch}(L_\Sigma)^k \mathrm{Td}(\Sigma)}{\mathcal{D}_{\mathbb{C}}^g(\nu_{\Sigma_g,\Sigma})} \mathrm{e}^{\langle \lambda, F_\theta\rangle}. \tag{3.13}$$

The right-hand-side of equation (3.13) is quasi-polynomial in (k, λ). Hence by Corollary 2.3, equation (3.13) holds on *all* of $C_{\mathfrak{p}}$ (and not only at points (k, λ) with $\lambda = k\xi$, ξ a rational, weakly regular value in the relative interior of $\mathfrak{p}$). This completes the proof of Theorem 1.2.

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