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Some Remarks on the Total CR Q and Q'-Curvatures

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Abstract. We prove that the total CR Q-curvature vanishes for any compact strictly pseudoconvex CR manifold. We also prove the formal self-adjointness of the P'-operator and the CR invariance of the total Q'-curvature for any pseudo-Einstein manifold without the assumption that it bounds a Stein manifold.

Key words: CR manifolds; Q-curvature; P'-operator; Q'-curvature

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1 Introduction

The Q-curvature, which was introduced by T. Branson [3], is a fundamental curvature quantity on even dimensional conformal manifolds. It satisfies a simple conformal transformation formula and its integral is shown to be a global conformal invariant. The ambient metric construction of the Q-curvature [9] also works for a CR manifold M of dimension $2n+1$, and we can define the CR Q-curvature, which we denote by Q. The CR Q-curvature is a CR density of weight $-n-1$ defined for a fixed contact form θ and is expressed in terms of the associated pseudo-hermitian structure. If we take another contact form $\widehat{\theta} = e^{\Upsilon}\theta$, $\Upsilon \in C^\infty(M)$, it transforms as

$$\widehat{Q} = Q + P\Upsilon,$$

where P is a CR invariant linear differential operator, called the (critical) CR GJMS operator. Since P is formally self-adjoint and kills constant functions, the integral

$$\overline{Q} = \int_M Q,$$

called the total CR Q-curvature, is invariant under rescaling of the contact form and gives a global CR invariant of M. However, it follows readily from the definition of the CR Q-curvature that Q vanishes identically for an important class of contact forms, namely the pseudo-Einstein contact forms. Since the boundary of a Stein manifold admits a pseudo-Einstein contact form [5], the CR invariant $\overline{Q}$ vanishes for such a CR manifold. Moreover, it has been shown that on a Sasakian manifold the CR Q-curvature is expressed as a divergence [1], and hence $\overline{Q}$ also vanishes in this case. Thus, it is reasonable to conjecture that the total CR Q-curvature vanishes for any CR manifold, and our first result is the confirmation of this conjecture:

Theorem 1.1. *Let M be a compact strictly pseudoconvex CR manifold. Then the total CR Q-curvature of M vanishes: $\overline{Q} = 0$.*

For three dimensional CR manifolds, Theorem 1.1 follows from the explicit formula of the CR Q-curvature; see [9]. In higher dimensions, we make use of the fact that a compact strictly pseudoconvex CR manifold M of dimension greater than three can be realized as the boundary of a complex variety with at most isolated singularities [2, 10, 11]. By resolution of singularities, we can realize M as the boundary of a complex manifold X which may not be Stein. In this setting, the total CR Q-curvature is characterized as the logarithmic coefficient of the volume expansion of the asymptotically Kähler–Einstein metric on X [15]. By a simple argument using Stokes' theorem, we prove that there is no logarithmic term in the expansion.

Although the vanishing of $\overline{Q}$ is disappointing, there is an alternative Q-like object on a CR manifold which admits pseudo-Einstein contact forms. Generalizing the operator of Branson–Fontana–Morpurgo [4] on the CR sphere, Case–Yang [7] (in dimension three) and Hirachi [12] (in general dimensions) introduced the P'-operator and the Q'-curvature for pseudo-Einstein CR manifolds. Let us denote the set of pseudo-Einstein contact forms by $\mathcal{PE}$ and the space of CR pluriharmonic functions by $\mathcal{P}$. Two pseudo-Einstein contact forms $\theta, \widehat{\theta} \in \mathcal{PE}$ are related by $\widehat{\theta} = e^{\Upsilon}\theta$ for some $\Upsilon \in \mathcal{P}$. For a fixed $\theta \in \mathcal{PE}$, the P'-operator is defined to be a linear differential operator on $\mathcal{P}$ which kills constant functions and satisfies the transformation formula

$$\widehat{P}' f = P' f + P(f\Upsilon)$$

under the rescaling $\widehat{\theta} = e^{\Upsilon}\theta$. The Q'-curvature is a CR density of weight $-n-1$ defined for $\theta \in \mathcal{PE}$, and satisfies

$$\widehat{Q}' = Q' + 2P'\Upsilon + P\big(\Upsilon^2\big)$$

for the rescaling. Thus, if P' is formally self-adjoint on $\mathcal{P}$, the total Q'-curvature

$$\overline{Q}' = \int_M Q'$$

gives a CR invariant of M. In dimension three and five, the formal self-adjointness of P' follows from the explicit formulas [6, 7]. In higher dimensions, Hirachi [12, Theorem 4.5] proved the formal self-adjointness under the assumption that M is the boundary of a Stein manifold X; in the proof he used Green's formula for the asymptotically Kähler–Einstein metric g on X, and the global Kählerness of g was needed to assure that a pluriharmonic function is harmonic with respect to g. In this paper, we slightly modify his proof and prove the self-adjointness of P' for general pseudo-Einstein manifolds:

Theorem 1.2. *Let M be a compact strictly pseudoconvex CR manifold. Then the P'-operator for a pseudo-Einstein contact form satisfies*

$$\int_M \big(f_1 P' f_2 - f_2 P' f_1\big) = 0$$

for any $f_1, f_2 \in \mathcal{P}$.

Consequently, the CR invariance of $\overline{Q}'$ holds for any CR manifold which admits a pseudo-Einstein contact form:

Theorem 1.3. *Let M be a compact strictly pseudoconvex CR manifold which admits a pseudo-Einstein contact form. Then the total Q'-curvature is independent of the choice of $\theta \in \mathcal{PE}$.*

We note that $\overline{Q}'$ is a nontrivial CR invariant since it has a nontrivial variational formula; see [13]. We also give an alternative proof of Theorem 1.3 by using the characterization [12, Theorem 5.6] of $\overline{Q}'$ as the logarithmic coefficient in the expansion of some integral over a complex manifold with boundary M.

2 Proof of Theorem 1.1

We briefly review the ambient metric construction of the CR Q-curvature; we refer the reader to [9, 12, 13] for detail.

Let $\overline{X}$ be an $(n+1)$-dimensional complex manifold with strictly pseudoconvex CR boundary M, and let $r \in C^\infty(\overline{X})$ be a boundary defining function which is positive in the interior X. The restriction of the canonical bundle $K_{\overline{X}}$ to M is naturally isomorphic to the CR canonical bundle $K_M := \wedge^{n+1}(T^{0,1}M)^\perp \subset \wedge^{n+1}(\mathbb{C}T^*M)$. We define the *ambient space* by $\widetilde{X} = K_{\overline{X}} \setminus \{0\}$, and set $\mathcal{N} = K_M \setminus \{0\} \cong \widetilde{X}|_M$. The density bundles over $\overline{X}$ and M are defined by

$$\widetilde{\mathcal{E}}(w) = \left(K_{\overline{X}} \otimes \overline{K}_{\overline{X}}\right)^{-w/(n+2)}, \qquad \mathcal{E}(w) = \left(K_M \otimes \overline{K}_M\right)^{-w/(n+2)} \cong \widetilde{\mathcal{E}}(w)|_M$$

for each $w \in \mathbb{R}$. We call $\mathcal{E}(w)$ the *CR density bundle* of weight w. The space of sections of $\widetilde{\mathcal{E}}(w)$ and $\mathcal{E}(w)$ are also denoted by the same symbols. We define a $\mathbb{C}^*$-action on $\widetilde{X}$ by $\delta_\lambda u = \lambda^{n+2} u$ for $\lambda \in \mathbb{C}^*$ and $u \in \widetilde{X}$. Then a section of $\widetilde{\mathcal{E}}(w)$ can be identified with a function on $\widetilde{X}$ which is homogeneous with respect to this action:

$$\widetilde{\mathcal{E}}(w) \cong \left\{ f \in C^\infty\left(\widetilde{X}\right) \mid \delta_\lambda^* f = |\lambda|^{2w} f \text{ for } \lambda \in \mathbb{C}^* \right\}.$$

Similarly, sections of $\mathcal{E}(w)$ are identified with homogeneous functions on $\mathcal{N}$.

Let $\boldsymbol{\rho} \in \widetilde{\mathcal{E}}(1)$ be a density on $\overline{X}$ and $(z^1, \dots, z^{n+1})$ local holomorphic coordinates. We set $\rho = |\mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^{n+1}|^{2/(n+2)} \boldsymbol{\rho} \in \widetilde{\mathcal{E}}(0)$ and define

$$\mathcal{J}[\boldsymbol{\rho}] := (-1)^{n+1} \det \begin{pmatrix} \rho & \partial_{z^{\bar{j}}} \rho \\ \partial_{z^i} \rho & \partial_{z^i} \partial_{z^{\bar{j}}} \rho \end{pmatrix}.$$

Since $\mathcal{J}[\boldsymbol{\rho}]$ is invariant under changes of holomorphic coordinates, $\mathcal{J}$ defines a global differential operator, called the *Monge–Ampère operator*. Fefferman [8] showed that there exists $\boldsymbol{\rho} \in \widetilde{\mathcal{E}}(1)$ unique modulo $O(r^{n+3})$ which satisfies $\mathcal{J}[\boldsymbol{\rho}] = 1 + O(r^{n+2})$ and is a defining function of $\mathcal{N}$. We fix such a $\boldsymbol{\rho}$ and define the *ambient metric* $\widetilde{g}$ by the Lorentz–Kähler metric on a neighborhood of $\mathcal{N}$ in $\widetilde{X}$ which has the Kähler form $-i\partial\overline{\partial}\boldsymbol{\rho}$.

Recall that there exists a canonical weighted contact form $\boldsymbol{\theta} \in \Gamma(T^*M \otimes \mathcal{E}(1))$ on M, and the choice of a contact form θ is equivalent to the choice of a positive section $\tau \in \mathcal{E}(1)$, called a *CR scale*; they are related by the equation $\boldsymbol{\theta} = \tau\theta$. For a CR scale $\tau \in \mathcal{E}(1)$, we define the CR Q-curvature by

$$Q = \widetilde{\Delta}^{n+1} \log \widetilde{\tau}\,|_{\mathcal{N}} \in \mathcal{E}(-n-1),$$

where $\widetilde{\Delta} = -\widetilde{\nabla}_I \widetilde{\nabla}^I$ is the Kähler Laplacian of $\widetilde{g}$ and $\widetilde{\tau} \in \widetilde{\mathcal{E}}(1)$ is an arbitrary extension of τ. It can be shown that Q is independent of the choice of an extension of τ, and the total CR Q-curvature $\overline{Q}$ is invariant by rescaling of τ.

The total CR Q-curvature has a characterization in terms of a complete metric on X. We note that the $(1,1)$-form $-i\partial\overline{\partial}\log\boldsymbol{\rho}$ descends to a Kähler form on X near the boundary. We extend this Kähler metric to a hermitian metric g on X. The Kähler Laplacian $\Delta = -g^{i\bar{j}}\nabla_i\nabla_{\bar{j}}$ of g is related to $\widetilde{\Delta}$ by the equation

$$\boldsymbol{\rho}\widetilde{\Delta} f = \Delta f, \qquad f \in \widetilde{\mathcal{E}}(0) \tag{2.1}$$

near $\mathcal{N}$ in $\widetilde{X} \setminus \mathcal{N}$. In the right-hand side, we have regarded f as a function on X.

For any contact form θ on M, there exists a boundary defining function ρ such that

$$\vartheta|_{TM} = \theta, \qquad |\partial \log \rho|_g = 1 \text{ near } M \text{ in } X, \tag{2.2}$$

where $\vartheta := \mathrm{Re}(i\partial\rho)$ ([15, Lemma 3.1]). Let ξ be the $(1,0)$-vector filed on $\overline{X}$ near M characterized by

$$\xi\rho = 1, \qquad \xi \perp_g \mathcal{H},$$

where $\mathcal{H} := \mathrm{Ker}\,\partial\rho \subset T^{1,0}\overline{X}$. Then, $N := \mathrm{Re}\,\xi$ is smooth up to the boundary and satisfies $N\rho = 1$, $\vartheta(N) = 0$. Moreover, $\nu := \rho N$ is $(\sqrt{2})^{-1}$ times the unit outward normal vector filed along the level sets of ρ. By Green's formula, for any function f on X we have

$$\int_{\rho>\epsilon} \Delta f \,\mathrm{vol}_g = \int_{\rho=\epsilon} \nu f \,\nu \lrcorner \,\mathrm{vol}_g\,. \tag{2.3}$$

Since the Monge–Ampère equation implies that g satisfies

$$\mathrm{vol}_g = -(n!)^{-1}(1+O(\rho))\rho^{-n-2}\mathrm{d}\rho \wedge \vartheta \wedge (\mathrm{d}\vartheta)^n,$$

the formula (2.3) is rewritten as

$$\int_{\rho>\epsilon} \Delta f \,\mathrm{vol}_g = -(n!)^{-1}\int_{\rho=\epsilon} Nf \cdot (1+O(\epsilon))\epsilon^{-n}\vartheta \wedge (\mathrm{d}\vartheta)^n. \tag{2.4}$$

With this formula, we prove the following characterization of $\overline{Q}$.

Lemma 2.1 ([15, Proposition A.3])**.** *For an arbitrary defining function ρ, we have*

$$\mathrm{lp}\int_{\rho>\epsilon} \mathrm{vol}_g = \frac{(-1)^n}{(n!)^2(n+1)!}\overline{Q},$$

where lp *denotes the coefficient of* $\log\epsilon$ *in the asymptotic expansion in* ϵ*.*

Proof. Since the coefficient of $\log\epsilon$ in the volume expansion is independent of the choice of ρ [15, Proposition 4.1], we may assume that ρ satisfies (2.2) for a fixed contact θ on M. We take $\widetilde{\tau} \in \widetilde{\mathcal{E}}(1)$ such that $\boldsymbol{\rho} = \widetilde{\tau}\rho$. Then, θ is the contact form corresponding to the CR scale $\widetilde{\tau}|_{\mathcal{N}}$. By the same argument as in the proof of [12, Lemma 3.1], we can take $F \in \widetilde{\mathcal{E}}(0)$, $\boldsymbol{G} \in \widetilde{\mathcal{E}}(-n-1)$ which satisfy

$$\widetilde{\Delta}\big(\log\widetilde{\tau} + F + \boldsymbol{G}\boldsymbol{\rho}^{n+1}\log\rho\big) = O\big(\rho^{\infty}\big), \qquad F = O(\rho), \qquad \boldsymbol{G}|_{\mathcal{N}} = \frac{(-1)^n}{n!(n+1)!}Q.$$

We set $G := \widetilde{\tau}^{n+1}\boldsymbol{G} \in \widetilde{\mathcal{E}}(0)$. By (2.1) and the equation $\boldsymbol{\rho}\widetilde{\Delta}\log\boldsymbol{\rho} = n+1$, we have

$$\Delta\big(\log\rho - F - G\rho^{n+1}\log\rho\big) = n+1+O(\rho^{\infty}).$$

Then, by using (2.4), we compute as

$$\begin{aligned}(n+1)\,\mathrm{lp}\int_{\rho>\epsilon}\mathrm{vol}_g &= \mathrm{lp}\int_{\rho>\epsilon}\Delta\big(\log\rho - F - G\rho^{n+1}\log\rho\big)\,\mathrm{vol}_g \\ &= -(n!)^{-1}\,\mathrm{lp}\int_{\rho=\epsilon} N\big(\log\rho - F - G\rho^{n+1}\log\rho\big)\cdot(1+O(\epsilon))\epsilon^{-n}\vartheta\wedge(\mathrm{d}\vartheta)^n \\ &= \frac{n+1}{n!}\int_M G\,\theta\wedge(\mathrm{d}\theta)^n \\ &= \frac{(-1)^n}{(n!)^3}\overline{Q}.\end{aligned}$$

Thus we complete the proof. ∎

Proof of Theorem 1.1. Let ρ be an arbitrary defining function of M, and $\widetilde{\tau} \in \widetilde{\mathcal{E}}(1)$ the density on $\overline{X}$ defined by $\boldsymbol{\rho} = \widetilde{\tau}\rho$. Then $\alpha := -i\partial\overline{\partial}\log\widetilde{\tau}$ is a closed $(1,1)$-form on $\overline{X}$. The volume form of g is given by $\mathrm{vol}_g = \omega^{n+1}/(n+1)!$ with the fundamental 2-form $\omega = ig_{j\overline{k}}\theta^j \wedge \theta^{\overline{k}}$. Near the boundary M in X, we have

$$\omega = -i\partial\overline{\partial}\log\boldsymbol{\rho} = -i\partial\overline{\partial}\log\rho + \alpha.$$

Since the logarithmic term in the volume expansion is determined by the behavior of vol_g near the boundary, we compute as

$$\begin{aligned}(n+1)!\,\mathrm{lp}\int_{\rho>\epsilon}\mathrm{vol}_g &= \mathrm{lp}\int_{\rho>\epsilon}(-i\partial\overline{\partial}\log\rho+\alpha)^{n+1}\\ &= \mathrm{lp}\int_{\rho>\epsilon}\alpha^{n+1} + \mathrm{lp}\int_{\rho>\epsilon}\sum_{k=1}^{n+1}\binom{n+1}{k}(-i\partial\overline{\partial}\log\rho)^k\wedge\alpha^{n+1-k}.\end{aligned}$$

The first term in the last line is 0 since α is smooth up to the boundary. Using $-i\partial\overline{\partial}\log\rho = \mathrm{d}(\vartheta/\rho)$ and $d\alpha = 0$, we also have

$$\mathrm{lp}\int_{\rho>\epsilon}(-i\partial\overline{\partial}\log\rho)^k\wedge\alpha^{n+1-k} = \mathrm{lp}\,\epsilon^{-k}\int_{\rho=\epsilon}\vartheta\wedge(\mathrm{d}\vartheta)^{k-1}\wedge\alpha^{n+1-k} = 0.$$

Thus, by Lemma 2.1 we obtain $\overline{Q} = 0$. ∎

3 Proof of Theorem 1.2

We will recall the definitions of the P'-operator and the Q'-curvature. A CR scale $\tau \in \mathcal{E}(1)$ is called *pseudo-Einstein* if it has an extension $\widetilde{\tau} \in \widetilde{\mathcal{E}}(1)$ such that $\partial\overline{\partial}\log\widetilde{\tau} = 0$ near $\mathcal{N}$ in $\widetilde{X}$. The corresponding contact form θ is called a *pseudo-Einstein contact form* and characterized in terms of associated pseudo-hermitian structure; see [12, 13, 14]. If τ is a pseudo-Einstein CR scale, another $\widehat{\tau}$ is pseudo-Einstein if and only if $\widehat{\tau} = e^{-\Upsilon}\tau$ for a CR pluriharmonic function $\Upsilon \in \mathcal{P}$. For any $f \in \mathcal{P}$, we take an extension $\widetilde{f} \in \widetilde{\mathcal{E}}(0)$ such that $\partial\overline{\partial}\widetilde{f} = 0$ near M in $\overline{X}$ and define

$$P'f = -\widetilde{\Delta}^{n+1}\bigl(\widetilde{f}\log\widetilde{\tau}\bigr)|_{\mathcal{N}} \in \mathcal{E}(-n-1).$$

We note that the germs of $\widetilde{\tau}$ and $\widetilde{f}$ along $\mathcal{N}$ is unique, and $P'f$ is assured to be a density by $\widetilde{\Delta}\widetilde{f}|_{\mathcal{N}} = 0$. The Q'-curvature is defined by

$$Q' = \widetilde{\Delta}^{n+1}(\log\widetilde{\tau})^2|_{\mathcal{N}} \in \mathcal{E}(-n-1).$$

Here, the homogeneity of Q' follows from the fact $\widetilde{\Delta}\log\widetilde{\tau}\,|_{\mathcal{N}} = 0$.

To prove the formal self-adjointness of P', we use its characterization in terms of the metric g. We define a differential operator Δ' by $\Delta' f = -g^{i\overline{j}}\partial_i\partial_{\overline{j}}f$. Since g is Kähler near the boundary, Δ' agrees with Δ near M in X.

Lemma 3.1 ([12, Lemma 4.4])**.** *Let $\tau \in \mathcal{E}(1)$ be a pseudo-Einstein CR scale and $\widetilde{\tau} \in \widetilde{\mathcal{E}}(1)$ its extension such that $\partial\overline{\partial}\log\widetilde{\tau} = 0$ near $\mathcal{N}$ in $\widetilde{X}$. Let $\rho = \boldsymbol{\rho}/\widetilde{\tau}$ be the corresponding defining function. Then, for any $f \in C^\infty(\overline{X})$ which is pluriharmonic in a neighborhood of M in $\overline{X}$, there exist $F, G \in C^\infty(\overline{X})$ such that $F = O(\rho)$ and*

$$\Delta'\bigl(f\log\rho - F - G\rho^{n+1}\log\rho\bigr) = (n+1)f + O\bigl(\rho^\infty\bigr).$$

Moreover, $\tau^{-n-1}G|_M = \frac{(-1)^{n+1}}{(n+1)!n!}P'f$ holds.

In the statement of [12, Lemma 4.4], the Laplacian Δ is used, but we may replace it by Δ' since they agree near the boundary in X.

Proof of Theorem 1.2. We extend f_j to a function on $\overline{X}$ such that $\partial\overline{\partial} f_j = 0$ in a neighborhood of M in $\overline{X}$. Let τ be a pseudo-Einstein CR scale and $\rho = \boldsymbol{\rho}/\widetilde{\tau}$ the corresponding defining function. Then we have $\omega = -i\partial\overline{\partial}\log\rho$ near M in X. We take F_j, G_j as in Lemma 3.1 so that $u_j := f_j \log\rho - F_j - G_j \rho^{n+1}\log\rho$ satisfies $\Delta' u_j = (n+1) f_j + O(\rho^\infty)$. We consider the coefficient of $\log\epsilon$ in the expansion of the integral

$$I_\epsilon = \operatorname{Re}\int_{\rho>\epsilon} \bigl(i\partial f_1 \wedge \overline{\partial} u_2 \wedge \omega^n + i\partial f_2 \wedge \overline{\partial} u_1 \wedge \omega^n - f_1 f_2\, \omega^{n+1}\bigr),$$

which is symmetric in the indices 1 and 2. Since $d\omega = 0, \partial\overline{\partial} f_2 = 0$ near M in $\overline{X}$, we have

$$\begin{aligned} i\partial f_1 \wedge \overline{\partial} u_2 \wedge \omega^n &= \mathrm{d}\bigl(i f_1 \overline{\partial} u_2 \wedge \omega^n\bigr) - i f_1 \partial\overline{\partial} u_2 \wedge \omega^n + i n f_1 \overline{\partial} u_2 \wedge \mathrm{d}\omega \wedge \omega^{n-1} \\ &= \mathrm{d}\bigl(i f_1 \overline{\partial} u_2 \wedge \omega^n\bigr) + \frac{1}{n+1} f_1 \Delta' u_2 \omega^{n+1} + (\text{cpt supp}), \\ i\partial f_2 \wedge \overline{\partial} u_1 \wedge \omega^n &= -\mathrm{d}\bigl(i u_1 \partial f_2 \wedge \omega^n\bigr) + (\text{cpt supp}), \end{aligned}$$

where (cpt supp) stands for a compactly supported form on X. Thus,

$$\begin{aligned} I_\epsilon = &\int_{\rho>\epsilon} \frac{1}{n+1} f_1 \bigl(\Delta' u_2 - (n+1) f_2\bigr)\omega^{n+1} \\ &+ \operatorname{Re}\int_{\rho=\epsilon} i(f_1 \overline{\partial} u_2 - u_1 \partial f_2) \wedge \omega^n + \int_{\rho>\epsilon} (\text{cpt supp}). \end{aligned}$$

The first and the third terms contain no log terms. Since $\omega = \mathrm{d}(\vartheta/\rho)$ near M in X, the second term is computed as

$$\begin{aligned} \operatorname{Re}\int_{\rho=\epsilon} i(f_1 \overline{\partial} u_2 - u_1 \partial f_2) \wedge \omega^n = \epsilon^{-n} \operatorname{Re}\int_{\rho=\epsilon} \Bigl(& i f_1 \overline{\partial}\bigl(f_2 \log\rho - F_2 - G_2 \rho^{n+1}\log\rho\bigr) \wedge (\mathrm{d}\vartheta)^n \\ & - i\bigl(f_1 \log\rho - F_1 - G_1 \rho^{n+1}\log\rho\bigr) \wedge \partial f_2 \wedge (\mathrm{d}\vartheta)^n\Bigr) + O\bigl(\epsilon^\infty\bigr). \end{aligned}$$

The logarithmic term in the right-hand side is

$$\log\epsilon \int_{\rho=\epsilon} (n+1) f_1 G_2 \vartheta \wedge (\mathrm{d}\vartheta)^n + 2\epsilon^{-n} \log\epsilon \operatorname{Re}\int_{\rho=\epsilon} i f_1 \overline{\partial} f_2 \wedge (\mathrm{d}\vartheta)^n + O(\epsilon\log\epsilon).$$

The coefficient of $\log\epsilon$ in the first term is

$$\frac{(-1)^{n+1}}{(n!)^2} \int_M f_1 P' f_2. \tag{3.1}$$

The second term is equal to

$$2\epsilon^{-n}\log\epsilon \operatorname{Re}\int_{\rho>\epsilon} i\partial f_1 \wedge \overline{\partial} f_2 \wedge (\mathrm{d}\vartheta)^n + \epsilon^{-n}\log\epsilon \int_{\rho>\epsilon} (\text{cpt supp}).$$

The first term in this formula is symmetric in the indices 1 and 2 while the second term gives no $\log\epsilon$ term. Therefore, (3.1) should also be symmetric in 1 and 2, which implies the formal self-adjointness of P'. ■

4 Proof of Theorem 1.3

The formal self-adjointness of the P'-operator implies the CR invariance of the total Q'-curvature. When $n \geq 2$, the CR invariance can also be proved by the following characterization of $\overline{Q}'$ in terms of the hermitian metric g on X whose fundamental 2-form $\omega = i g_{j\overline{k}} \theta^j \wedge \theta^{\overline{k}}$ agrees with $-i\partial\overline{\partial} \log \boldsymbol{\rho}$ near M in X:

Theorem 4.1 ([12, Theorem 5.6])**.** *Let $\tau \in \mathcal{E}(1)$ be a pseudo-Einstein CR scale and $\widetilde{\tau} \in \widetilde{\mathcal{E}}(1)$ its extension such that $\partial\overline{\partial} \log \widetilde{\tau} = 0$ near $\mathcal{N}$ in $\widetilde{X}$. Let $\rho = \boldsymbol{\rho}/\widetilde{\tau}$ be the corresponding defining function. Then we have*

$$\operatorname{lp} \int_{r>\epsilon} i\partial \log \rho \wedge \overline{\partial} \log \rho \wedge \omega^n = \frac{(-1)^n}{2(n!)^2} \overline{Q}' \tag{4.1}$$

for any defining function r.

In [12, Theorem 5.6], it is assumed that X is Stein and $\omega = -i\partial\overline{\partial} \log \rho$ globally on X, but as the logarithmic term is determined by the boundary behavior, it is sufficient to assume $\omega = -i\partial\overline{\partial} \log \rho$ near M in X as above.

Proof of Theorem 1.3. Let τ, ρ be as in Theorem 4.1 and let $\widehat{\rho}$ be the defining function corresponding to another pseudo-Einstein CR scale $\widehat{\tau}$. Then we can write as $\widehat{\rho} = e^{\Upsilon} \rho$ with $\Upsilon \in C^\infty(\overline{X})$ such that $\partial\overline{\partial}\Upsilon = 0$ near M in $\overline{X}$.

Using the defining function ρ for r in the formula (4.1), we compute as

$$\begin{aligned} \operatorname{lp} \int_{\rho>\epsilon} i\partial \log \widehat{\rho} \wedge \overline{\partial} \log \widehat{\rho} \wedge \omega^n &= \operatorname{lp} \int_{\rho>\epsilon} i(\partial \log \rho + \partial\Upsilon) \wedge (\overline{\partial} \log \rho + \overline{\partial}\Upsilon) \wedge \omega^n \\ &= \operatorname{lp} \int_{\rho>\epsilon} i\partial \log \rho \wedge \overline{\partial} \log \rho \wedge \omega^n + \operatorname{lp} \int_{\rho>\epsilon} i\partial\Upsilon \wedge \overline{\partial}\Upsilon \wedge \omega^n \\ &\quad + 2 \operatorname{Re} \operatorname{lp} \int_{\rho>\epsilon} i\partial \log \rho \wedge \overline{\partial}\Upsilon \wedge \omega^n. \end{aligned}$$

The second term in the last line is

$$\operatorname{lp} \int_{\rho>\epsilon} i\partial\Upsilon \wedge \overline{\partial}\Upsilon \wedge \omega^n = \operatorname{lp} \int_{\rho=\epsilon} i\Upsilon\overline{\partial}\Upsilon \wedge \omega^n + \operatorname{lp} \int_{\rho>\epsilon} (\text{cpt supp}) = 0.$$

Since $\omega = \mathrm{d}(\vartheta/\rho)$ near M in X, we have

$$\begin{aligned} \int_{\rho>\epsilon} i\partial \log \rho \wedge \overline{\partial}\Upsilon \wedge \omega^n &= \log \epsilon \int_{\rho=\epsilon} i\overline{\partial}\Upsilon \wedge \omega^n + \int_{\rho>\epsilon} (\text{cpt supp}) \\ &= \epsilon^{-n} \log \epsilon \int_{\rho=\epsilon} i\overline{\partial}\Upsilon \wedge (d\vartheta)^n + \int_{\rho>\epsilon} (\text{cpt supp}) \\ &= \epsilon^{-n} \log \epsilon \int_{\rho>\epsilon} (\text{cpt supp}) + \int_{\rho>\epsilon} (\text{cpt supp}), \end{aligned}$$

which implies that the third term is also 0. Thus, $\overline{Q}'$ is independent of the choice of a pseudo-Einstein CR scale τ. ∎

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References

[1] Alexakis S., Hirachi K., Integral Kähler invariants and the Bergman kernel asymptotics for line bundles, *Adv. Math.* **308** (2017), 348–403, arXiv:1501.02463.

[2] Boutet de Monvel L., Intégration des équations de Cauchy–Riemann induites formelles, in Séminaire Goulaouic–Lions–Schwartz 1974–1975; Équations aux derivées partielles linéaires et non linéaires, Centre Math., École Polytech., Paris, 1975, Exp. No. 9, 14 pages.

[3] Branson T.P., The functional determinant, *Lecture Notes Series*, Vol. 4, Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1993.

[4] Branson T.P., Fontana L., Morpurgo C., Moser–Trudinger and Beckner–Onofri's inequalities on the CR sphere, *Ann. of Math.* **177** (2013), 1–52, arXiv:0712.3905.

[5] Cao J., Chang S.C., Pseudo-Einstein and Q-flat metrics with eigenvalue estimates on CR-hypersurfaces, *Indiana Univ. Math. J.* **56** (2007), 2839–2857, math.DG/0609312.

[6] Case J.S., Gover A.R., The P'-operator, the Q'-curvature, and the CR tractor calculus, arXiv:1709.08057.

[7] Case J.S., Yang P., A Paneitz-type operator for CR pluriharmonic functions, *Bull. Inst. Math. Acad. Sin. (N.S.)* **8** (2013), 285–322, arXiv:1309.2528.

[8] Fefferman C.L., Monge–Ampère equations, the Bergman kernel, and geometry of pseudoconvex domains, *Ann. of Math.* **103** (1976), 395–416.

[9] Fefferman C.L., Hirachi K., Ambient metric construction of Q-curvature in conformal and CR geometries, *Math. Res. Lett.* **10** (2003), 819–831, math.DG/0303184.

[10] Harvey F.R., Lawson Jr. H.B., On boundaries of complex analytic varieties. I, *Ann. of Math.* **102** (1975), 223–290.

[11] Harvey F.R., Lawson Jr. H.B., On boundaries of complex analytic varieties. II, *Ann. of Math.* **106** (1977), 213–238.

[12] Hirachi K., Q-prime curvature on CR manifolds, *Differential Geom. Appl.* **33** (2014), suppl., 213–245, arXiv:1302.0489.

[13] Hirachi K., Marugame T., Matsumoto Y., Variation of total Q-prime curvature on CR manifolds, *Adv. Math.* **306** (2017), 1333–1376, arXiv:1510.03221.

[14] Lee J.M., Pseudo-Einstein structures on CR manifolds, *Amer. J. Math.* **110** (1988), 157–178.

[15] Seshadri N., Volume renormalization for complete Einstein–Kähler metrics, *Differential Geom. Appl.* **25** (2007), 356–379, math.DG/0404455.