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Positive Definite Functions on Complex Spheres and their Walks through Dimensions

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Abstract. We provide walks through dimensions for isotropic positive definite functions defined over complex spheres. We show that the analogues of Montée and Descente operators as proposed by Beatson and zu Castell [*J. Approx. Theory* **221** (2017), 22–37] on the basis of the original Matheron operator [Les variables régionalisées et leur estimation, Masson, Paris, 1965], allow for similar walks through dimensions. We show that the Montée operators also preserve, up to a constant, strict positive definiteness. For the Descente operators, we show that strict positive definiteness is preserved under some additional conditions, but we provide counterexamples showing that this is not true in general. We also provide a list of parametric families of (strictly) positive definite functions over complex spheres, which are important for several applications.

Key words: Descente; disk polynomials; Montée; positive definite functions

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1 Introduction and main results

Positive definite functions have a long history which can be traced back to papers by Carathéodory, Herglotz, Bernstein and Matthias, culminating in Bochner's theorem from 1932–1933. See Berg [6] for details. In the last twenty years several results related to this topic were obtained in fields as diverse as mathematical analysis, numerical analysis, potential theory, probability theory and geostatistics: we refer the reader to the surveys in Schaback [35, 36], Berg [6] and Fasshauer [14] for a complete list of references in this direction.

Positive definite radial functions have been known since the two seminal papers by Schoenberg [39, 40]. The former is devoted to radially symmetric functions depending on the Euclidean distance, and the latter to isotropic functions on unit spheres $\mathbb{S}^d$ of $\mathbb{R}^{d+1}$. Literature on radially symmetric functions on Euclidean spaces has been especially fervent. In his essay devoted to the *clavier spherique*, Matheron [24] proposed operators called *Montée* and *Descente* that preserve the property of positive definiteness but changing the dimension of the space initially considered. Such a property has been called *walk through dimensions*. It is worth noting that the walk through dimensions is achieved at the expense of modifying the differentiability at the origin of a given candidate function. Wendland [45] used the Montée operator with a class of compactly supported radial basis functions, termed Wendland's functions after his works. Schaback [37] covered the missing cases of walks through dimensions. Porcu et al. [30] used a fractional version of the Montée operator to obtain generalized versions of Wendland's functions. For a reference on walks through dimensions in the geostatistical setting, the reader is referred to Gneiting [16] and to the more recent work of Porcu and Zastavnyi [29].

Positive definite functions as well as strictly positive definite functions in several contexts have been deeply studied by the mathematical analysis literature, and the reader is referred to the works by Menegatto et al. (see Chen et al. [11], Menegatto and Peron [26], Guella et al. [19], and references therein). The use of positive definite functions on real spheres for geostatisticians has arrived recently, thanks to the survey by Gneiting [17] and the recent developments by Berg and Porcu [7] and Porcu et al. [28]. In particular, Berg and Porcu [7] characterized the class of the positive definite functions on the product of $\mathbb{S}^d$ with a locally compact group, extending the Schoenberg's class Ψ_d of the positive definite functions on $\mathbb{S}^d$ (Schoenberg [40]).

A continuous function $f\colon [-1,1] \to \mathbb{R}$ belongs to the class Ψ_d when the kernel

$$K\colon \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{R}\colon K(\xi,\eta) = f(\langle \xi, \eta\rangle)$$

is positive definite. Schoenberg [40] proved that $f \in \Psi_d$ if, and only if,

$$f(x) = \sum_{k\geq 0} a_k^d c_k(d,x), \qquad \sum_{k\geq 0} a_k^d < \infty, \qquad a_k^d \geq 0, \quad \forall\, k \geq 0, \tag{1.1}$$

where $c_k(d,\cdot)$ are the normalized Gegenbauer polynomials associated to the index d (see Szegő [44, p. 80]). The coefficients in the above series are called *d-Schoenberg coefficients*. On the other hand, the subclass Ψ_d^+ of Ψ_d of the strict positive definite functions on $\mathbb{S}^d$, $d \geq 2$, was characterized by Chen et al. [11]: $f \in \Psi_d^+$ if, and only if, the set $\{k\colon a_k^d > 0\}$ contains infinitely many odd and infinitely many even integers.

The class Ψ_d has received special interest in the last twenty years, while walks through dimensions for positive definite functions on real spheres have been studied in the recent tour de force by Beatson and zu Castell [3, 4]. In particular, Beatson and zu Castell [4] define the *Montée* operator

$$(If)(x) = \int_{-1}^{x} f(u)\mathrm{d}u, \qquad x \in [-1,1],$$

for f integrable in $[-1,1]$, and the *Descente* operator

$$(Df)(x) = \frac{\mathrm{d}}{\mathrm{d}x} f(x), \qquad x \in [-1,1],$$

for f absolutely continuous in $[-1,1]$. They prove that, for $d \geq 2$:

(i) if $f \in \Psi_{d+2}$, then there exists a constant c such that $c + If \in \Psi_d$;

(ii) if $f \in \Psi_{d+2}^+$, then there exists a constant c such that $c + If \in \Psi_d^+$;

(iii) if $f \in \Psi_{d+2}$, $f \geq 0$ and all $(d+2)$-Schoenberg coefficients are positive, then $If \in \Psi_d$ and all its d-Schoenberg coefficients are positive;

(iv) if $f \in \Psi_d$ and Df is continuous, then $Df \in \Psi_{d+2}$;

(v) if $f \in \Psi_d^+$ and Df is continuous, then $Df \in \Psi_{d+2}^+$.

Observe that the property of (strict) positive definiteness of f is preserved by the operators *Montée* I and *Descente* D.

In this paper, inspired by the work of Beatson and zu Castell [4], we study positive definite functions on complex unit spheres Ω_{2q} of $\mathbb{C}^q$. In particular, we provide walks through dimensions over complex spheres.

Below, we state our main results and we refer to Section 2 for the necessary background.

We denote the class of positive definite functions on Ω_{2q} by $\Psi(\Omega_{2q})$. A characterization of such functions was proposed in Menegatto and Peron [26]: let $\mathbb{D} := \{z \in \mathbb{C} \colon |z| \le 1\} \subset \mathbb{C}$, when a continuous function $f \colon \mathbb{D} \to \mathbb{C}$ belongs to $\Psi(\Omega_{2q})$, an expansion similar to (1.1) exists, namely

$$f(z) = \sum_{m,n \ge 0} a_{m,n}^{q-2} R_{m,n}^{q-2}(z), \qquad z \in \mathbb{D},$$

(see equation (2.3) and Theorem 2.1). We will call the coefficients $a_{m,n}^{q-2}$ as $(2q)$-*complex Schoenberg coefficients*.

In order to make the statements clear, it is convenient to introduce the Descente and Montée operators in the complex context.

Given $f \colon \mathbb{D} \to \mathbb{C}$, we say that f is differentiable if, writing $z = x + iy \in \mathbb{D}$, f is differentiable as a function of x and y. Then, we denote by $\mathcal{D}_x f$ and $\mathcal{D}_y f$ the partial derivatives with respect to x and y, respectively, and we define the Descente operators through the following *Wirtinger* derivatives:

$$\mathcal{D}_z f = \frac{1}{2}(\mathcal{D}_x f - i\mathcal{D}_y f), \qquad \mathcal{D}_{\overline{z}} f = \frac{1}{2}(\mathcal{D}_x f + i\mathcal{D}_y f). \tag{1.2}$$

We observe that f might not be complex differentiable, actually it is so only when $\mathcal{D}_{\overline{z}} f = 0$, and in this case $\mathcal{D}_z f = f'$, the complex derivative of f.

If f admits a z-primitive F and a $\overline{z}$-primitive G in $\mathbb{D}$, that is, $\mathcal{D}_z F = \mathcal{D}_{\overline{z}} G = f$, then we can define the Montée operators $\mathcal{I}$ and $\overline{\mathcal{I}}$ by

$$\mathcal{I}(f)(z) := F(z) - F(0) \qquad \text{and} \qquad \overline{\mathcal{I}}(f)(z) := G(z) - G(0), \qquad z \in \mathbb{D}.$$

By definition,

$$\mathcal{D}_z(\mathcal{I} f) = f \qquad \text{and} \qquad \mathcal{D}_{\overline{z}}(\overline{\mathcal{I}} f) = f. \tag{1.3}$$

Moreover,

$$\mathcal{I}(\mathcal{D}_z(f))(z) = f(z) - f(0) \qquad \text{and} \qquad \overline{\mathcal{I}}(\mathcal{D}_{\overline{z}}(f))(z) = f(z) - f(0), \qquad z \in \mathbb{D}.$$

Our main results are related with walks through dimensions for Descente and Montée operators over complex spheres:

Theorem 1.1. *Let $f \colon \mathbb{D} \to \mathbb{C}$ be continuously differentiable.*

(i) If f belongs to the class $\Psi(\Omega_{2q})$, then $\mathcal{D}_z f$, $\mathcal{D}_{\overline{z}} f$ and $\mathcal{D}_x f$ belong to the class $\Psi(\Omega_{2q+2})$.

(ii) If f belongs to the class $\Psi(\Omega_{2q})$ and has all positive $(2q)$-complex Schoenberg coefficients, then $\mathcal{D}_z f$, $\mathcal{D}_{\overline{z}} f$ and $\mathcal{D}_x f$ belong to the class $\Psi^+(\Omega_{2q+2})$.

Theorem 1.2. *Let $f \colon \mathbb{D} \to \mathbb{C}$ be a continuous function admitting a z-primitive and a $\overline{z}$-primitive in $\mathbb{D}$.*

(i) If f belongs to the class $\Psi(\Omega_{2q+2})$, then there exist real constants c and C such that $c + \mathcal{I} f$ and $C + \overline{\mathcal{I}} f$ belong to the class $\Psi(\Omega_{2q})$.

(ii) If f belongs to the class $\Psi^+(\Omega_{2q+2})$, then there exist real constants c and C such that $c + \mathcal{I} f$ and $C + \overline{\mathcal{I}} f$ belong to the class $\Psi^+(\Omega_{2q})$.

Observe that in Theorem 1.1(ii) we assumed the additional condition that all $(2q)$-complex Schoenberg coefficients are positive. This condition can be weakened (see Remark 1.4 below), but not completely removed.

In fact, the following counterexamples show that the Descente operators over complex spheres do not preserve, in general, strict positive definiteness, in contrast to the real case of Beatson and zu Castell.

Counterexample 1.3. *Let $q \geq 2$ be an integer.*

(i) *If $f(z) = \sum_{m=0}^{\infty} a_{m,0}^{q-2} R_{m,0}^{q-2}(z)$, where $\sum_{m=0}^{\infty} a_{m,0}^{q-2} < \infty$ and $a_{m,0}^{q-2} > 0$ for all m, then $f \in \Psi^+(\Omega_{2q})$ and $\mathcal{D}_z f, \mathcal{D}_x f \in \Psi^+(\Omega_{2q+2})$ but $\mathcal{D}_{\overline{z}} f \notin \Psi^+(\Omega_{2q+2})$.*

(ii) *If $f(z) = \sum_{n=0}^{\infty} a_{0,n}^{q-2} R_{0,n}^{q-2}(z)$, where $\sum_{n=0}^{\infty} a_{0,n}^{q-2} < \infty$ and $a_{0,n}^{q-2} > 0$ for all n, then $f \in \Psi^+(\Omega_{2q})$ and $\mathcal{D}_{\overline{z}} f, \mathcal{D}_x f \in \Psi^+(\Omega_{2q+2})$ but $\mathcal{D}_z f \notin \Psi^+(\Omega_{2q+2})$.*

(iii) *If $f(z) = \sum_{n=0}^{\infty} a_{0,n}^{q-2} R_{0,n}^{q-2}(z) + \sum_{m=0}^{\infty} a_{m,0}^{q-2} R_{m,0}^{q-2}(z)$, where $a_{0,n}^{q-2}, a_{m,0}^{q-2} \geq 0$ for all m, n, and*

$$
\begin{aligned}
&a_{0,n}^{q-2} > 0 \Longleftrightarrow n \in 5\mathbb{Z}_+ + 4, \\
&a_{m,0}^{q-2} > 0 \Longleftrightarrow m \in (5\mathbb{Z}_+ \setminus \{0\}) \cup (5\mathbb{Z}_+ + 2) \cup (5\mathbb{Z}_+ + 3) \cup (5\mathbb{Z}_+ + 4),
\end{aligned}
$$

then $f \in \Psi^+(\Omega_{2q})$ but $\mathcal{D}_z f, \mathcal{D}_{\overline{z}} f, \mathcal{D}_x f \notin \Psi^+(\Omega_{2q+2})$.

Remark 1.4. In the real case, the condition that all d-Schoenberg coefficients are positive is satisfied by most of the functions in the class Ψ_d^+ which appear in applications such as in statistics and geostatistics.

In the complex case, among the examples that we provide in Section 2.1, only the exponential function satisfies this condition. On the other hand, the Aktaş–Taşdelen–Yavuz, Horn and Lauricella families, satisfy the following simple weaker condition, which is also sufficient to obtain the conclusion of Theorem 1.1(ii):

- *if $a_{m,n}^{q-2}$ are the $(2q)$-complex Schoenberg coefficients of f, then for some $c, d \in \mathbb{N}$, the set*

  $$\left\{m - n \colon a_{m,n}^{q-2} > 0,\ m, n \geq c\right\}$$

  contains $(d + \mathbb{Z}_+)$ or $(-d - \mathbb{Z}_+)$.

In fact, the weakest possible condition to be used in Theorem 1.1(ii) follows from Guella and Menegatto [18] and reads as follows:

$$\left\{m - n \colon a_{m,n}^{q-2} > 0,\ m, n \geq 1\right\} \cap (N\mathbb{Z} + j) \neq \varnothing, \tag{1.4}$$

for every $N \geq 1$, $j = 0, 1, \ldots, N-1$. We will prove Theorem 1.1 with this last condition, since the previous ones are stronger.

This paper is organized as follows: in Section 2, we provide the necessary background about positive definite functions on complex spheres and we give a list of parametric families of these functions, which are of interest for both numerical analysis and geostatistical communities. Finally, in Section 3, we obtain all necessary technical lemmas, we give the proofs of Theorems 1.1 and 1.2, and we show the Counterexample 1.3.

2 The classes $\Psi(\Omega_{2q})$ and $\Psi^+(\Omega_{2q})$: a brief survey

This section is largely expository and presents some basic facts and background needed for a self contained exposition.

For q being a positive integer, we denote by Ω_{2q} the unit sphere of $\mathbb{C}^q$ and by $B_{2q} := \{z \in \mathbb{C}^q : |z| \leq 1\}$ the closed disk in $\mathbb{C}^q$. Also, we define the Pochhammer symbol $(a)_n := a(a+1)\cdots(a+n-1)$, with $(a)_0 := 1$.

Let A be a nonempty set. A continuous kernel $K: A^2 \to \mathbb{C}$ is *positive definite* if and only if

$$\sum_{\mu,\nu=1}^{l} c_\mu \overline{c_\nu} K(\xi_\mu, \xi_\nu) \geq 0, \tag{2.1}$$

for all $l \in \mathbb{Z}_+ := \{0,1,2,\ldots\}$, $\{\xi_1, \xi_2, \ldots, \xi_l\} \subset A$ and $\{c_1, c_2, \ldots, c_l\} \subset \mathbb{C}$. If the inequality in (2.1) is strict when at least one c_μ is nonzero, then K is called *strictly positive definite*. For q a strictly positive integer, we define $A_q := \Omega_2$ when $q = 1$ and $A_q := \mathbb{D}$ for $q > 1$. Throughout we shall work with the class $\Psi(\Omega_{2q})$ of continuous functions $f: A_q \to \mathbb{C}$ such that the kernel $K: \Omega_{2q} \times \Omega_{2q} \to \mathbb{C}$ defined as

$$K(\xi, \eta) = f(\langle \xi, \eta \rangle), \qquad (\xi, \eta) \in \Omega_{2q} \times \Omega_{2q}, \tag{2.2}$$

where the symbol $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{C}^q$, is positive definite.

Observe that an immediate consequence of the definition is that f satisfies $f(\overline{z}) = \overline{f(z)}$. We shall use the notation $\Psi^+(\Omega_{2q})$ if the kernel K associated to f through (2.2) is strictly positive definite. Positive definite kernels satisfying the identity above are called isotropic. The class $\Psi(\Omega_{2q})$ is parenthetical to the class Ψ_d introduced by Schoenberg [40], and we refer the reader to the recent review in Gneiting [17] for a thorough description of the properties of this class. Further, the class Ψ_d represents the building block for extension to product spaces, and the reader is referred to Berg and Porcu [7] as well as to Guella et al. [19] for recent efforts in this direction. The classes $\Psi(\Omega_{2q})$ are nested, with the following inclusion relation being strict:

$$\Psi(\Omega_4) \supset \Psi(\Omega_6) \supset \cdots \supset \Psi(\Omega_\infty),$$

where Ω_∞ is the unit sphere in the Hilbert space $\ell_2(\mathbb{C})$. Analogous relations apply to $\Psi^+(\Omega_{2q})$.

Observe that the class $\Psi(\Omega_2)$ is a different class and it can not be added to the inclusions above (see Menegatto and Peron [26]). For this reason, in this work we always consider $q \geq 2$. Actually the main purpose here is to study the walks through dimensions considering functions in the classes $\Psi(\Omega_{2q})$.

Characterization theorems for the classes $\Psi(\Omega_{2q})$ are available in recent literature, and some ingredients are needed for a detailed exposition. We refer to Boyd and Raychowdhury [10], Dreseler and Hrach [13], and Koornwinder [22, 23] for more information concerning this necessary material.

The *disc polynomial* $R^\alpha_{m,n}$ of degree $m+n$ in x and y associated to a real number $\alpha > -1$ was introduced by Zernike [47] and Zernike and Brinkman [48], see also Koornwinder [22], as the polynomial given by

$$R^\alpha_{m,n}(z) := r^{|m-n|} e^{i(m-n)\theta} R^{(\alpha,|m-n|)}_{\min\{m,n\}}(2r^2-1), \qquad z = re^{i\theta} = x+iy \in \mathbb{D}, \tag{2.3}$$

where $R^{(\alpha,\beta)}_k$ is the usual Jacobi polynomial of degree k associated to the numbers $\alpha, \beta > -1$ and normalized by $R^{(\alpha,\beta)}_k(1) = 1$ (see Szegő [44, p. 58]). Note that the function $R^\alpha_{m,n}$ is a polynomial of degrees m and n with respect to the arguments z and $\overline{z}$, respectively. Moreover it satisfies $R^\alpha_{m,n}(\overline{z}) = \overline{R^\alpha_{m,n}(z)}$.

Let $\mathrm{d}\nu_\alpha$ be the positive measure having total mass identically equal to one on $\mathbb{D}$, and given by

$$\mathrm{d}\nu_\alpha(z) = \frac{\alpha+1}{\pi}\big(1-x^2-y^2\big)^\alpha \mathrm{d}x\mathrm{d}y, \qquad z = x+iy. \tag{2.4}$$

Due to the orthogonality relations for Jacobi polynomials, the set $\{R^\alpha_{m,n} \colon 0 \le m,n < \infty\}$ forms a complete orthogonal system in $L^2(\mathbb{D}, \mathrm{d}\nu_\alpha)$ with

$$\int_{\mathbb{D}} R^\alpha_{m,n}(z)\overline{R^\alpha_{k,l}(z)}\mathrm{d}\nu_\alpha(z) = \frac{1}{h^\alpha_{m,n}}\delta_{m,k}\delta_{n,l}, \tag{2.5}$$

where

$$h^\alpha_{m,n} = \frac{m+n+\alpha+1}{\alpha+1}\binom{\alpha+m}{\alpha}\binom{\alpha+n}{\alpha}, \tag{2.6}$$

and $\delta_{n,l}$ denotes the Kronecker delta. Thus, a function $f \in L^1(\mathbb{D}, \nu_\alpha)$, $\alpha \ge 0$, has an expansion in terms of disc polynomials $R^\alpha_{m,n}$ defined through

$$f(z) \sim \sum_{m,n\ge 0} a^\alpha_{m,n} R^\alpha_{m,n}(z), \tag{2.7}$$

where

$$a^\alpha_{m,n} = h^\alpha_{m,n}\int_{\mathbb{D}} f(z)\overline{R^\alpha_{m,n}(z)}\mathrm{d}\nu_a(z). \tag{2.8}$$

The Poisson–Szegő kernel will be a fundamental tool for the proof of Theorem 2.1(1) below: the characterization of the class $\Psi(\Omega_{2q})$. We give here a brief presentation of it, since this kernel will also be used ahead. The Poisson–Szegő kernel is defined by

$$\mathcal{P}_q(r\xi,\eta) := \frac{1}{\sigma_{2q}}\frac{(1-|r\xi|^2)^q}{|1-\langle r\xi,\eta\rangle|^{2q}}, \qquad r\in[0,1), \quad \xi,\eta\in\Omega_{2q}, \tag{2.9}$$

where σ_{2q} is the total surface of Ω_{2q}. Folland [15] proved that it has an expansion in terms of disc polynomials as

$$\mathcal{P}_q(r\xi,\eta) = \sum_{m,n\ge 0}\frac{h^{q-2}_{m,n}}{\sigma_{2q}} S^q_{m,n}(r) R^{q-2}_{m,n}(\langle\xi,\eta\rangle), \qquad \xi,\eta\in\Omega_{2q}, \quad r\in[0,1), \tag{2.10}$$

where $S^q_{m,n}(r) \ge 0$, $\lim\limits_{r\to 1^-} S^q_{m,n}(r) = 1$ and the series converges absolutely and uniformly for $\xi,\eta\in\Omega_{2q}$ and $0\le r\le R$, for each $R<1$.

The Poisson–Szegő kernel also appears in the solution of the following Dirichlet problem for the Laplace–Beltrami operator Δ_{2q} (see Stein [43]): given a continuous function $h\colon \Omega_{2q}\to\mathbb{C}$, there exists a continuous function $u\colon B_{2q}\to\mathbb{C}$ such that $\Delta_{2q}u = 0$ and $u|_{\Omega_{2q}} = h$. The solution u can be computed through

$$u(z) = \int_{\Omega_{2q}} \mathcal{P}_q(z,\rho)h(\rho)\mathrm{d}\omega_{2q}(\rho), \qquad z\in B_{2q}, \tag{2.11}$$

where $\mathrm{d}\omega_{2q}$ denotes the rotation-invariant surface element on Ω_{2q}.

In fact, using this, if $\alpha = q-2 \ge 0$ is an integer and f is a continuous function on $\mathbb{D}$, the coefficients in the series in (2.7), can be written as (see Menegatto and Peron [26]):

$$a^{q-2}_{m,n} = \frac{h^{q-2}_{m,n}}{\sigma_{2q}}\int_{\Omega_{2q}} f(\langle\rho,e_1\rangle)R^{q-2}_{m,n}(\langle e_1,\rho\rangle)\mathrm{d}\omega_{2q}(\rho), \tag{2.12}$$

where $e_1 = (1,0,\dots,0)\in\Omega_{2q}$.

We give now the representations for the elements of the classes $\Psi(\Omega_{2q})$ and $\Psi^+(\Omega_{2q})$ that were proved by Menegatto and Peron [25, 26] and Guella and Menegatto [18]:

Theorem 2.1. *Let $f\colon \mathbb{D} \to \mathbb{C}$ be a continuous function. The following assertions are true:*

(1) *$f \in \Psi(\Omega_{2q})$ if, and only if,*

$$f(z) = \sum_{m,n\geq 0} a_{m,n}^{q-2} R_{m,n}^{q-2}(z), \qquad z \in \mathbb{D}, \tag{2.13}$$

where $\sum_{m,n\geq 0} a_{m,n}^{q-2} < \infty$ and $a_{m,n}^{q-2} \geq 0$ for all (m,n);

(2) *$f \in \Psi^+(\Omega_{2q})$ if, and only if, $f \in \Psi(\Omega_{2q})$ and*

$$\left\{m-n\colon a_{m,n}^{q-2} > 0,\ m,n \geq 0\right\} \cap (N\mathbb{Z} + j) \neq \varnothing, \tag{2.14}$$

for every $N \geq 1$, $j = 0, 1, \ldots, N-1$.

Note that the index $\alpha = q-2$ of the disc polynomials is related to the sphere Ω_{2q} and consequently $\alpha + 1 = q - 1$ is related to Ω_{2q+2}.

The coefficients $a_{m,n}^{q-2}$ are the analogue of the d-Schoenberg coefficients a_k^d as in Daley and Porcu [12] and Ziegel [49], referring to the expansion of the members of the Schoenberg class Ψ_d. In analogy, we will call $a_{m,n}^{q-2}$ as $(2q)$-*complex Schoenberg coefficients*.

2.1 Families within the classes $\Psi(\Omega_{2q})$ and $\Psi^+(\Omega_{2q})$

It is well known that there exist many examples of functions in the class Ψ_d, some of them widely used in applications (see for example Gneiting [17] and Porcu et al. [28]).

In the literature it is also possible to find examples of functions that satisfy the conditions in Theorem 2.1, or those in Remark 1.4, and therefore they belong to the classes $\Psi(\Omega_{2q})$ and $\Psi^+(\Omega_{2q})$. Some of them, as well as their use in applications, appeared recently, probably originated by the work of Wünsche [46], that deals with disc polynomials: a fundamental tool for studying the functions in these classes. We give below a collection of such functions.

1. Disk Polynomials and related families. The product kernel (Boyd and Raychowdhury [10]),

$$f_{m,n}(z) = z^m \overline{z}^n = \sum_{j=0}^{\min\{m,n\}} c_{q,m,n}^j R_{m-j,n-j}^{q-2}(z), \qquad c_{q,m,n}^j \geq 0, \quad z \in \mathbb{D},$$

is an element of the class $\Psi(\Omega_{2q})$, for each $m, n \geq 0$.

2. Poisson–Szegő kernel and related families. An application of (2.9) and (2.10) shows that

$$f_r(z) := \frac{1}{\sigma_{2q}} \frac{(1-r^2)^q}{|1-rz|^{2q}} = \sum_{m,n\geq 0} \frac{h_{m,n}^{q-2}}{\sigma_{2q}} S_{m,n}^q(r) R_{m,n}^{q-2}(z), \qquad z \in \mathbb{D},$$

and hence it is a member of the class $\Psi(\Omega_{2q})$, for each $r \in [0,1)$.

3. Exponential function. The function (Menegatto et al. [27])

$$e^{z+\overline{z}} = \sum_{m+n=0}^{\infty} \frac{(m+1)_{q-2}(n+1)_{q-2}}{(q-2)!} \left(\sum_{j=0}^{\infty} \frac{1}{j!(m+n+q-1)_j} \right) R_{m,n}^{q-2}(z), \qquad z \in \mathbb{D},$$

belongs to the class $\Psi^+(\Omega_{2q})$.

4. Aktaş, Taşdelen and Yavuz family. The function (Aktaş et al. [2])

$$f_t(z) := \frac{1}{R}\left(\frac{2}{1-t+R}\right)^{q-2} e^{(2tz)/(1+t+R)} = \sum_{m,n\geq 0} (q-1)_n \frac{t^{m+n}}{m!n!} R^{q-2}_{m+n,n}(z), \qquad z \in \mathbb{D},$$

where $R := \big(1 - 2\big(2|z|^2-1\big)t + t^2\big)^{1/2}$, is a member of $\Psi^+(\Omega_{2q})$, for each $t \in (0,1)$.

5. Horn family. Let r, R be positive integers such that $4r = (R-1)^2$. Horn's function H_4 is defined on p. 57 of Srivastava and Manocha [42] by

$$H_4(a,b;c,d;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{2m+n}(b)_n}{(c)_m(d)_n} \frac{x^m y^n}{m!n!},$$

where $|x| < r$ and $|y| < R$. An application of Theorem 2.2 in Aktaş et al. [2] shows that

$$\begin{aligned} f_{t,s,b}(z) &:= \frac{1}{(1-s)^{q-1}} H_4\left(q-1, b; q-1, q-1; \frac{s(|z|^2-1)}{(1-s)^2}, \frac{t\overline{z}}{1-s}\right) \\ &= \sum_{m,n\geq 0} (q+n-1)_m (b)_n \frac{t^n s^m}{m!n!} R^{q-2}_{m,m+n}(z), \qquad z \in \mathbb{D}. \end{aligned}$$

Hence it is a member of $\Psi^+(\Omega_{2q})$, for each b, a positive integer, and t, s positive numbers satisfying

$$|s| < 1, \qquad \frac{|s|}{(1-s)^2} < r, \qquad \text{and} \qquad \frac{|t|}{1-s} < R.$$

6. Lauricella family. Let r_1, r_2 and r_3 be positive integers such that $r_1 r_2 = (1-r_2)(r_2-r_3)$. The Lauricella hypergeometric function of three variables F_{14} (Saran's notation F_F is also used (Saran [32])) is defined by (see p. 67 of Srivastava and Manocha [42])

$$F_{14}(a_1,a_1,a_1,b_1,b_2,b_1;c_1,c_2,c_2;x_1,x_2,x_3) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n+p}(b_1)_{m+p}(b_2)_n}{(c_1)_m(c_2)_{n+p}} \frac{x_1^m x_2^n x_3^p}{m!n!p!},$$

where $|x_1| < r_1$, $|x_2| < r_2$ and $|x_3| < r_3$. For $t, s \in \mathbb{R}$ such that $|s| < r_1$ and $|t| < r_2$, where $r_1 = r_2(1-r_2)$, define

$$f_{t,s,b}(z) := F_{14}\big(1,1,1,q-1,b,q-1;q-1,1,1;s\big(|z|^2-1\big),tz,s|z|^2\big), \qquad z \in \mathbb{D}.$$

From Theorem 2.3 in Aktaş et al. [2] we get

$$f_{t,s,b}(z) = \sum_{m,n\geq 0} (q-1)_n (b)_m \frac{t^m s^n}{m!n!} R^{q-2}_{m+n,n}(z), \qquad z \in \mathbb{D},$$

and hence, $f_{t,s,b}$ is a member of $\Psi^+(\Omega_{2q})$, for each b, a positive integer, and t, s positive numbers satisfying the relevant conditions above.

Some comments are in order. Lauricella functions are generalizations of the Gauss hypergeometric functions to multiple variables and were introduced by Lauricella in 1893. Recursion formulas and integral representation for Lauricella functions, including F_{14} (F_F), have been studied and can be found, for example, in Sahai and Verma [31] and Saran [33, 34]. In 1873, Schwarz [41] found a list of 15 cases where hypergeometric functions can be expressed algebraically. More precisely, Schwarz gave a list of parameters determining the cases where the hypergeometric differential equation has two independent solutions that are algebraic functions. Between 1989 and 2009 several researchers extended this list: to general one-variable hypergeometric functions ${}_{p+1}F_p$ (Beukers and Heckman [8]), the Appell–Lauricella functions F_1 and F_D (Beazley Cohen and Wolfart [5]), the Appell functions F_2 and F_4 (Kato [20, 21]), and the Horn function G_3 (Schipper [38]). In 2012, Bod [9] extended Schwarz' list to the four classes of Appell–Lauricella functions and the 14 complete Horn functions, including H_4.

3 Proof of the results

In this section we first prove some technical lemmas. Then, we shall be able to give the proof of our main results and to present the counterexamples.

The first lemma contains recurrence formulas connecting disc polynomials of different indexes and degrees. They are obtained from equation (5.5) in Aharmim et al. [1] and the following properties of the disc polynomials

$$\overline{R^{\alpha}_{m,n}(z)} = R^{\alpha}_{n,m}(z), \qquad \overline{\mathcal{D}_z R^{\alpha}_{m,n}(z)} = \mathcal{D}_{\overline{z}} R^{\alpha}_{n,m}(z), \qquad \alpha > -1, \quad m,n \geq 0, \quad z \in \mathbb{D}.$$

We observe that the normalization adopted in Aharmim et al. [1] for the disc polynomials is different from the one we use here.

Lemma 3.1. *Let m, n be non negative integers and $\alpha > -1$ be a real number. Then, for any $z \in \mathbb{D}$, we have*

$$(\alpha+1)R^{\alpha}_{m,n+1}(z) = (\alpha+1)\overline{z}R^{\alpha+1}_{m,n}(z) - \left(1-|z|^2\right)\mathcal{D}_z R^{\alpha+1}_{m,n}(z), \tag{3.1}$$

and

$$(\alpha+1)R^{\alpha}_{n+1,m}(z) = (\alpha+1)zR^{\alpha+1}_{n,m}(z) - \left(1-|z|^2\right)\mathcal{D}_{\overline{z}} R^{\alpha+1}_{n,m}(z). \tag{3.2}$$

Below we prove an important technical result, that connects the expansion of a continuously differentiable function f in terms of the disc polynomials $R^{\alpha}_{m,n}$ with the expansion of its derivatives in terms of the disc polynomials $R^{\alpha+1}_{m,n}$.

Since $R^{q-2}_{m,n}$ belongs to $\Psi(\Omega_{2q})$ when $q \geq 2$ is an integer, this connection will be the main ingredient in order to obtain preservation of positive definiteness for the Descente operators, when walks through dimensions over complex spheres are provided.

Lemma 3.2. *Let $f\colon \mathbb{D} \to \mathbb{C}$ be continuously differentiable and let $\alpha > -1$ be a real number. Consider the expansion of f in terms of the disc polynomials $R^{\alpha}_{m,n}$ and the expansions of $\mathcal{D}_z f$ and $\mathcal{D}_{\overline{z}} f$ in terms of the disc polynomials $R^{\alpha+1}_{m,n}$*

$$f(z) \sim \sum_{m,n=0}^{\infty} a^{\alpha}_{m,n} R^{\alpha}_{m,n}(z), \qquad z \in \mathbb{D},$$

$$\mathcal{D}_z f(z) \sim \sum_{m,n=0}^{\infty} b^{\alpha+1}_{m,n} R^{\alpha+1}_{m,n}(z) \qquad \text{and} \qquad \mathcal{D}_{\overline{z}} f(z) \sim \sum_{m,n=0}^{\infty} \tilde{b}^{\alpha+1}_{m,n} R^{\alpha+1}_{m,n}(z), \qquad z \in \mathbb{D}.$$

Then,

$$b^{\alpha+1}_{m,n} = \frac{(m+1)(n+\alpha+1)}{(\alpha+1)} a^{\alpha}_{m+1,n}, \qquad m,n \geq 0,$$

and

$$\tilde{b}^{\alpha+1}_{m,n} = \frac{(n+1)(m+\alpha+1)}{(\alpha+1)} a^{\alpha}_{m,n+1}, \qquad m,n \geq 0.$$

It is worth noting that this result is not surprising if we consider the identities obtained in Koornwinder [23]: for $\alpha > -1$,

$$\mathcal{D}_z R^{\alpha}_{m,n} = c_{\alpha}(m,n) R^{\alpha+1}_{m-1,n} \qquad \text{and} \qquad \mathcal{D}_{\overline{z}} R^{\alpha}_{m,n} = c_{\alpha}(n,m) R^{\alpha+1}_{m,n-1}, \tag{3.3}$$

where $c_{\alpha}(m,n) := (m(n+\alpha+1))/(\alpha+1)$. These are, in the complex case, the analogue of the identities for the derivative of the Gegenbauer polynomials (see Szegő [44, equation (4.7.14)]).

Actually, Lemma 3.2 shows that the coefficients in the expansions are linked as if the series could be derived term by term.

Proof of Lemma 3.2. The coefficients $b^{\alpha+1}_{m,n}$ are given by the formula

$$b^{\alpha+1}_{m,n} = h^{\alpha+1}_{m,n} \int_{\mathbb{D}} \mathcal{D}_z f(z) \overline{R^{\alpha+1}_{m,n}(z)} \mathrm{d}\nu_{\alpha+1}(z),$$

where the constants $h^{\alpha+1}_{m,n}$ are given in (2.6). Define

$$I := \int_{\mathbb{D}} \mathcal{D}_z f(z) R^{\alpha+1}_{n,m}(z) \mathrm{d}\nu_{\alpha+1}(z) = \frac{\alpha+2}{\pi} \int_{\mathbb{D}} \mathcal{D}_z f(z) R^{\alpha+1}_{n,m}(z) \big(1 - x^2 - y^2\big)^{\alpha+1} \mathrm{d}x\mathrm{d}y.$$

Integration by parts and direct inspection shows that

$$\begin{aligned} I = \frac{\alpha+2}{\pi} \bigg\{ &\int_{\mathbb{D}} \mathcal{D}_z \big[f(z) R^{\alpha+1}_{n,m}(z) \big(1 - |z|^2\big)^{\alpha+1} \big] \mathrm{d}x\mathrm{d}y \\ &- \int_{\mathbb{D}} f(z) \mathcal{D}_z \big[R^{\alpha+1}_{n,m}(z) \big(1 - |z|^2\big)^{\alpha+1} \big] \mathrm{d}x\mathrm{d}y \bigg\}. \end{aligned}$$

Using Green's theorem and (1.2) we have

$$\int_{\Omega_2} g(z) \mathrm{d}\overline{z} = -2i \int_{\mathbb{D}} \mathcal{D}_z(g)(z) \mathrm{d}x\mathrm{d}y,$$

for any continuously differentiable function g. Thus,

$$\begin{aligned} I &= \frac{\alpha+2}{\pi} \left\{ \frac{i}{2} \int_{\Omega_2} f(z) R^{\alpha+1}_{n,m}(z) \big(1 - |z|^2\big)^{\alpha+1} \mathrm{d}\overline{z} - \int_{\mathbb{D}} f(z) \mathcal{D}_z \left[R^{\alpha+1}_{n,m}(z) \big(1 - |z|^2\big)^{\alpha+1} \right] \mathrm{d}x\mathrm{d}y \right\} \\ &= -\frac{\alpha+2}{\pi} \int_{\mathbb{D}} f(z) \mathcal{D}_z \left[R^{\alpha+1}_{n,m}(z) \big(1 - |z|^2\big)^{\alpha+1} \right] \mathrm{d}x\mathrm{d}y. \end{aligned}$$

Now, by noting that

$$\mathcal{D}_z \big[R^{\alpha+1}_{n,m}(z) \big(1 - |z|^2\big)^{\alpha+1} \big] = \mathcal{D}_z R^{\alpha+1}_{n,m}(z) \big(1 - |z|^2\big)^{\alpha+1} - (\alpha+1) \big(1 - |z|^2\big)^{\alpha} \overline{z} R^{\alpha+1}_{n,m}(z),$$

we get

$$I = \frac{\alpha+2}{\pi} \int_{\mathbb{D}} f(z) \big(1 - |z|^2\big)^{\alpha} \left[(\alpha+1) \overline{z} R^{\alpha+1}_{n,m}(z) - \big(1 - |z|^2\big) \mathcal{D}_z R^{\alpha+1}_{n,m}(z) \right] \mathrm{d}x\mathrm{d}y.$$

Hence, using Lemma 3.1, we have

$$I = \frac{\alpha+2}{\pi} \int_{\mathbb{D}} f(z) \big(1 - |z|^2\big)^{\alpha} (\alpha+1) R^{\alpha}_{n,m+1}(z) \mathrm{d}x\mathrm{d}y = (\alpha+2) \int_{\mathbb{D}} f(z) \overline{R^{\alpha}_{m+1,n}(z)} \mathrm{d}\nu_{\alpha}(z).$$

Thus,

$$b^{\alpha+1}_{m,n} = h^{\alpha+1}_{m,n} I = (\alpha+2) h^{\alpha+1}_{m,n} \frac{1}{h^{\alpha}_{m+1,n}} a^{\alpha}_{m+1,n}.$$

Replacing the values of $h^{\alpha+1}_{m,n}$ and $h^{\alpha}_{m+1,n}$ given in equation (2.6), we obtain

$$b^{\alpha+1}_{m,n} = (\alpha+2) \frac{(m+1)(\alpha+n+1)}{(\alpha+2)(\alpha+1)} a^{\alpha}_{m+1,n} = \frac{(m+1)(\alpha+n+1)}{(\alpha+1)} a^{\alpha}_{m+1,n}.$$

The proof for the case of the operator $\mathcal{D}_{\overline{z}}$ is analogous observing that

$$\int_{\Omega_2} g(z) \mathrm{d}z = 2i \int_{\mathbb{D}} \mathcal{D}_{\overline{z}}(g)(z) \mathrm{d}x\mathrm{d}y. \qquad \blacksquare$$

The last technical lemma gives a condition for the expansion of a continuous function in terms of the disc polynomials to be uniformly convergent.

Lemma 3.3. *Let $g\colon \mathbb{D} \to \mathbb{C}$ be a continuous function and consider its expansion*

$$g(z) \sim \sum_{m,n\geq 0} d^{q-2}_{m,n} R^{q-2}_{m,n}(z), \qquad z \in \mathbb{D}, \tag{3.4}$$

where $d^{q-2}_{m,n}$ are given as in (2.12). If $d^{q-2}_{m,n} \geq 0$ for all $m,n\geq 0$, then $\sum_{m,n\geq 0} d^{q-2}_{m,n} < \infty$. In particular, the series in (3.4) converges uniformly in $\mathbb{D}$.

Proof. The argument is similar to the one used in the proof of Theorem 4.1 in Menegatto and Peron [26]. Given $\xi \in \Omega_{2q}$, consider the continuous function $h(\rho) := g(\langle \rho, \xi\rangle)$, $\rho \in \Omega_{2q}$. By equation (2.11), the solution of the Dirichlet problem $\Delta_{2q} u = 0$ in the interior of B_{2q} with boundary condition h, evaluated on the segment $r\xi$, $r \in [0,1)$, is

$$u(r\xi) = \int_{\Omega_{2q}} \mathcal{P}_q(r\xi,\rho) g(\langle \rho,\xi\rangle) \mathrm{d}\omega_{2q}(\rho) = \sum_{m,n\geq 0} S^q_{m,n}(r) d^{q-2}_{m,n},$$

where the last equality is obtained from (2.10), (2.12).

Since u is continuous up to the boundary and coincides with h on Ω_{2q}, we obtain

$$\lim_{r\to 1^-} \sum_{m,n\geq 0} d^{q-2}_{m,n} S^q_{m,n}(r) = \lim_{r\to 1^-} u(r\xi) = u(\xi) = g(\langle \xi,\xi\rangle) = g(1).$$

Now, note that

$$0 \leq \sum_{m=0}^{k}\sum_{n=0}^{l} d^{q-2}_{m,n} S^q_{m,n}(r) \leq \sum_{m,n\geq 0} d^{q-2}_{m,n} S^q_{m,n}(r), \qquad 0 \leq r < 1.$$

Letting $r \to 1^-$, we get

$$0 \leq s_{k,l} := \sum_{m=0}^{k}\sum_{n=0}^{l} d^{q-2}_{m,n} \leq \lim_{r\to 1^-} \sum_{m,n\geq 0} d^{q-2}_{m,n} S^q_{m,n}(r) = g(1), \qquad k,l \in \mathbb{Z}_+.$$

Hence, the sequence $\{s_{k,l}\}_{k,l\in\mathbb{Z}_+}$ is bounded and increasing. Thus, the series $\sum_{m,n\geq 0} d^{q-2}_{m,n}$ is convergent. Using the fact that $|R^{q-2}_{m,n}(z)| \leq 1$ for all $z \in \mathbb{D}$ and using the Weierstrass M-Test, the proof is completed. ■

At this point, we are able to prove our main results.

Proof of Theorem 1.1. Let f be a function in the class $\Psi(\Omega_{2q})$. Then, by Theorem 2.1(1),

$$f(z) = \sum_{m,n\geq 0} a^{\alpha}_{m,n} R^{\alpha}_{m,n}(z), \qquad z \in \mathbb{D},$$

where $\alpha = q-2$, $a^{\alpha}_{m,n} \geq 0$, for all $m,n \geq 0$, and $\sum_{m,n\geq 0} a^{\alpha}_{m,n} < \infty$. Consider the expansion in terms of disc polynomials of $\mathcal{D}_z f$:

$$\mathcal{D}_z f(z) \sim \sum_{m,n\geq 0} b^{\alpha+1}_{m,n} R^{\alpha+1}_{m,n}(z), \qquad z \in \mathbb{D}.$$

By Lemma 3.2 and equation (3.3),

$$b^{\alpha+1}_{m,n} = c_\alpha(m+1,n)a^\alpha_{m+1,n}, \qquad m,n \geq 0. \tag{3.5}$$

Roughly speaking, (3.5) means that the coefficients $\{b^{\alpha+1}_{m,n}\}$ are obtained from the $\{a^\alpha_{m,n}\}$ by suppressing the $a^\alpha_{0,n}$, translating in the first index and multiplying by the positive constants $\{c_\alpha(m+1,n)\}$.

Then, by equation (3.3), we have

$$\sum_{m,n\geq 0} a^\alpha_{m,n}\mathcal{D}_z R^\alpha_{m,n}(z) = \sum_{m\geq -1}\sum_{n\geq 0} a^\alpha_{m+1,n}\mathcal{D}_z R^\alpha_{m+1,n}(z) = \sum_{m,n\geq 0} b^{\alpha+1}_{m,n}R^{\alpha+1}_{m,n}(z).$$

Now, since $c_\alpha(m+1,n)$ are positive constants, we have that $b^{\alpha+1}_{m,n} \geq 0$ for all $m,n\geq 0$. By Lemma 3.3, the series

$$\sum_{m,n\geq 0} a^\alpha_{m,n}\mathcal{D}_z R^\alpha_{m,n}(1) = \sum_{m,n\geq 0} b^{\alpha+1}_{m,n}$$

is convergent and the series $\sum_{m,n\geq 0} b^{\alpha+1}_{m,n}R^{\alpha+1}_{m,n}(z)$ converges uniformly in $\mathbb{D}$. It follows, by term by term differentiation, that

$$\mathcal{D}_z f(z) = \sum_{m,n\geq 0} b^{\alpha+1}_{m,n}R^{\alpha+1}_{m,n}(z).$$

Hence, by Theorem 2.1(1), $\mathcal{D}_z f$ belongs to the class $\Psi(\Omega_{2q+2})$.

Similarly, we can conclude the same for the operator $\mathcal{D}_{\overline{z}}$.

For the item (ii), observe that, as a consequence of (3.5), if the $(2q)$-complex Schoenberg coefficients $a^\alpha_{m,n}$ of f satisfy (1.4), then the $(2q+2)$-complex Schoenberg coefficients $b^{\alpha+1}_{m,n}$ of $\mathcal{D}_z f$ (and similarly for $\mathcal{D}_{\overline{z}} f$) satisfy (2.14). Actually, the condition $m,n\geq 1$ in the set considered in (1.4) guarantees that the intersections with the arithmetic progressions in $\mathbb{Z}$ do not depend on the coefficients $a^\alpha_{m,0}$ or $a^\alpha_{0,n}$, which are suppressed by the Descente operators.

The results for $\mathcal{D}_x f$ follow immediately by (1.2). ■

Proof of Theorem 1.2. Suppose that f belongs to the class $\Psi(\Omega_{2q+2})$. By Theorem 2.1,

$$f(z) = \sum_{m,n\geq 0} a^{\alpha+1}_{m,n}R^{\alpha+1}_{m,n}(z), \qquad z\in\mathbb{D},$$

where $\alpha = q-2$ and $a^{\alpha+1}_{m,n}\geq 0$ for all $m,n\geq 0$ and $\sum_{m,n\geq 0} a^{\alpha+1}_{m,n} < \infty$. By equation (3.3), we have

$$\mathcal{I}(R^{\alpha+1}_{m-1,n})(z) = \frac{1}{c_\alpha(m,n)}\big(R^\alpha_{m,n}(z) - R^\alpha_{m,n}(0)\big), \tag{3.6}$$

where $R^\alpha_{n,n}(0) = (-1)^n n!\alpha!/(n+\alpha)!$ and $R^\alpha_{m,n}(0) = 0$, $m\neq n$ (Wünsche [46, equation (2.9)]). Thus consider

$$F(z) := \sum_{m,n\geq 0} a^{\alpha+1}_{m,n}\mathcal{I}(R^{\alpha+1}_{m,n})(z) = \sum_{m\geq 1}\sum_{n\geq 0}\frac{a^{\alpha+1}_{m-1,n}}{c_\alpha(m,n)}\left(R^\alpha_{m,n}(z) - R^\alpha_{m,n}(0)\right), \qquad z\in\mathbb{D}. \tag{3.7}$$

Since $c_\alpha(m,n)\geq 1$ for all $m\geq 1$, $n\geq 0$ and $|R^\alpha_{m,n}(0)|\leq 1$, for all m, n, we have that the series

$$\sum_{m\geq 1}\sum_{n\geq 0}\frac{a^{\alpha+1}_{m-1,n}}{c_\alpha(m,n)} \qquad \text{and} \qquad c := \sum_{m\geq 1}\sum_{n\geq 0}\frac{a^{\alpha+1}_{m-1,n}}{c_\alpha(m,n)}R^\alpha_{m,n}(0)$$

are convergent. Furthermore, since

$$\left|\frac{a^{\alpha+1}_{m-1,n}}{c_\alpha(m,n)}\left(R^\alpha_{m,n}(z)-R^\alpha_{m,n}(0)\right)\right| \le 2\frac{a^{\alpha+1}_{m-1,n}}{c_\alpha(m,n)}, \qquad m,n\ge 0, \quad z\in\mathbb{D},$$

the series in (3.7) converges uniformly in $\mathbb{D}$. On the other hand, by applying the derivation operator $\mathcal{D}_z$ term by term in (3.7), one obtains the uniformly convergent series of f. Then F is a z-primitive of f. Since $F(0)$=0, we conclude that (3.7) converges to $\mathcal{I}(f)(z)$.

We can now write

$$\mathcal{I}(f)(z) = \sum_{m,n\ge 0} b^\alpha_{m,n} R^\alpha_{m,n}(z),$$

where

$$\begin{aligned} &b^\alpha_{0,0} := -c; \qquad b^\alpha_{0,n} := 0, \qquad n\ge 1;\\ &b^\alpha_{m,n} := \frac{a^{\alpha+1}_{m-1,n}}{c_\alpha(m,n)}, \qquad m\ge 1, \quad n\ge 0; \end{aligned} \tag{3.8}$$

and $\sum_{m,n\ge 0} b^\alpha_{m,n} < \infty$. Now we can write

$$c+\mathcal{I}f(z) = \sum_{m,n\ge 0} \widehat{b}^\alpha_{m,n} R^\alpha_{m,n}(z),$$

where

$$\widehat{b}^\alpha_{0,0} := 0; \qquad \widehat{b}^\alpha_{0,n} := b^\alpha_{0,n}, \qquad n\ge 1 \qquad \text{and} \qquad \widehat{b}^\alpha_{m,n} := b^\alpha_{m,n}, \qquad m\ge 1, \quad n\ge 0 \tag{3.9}$$

are nonnegative constants and $\sum_{m,n\ge 0} \widehat{b}^\alpha_{m,n} < \infty$.

Equations (3.8) and (3.9) mean that the coefficients $\left\{\widehat{b}^\alpha_{m,n}\right\}$ are obtained from the $\{a^{\alpha+1}_{m,n}\}$, by translating in the first index, adding the new coefficients $\widehat{b}^\alpha_{0,n}=0$, and dividing by the positive constants $\{c_\alpha(m,n)\}$.

Hence, applying Theorem 2.1(1) again, we have that $c+\mathcal{I}f$ belongs to the class $\Psi(\Omega_{2q})$.

For the item (ii), it is enough to observe that the $(2q{+}2)$-complex Schoenberg coefficients $a^{\alpha+1}_{m,n}$ of f satisfy (2.14) by the assumption $f\in\Psi^+(\Omega_{2q+2})$, then, as a consequence of (3.8), (3.9), also the $(2q)$-complex Schoenberg coefficients $\widehat{b}^\alpha_{m,n}$ of $c{+}\mathcal{I}f$ satisfy (2.14), implying $c{+}\mathcal{I}f\in\Psi^+(\Omega_{2q})$.

For the operator $\overline{\mathcal{I}}$, one uses the relation

$$\overline{\mathcal{I}}(R^{\alpha+1}_{m,n-1})(z) = \frac{1}{c_\alpha(n,m)}\left(R^\alpha_{m,n}(z)-R^\alpha_{m,n}(0)\right),$$

and follows the same arguments. In fact, the $(2q)$-complex Schoenberg coefficients of $C+\overline{\mathcal{I}}f$ are given by

$$\begin{aligned} &\check{b}^\alpha_{0,0} := C - \sum_{\mu\ge 1}\sum_{\nu\ge 0} \frac{a^{\alpha+1}_{\mu,\nu-1}}{c_\alpha(\nu,\mu)} R^\alpha_{\mu,\nu}(0); \qquad \check{b}^\alpha_{m,0} := 0, \qquad m\ge 1;\\ &\check{b}^\alpha_{m,n} := \frac{a^{\alpha+1}_{m,n-1}}{c_\alpha(m,n)}, \qquad m\ge 0, \quad n\ge 1. \end{aligned}$$

■

Proof of Counterexample 1.3. Let us denote by $a^{q-2}_{m,n}(g)$ the $(2q)$-complex Schoenberg coefficients of a positive definite function g. Theorem 2.1(2) is required.

(i) For a function f as in the statement, we have $\mathcal{D}_x f = \mathcal{D}_z f$ and

$$\big\{m-n\colon a^{q-1}_{m,n}(\mathcal{D}_z f) > 0\big\} = \big\{m-n\colon a^{q-2}_{m,n}(f) > 0\big\} = \mathbb{Z}_+.$$

Hence the above set intercepts every arithmetic progression in $\mathbb{Z}$, that is $f \in \Psi^+(\Omega_{2q})$ and $\mathcal{D}_z f, D_x f \in \Psi^+(\Omega_{2q+2})$. However, $\mathcal{D}_{\overline{z}} f \equiv 0$, so that $\mathcal{D}_{\overline{z}} f \notin \Psi^+(\Omega_{2q+2})$.

(ii) Analogous to (i).

(iii) For a function f as in the statement, we have

$$\big\{m-n\colon a^{q-2}_{m,n}(f) > 0,\, m,n \geq 0\big\} = \left(\bigcup_{j=2}^{5} 5\mathbb{Z}_+ + j\right) \cup (-5\mathbb{Z}_+ - 4),$$

which intercepts every arithmetic progression in $\mathbb{Z}$ and then $f \in \Psi^+(\Omega_{2q})$. However

$$\big\{m-n\colon a^{q-1}_{m,n}(\mathcal{D}_z f) > 0,\, m,n \geq 0\big\} = \mathbb{Z}_+ \setminus 5\mathbb{Z}$$

and

$$\big\{m-n\colon a^{q-1}_{m,n}(\mathcal{D}_{\overline{z}} f) > 0,\, m,n \geq 0\big\} = -5\mathbb{Z}_+ - 3,$$

that is, $\mathcal{D}_z f, \mathcal{D}_{\overline{z}} f \notin \Psi^+(\Omega_{2q+2})$. To see that $\mathcal{D}_x f \notin \Psi(\Omega_{2q+2})$, note that $\{m-n\colon a^{q-1}_{m,n}(\mathcal{D}_x f) > 0$, $m,n \geq 0\}$ is the union of the previous two sets, so it does not intersect the progression $5\mathbb{Z}$. ■

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