The Quaternions and Bott Periodicity Are Quantum Hamiltonian Reductions*

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Abstract. We show that the Morita equivalences $\mathrm{Cliff}(4) \simeq \mathbb{H}$, $\mathrm{Cliff}(7) \simeq \mathrm{Cliff}(-1)$, and $\mathrm{Cliff}(8) \simeq \mathbb{R}$ arise from quantizing the Hamiltonian reductions $\mathbb{R}^{0|4}//\mathrm{Spin}(3)$, $\mathbb{R}^{0|7}//G_2$, and $\mathbb{R}^{0|8}//\mathrm{Spin}(7)$, respectively.

Key words: Clifford algebras; quaternions; Bott periodicity; Morita equivalence; quantum Hamiltonian reduction; super symplectic geometry

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This note provides (super) symplectic origins for the quaternion algebra $\mathbb{H}$ and for the eightfold "Bott periodicity" of Clifford algebras (due originally to Cartan [3]) in terms of quantum Hamiltonian reduction. Clifford algebras arise in symplectic supergeometry as the Weyl (aka canonical commutation) algebras of purely-odd symplectic supermanifolds $\mathbb{R}^{0|n}$. As we explain, Hamiltonian reductions quantize to bimodules, which are often Morita equivalences. In particular, we will show that the well-known Morita equivalence $\mathbb{H} \simeq \mathrm{Cliff}(4)$ is the quantization of the Hamiltonian reduction $\mathbb{R}^{0|4}//\mathrm{Spin}(3)$, where $\mathrm{Spin}(3) = \mathrm{SU}(2)$ acts on $\mathbb{R}^{0|4}$ as the underlying real module of the defining action of $\mathrm{SU}(2)$ on $\mathbb{C}^2$, and that the reduction $\mathbb{R}^{0|8}//\mathrm{Spin}(7)$ coming from the spin representation quantizes to the "Bott periodicity" Morita equivalence $\mathrm{Cliff}(8) \simeq \mathbb{R}$. We also show that the Morita equivalence $\mathrm{Cliff}(7) \simeq \mathrm{Cliff}(-1)$ arises from the Hamiltonian reduction $\mathbb{R}^{0|7}//G_2$, where $G_2 \subseteq \mathrm{SO}(7)$ is the exceptional Lie group of automorphisms of the octonion algebra $\mathbb{O}$.

1 Symplectic supermanifolds and Clifford algebras

A *superalgebra* is a $\mathbb{Z}/2$-graded associative algebra (meaning, in particular, that the multiplication adds degree modulo 2); morphisms are grading-preserving. A *supermodule* is a $\mathbb{Z}/2$-graded module. If M is a left A-supermodule, the algebra $\mathrm{End}_A(M)$ of all A-linear endomorphisms of M is naturally a superalgebra acting on M from the right (with multiplication $fg = g \circ f$). Two superalgebras A and B are *super Morita equivalent* if there are $\mathbb{Z}/2$-graded bimodules ${}_AM_B$ and ${}_BN_A$ with grading-preserving bimodule isomorphisms $M \otimes_B N \cong A$ and $N \otimes_A M \cong B$. We will generally suppress the word "super": for example, "module" and "Morita equivalence" will henceforth always be meant in the super sense.

A superalgebra A is *commutative* if for homogeneous elements x and y (with degrees $|x|$ and $|y|$), $yx = (-1)^{|x|\cdot|y|}xy$. Note in particular that for an odd element x in a commutative superalgebra, $x^2 = -x^2$, and so $x^2 = 0$. By definition, *odd n-dimensional space* $\mathbb{R}^{0|n}$ is the "spectrum" of the commutative superalgebra $\mathcal{O}(\mathbb{R}^{0|n}) = \mathbb{R}[x_1, \dots, x_n]$ where the *coordinate functions* $x_1, \dots, x_n$ are odd, and $\mathbb{R}[\dots]$ denotes the free commutative superalgebra on "$\dots$".

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Thus $\mathcal{O}(\mathbb{R}^{0|n}) \cong \bigwedge^\bullet \mathbb{R}^n$. In general, a *supermanifold* is a "space" that looks locally like $\mathbb{R}^{m|n} = \mathbb{R}^m \times \mathbb{R}^{0|n}$. See [5] for details on superalgebras and supermanifolds.

A *symplectic structure* on a supermanifold is an even nondegenerate closed de Rham 2-form. De Rham forms can be defined for commutative superalgebras just like for commutative algebras, but behave differently in one important way: if x is an odd coordinate, then $\mathrm{d}x$ is even, and so $\mathrm{d}x \wedge \mathrm{d}x \neq 0$, and if x and y are both odd, then $\mathrm{d}x \wedge \mathrm{d}y = \mathrm{d}y \wedge \mathrm{d}x$ with no sign. A side effect of this is that symplectic structures on odd manifolds behave somewhat like metrics on even manifolds.

Equip $\mathbb{R}^{0|n}$ with the positive-definite symplectic form $\omega = \sum_i \frac{(\mathrm{d}x_i)^2}{2}$. The corresponding Poisson structure on $\mathbb{R}^{0|n}$ is given by the Poisson brackets $\{x_i, x_j\} = -2\delta_{ij}$. (The sign depends on an essentially-arbitrary choice of convention for inverse matrices in superalgebra.) The symplectic form ω on $\mathbb{R}^{0|n}$ is translation-invariant and so admits a *canonical quantization* to the Weyl algebra $\mathcal{W}(\mathbb{R}^{0|n}) = \mathbb{R}\langle x_1, \dots, x_n\rangle/([x_i, x_j] = \{x_i, x_j\})$, where by definition in a superalgebra the commutator is defined on homogeneous elements by $[x, y] = xy - (-1)^{|x|\cdot|y|} yx$. Thus $\mathcal{W}(\mathbb{R}^{0|n})$ is the *Clifford algebra* $\mathrm{Cliff}(n) = \mathbb{R}\langle x_1, \dots, x_n\rangle/(x_i^2 = -1, x_i x_j = -x_j x_i \text{ for } i \neq j)$ with its usual $\mathbb{Z}/2$-grading in which all generators x_i are odd. The Weyl algebra of $\mathbb{R}^{0|n}$ equipped with symplectic form $-\omega$ is $\mathrm{Cliff}(-n) = \mathbb{R}\langle x_1, \dots, x_n\rangle/(x_i^2 = 1, x_i x_j = -x_j x_i)$.

2 Quantum Hamiltonian reduction

A *moment map* for the action of a super Lie group G on a symplectic supermanifold M is a map $\mu\colon M \to \mathfrak{g}^* = \mathrm{Lie}(G)^*$ of Poisson supermanifolds such that the infinitesimal action of an element $a \in \mathfrak{g}$ is given by the Hamiltonian vector field for the function $m \mapsto \langle \mu(m), a\rangle$, where $\langle\,,\,\rangle$ denotes the pairing of the vector space $\mathfrak{g}$ with its dual; such data is equivalent to a Lie algebra map $\mu^*\colon \mathfrak{g} \to \mathcal{O}(M)$, where the latter is treated as a super Lie algebra with its Poisson bracket. When certain cohomology groups of M and $\mathfrak{g}$ vanish, μ exists and is unique. The *Hamiltonian reduction* $M//G$ of this data is the quotient space $\mu^{-1}(0)/G$. This can be defined in the super case via its algebra of functions $(\mathcal{O}(M)/\langle \mu^*\mathfrak{g}\rangle)^G$, where $\langle \mu^*\mathfrak{g}\rangle$ denotes the ideal generated by the image of μ^*. As Marsden and Weinstein explained in the even case [7], when 0 is a regular value of μ and the action of G on $\mu^{-1}(0)$ is free and proper, the manifold $M//G$ is naturally symplectic. The natural maps $\mu^{-1}(0) \hookrightarrow M$ and $\mu^{-1}(0) \twoheadrightarrow M//G$ are together a Lagrangian correspondence between M and $M//G$. Super Hamiltonian reduction can be cleanly expressed as an example of coisotropic reduction of super Poisson algebras [4, 8].

Suppose that G acts instead on an associative superalgebra A. A *comoment map* is a Lie algebra map $\mu^*\colon \mathfrak{g} \to A$, where A is treated as a super Lie algebra with its commutator bracket, such that the infinitesimal action of $a \in \mathfrak{g}$ is given by the inner derivation $[\mu^*(a), -]$. Corresponding to the zero section $\mu^{-1}(0)$ is the quotient module $A/\langle \mu^*\mathfrak{g}\rangle$, where $\langle \mu^*\mathfrak{g}\rangle$ denotes the left ideal generated by the image of μ^*. Corresponding to the quotient $M//G = \mu^{-1}(0)/G$ is the *quantum Hamiltonian reduction* $A//G = (A/\langle \mu^*\mathfrak{g}\rangle)^G$. This is naturally an algebra because it is isomorphic to $\mathrm{End}_A(A/\langle \mu^*\mathfrak{g}\rangle)$. When G is compact, $A//G \cong A^G/\big(A^G \cap \langle \mu^*\mathfrak{g}\rangle\big)$, where A^G denotes the G-invariant subalgebra of A. From this perspective, the algebra structure on $A//G$ arises because, although $\langle \mu^*\mathfrak{g}\rangle$ is merely a left ideal in A, its intersection with A^G is a two-sided ideal, as $\mu^*\mathfrak{g}$ is central in A^G. The module $A/\langle \mu^*\mathfrak{g}\rangle$ is by construction a bimodule between A and $A//G$.

Example 1. Suppose that M is a linear symplectic supermanifold and $C \subseteq M$ is a coisotropic submanifold cut out by linear equations $r_1 = \dots = r_{p+q} = 0$, where $r_1, \dots, r_p$ are even and $r_{p+1}, \dots, r_{p+q}$ are odd. The Hamiltonian flows for $r_1, \dots, r_{p+q}$ define an action on M of the abelian Lie supergroup $\mathbb{R}^{p|q}$. Let $C^\perp \subseteq C$ denote the symplectic orthogonal to C. The Hamiltonian reduction $M//\mathbb{R}^{p|q}$ is then canonically linearly symplectomorphic to $C/C^\perp$.

Since M is linear, it admits a canonical quantization to the Weyl algebra $\mathcal{W}(M) = \mathcal{T}(M^*)/([a,b] = \{a, b\}, a, b \in M^*)$. The quotient $\mathcal{W}(C) = \mathcal{W}(M)/\langle r_1, \dots, r_c\rangle$ is the canonical quantization of C, and $\mathcal{W}(M)//\mathbb{R}^{p|q} \cong \mathcal{W}(C/C^\perp)$. In the purely-odd case, which is the only case of concern in this paper, $\mathcal{W}(C)$ is a Morita equivalence between $\mathcal{W}(M)$ and $\mathcal{W}(C/C^\perp)$: it suffices to consider the case $M = \mathbb{R}^{0|2}$ with "split" symplectic form $\frac{(\mathrm{d}x)^2}{2} - \frac{(\mathrm{d}y)^2}{2}$ and Lagrangian $C \cong \mathbb{R}^{0|1}$ spanned by the vector $(1,1)$; then $\mathcal{W}(M) \cong \mathrm{Mat}(1|1)$ is the algebra of 2×2 matrices in which $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)$ are even and $\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$ are odd, and $\mathcal{W}(C)$ is the defining $(1|1)$-dimensional module.

(When there are even coordinates, $\mathcal{W}(C)$ is not a Morita equivalence. The Stone–von Neumann theorem can be understood as saying that for purely even M, $\mathcal{W}(C)$ becomes a Morita equivalence after appropriate functional analytic completions. The mixed case can be handled by decomposing M and C into even and odd parts.)

In particular, linear Lagrangians provide Morita equivalences $\mathcal{W}(M) \simeq \mathbb{R}$. This does not explain why $\mathrm{Cliff}(8) = \mathcal{W}(\mathbb{R}^{0|8}) \simeq \mathbb{R}$, because the positive-definiteness of the symplectic form prevents $\mathbb{R}^{0|n}$ from admitting Lagrangian sub-supermanifolds, linear or not.

Lemma 1. *If the Hamiltonian reduction* $\mathrm{Cliff}(n)//G$ *is not the zero algebra, then* $\mathrm{Cliff}(n)/\langle \mu^*\mathfrak{g}\rangle$ *is a Morita equivalence between* $\mathrm{Cliff}(n)$ *and* $\mathrm{Cliff}(n)//G$.

Proof. For any superalgebra A, an A-module X is a Morita equivalence between A and $\mathrm{End}_A(X)$ if and only if X is a finitely-generated projective generator of the supercategory of A-modules. The holomorphic symplectic supermanifold $\mathbb{C}^{0|n} = \mathbb{R}^{0|n} \otimes \mathbb{C}$ admits a linear Lagrangian L if n is even and an $(n+1)/2$-dimensional linear coisotropic C if n is odd. Via Example 1, these linear coisotropics provide Morita equivalences $\mathrm{Cliff}(n)\otimes\mathbb{C} \simeq \mathbb{C} = \mathcal{W}(L/L^\perp)$ or $\mathrm{Cliff}(1)\otimes\mathbb{C} = \mathcal{W}(C/C^\perp)$. For $\mathbb{C}$ and $\mathrm{Cliff}(1)\otimes\mathbb{C}$, any non-zero finitely-generated module is a projective generator. But "non-zero", "finitely-generated", and "projective" are Morita-invariant notions, so these properties hold also for $\mathrm{Cliff}(n)\otimes\mathbb{C}$ and hence for $\mathrm{Cliff}(n)$. ■

3 Cliff(4) and $\mathbb{H}$

Corresponding to the exceptional isomorphism $\mathrm{SO}(4) \cong \mathrm{Spin}(3)\times_{\mathbb{Z}/2}\mathrm{Spin}(3)$ are two commuting actions of $\mathrm{Spin}(3)$ on $\mathbb{R}^{0|4}$ by linear symplectic automorphisms. (Odd symplectic groups are even orthogonal groups; metaplectic groups correspond to spin groups.) Denote the coordinates on $\mathbb{R}^{0|4}$ by $\{w,x,y,z\}$ and the bases for two copies of $\mathfrak{so}(3)$ by $\{a_+,b_+,c_+\}$ and $\{a_-,b_-,c_-\}$, normalized so that their brackets are $[a_\pm,b_\pm] = \pm 2c_\pm$, $[b_\pm,c_\pm] = \pm 2a_\pm$, $[c_\pm,a_\pm] = \pm 2b_\pm$. The comoment maps for the actions are:

$$a_\pm \mapsto \tfrac{1}{2}(wx \pm yz), \qquad b_\pm \mapsto \tfrac{1}{2}(wy \pm zx), \qquad c_\pm \mapsto \tfrac{1}{2}(wz \pm xy).$$

Together these six elements are a basis for the space of homogeneous-quadratic functions on $\mathbb{R}^{0|4}$. The quadratic Casimir for both $\mathfrak{so}(3)$s is $\theta = \pm 2a_\pm^2 = \pm 2b_\pm^2 = \pm 2c_\pm^2 = wxyz$. Completing the basis for $\mathcal{O}(\mathbb{R}^{0|4})$ are the unit 1 and xyz, wyz, wzx, and wxy. Basis vectors 1, $a_\pm$, $b_\pm$, $c_\pm$, and θ are even, and x, y, z, w, xyz, wyz, wzx, and wxy are odd.

We now consider the quantization $\mathrm{Cliff}(4) = \mathcal{W}(\mathbb{R}^{0|4})$, for which we can use the same basis $\{1,w,x,y,z,a_+,b_+,c_+,a_-,b_-,c_-,xyz,wyz,wzx,wxy,\theta\}$ (with the same grading). Note that, whereas in $\mathcal{O}(\mathbb{R}^{0|4})$ we had $a_\pm^2 = b_\pm^2 = c_\pm^2 = \pm\theta/2$, in $\mathrm{Cliff}(4)$ we have $a_\pm^2 = b_\pm^2 = c_\pm^2 = \frac{1}{2}(\pm\theta - 1)$. Since the actions of $\mathrm{Spin}(3)$ on $\mathbb{R}^{0|4}$ are linear, they lift to $\mathrm{Cliff}(4)$, and the same assignments $a_+,\dots,c_-$ provide the quantum comoment maps.

Theorem 1. *The quantum Hamiltonian reduction of either of the* $\mathfrak{so}(3)$*-actions on* $\mathbb{R}^{0|4}$ *produces a Morita equivalence* $\mathrm{Cliff}(4) \simeq \mathbb{H} = \mathbb{R}\langle i,j,k\rangle/(i^2 = j^2 = k^2 = ijk = -1)$ (*where the quaternion algebra* $\mathbb{H}$ *is purely even*).

Proof. There is a manifest symmetry interchanging the two $\mathfrak{so}(3)$-actions; we will work with the action of the a_-, b_-, and c_-. It is not hard to see that $[a_-,-]$, $[b_-,-]$, and $[c_-,-]$ preserve polynomial degree. We have observed already that the quadratic elements a_+, b_+, c_+ commute with the generators a_-, b_-, c_- of the action, as well as with $\theta = wxyz$. The subspace of Cliff(4) spanned by $\{a_-, b_-, c_-\}$ is a submodule for the action of $\{a_-, b_-, c_-\}$ isomorphic to the adjoint action. The subspaces spanned by $\{w, x, y, z\}$ and $\{xyz, wyz, wzx, wxy\}$ are each isomorphic to the underlying real module of the defining module of $\mathfrak{su}(2)$. Thus a basis for the $\mathfrak{so}(3)$-fixed subalgebra $\mathrm{Cliff}(4)^{\mathrm{Spin}(3)}$ is given by the five even elements $\{1, a_+, b_+, c_+, \theta\}$.

The left ideal $\langle \mu^* \mathfrak{so}(3) \rangle$ in Cliff(4) generated by $\{a_-, b_-, c_-\}$ is eight-dimensional with basis $\{a_-, b_-, c_-, w - xyz, x + wyz, y + wzx, z + wxy, \theta + 1\}$. This ideal intersects $\mathrm{Cliff}(4)^{\mathrm{Spin}(3)}$ only in the one-dimensional space spanned by $\theta + 1$. It follows that in the quotient $\mathrm{Cliff}(4)^{\mathrm{Spin}(3)}/\big(\mathrm{Cliff}(4)^{\mathrm{Spin}(3)} \cap \langle \mu^* \mathfrak{so}(3) \rangle\big)$ we have $a_+^2 = \frac{1}{2}(\theta - 1) \equiv \frac{1}{2}(-2) = -1$, and so the map $(a_+, b_+, c_+) \mapsto (i, j, k)$ identifies the quantum Hamiltonian reduction Cliff(4)//Spin(3) with the quaternion algebra $\mathbb{H}$. Lemma 1 completes the proof. ■

Theorem 1 suggests that the "classical limit" of $\mathbb{H}$ is the Hamiltonian reduction $\mathbb{R}^{0|4}//\mathrm{Spin}(3)$. Since 0 is not a regular value of the classical moment map, $\mathbb{R}^{0|4}//\mathrm{Spin}(3)$ is not a supermanifold. It does make sense as an affine super scheme: its algebra of functions is the purely even Poisson algebra $\mathbb{R}[a, b, c]/(a^2 = b^2 = c^2 = ab = bc = ca = 0)$ with Poisson brackets $\{a, b\} = 2c$, $\{b, c\} = 2a$, and $\{c, a\} = 2b$. Thus the "classical limit" of $\mathbb{H}$ is the first-order neighborhood of 0 in $\mathfrak{so}(3)^*$.

4 Cliff(7) and G_2

The 14-dimensional exceptional Lie group G_2 is the subgroup of SO(7) preserving the alternating 3-form ϵ on $\mathbb{R}^7$ defined by identifying $\mathbb{R}^7$ with the pure-imaginary octonions and setting $\epsilon(a, b, c) = (ab)c - a(bc) \in \mathbb{R}$ [2]. Since SO(7) acts by linear symplectic automorphisms of $\mathbb{R}^{0|7}$, we get an induced symplectic action of G_2.

Theorem 2. *The quantum Hamiltonian reduction* Cliff(7)//G_2 *provides the Morita equivalence* Cliff(7) $\simeq$ Cliff(−1).

Proof. By Lemma 1, it suffices to compute Cliff(7)//G_2. As in Theorem 1, the Poincaré–Birkoff–Witt isomorphism $\mathcal{O}(\mathbb{R}^{0|7}) = \bigwedge^\bullet(\mathbb{R}^7) \cong \mathrm{Cliff}(7)$ is SO(7)-equivariant by the functoriality of the Weyl algebra construction. The G_2-fixed algebra has as its basis a set of the form $\{1, \epsilon, \bar{\epsilon}, \theta\}$, where ϵ is the cubic 3-form defining G_2, $\bar{\epsilon}$ is its dual quartic, and θ is the generator of $\bigwedge^7 \mathbb{R}^7$. In Cliff(7), $\bar{\epsilon} = \theta\epsilon$ and $\theta^2 = 1$.

An explicit presentation of the action of $\mathrm{Lie}(G_2) = \mathfrak{g}_2$ is given in [1] as follows. Denote the coordinates on $\mathbb{R}^{0|7}$ by $x_1, x_2, \ldots, x_7$. The cubic function ϵ is

$$\epsilon = x_1x_2x_3 + x_1x_4x_5 + x_1x_6x_7 + x_2x_4x_6 + x_2x_7x_5 + x_3x_7x_4 + x_3x_6x_5.$$

Consider the quadratic functions $e_1, \ldots, e_7$ defined by $e_i = \frac{\partial}{\partial x_i}\epsilon$. For example, $e_1 = x_2x_3 + x_4x_5 + x_6x_7$. Orthogonal to each e_i is a two-dimensional vector space of commuting quadratics given by the differences of the monomials in e_i. For example, orthogonal to e_1 are $x_2x_3 - x_4x_5$, $x_4x_5 - x_6x_7$, and their sum $x_2x_3 - x_6x_7$. A basis for the image of $\mathfrak{g}_2$ under μ^* is given by choosing for each $i = 1, \ldots, 7$ two quadratics orthogonal to e_i. For example:

$$\begin{array}{lllll} x_2x_3 - x_4x_5, & x_4x_5 - x_6x_7, & x_3x_1 - x_4x_6, & x_4x_6 - x_7x_5, & x_1x_2 - x_7x_4, \\ x_7x_4 - x_6x_5, & x_5x_1 - x_6x_2, & x_6x_2 - x_3x_7, & x_1x_4 - x_2x_7, & x_2x_7 - x_3x_6, \\ x_7x_1 - x_2x_4, & x_2x_4 - x_5x_3, & x_1x_6 - x_5x_2, & x_5x_2 - x_4x_3. & \end{array}$$

We wish to compute $\mathrm{End}_{\mathrm{Cliff}(7)}(\mathrm{Cliff}(7)/\langle \mu^*\mathfrak{g}_2 \rangle) = \mathrm{Cliff}(7)^{G_2}/(\langle \mu^*\mathfrak{g}_2 \rangle \cap \mathrm{Cliff}(7)^{G_2})$, where $\langle \mu^*\mathfrak{g}_2 \rangle$ is the left ideal generated by these 14 elements.

Note that $x_1x_4x_5x_6x_7(x_2x_3 - x_4x_5) = \theta + x_1x_6x_7$, where $\theta = x_1x_2\cdots x_7 \in \mathrm{Cliff}(7)^{G_2}$. The numerics of the second summand are: $x_2x_3 - x_4x_5$ was orthogonal to e_1; x_6x_7 is the unused monomial in e_1. Similarly, for each monomial μ in the cubic ϵ one can find $\theta + \mu \in \langle \mu^*\mathfrak{g}_2\rangle$, and summing shows that $7\theta + \epsilon \in \langle \mu^*\mathfrak{g}_2\rangle \cap \mathrm{Cliff}(7)^{G_2}$; hence also $7 + \bar\epsilon \in \langle \mu^*\mathfrak{g}_2\rangle \cap \mathrm{Cliff}(7)^{G_2}$. It follows that $\mathrm{Cliff}(7)^{G_2}/(\langle \mu^*\mathfrak{g}_2\rangle \cap \mathrm{Cliff}(7)^{G_2})$ is a quotient of the copy of $\mathrm{Cliff}(-1)$ spanned by the classes of 1 and θ.

Finally, for any basis element $\alpha \in \mu^*\mathfrak{g}_2$, we have $\alpha(\theta - \epsilon) = 0$, from which it follows that $1 \notin \langle \mu^*\mathfrak{g}_2\rangle$. The ideal cannot mix even and odd terms without setting both to 0, and so we find $\mathrm{Cliff}(7)^{G_2}/(\langle \mu^*\mathfrak{g}_2\rangle \cap \mathrm{Cliff}(7)^{G_2}) \cong \mathrm{Cliff}(-1)$. ■

5 Spin(7) and Bott periodicity

We conclude by providing a Hamiltonian reduction whose quantization is the famous "Bott periodicity" equivalence $\mathrm{Cliff}(8) \simeq \mathbb{R}$. The irreducible real spin representations of all four groups Spin(5), Spin(6), Spin(7), and Spin(8) are eight-real-dimensional. The reduction Cliff(8)//Spin(8) vanishes since the image of the comoment map consists of all quadratic elements of Cliff(8), including some which are invertible, and so the Spin(8)-action does not induce a Morita equivalence. The reader is invited to compute Cliff(8)//Spin(5) and Cliff(8)//Spin(6). We will show:

Theorem 3. *$\mathrm{Cliff}(8)//\mathrm{Spin}(7) \cong \mathbb{R}$.*

By Lemma 1, Theorem 3 establishes that the cyclic module $\mathrm{Cliff}(8)/\langle \mu^*\mathfrak{so}(7)\rangle$ is a Morita equivalence between Cliff(8) and $\mathbb{R}$.

Proof. The following construction of Spin(7), and its eight-dimensional spin representation, are developed in [6]. Consider the octonion algebra $\mathbb{O}$ and the 4-form $\phi \in \bigwedge^4 \mathbb{O}^*$ defined by $\phi(a, b, c, d) = \langle a, b \times c \times d\rangle$, where the triple cross product is by definition $b \times c \times d = \frac{1}{2}\big(b(\bar c d) - d(\bar c b)\big)$ and $\bar c$ is the octonionic conjugate of c. Then Spin(7) is precisely the subgroup of SO(8) fixing ϕ. In terms of coordinates $x_1, x_2, \ldots, x_8$ on $\mathbb{R}^{0|8}$, ϕ corresponds to the function

$$\begin{aligned}\phi = {}& x_{1234} + x_{1256} + x_{1278} + x_{1357} - x_{1368} - x_{1458} - x_{1467}\\ &+ x_{5678} + x_{3478} + x_{3456} + x_{2468} - x_{2457} - x_{2367} - x_{2358},\end{aligned}$$

where we have abbreviated $x_{ij\ldots k} = x_ix_j\cdots x_k$. Let $\theta = x_{12345678}$. The fixed algebra $\mathrm{Cliff}(8)^{\mathrm{Spin}(7)}$ has basis $\{1, \phi, \theta\}$ and multiplication $\theta^2 = 1$, $\theta\phi = \phi\theta = \phi$, and $\phi^2 = 14\theta + 14 - 12\phi$.

The image $\mu^*\mathfrak{so}(7) \subseteq \bigwedge^2 \mathbb{R}^8 \subseteq \mathrm{Cliff}(8)$ of the comoment map is spanned by quadratic elements of the form $\alpha = x_{ij} \pm x_{kl}$ such that $-\frac{1}{2}\alpha^2 - 1 = \mp x_{ijkl}$ is (with the given sign) a monomial in ϕ. (There are $3 \cdot 14$ such αs; given $\{i, j\}$, there are three αs that include the monomial x_{ij} and three that are disjoint from $\{i, j\}$, and these six span a three-dimensional space; this counts correctly the 21-dimensional space $\mathfrak{so}(7)$.) It follows that $\phi+14$, being a sum of 14 terms of the form $-\frac{1}{2}\alpha^2$, is in the ideal $\langle \mu^*\mathfrak{so}(7)\rangle$, and so Cliff(8)//Spin(7) is a quotient of $\mathbb{R}\{1, \phi, \theta\}/\langle \phi+14\rangle \cong \mathbb{R}$.

Each $\alpha = x_{ij} \pm x_{kl} \in \mu^*\mathfrak{so}(7)$ determines a splitting $\phi = x_{ijkl}(1 + \theta) + \kappa + \lambda$ where κ is a sum of four quartic monomials each of which has indices containing either $\{i, j\}$ or $\{k, l\}$ but not both, and λ is a sum of eight quartic monomials each of which has indices intersecting the sets $\{i, j\}$ and $\{k, l\}$ at one element each. For example, when $\alpha = x_{13} - x_{57}$, we have

$$\begin{aligned}\phi = {}& \underbrace{x_{1357} + x_{2468}}_{x_{ijkl}(1+\theta)} + \underbrace{x_{1234} - x_{1368} + x_{5678} - x_{2457}}_{\kappa}\\ &+ \underbrace{x_{1256} + x_{1278} - x_{1458} - x_{1467} + x_{3478} + x_{3456} - x_{2367} - x_{2358}}_{\lambda}.\end{aligned}$$

We see that $[\alpha, x_{ijkl}(1+\theta)] = 0$ and $[\alpha, \kappa] = 0$. Since we know that $[\alpha, \phi] = 0$, we find $[\alpha, \lambda] = 0$ as well. Suppose β is a quadratic monomial and ν a quartic monomial such that the indices in β and ν overlap at one element. Then $\beta\nu = \frac{1}{2}[\beta, \nu]$, from which it follows that $\alpha\lambda = \frac{1}{2}[\alpha, \lambda] = 0$. It's also clear that $\alpha\kappa = 0$, since κ factors as $(x_{ij} \mp x_{kl})(\dots)$ where $\alpha = x_{ij} \pm x_{kl}$ and $(\dots)$ is a sum of two quadratic monomials that have no overlap with α. Finally, $\alpha x_{ijkl}(1+\theta) = \alpha(-1-\theta)$. All together, we find:

$$\alpha(\phi + 1 + \theta) = \alpha\big(x_{ijkl}(1+\theta) + \kappa + \lambda + (1+\theta)\big) = \alpha(-1-\theta) + 0 + 0 + \alpha(1+\theta) = 0.$$

It follows that $1 \notin \langle \mu^* \mathfrak{so}(7) \rangle$, and so $\mathrm{Cliff}(8)//\mathrm{Spin}(7) \ncong 0$, completing the proof. ■

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