

Nonstandard Deformed Oscillators from q - and p, q -Deformations of Heisenberg Algebra

Alexandre M. GAVRILIK [†] and Ivan I. KACHURIK ^{†‡}

[†] *Bogolyubov Institute for Theoretical Physics, 14-b Metrolohichna Str., Kyiv, 03680 Ukraine*
E-mail: omgavr@bitp.kiev.ua, kachurik@bitp.kiev.ua

[‡] *Khmelnytskyi National University, 11 Instytutska Str., Khmelnytskyi, 29016 Ukraine*

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Abstract. For the two-parameter p, q -deformed Heisenberg algebra introduced recently and in which, instead of usual commutator of X and P in the l.h.s. of basic relation $[X, P] = i\hbar$, one uses the p, q -commutator, we established interesting properties. Most important is the realizability of the p, q -deformed Heisenberg algebra by means of the appropriate deformed oscillator algebra. Another uncovered property is special extension of the usual mutual Hermitian conjugation of the creation and annihilation operators, namely the so-called $\eta(N)$ -pseudo-Hermitian conjugation rule, along with the related $\eta(N)$ -pseudo-Hermiticity property of the position or momentum operators. In this work, we present some new solutions of the realization problem yielding new (nonstandard) deformed oscillators, and show their inequivalence to the earlier known solution and the respective deformed oscillator algebra, in particular what concerns ground state energy.

Key words: deformed Heisenberg algebra; position and momentum operators; deformed oscillators; structure function of deformation; deformation parameters; ground state energy

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1 Introduction

The obtaining and analysis of modified/generalized versions of the famous Heisenberg uncertainty relation or principle corresponds to diverse generalizations of the standard Heisenberg algebra (HA) with basic relation $[X, P] = i\hbar$ for the position and momentum operators. This direction of research is under development for more than three decades (with some early works quoted as [7, 12, 28, 29, 34, 37, 38]), and still remains to be a hot topic, see, e.g., [5, 8, 13, 16, 17, 32] for more recent papers. Physically, deformations of the Heisenberg algebra and the related generalized uncertainty relation find their motivation in quantum gravity, in string theory, non-commutative geometry etc. An overview of different approaches can be found, e.g., in [14, 27]. What concerns the particular existing variants of deformed Heisenberg algebra (DHA), we have to note that predominantly there were DHAs with deformed r.h.s. of the basic relation for X and P that involved a function of momentum, or some function of the Hamiltonian. However, within more exotic approach followed in [12, 38], deformation of HA emerged in the l.h.s. of basic relation, by using “ q -commutator” instead of the usual commutator.

Recently, a hybrid so-called “two-sided” deformation of HA has been introduced [16]. In that DHA, first, the commutator in the l.h.s. of its defining relation is replaced with q -commutator like in [12, 38] or with q, p -commutator as in [16]. In addition, the r.h.s. of basic relation is modified by an extra term involving Hamiltonian times a pre-factor denoted as μ . For this two-sided or q, p, μ -deformed DHA it was shown that, similarly to the “left-handed” q - and q, p -deformed ones, this more general deformed algebra of the operators X and P can as well be realized by (i.e., mapped onto) definite nonstandard deformed oscillator algebra (DOA).

Unusual thing concerning just the q, p, μ -DHA is that the deformation factor (parameter) μ in front of the Hamiltonian H in the r.h.s. of basic relation becomes *inevitably* depending not only on the deformation parameter(s) q or q, p of the q - or q, p -commutator in the l.h.s., but also on the particle number operator N , see [16].

The explicit connection of the DHA with the respective DOA discloses an unexpected property [17] of the operators involved. We mean the option that the creation and annihilation operators a^+ and a^- may be not the (mutual) Hermitian conjugates, but the so-called η -pseudo-Hermitian conjugates of one another, with $\eta \equiv \eta(N)$ – certain operator function of the number operator N . Accordingly, the position operator X or the momentum operator P , or both, were shown to possess not the customary Hermiticity, but the property of being $\eta(N)$ -pseudo-Hermitian [17].

The latter fact was uncovered in the framework of the particular DOA presented explicitly through its structure function of deformation (DSF) $\Phi(N)$. That was found in closed form in [16]. However, further analysis shows that this solution given by the DSF $\Phi(N)$ is not unique, and other possibilities may exist. The goal of this paper is to present other admissible solutions which as well provide realization of DHA by some (different) DOA. After presenting the new solutions, we demonstrate that these are inequivalent to the first one, found in [16] and given by $\Phi(N)$. As a major feature, they show differing values of the ground state energy. The latter can be either lower or higher than the familiar ground state energy $E_0 = \frac{1}{2}\hbar\omega$ of the usual quantum harmonic oscillator¹.

2 q -deformed Heisenberg algebra

An alternative approach to deform the Heisenberg algebra was considered in the literature. Namely, in [12, 38] different approach of deforming HA was studied such that a deformation is introduced in the commutator in the l.h.s. of basic relation (this is the q -deformed HA)

$$XP - qPX = i\hbar. \quad (2.1)$$

For convenience, in all our treatment we put $\hbar = 1$. Below, in the first subsection we mainly follow [12]. We require that the equality (2.1) is connected with special deformed oscillator algebra whose generating elements a^+ , a^- and N (the creation/annihilation operators, which are not necessarily strict conjugates of each other, and the excitation number operator) satisfy

$$[N, a^+] = a^+, \quad [N, a^-] = -a^-, \quad (2.2)$$

$$H(N)a^-a^+ - G(N)a^+a^- = 1. \quad (2.3)$$

It is meant that the operator functions $H(N)$ and $G(N)$ admit formal power series expansion.

2.1 From DHA to DOA

Like in [12] we express the position and momentum operators in terms of a^- , a^+ as

$$X \equiv f(N)a^- + g(N)a^+, \quad P \equiv i(k(N)a^+ - h(N)a^-) \quad (2.4)$$

where $f(N)$, $g(N)$, $h(N)$, $k(N)$ are some functions of the operator N .

On the base of (2.2), for any function $\mathcal{F}(N)$ possessing formal power series expansion we have

$$\mathcal{F}(N)a^\pm = a^\pm \mathcal{F}(N \pm 1), \quad [\mathcal{F}(N), a^\pm a^\mp] = 0. \quad (2.5)$$

¹Note that this same ground state energy $E_0 = \frac{1}{2}\hbar\omega$ is shared by such well-known deformed oscillators as Arik–Coon (AC) q -oscillator [3], Biedenharn–Macfarlane (BM) q -oscillator [6, 31], Tamm–Dancoff (TD) type q -oscillator [10], and the p, q -deformed oscillator [4, 9].

Proceeding like in [12] and using (2.3)–(2.5), we deduce

$$\frac{h(N+1)}{h(N)} = q \frac{f(N+1)}{f(N)}, \quad \frac{k(N-1)}{k(N)} = q \frac{g(N-1)}{g(N)} \quad (2.6)$$

as well as the expressions

$$H(N) = f(N)k(N+1) + qh(N)g(N+1), \quad (2.7)$$

$$G(N) = g(N)h(N-1) + qk(N)f(N-1). \quad (2.8)$$

The DSF $\Phi(N)$, see, e.g., [33], determines both the bilinears (cf. (2.5))

$$a^+ a^- = \Phi(N), \quad a^- a^+ = \Phi(N+1),$$

and the commutation relation

$$[a^-, a^+] = \Phi(N+1) - \Phi(N). \quad (2.9)$$

It also gives the action formulas in the $\Phi(n)$ -deformed analog of Fock space

$$N|n\rangle = n|n\rangle, \quad |n\rangle = \frac{(a^+)^n}{\sqrt{\Phi(n)!}}|0\rangle, \quad a^-|0\rangle = 0,$$

where $\Phi(n)! \equiv \Phi(n)\Phi(n-1)\cdots\Phi(2)\Phi(1)$ and, in addition, $\Phi(0)! = 1$. In that space we have

$$a^+|n\rangle = \sqrt{\Phi(n+1)}|n+1\rangle, \quad a^-|n\rangle = \sqrt{\Phi(n)}|n-1\rangle. \quad (2.10)$$

The DSF is obtainable from the above functions $H(n)$ and $G(n)$ using the recipe [33]

$$\Phi(n) = \frac{G(n-1)!}{H(n-1)!} \left(\frac{1}{H(0)} + \sum_{j=1}^{n-1} \frac{H(j-1)!}{G(j)!} \right). \quad (2.11)$$

Here the factorials are defined similarly to $\Phi(n)!$.

2.2 Solutions of the relations (2.6)

We need the solutions of (2.6) which then, using (2.7) and (2.8), yield the corresponding operator functions $G(N)$ and $H(N)$. So let us list some of them.

Solution A. This is

$$f(N) = k(N) = \frac{1}{\sqrt{2}}q^N, \quad h(N) = g(N) = \frac{1}{\sqrt{2}}q^{2N}, \quad (2.12)$$

from which

$$H(N) = \frac{1}{2}q^{2N+1}(1+q^{2N+2}), \quad G(N) = \frac{1}{2}q^{2N}(1+q^{2N-2}) = q^3H(N-2). \quad (2.13)$$

From (2.11) and (2.13) we obtain the DSF

$$\Phi_q^{(1)}(n) = \frac{2q^{-n}}{(1+q^{2n-2})(1+q^{2n})} \left(1 + \frac{q^n - q^{-n+1}}{q-1} \right) = \frac{2q^{-n}[n]_q(1+q^{-n+1})}{(1+q^{2n-2})(1+q^{2n})}, \quad (2.14)$$

where $[n]_q \equiv (1-q^n)/(1-q)$. The obtained DSF (2.14) implies that now we have, besides the relation (2.3), also the alternative form (2.9) of the commutation relation.

In addition, putting (2.12) into (2.4), for the operators X and P we obtain

$$X = \frac{1}{\sqrt{2}}(q^{2N}a^+ + q^N a^-), \quad P = \frac{i}{\sqrt{2}}(q^N a^+ - q^{2N} a^-). \quad (2.15)$$

Note that this DSF (2.14) coincides with the deformation structure function formerly found in [16] (see also [12]). However, new solutions are possible, see the cases **B–D**.

Solution B. This has the form

$$f(N) = k(N) = \frac{1}{\sqrt{2}}q^{-2N}, \quad h(N) = g(N) = \frac{1}{\sqrt{2}}q^{-N}, \quad (2.16)$$

which then yields

$$H(N) = \frac{1}{2}q^{-2N}(1 + q^{-2N-2}), \quad G(N) = \frac{1}{2}q^{-2N+1}(1 + q^{-2N+2}) = q^{-3}H(N-2). \quad (2.17)$$

Then from (2.11) and (2.17) we obtain the DSF (which casts (2.3) into the form (2.9))

$$\Phi_q^{(2)}(n) = \frac{2q^{5n-3}[n]_q(1 + q^{-n+1})}{(1 + q^{2n-2})(1 + q^{2n})} = q^{3(2n-1)}\Phi_q^{(1)}(n). \quad (2.18)$$

Moreover, putting (2.16) into (2.4), for the operators X and P we obtain

$$X = \frac{1}{\sqrt{2}}(q^{-N}a^+ + q^{-2N}a^-), \quad P = \frac{i}{\sqrt{2}}(q^{-2N}a^+ - q^{-N}a^-). \quad (2.19)$$

Solution C. In this case we find

$$f(N) = \frac{1}{\sqrt{2}}q^{-2N}, \quad k(N) = \frac{1}{\sqrt{2}}q^N, \quad g(N) = \frac{1}{\sqrt{2}}q^{2N}, \quad h(N) = \frac{1}{\sqrt{2}}q^{-N}, \quad (2.20)$$

from which we infer

$$H(N) = \frac{1}{2}q^{-N+1}(1 + q^{2N+2}), \quad G(N) = \frac{1}{2}q^{N+1}(1 + q^{-2N+2}) = H(N-2). \quad (2.21)$$

From (2.11) and (2.21) we obtain the DSF (which casts (2.3) into the form (2.9))

$$\Phi_q^{(3)}(n) = \frac{2q^{2n-3}[n]_q(1 + q^{-n+1})}{(1 + q^{2n-2})(1 + q^{2n})} = q^{3(n-1)}\Phi_q^{(1)}(n). \quad (2.22)$$

In addition, putting (2.20) into (2.4), for the operators X and P we find

$$X = \frac{1}{\sqrt{2}}(q^{2N}a^+ + q^{-2N}a^-), \quad P = \frac{i}{\sqrt{2}}(q^N a^+ - q^{-N} a^-). \quad (2.23)$$

Solution D. That reads

$$f(N) = \frac{1}{\sqrt{2}}q^N, \quad k(N) = \frac{1}{\sqrt{2}}q^{-2N}, \quad g(N) = \frac{1}{\sqrt{2}}q^{-N}, \quad h(N) = \frac{1}{\sqrt{2}}q^{2N}, \quad (2.24)$$

so that

$$H(N) = \frac{1}{2}q^N(1 + q^{-2N-2}), \quad G(N) = \frac{1}{2}q^{-N}(1 + q^{2N-2}) = H(N-2). \quad (2.25)$$

From (2.11) and (2.25) we obtain the DSF (which casts (2.3) into the form (2.9))

$$\Phi_q^{(4)}(n) = \frac{2q^{2n}[n]_q(1 + q^{-n+1})}{(1 + q^{2n-2})(1 + q^{2n})} = q^{3n}\Phi_q^{(1)}(n) = q^3\Phi_q^{(3)}(n). \quad (2.26)$$

At last, putting (2.24) into (2.4), for the operators X and P we obtain

$$X = \frac{1}{\sqrt{2}}(q^{-N}a^+ + q^N a^-), \quad P = \frac{i}{\sqrt{2}}(q^{-2N}a^+ - q^{2N}a^-). \quad (2.27)$$

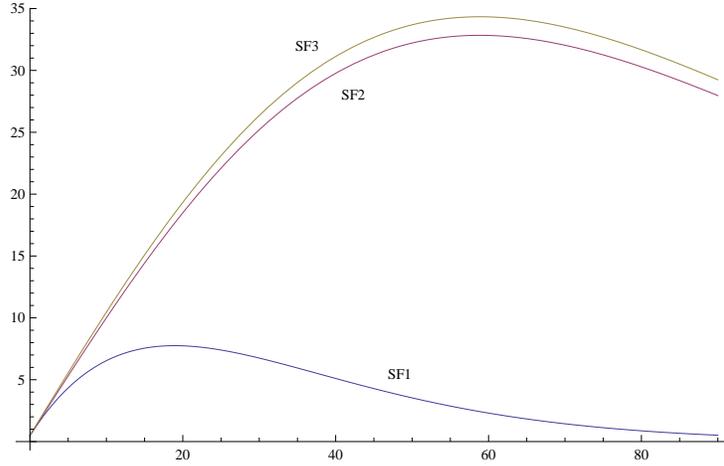


Figure 1. Deformation structure functions versus n at fixed value $q = 1.015$: here SF1, SF2 and SF3 denote respectively $\Phi_q^{(1)}(n)$, $\Phi_q^{(2)}(n)$ and $\Phi_q^{(3)}(n)$ from (2.15), (2.19) and (2.23).

Remark 1. Each of the solutions **A–D** in the limit $q \rightarrow 1$ yields $f = g = h = k = 1/\sqrt{2}$, $G = H = 1$. Then we recover the structure function $\Phi(n) = n$ of the usual oscillator, along with known relations $X = (a^+ + a^-)/\sqrt{2}$, $P = i(a^+ - a^-)/\sqrt{2}$.

Remark 2 (concerning pseudo-Hermiticity). From any of the relations (2.15), (2.19), (2.23), (2.27) it can be deduced that the position operator and the momentum operator cannot be both Hermitian simultaneously. Say, using the relation identical to (2.15) it has been shown in [17] that (i) if P is chosen to be Hermitian, then X turns out to be $\eta_X(N)$ -pseudo-Hermitian; (ii) if X is fixed to be Hermitian, then P is $\eta_P(N)$ -pseudo-Hermitian; (iii) in general, the both of X , P are non-Hermitian and, moreover, X is $\eta_X(N)$ -pseudo-Hermitian and P is $\eta_P(N)$ -pseudo-Hermitian. For more details see [17].

Now let us examine the behavior of all the structure functions as reflected in the energy eigenspectrum. With the Hamiltonian and the energy eigenspectrum given as

$$\mathcal{H} = \frac{1}{2}(aa^+ + a^+a) = \frac{1}{2}(\Phi_q(N+1) + \Phi_q(N)), \quad E_q(n) = \frac{1}{2}(\Phi_q(n+1) + \Phi_q(n)),$$

we obtain the energies $E(n)$ for the respective DSFs. In Fig. 1, the structure functions $\Phi_q^{(1)}(n)$, $\Phi_q^{(2)}(n)$ and $\Phi_q^{(3)}(n)$ from (2.15), (2.19) and (2.23) are shown for the particular value $q = 1.015$ of the deformation parameter. Note that the 4th DSF $\Phi_q^{(4)}(n)$, due to the simple relation with $\Phi_q^{(3)}(n)$, see (2.26), and the chosen $q = 1.015$, looks almost the same as $\Phi_q^{(3)}(n)$ and so is not shown in Fig. 1.

Remark 3 (ground state energies). It is of interest to examine the ground state (at $n = 0$) energy eigenvalues for each of the cases **A–D**. As seen from (2.14), (2.18), (2.22), (2.26), each of the DSFs $\Phi_q^{(1)}(n), \dots, \Phi_q^{(4)}(n)$ give zero at $n = 0$, due to the common factor $[n]_q$ such that $[0]_q = 0$. Therefore the ground state energy is given by (one half of) the corresponding DSF value at $n = 1$. This way we obtain the following data for $E_q^{(j)}(0)$

$$\begin{aligned} E_q^{(1)}(0) &= \frac{1}{2}\Phi_q^{(1)}(1) = \frac{q^{-1}}{1+q^2}, & E_q^{(2)}(0) &= \frac{1}{2}\Phi_q^{(2)}(1) = \frac{q^2}{1+q^2}, \\ E_q^{(3)}(0) &= \frac{1}{2}\Phi_q^{(3)}(1) = \frac{q^{-1}}{1+q^2}, & E_q^{(4)}(0) &= \frac{1}{2}\Phi_q^{(4)}(1) = \frac{q^2}{1+q^2}. \end{aligned}$$

Assuming $q > 1$ (like in Fig. 1) we find that, in comparison with the zero-level energy $E_q(0)|_{q=1} = \frac{1}{2}$ of the usual quantum oscillator, here we have both the increased and the lowered ground state energies of deformed oscillators

$$E_q^{(1)}(0) = E_q^{(3)}(0) < \frac{1}{2}, \quad \text{whereas} \quad E_q^{(2)}(0) = E_q^{(4)}(0) > \frac{1}{2}.$$

On the other hand, if $q < 1$ we find

$$E_q^{(1)}(0) = E_q^{(3)}(0) > \frac{1}{2}, \quad \text{while} \quad E_q^{(2)}(0) = E_q^{(4)}(0) < \frac{1}{2}.$$

Remark 4 (concerning accidental degeneracy). As seen from the behavior of the structure functions pictured in Fig. 1, the considered deformed oscillators may possess, at certain corresponding values of the deformation parameter q , diverse cases of accidental pairwise energy level degeneracy (note that such kind of degeneracy in one dimension is peculiar for certain class of *deformed* oscillators that was formerly studied in [21, 22, 24], along with the modified versions in [11, 18, 23]). For instance, for the deformed oscillator given by the DSF $\Phi_q^{(1)}(n)$ from (2.14) there exists certain value $q = q(n)|_{n=2}$ such that the degeneracy $E_q(0) = E_q(2)$ is realized. In the case of general n such value is to be found by solving the equation

$$E_q^{(1)}(n) - E_q^{(1)}(0) = \frac{1}{2} [\Phi_q^{(1)}(n+1) + \Phi_q^{(1)}(n)] - \frac{q^{-1}}{1+q^2} = 0$$

or, in a more expanded form, the equation

$$q^{4n}(q^2 + q^{-2}) + q^{3n}(q-1)(q^2 + q^{-3}) - q^{2n}(q^3 + q^{-3} - 2) + q^n(q - q^{-1}) - q - q^{-1} - \frac{1 - q^{-1}}{1 + q^2} q^{2n}(1 + q^{2n-2})(1 + q^{2n})(1 + q^{2n+2}) = 0.$$

For instance, if one fixes $n = 10$, the solution for $E_q^{(1)}(10) - E_q^{(1)}(0) = 0$ is $q_{10} = 1.0913$. Likewise, if $n = 90$ the degeneracy $E_q^{(1)}(90) = E_q^{(1)}(0)$ occurs at $q_{90} = 1.015148$. Note that $q_{90} < q_{10}$, i.e., the larger is n the lesser is respective value q_n .

In a similar fashion, for each of the DSFs $\Phi^{(j)}(n)$, $j = 1, 2, 3, 4$, one can consider these and many other cases of degeneracies (and find the relevant values of q), e.g., such as $E_q(1) = E_q(7)$, $E_q(2) = E_q(5)$, the neighboring $E_q(3) = E_q(4)$ and so on.

2.3 General approach to the relations (2.6)

Let us take (2.6) in the equivalent form

$$\frac{h(n+1)}{f(n+1)} = q \frac{h(n)}{f(n)}.$$

The latter, by denoting the ratio as $\frac{h(n)}{f(n)} \equiv d(n)$, implies $d(n+1) = qd(n)$ from which we infer $d(n) = q^n d(0)$. That yields the relation

$$h(n) = q^n f(n) d(0),$$

and similarly (putting $c(0) \equiv g(0)/k(0)$) we find the relation

$$g(n) = q^n k(n) c(0).$$

With account of the latter two relations, from (2.7) and (2.8) we infer the desired functions $G(N)$ and $H(N)$:

$$G(N) = qf(N-1)k(N)(1+c(0)d(0)q^{2N-2}), \quad (2.28)$$

$$H(N) = f(N)k(N+1)(1+c(0)d(0)q^{2N+2}), \quad (2.29)$$

along with

$$H(0) = f(0)k(1)(1+c(0)d(0)q^2)$$

(of course, there is natural simplifying choice $c(0) = d(0) = 1$). The expressions (2.28) and (2.29) can be used for obtaining the corresponding DSF.

By introducing $\mathcal{R}(N) \equiv f(N-1)k(N)$ we can present (2.28) and (2.29) in the form

$$G(N) = q\mathcal{R}(N)(1+c(0)d(0)q^{2N-2}), \quad H(N) = \mathcal{R}(N+1)(1+c(0)d(0)q^{2N+2}),$$

with the condition $G(N)|_{q \rightarrow 1} = H(N)|_{q \rightarrow 1} = 1$, see (2.3) and Remark 1. But that means $\mathcal{R}(N) = \mathcal{R}(N+1)$ if $q = 1$, or $\mathcal{R}(N)|_{q \rightarrow 1} = \text{const.}$

3 A q, p -deformed HA and q, p -oscillators

An extended two-parameter deformation of HA, see [16], obeys the basic relation

$$pXP - qPX = i\hbar, \quad p \neq q, \quad p \neq 1, \quad q \neq 1. \quad (3.1)$$

Denote $Q \equiv \frac{q}{p}$. In analogy to (2.6), (2.7) and (2.8) we obtain the formulas

$$\frac{\tilde{f}(N+1)}{\tilde{f}(N)} = Q^{-1} \frac{\tilde{h}(N+1)}{\tilde{h}(N)}, \quad \frac{\tilde{g}(N-1)}{\tilde{g}(N)} = Q^{-1} \frac{\tilde{k}(N-1)}{\tilde{k}(N)}, \quad (3.2)$$

and the relations

$$\begin{aligned} \tilde{H}(N) &= p\tilde{f}(N)\tilde{k}(N+1) + q\tilde{h}(N)\tilde{g}(N+1), \\ \tilde{G}(N) &= p\tilde{g}(N)\tilde{h}(N-1) + q\tilde{k}(N)\tilde{f}(N-1). \end{aligned}$$

Now we seek solutions of (3.2). For each of them, we also give the corresponding $\tilde{H}(N)$, $\tilde{G}(N)$, the operators X , P , and the DSF.

Solution $\tilde{\mathbf{A}}$. The first solution is

$$\tilde{f}(N) = \tilde{k}(N) = \frac{1}{\sqrt{2}}Q^N, \quad \tilde{h}(N) = \tilde{g}(N) = \frac{1}{\sqrt{2}}Q^{2N}$$

that leads to the result

$$\tilde{H}(N) = \frac{1}{2}pQ^{2N+1}(1+Q^{2N+2}), \quad \tilde{G}(N) = \frac{1}{2}pQ^{2N}(1+Q^{2N-2}) = Q^3\tilde{H}(N-2), \quad (3.3)$$

along with

$$X = \frac{1}{\sqrt{2}}[Q^{2N}a^+ + Q^N a^-], \quad P = \frac{i}{\sqrt{2}}[Q^N a^+ - Q^{2N} a^-].$$

The DSF $\Phi_{q,p}(N)$ (it determines the relation of $a^\pm a^\mp$ with N like in (2.9) as well as the action formulas for a^+ , a^- , see (2.10)) is inferred from equation (2.11) using the functions $\tilde{H}(n)$ and $\tilde{G}(n)$ in (3.3). The result is

$$\Phi_{q,p}^{(1)}(n) = \frac{2p^{-1}Q^{-n}}{(1+Q^{2n-2})(1+Q^{2n})} \left(1 + \frac{Q^n - Q^{-n+1}}{Q-1} \right)$$

$$= \frac{2q^{-n}p^{5n-3}}{(q^{2n-2} + p^{2n-2})(q^{2n} + p^{2n})} \left(1 + \frac{[2n-1]_{q,p}}{(qp)^{n-1}} \right) = \frac{2p^{-1}Q^{-n}[n]_Q(1+Q^{1-n})}{(1+Q^{2n-2})(1+Q^{2n})}, \quad (3.4)$$

where

$$[x]_{q,p} \equiv \frac{p^x - q^x}{p - q}, \quad [x]_Q \equiv \frac{1 - Q^x}{1 - Q}.$$

This DSF determines the nonstandard two-parameter deformed oscillator coinciding with that found in [16]. It is $(q \leftrightarrow p)$ -nonsymmetric and thus obviously differs from the well known q, p -oscillator [9, 4] whose structure function $\Phi_{q,p}(n) = [n]_{q,p}$ is $(q \leftrightarrow p)$ -symmetric.

Solution $\tilde{\mathbf{B}}$. The next solution is

$$\tilde{f}(N) = \tilde{k}(N) = \frac{1}{\sqrt{2}}Q^{-2N}, \quad \tilde{h}(N) = \tilde{g}(N) = \frac{1}{\sqrt{2}}Q^{-N},$$

that leads to the result

$$\begin{aligned} \tilde{H}(N) &= \frac{1}{2}pQ^{-2N}(1 + Q^{-2N-2}), \\ \tilde{G}(N) &= \frac{1}{2}pQ^{-2N+1}(1 + Q^{-2N+2}) = Q^{-3}\tilde{H}(N-2), \end{aligned} \quad (3.5)$$

along with

$$X = \frac{1}{\sqrt{2}}[Q^{-N}a^+ + Q^{-2N}a^-], \quad P = \frac{i}{\sqrt{2}}[Q^{-2N}a^+ - Q^{-N}a^-].$$

The DSF $\Phi_{q,p}(N)$ (which determines the relation of $a^\pm a^\mp$ with N like in (2.9) as well as the action formulas for a^+ , a^- , see (2.10)) is inferred from equation (2.11) using the functions $\tilde{H}(n)$ and $\tilde{G}(n)$ in (3.5). The result is

$$\Phi_{q,p}^{(2)}(n) = \left(\frac{q}{p}\right)^{3(2n-1)} \Phi_{q,p}^{(1)}(n). \quad (3.6)$$

This DSF determines the second, nonstandard $(q \leftrightarrow p)$ -nonsymmetric two-parameter deformed oscillator.

Solution $\tilde{\mathbf{C}}$. The next solution is

$$\tilde{f}(N) = \frac{1}{\sqrt{2}}Q^{-2N}, \quad \tilde{k}(N) = \frac{1}{\sqrt{2}}Q^N, \quad \tilde{g}(N) = \frac{1}{\sqrt{2}}Q^{2N}, \quad \tilde{h}(N) = \frac{1}{\sqrt{2}}Q^{-N}, \quad (3.7)$$

that leads to the result

$$\tilde{H}(N) = \frac{1}{2}pQ^{1-N}(1 + Q^{2N+2}), \quad \tilde{G}(N) = \frac{1}{2}pQ^{N+1}(1 + Q^{-2N+2}) = \tilde{H}(N-2), \quad (3.8)$$

along with

$$X = \frac{1}{\sqrt{2}}[Q^{2N}a^+ + Q^{-2N}a^-], \quad P = \frac{i}{\sqrt{2}}[Q^N a^+ - Q^{-N} a^-].$$

The DSF $\Phi_{p,q}(N)$ (it provides the relation of $a^\pm a^\mp$ with N like in (2.9) and the action formulas for a^+ , a^- , see (2.10)) is inferred from equation (2.11) using the functions $\tilde{H}(n)$ and $\tilde{G}(n)$ from (3.8). The result is

$$\Phi_{q,p}^{(3)}(n) = \left(\frac{q}{p}\right)^{3(n-1)} \Phi_{q,p}^{(1)}(n). \quad (3.9)$$

This DSF determines yet another nonstandard, $(q \leftrightarrow p)$ -nonsymmetric deformed oscillator.

Solution $\tilde{\mathbf{D}}$. This solution is of the form

$$\tilde{f}(N) = \frac{1}{\sqrt{2}}Q^N, \quad \tilde{k}(N) = \frac{1}{\sqrt{2}}Q^{-2N}, \quad \tilde{g}(N) = \frac{1}{\sqrt{2}}Q^{-N}, \quad \tilde{h}(N) = \frac{1}{\sqrt{2}}Q^{2N},$$

that leads to the result

$$\begin{aligned} \tilde{H}(N) &= \frac{1}{2}pQ^N(1 + Q^{-2N-2}), & \tilde{G}(N) &= \frac{1}{2}pQ^{-N}(1 + Q^{2N-2}) = \tilde{H}(N-2), \\ X &= \frac{1}{\sqrt{2}}[Q^{-N}a^+ + Q^Na^-], & P &= \frac{i}{\sqrt{2}}[Q^{-2N}a^+ - Q^{2N}a^-]. \end{aligned} \quad (3.10)$$

The DSF $\Phi_{q,p}(N)$ (it determines the relation of $a^\pm a^\mp$ with N like in (2.9), and the action formulas for a^+ , a^- , see (2.10)) is inferred using equation (2.11) with the functions $\tilde{H}(n)$ and $\tilde{G}(n)$ from (3.10). The result is

$$\Phi_{q,p}^{(4)}(n) = \left(\frac{q}{p}\right)^{3n} \Phi_{q,p}^{(1)}(n). \quad (3.11)$$

The latter DSF determines the forth, nonstandard, two-parameter deformed oscillator obviously nonsymmetric under $q \leftrightarrow p$.

Note that at $p \rightarrow 1$ the results obtained here for the p, q -deformed HA (3.1) reduce to those of the preceding section (say, (3.4) reduces to (2.14), etc.), whereas for the case $p = q \neq 1$ we come to the structure function $\phi(n) = \frac{n}{q}$, the familiar operators $X = \frac{1}{\sqrt{2}}(a^+ + a^-)$ and $P = \frac{i}{\sqrt{2}}(a^+ - a^-)$ along with $[a^-, a^+] = 1/q$ and $\tilde{H}(N) = H = q$, $\tilde{G}(N) = G = q$. Obviously, we deal again with the usual harmonic oscillator, but the spacing in its (linear) energy spectrum is $\frac{1}{q}$ -scaled.

Remark 5. Using the two-parameter family of $(p \leftrightarrow q)$ -symmetric p, q -oscillators from [4, 9] one can infer, see [24], a whole ‘‘plethora’’ of one-parameter q -deformed oscillators. This variety includes such well-known or ‘‘standard’’ ones as Biedenharn–Macfarlane [6, 31] (if $p = q^{-1}$), Arik–Coon [3] (if $p = 1$), and Tamm–Dancoff [10] (if $p = q$) q -oscillators. Now, quite analogously, by imposing diverse functional dependences $p = \xi(q)$ it is possible to deduce from each of the new $(p \leftrightarrow q)$ -nonsymmetric $\tilde{\mathbf{A}}$ – $\tilde{\mathbf{D}}$ families found in this section, see (3.4), (3.6), (3.9) and (3.11), the corresponding alternative ‘‘plethoras’’ of *non-standard* q -deformed oscillators, of which only relatively simple examples (got by setting $p = 1$) are given as the solutions \mathbf{A} – \mathbf{D} above (in Section 2.2).

Remark 6. The parameters p and q in the defining relation (3.1) of the q, p -DHA may be either real or complex. The issue of which particular (complex) values of p, q are admissible was discussed in [17, Section 4] – clearly that depends on the adopted rules of (pseudo)Hermiticity of X and P . On the other hand, the deformed oscillators obtained in Section 2.2 and given by the DSFs (2.14), (2.18), (2.22) and (2.26) admit only real values of the deformation parameter q . However, from each of these DSFs, say $\Phi_q^{(1)}(n)$ from (2.14), we can construct the related $(q \leftrightarrow q^{-1})$ -symmetric deformed oscillator by combining this q -DSF with its ‘‘cousin’’ $\Phi_{q^{-1}}^{(1)}(n)$. That yields the symmetrized (and factorized) DSF

$$\begin{aligned} \Phi_{\text{symm.}}^{(1)}(n) &\equiv \frac{1}{2}(\Phi_q^{(1)}(n) + \Phi_{q^{-1}}^{(1)}(n)) = \frac{q^{-n}[n]_q(1 + q^{-n+1})}{(1 + q^{2n-2})(1 + q^{2n})} + \frac{q^n[n]_{q^{-1}}(1 + q^{n-1})}{(1 + q^{-2n+2})(1 + q^{-2n})} \\ &= \frac{(q^{3n-1} + q^{-3n+1})(q^{\frac{n-1}{2}} + q^{\frac{-n+1}{2}})}{(q^n + q^{-n})(q^{n-1} + q^{-n+1})} [[n]]_{q^{1/2}}, \quad [[X]]_q \equiv \frac{q^X - q^{-X}}{q - q^{-1}}, \end{aligned}$$

which admits (real and) the complex form² of q , namely $q = e^{i\theta}$, $0 \leq \theta < \pi$. The same recipe applies to $\Phi_q^{(j)}(n)$, $j = 2, 3, 4$. It is also clear how to proceed in the case of two-parameter deformed oscillators of Section 3. Namely, each of those DSFs yields the corresponding ($q \leftrightarrow p$)-symmetric DSF and its deformed oscillator, by adding $\Phi_{q,p}^{(j)}(n)$ to its “cousin” $\Phi_{p,q}^{(j)}(n)$, $j = 1, \dots, 4$. Then the parameters may be either real or complex such that $p = \bar{q} = re^{-i\theta}$.

At last let us note that the DSF $\Phi_q^{(j)}(n)$ and its “cousin” DSF $\Phi_{q^{-1}}^{(j)}(n)$ are inferred from different copies of DHA. Likewise, $\Phi_{q,p}^{(j)}(n)$ and its “cousin” $\Phi_{p,q}^{(j)}(n)$ are linked with the (differing) q, p -DHA and p, q -DHA respectively.

4 Discussion and outlook

Our main results are contained in Sections 2, 3 and give solutions of the mapping DHA to DOA problem which allow to present the q, p -deformed (“left-sided”) HA in terms of respective *non-standard* deformed oscillators determined through their respective structure functions $\Phi_q^j(n)$ and $\Phi_{q,p}^j(n)$ where $j = 1, 2, 3, 4$. Note that the aspects concerning (pseudo-Hermitian) mutual conjugation of a^- and a^+ , as well as special non-Hermiticity (i.e., $\eta(N)$ -pseudo-Hermiticity) of X, P can be examined by a detailed analysis, in analogy to what was done in [17].

Remembering that in [17] we used as starting point not only q - and p, q -deformed Heisenberg algebras but also the two-sided p, q, μ -deformed DHA, it would be useful to undertake more detailed and complete (than in [17]) study of the problem of finding solutions which map the p, q, μ -deformed DHA onto certain p, q, μ -deformed oscillators.

Both the BM-type, AC-type q -oscillators and the p, q -oscillators [3, 6, 9, 10, 31], along with more exotic deformed oscillators [18, 24, 28], are used to construct the respective one- and two-parameter deformed analogs of Bose gas model (see [1, 2, 19, 25, 26] and references therein) which find interesting applications including phenomenological ones [15, 20, 30]. So it is of interest to develop, starting from the q - or p, q -deformed oscillators explored in this paper, the corresponding new deformed (certainly *non-standard*) models with either thermodynamics or statistical distributions and correlations in the focus. Also, there is an interesting issue of modified versions of the Heisenberg uncertainty relation (with its expected physical implications) for the above studied deformations of HA. All these topics are worth of detailed future studies.

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References

- [1] Adamska L.V., Gavrilik A.M., Multi-particle correlations in qp -Bose gas model, *J. Phys. A: Math. Gen.* **37** (2004), 4787–4795, [hep-ph/0312390](#).
- [2] Algin A., Bose–Einstein condensation in a gas of Fibonacci oscillators, *J. Stat. Mech. Theory Exp.* **2008** (2008), P1009, 23 pages, [arXiv:0810.1827](#).
- [3] Arik M., Coon D.D., Hilbert spaces of analytic functions and generalized coherent states, *J. Math. Phys.* **17** (1976), 524–527.
- [4] Arik M., Demircan E., Turgut T., Ekinçi L., Mungan M., Fibonacci oscillators, *Z. Phys. C* **55** (1992), 89–95.

²For some profits of dealing with complex deformation parameter(s) see the last paragraph in [11, Section 7] (with references therein), and also the works [26, 35, 36].

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- [5] Bagchi B., Fring A., Minimal length in quantum mechanics and non-Hermitian Hamiltonian systems, *Phys. Lett. A* **373** (2009), 4307–4310, [arXiv:0907.5354](#).
- [6] Biedenharn L.C., The quantum group $SU_q(2)$ and a q -analogue of the boson operators, *J. Phys. A: Math. Gen.* **22** (1989), L873–L878.
- [7] Brodimas G., Jannussis A., Mignani R., Bose realization of a noncanonical Heisenberg algebra, *J. Phys. A: Math. Gen.* **25** (1992), L329–L334.
- [8] Calisto H., Leiva C., Generalized commutation relations and nonlinear momenta theories: a close relationship, *Internat. J. Modern Phys. D* **16** (2007), 927–940, [hep-th/0509227](#).
- [9] Chakrabarti A., Jagannathan R., A (p, q) -oscillator realization of two-parameter quantum algebras, *J. Phys. A: Math. Gen.* **24** (1991), L711–L718.
- [10] Chaturvedi S., Srinivasan V., Jagannathan R., Tamm–Dancoff deformation of bosonic oscillator algebras, *Modern Phys. Lett. A* **8** (1993), 3727–3734.
- [11] Chung W.S., Gavrilik A.M., Kachurik I.I., Rebesh A.P., The symmetric Tamm–Dancoff q -oscillator: the representation, quasi-Fibonacci nature, accidental degeneracy and coherent states, *J. Phys. A: Math. Theor.* **47** (2014), 305304, 14 pages, [arXiv:1402.7241](#).
- [12] Chung W.S., Klimyk A.U., On position and momentum operators in the q -oscillator algebra, *J. Math. Phys.* **37** (1996), 917–932.
- [13] Dorsch G.C., Nogueira J.A., Maximally localized states in quantum mechanics with a modified commutation relation to all orders, *Internat. J. Modern Phys. A* **27** (2012), 1250113, 18 pages, [arXiv:1106.2737](#).
- [14] Garay L.J., Quantum gravity and minimum length, *Internat. J. Modern Phys. A* **10** (1995), 145–166, [gr-qc/9403008](#).
- [15] Gavrilik A.M., Combined analysis of two- and three-particle correlations in q, p -Bose gas model, *SIGMA* **2** (2006), 074, 12 pages, [hep-ph/0512357](#).
- [16] Gavrilik A.M., Kachurik I.I., Three-parameter (two-sided) deformation of Heisenberg algebra, *Modern Phys. Lett. A* **27** (2012), 1250114, 12 pages, [arXiv:1204.2817](#).
- [17] Gavrilik A.M., Kachurik I.I., New version of pseudo-hermiticity in the two-sided deformation of Heisenberg algebra, *Modern Phys. Lett. A* **31** (2016), 1650024, 15 pages, [arXiv:1503.04143](#).
- [18] Gavrilik A.M., Kachurik I.I., Rebesh A.P., Quasi-Fibonacci oscillators, *J. Phys. A: Math. Theor.* **43** (2010), 245204, 16 pages, [arXiv:1002.0601](#).
- [19] Gavrilik A.M., Mishchenko Yu.A., Deformed Bose gas models aimed at taking into account both compositeness of particles and their interaction, *Ukr. J. Phys.* **58** (2013), 1171–1177, [arXiv:1312.1573](#).
- [20] Gavrilik A.M., Mishchenko Yu.A., Correlation function intercepts for $\tilde{\mu}, q$ -deformed Bose gas model implying effective accounting for interaction and compositeness of particles, *Nuclear Phys. B* **891** (2015), 466–481, [arXiv:1411.5955](#).
- [21] Gavrilik A.M., Rebesh A.P., A q -oscillator with “accidental” degeneracy of energy levels, *Modern Phys. Lett. A* **22** (2007), 949–960, [quant-ph/0612122](#).
- [22] Gavrilik A.M., Rebesh A.P., Deformed oscillators with two double (pairwise) degeneracies of energy levels, *SIGMA* **3** (2007), 112, 11 pages, [arXiv:0710.0841](#).
- [23] Gavrilik A.M., Rebesh A.P., Occurrence of pairwise energy level degeneracies in q, p -oscillator model, *Ukr. J. Phys.* **53** (2008), 586–594, [arXiv:0805.4173](#).
- [24] Gavrilik A.M., Rebesh A.P., Plethora of q -oscillators possessing pairwise energy level degeneracy, *Modern Phys. Lett. A* **23** (2008), 921–932, [arXiv:1306.6573](#).
- [25] Gavrilik A.M., Rebesh A.P., Intercepts of the momentum correlation functions in μ -Bose gas model and their asymptotics, *Eur. Phys. J. A* **47** (2011), 55, 8 pages, [arXiv:1007.5187](#).
- [26] Gavrilik A.M., Rebesh A.P., Deformed gas of p, q -bosons: virial expansion and virial coefficients, *Modern Phys. Lett. B* **26** (2012), 1150030, 13 pages, [arXiv:1111.7216](#).
- [27] Hossenfelder S., A note on theories with a minimal length, *Classical Quantum Gravity* **23** (2006), 1815–1821, [hep-th/0510245](#).
- [28] Jannussis A., New deformed Heisenberg oscillator, *J. Phys. A: Math. Gen.* **26** (1993), L233–L237.
- [29] Kempf A., Uncertainty relation in quantum mechanics with quantum group symmetry, *J. Math. Phys.* **35** (1994), 4483–4496, [hep-th/9311147](#).

-
- [30] Liu Y.-X., Sun C.P., Yu S.X., Zhou D.L., Semiconductor-cavity QED in high- Q regimes with q -deformed bosons, *Phys. Rev. A* **63** (2001), 023802, 5 pages.
 - [31] Macfarlane A.J., On q -analogues of the quantum harmonic oscillator and the quantum group $SU(2)_q$, *J. Phys. A: Math. Gen.* **22** (1989), 4581–4588.
 - [32] Masłowski T., Nowicki A., Tkachuk V.M., Deformed Heisenberg algebra and minimal length, *J. Phys. A: Math. Theor.* **45** (2012), 075309, 5 pages, [arXiv:1201.5545](https://arxiv.org/abs/1201.5545).
 - [33] Meljanac S., Mileković M., Pallua S., Unified view of deformed single-mode oscillator algebras, *Phys. Lett. B* **328** (1994), 55–59, [hep-th/9404039](https://arxiv.org/abs/hep-th/9404039).
 - [34] Plyushchay M.S., Deformed Heisenberg algebra, fractional spin fields, and supersymmetry without fermions, *Ann. Physics* **245** (1996), 339–360, [hep-th/9601116](https://arxiv.org/abs/hep-th/9601116).
 - [35] Plyushchay M.S., Deformed Heisenberg algebra with reflection, *Nuclear Phys. B* **491** (1997), 619–634, [hep-th/9701091](https://arxiv.org/abs/hep-th/9701091).
 - [36] Rovenchak A., Complex-valued fractional statistics for D -dimensional harmonic oscillators, *Phys. Lett. A* **378** (2014), 100–108.
 - [37] Saavedra I., Utreras C., A generalization of quantum mechanics for high energies and quark physics, *Phys. Lett. B* **98** (1981), 74–76.
 - [38] Schwenk J., Wess J., A q -deformed quantum mechanical toy model, *Phys. Lett. B* **291** (1992), 273–277.