

Structure Preserving Discretizations of the Liouville Equation and their Numerical Tests^{*}

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Abstract. The main purpose of this article is to show how symmetry structures in partial differential equations can be preserved in a discrete world and reflected in difference schemes. Three different structure preserving discretizations of the Liouville equation are presented and then used to solve specific boundary value problems. The results are compared with exact solutions satisfying the same boundary conditions. All three discretizations are on four point lattices. One preserves linearizability of the equation, another the infinite-dimensional symmetry group as higher symmetries, the third one preserves the maximal finite-dimensional subgroup of the symmetry group as point symmetries. A 9-point invariant scheme that gives a better approximation of the equation, but significantly worse numerical results for solutions is presented and discussed.

Key words: Lie algebras of Lie groups; integrable systems; partial differential equations; discretization procedures for PDEs

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Dedicated to Luc Vinet on the occasion of his 60th birthday.

1 Introduction

This article is part of a general program the aim of which is to make full use of the theory of Lie groups to study the solution space of discrete equations and in particular to solve difference equations [7, 8, 21, 24, 28, 29, 30, 42]. This is one of the areas to which Luc Vinet made important contributions [14, 15, 16, 27].

S. Lie introduced what is now called Lie groups as groups of transformations of the independent and dependent variables figuring in a system of differential equations [31, 36]. Of special importance are symmetry groups, transforming solutions into solutions. These may be point transformations, where new variables depend only on the old ones. They may be contact transformations, where the new variables depend also on the first derivatives of the dependent variables. They may also be generalized symmetries where the new variables can also depend on all derivatives of the old ones.

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Lie's method is particularly powerful for ordinary differential equations (ODEs). A one-dimensional point symmetry group can be used to lower the order of the ODE by one. An n -dimensional (solvable) Lie point symmetry group can be used to decrease the order by n . Thus, if the ODE is of order $n' \leq n$ Lie group theory can provide the general solution (i.e., one that satisfies arbitrary initial conditions) in explicit or implicit analytic form. For partial differential equations (PDEs) the Lie point symmetry group is used to decrease the number of independent variables in the equation and to provide special solutions (*group invariant solutions*), satisfying particularly symmetrical boundary conditions.

The aim of this general program is to extend the use of Lie symmetry groups to difference systems (ΔS), i.e., to difference equations together with the lattice they are written on.

The program has two complementary aspects, an analytical and a numerical one.

The aim of the *analytical aspect* is to determine the maximal symmetry group of the ΔS , i.e., the group of transformations that takes solutions into solutions, and then to use it to obtain exact analytic solutions, at least special ones, if possible general ones. The ΔS to which the approach is applied can come from the study of discrete physical, chemical, biological or other systems, for which symmetries play an important role. Among them we mention phenomena in crystals, or in atomic or molecular chains.

On the other hand ΔS can be obtained by discretizing ODEs, or PDEs, that have nontrivial symmetry groups reflecting fundamental physical laws such as Galilei, Lorentz, or conformal invariance. At the scale of the Planck length space-time may very well be discrete. In this case continuous equations are approximations (continuous limits) of discrete ones. From the physical point of view the symmetries are very important and should be preserved, e.g., when studying quantum field theories on lattices.

One way of preserving symmetries in a discretization of continuous equations (the one used in this article) is to use symmetry adapted lattices that themselves transform under the group action. This greatly enlarges the set of equations for which symmetry preserving discretization is possible. We will however see that in some cases only a subgroup of the Lie point symmetry group can be preserved as point symmetries.

The *numerical aspect* of our program is the following. When solving an ODE or PDE numerically it is always necessary to replace the continuous equation by a difference system. This can be done in a standard manner, applicable to all equations, simply by replacing derivatives by discrete derivatives. The other possibility takes us directly into the field of geometric integration [20, 22, 33, 34]. The idea is to focus on some important feature of the underlying problem and to preserve it in the discretization. Such a feature may be, for instance linearizability, hamiltonian structure, integrability in the sense of the existence of a Lax pairs and generalized symmetries or point and contact symmetries. We are concentrating on point symmetries and exploring the possibility and usefulness of including them in numerical calculations.

Earlier work has shown that for first-order ODEs preserving a one-dimensional symmetry group provides an exact discretization [40]. For second-order ODEs preserving a 3-dimensional symmetry group often provides analytically solvable schemes (either via a Lagrangian [11, 12] or via the adjoint equation method [9]). For third- and higher-order ODEs symmetry preserving discretization provides numerical solutions that are, usually, closer to exact ones than those obtained by other methods, specially near to the singularities [5, 39]. For previous work on PDEs see [2, 3, 4, 6, 10, 19, 25, 26, 37, 38, 41].

Several recent articles [1, 23, 38] were devoted to discretizations of the Liouville equation [32]

$$z_{xy} = e^z, \quad (1.1)$$

or its algebraic version

$$uu_{xy} - u_x u_y = u^3, \quad u = e^z. \quad (1.2)$$

The Liouville equation is of interest for many reasons. In differential geometry it is the equation satisfied by the conformal factor $z(x, y)$ of the metric $ds^2 = z^2(dx^2 + dy^2)$ of a two-dimensional space of constant curvature [13]. In the theory of infinite-dimensional nonlinear integrable systems it is the prototype of a nonlinear partial differential equation (PDE) linearizable by a transformation of variables, involving the dependent variables (and their first derivatives) alone [32]

$$u = 2\frac{\phi_x\phi_y}{\phi^2}, \quad \phi_{xy} = 0. \quad (1.3)$$

In Lie theory this is probably the simplest PDE that has an infinite-dimensional Lie point symmetry group [35]. The symmetry algebra of the algebraic Liouville equation (1.2) is given by the vector fields

$$X(f(x)) = f(x)\partial_x - f_x(x)u\partial_u, \quad Y(g(y)) = g(y)\partial_y - g_y(y)u\partial_u, \quad (1.4)$$

where $f(x)$ and $g(y)$ are arbitrary smooth functions.

Equation (1.4) is a standard realization of the direct product of two centerless Virasoro algebras and we shall denote the corresponding Lie group $\text{VIR}(x) \otimes \text{VIR}(y)$. Restricting $f(x)$ and $g(y)$ to second-order polynomials we obtain the maximal finite-dimensional subalgebra $\mathfrak{sl}_x(2, \mathbb{R}) \oplus \mathfrak{sl}_y(2, \mathbb{R})$ and the corresponding finite-dimensional subgroup $\text{SL}_x(2, \mathbb{R}) \otimes \text{SL}_y(2, \mathbb{R})$ of the symmetry group.

The Liouville equation is also an excellent tool for testing numerical methods for solving PDE's, since equation (1.3) provides a very large class of exact analytic solutions, obtained by putting

$$\phi(x, y) = \phi_1(x) + \phi_2(y), \quad (1.5)$$

where $\phi_1(x)$ and $\phi_2(x)$ are arbitrary $\mathcal{C}^{(2)}(I)$ functions on some interval I .

In [1] Adler and Startsev presented a discrete Liouville equation that preserves the property of being linearizable and exactly solvable. In [38] Rebelo and Valiquette wrote a discrete Liouville equation that has the same infinite-dimensional $\text{VIR}(x) \otimes \text{VIR}(y)$ symmetry group as the continuous Liouville equation. The transformations are however generalized symmetries, rather than point ones. In our article [23] we presented a discretization on a four-point stencil that preserves the maximal finite-dimensional subgroup of the $\text{VIR}(x) \otimes \text{VIR}(y)$ group as point symmetries. It was also shown that it is not possible to conserve the entire infinite-dimensional Lie group of the Liouville equation as *point* symmetries. In [23] we also compared numerical solutions obtained using standard (non invariant) discretizations, the Rebelo–Valiquette invariant discretization [38] and our discretization with exact solutions (for 3 different specific solutions). It turned out that the discretization based on preserving the maximal subgroup of point transformations always gave the most accurate results for the considered solutions (all of them strictly positive in the area of integration).

The purpose of this article is to further explore and compare the different discretizations of the Liouville equation from two points of view. One is a theoretical one, namely to investigate the degree to which different discretizations preserve the qualitative feature of the equation: its exact linearizability, its infinite-dimensional Lie point symmetry algebra, the behavior of the zeroes of the solutions. The other point of view is that of geometric integration: what are the advantages and disadvantages of the different discretizations as tools for obtaining numerical solutions.

In Section 2 we reproduce our previous [23] $\text{SL}_x(2, \mathbb{R}) \otimes \text{SL}_y(2, \mathbb{R})$ symmetry preserving discretization using a 4-point stencil and show that after a slight modification it can reproduce solutions that have horizontal or vertical lines of zeroes (or both). In Section 3 we propose an

alternative discretization, using a 9-point stencil, instead of the 4-point one. It approximates the continuous Liouville equation with ϵ^2 precision, as opposed to the ϵ precision of the 4-point discretization. We show that increasing the number of points does not allow us to preserve the entire infinite-dimensional symmetry algebra, nor to treat the lines of zeroes of solutions in a satisfactory manner. Further, in Section 3 we take a specific exact solution of the continuous algebraic Liouville equation (1.2) and approximate it on a 9-point lattice by a numerical solution. The Adler–Startsev discretization [1] is reproduced in Section 4 in a form suitable for numerical calculations. Section 5 is devoted to numerical tests of the invariant 4-point scheme. Five different exact solution of the algebraic Liouville equation are presented and then used to calculate boundary conditions on two lines parallel to the x and y coordinate axes, respectively. The solutions are then calculated numerically using four different discretizations. We compare the validity of the different methods and their qualitative features. Some general conclusions, placing this article in the context of geometric integration, are presented in the final Section 6.

2 Point symmetries on a four point lattice and solutions with zeroes

In our previous article [23] we discretized the algebraic Liouville equation (1.2) on a four point regular orthogonal lattice preserving the $\mathrm{SL}_x(2, \mathbb{R}) \otimes \mathrm{SL}_y(2, \mathbb{R})$ subgroup of its Lie point symmetries. The discretization was shown to provide good numerical results for solutions that were strictly positive in the entire integration region (a quadrant to the right and above a chosen point (x_0, y_0) , i.e., for $x \geq x_0, y \geq y_0$).

A particular property of the Liouville equation is that the zeroes of its solutions are not isolated. They occur on lines parallel to the x or y axes. Indeed, consider the infinite family of solutions of (1.2) parametrized by two arbitrary smooth functions of one variable $\phi_1(x), \phi_2(y)$ (1.5). We take a region in which we have $\phi_1(x) + \phi_2(y) \neq 0$. Zeroes of $u(x, y)$ occur if $\phi_{1,x}(x)$, or $\phi_{2,y}(y)$ are zero at some point x_s , or y_s (or both), respectively. We then have

$$u(x_s, y) = 0, \quad \forall y, \quad \text{or} \quad u(x, y_s) = 0, \quad \forall x. \quad (2.1)$$

This must be reflected in any computational scheme and the value $u(x, y) = 0$ will also occur on the intersection with the corresponding coordinate axis.

In [23] we considered several different boundary value problems. Here we restrict to the case of boundary conditions given on the lines $x \geq x_0, y \geq y_0$ parallel to the coordinate axes. We can impose

$$u(x_s, 0) = 0 \quad \text{and/or} \quad u(0, y_s) = 0$$

in order to obtain a solution satisfying (2.1).

The $\mathrm{SL}_x(2, \mathbb{R}) \otimes \mathrm{SL}_y(2, \mathbb{R})$ invariants used in [23] to describe both the lattice and the discrete algebraic Liouville equation on a four point stencil were

$$\xi_1 = \frac{(x_{m,n+1} - x_{m,n})(x_{m+1,n+1} - x_{m+1,n})}{(x_{m,n} - x_{m+1,n})(x_{m,n+1} - x_{m+1,n+1})},$$

$$\eta_1 = \frac{(y_{m,n} - y_{m+1,n})(y_{m,n+1} - y_{m+1,n+1})}{(y_{m,n+1} - y_{m,n})(y_{m+1,n+1} - y_{m+1,n})}, \quad (2.2)$$

$$J_1 = u_{m+1,n} u_{m,n+1} h^2 k^2, \quad J_2 = u_{m,n} u_{m+1,n+1} h^2 k^2. \quad (2.3)$$

The lattice equations

$$\xi_1 = 0, \quad \eta_1 = 0 \quad (2.4)$$

are satisfied by the uniform orthogonal lattice

$$x_{m,n} = hm + x_0, \quad y_{m,n} = kn + y_0, \quad (2.5)$$

where the scale factors h and k are the same as in (2.3). The continuous limit corresponds to $h \rightarrow 0, k \rightarrow 0$. Two further independent $\mathrm{SL}_x(2, \mathbb{R}) \otimes \mathrm{SL}_y(2, \mathbb{R})$ invariants exist on the four point stencil but any combination of them will either vanish, or be infinite on the lattice given by (2.4) (see [23]).

The Liouville equation (1.2) was approximated in [23] by the difference scheme

$$\begin{aligned} J_2 - J_1 &= a \operatorname{sign}(J_1) |J_1|^{3/2} + b \operatorname{sign}(J_2) J_1 |J_2|^{1/2} \\ &\quad + c \operatorname{sign}(J_1) |J_1|^{1/2} J_2 + d \operatorname{sign}(J_2) |J_2|^{3/2}, \\ \xi_1 = \eta_1 &= 0, \quad a + b + c + d = 1. \end{aligned} \quad (2.6)$$

The symbols $\operatorname{sign}(J_1)$ and $\operatorname{sign}(J_2)$ were omitted in [23] and were not necessary as we restricted our formulation to strictly positive solutions. Equation (2.6) can be solved for $u_{m+1,n+1}$ in terms of $u_{n,m}, u_{n+1,m}$ and $u_{n,m+1}$. On the first stencil we have $m = n = 0$. The boundary conditions are $u_{m,0} = f(m)$ and $u_{0,n} = g(n)$ with f and g given.

Let us now rewrite the recurrence relation (2.6) in terms of $u_{m,n}$, choose $b = d = 0, c = 1 - a, a \in \mathbb{R}$ (in order to have an explicit scheme) and solve for $u_{m+1,n+1}$. We have

$$u_{m+1,n+1} = \frac{u_{m,n+1} u_{m+1,n}}{u_{m,n}} A_{m,n+1;m+1,n}, \quad (2.7)$$

$$A_{m,n+1;m+1,n} = \frac{1 + ahk \operatorname{sign}(u_{m+1,n} u_{m,n+1}) \sqrt{|u_{m,n+1} u_{m+1,n}|}}{1 + (a-1)hk \operatorname{sign}(u_{m+1,n} u_{m,n+1}) \sqrt{|u_{m,n+1} u_{m+1,n}|}}. \quad (2.8)$$

The expression $\operatorname{sign}(u_{m+1,n} u_{m,n+1})$ follows from (2.6). Here the *sign* before the square root is important since it will change when the sign of u changes in the recurrence relation.

We shall use (2.7), (2.8) to investigate the behaviour of the numerical schemes for solutions that have rows (horizontal lines) or columns (vertical lines) of zeroes. We impose boundary conditions on the lines $x = x_s$ and $y = y_s$. To see the influence of the boundary conditions we introduce small quantities μ and ν on the coordinate axes that will later be set to zero. We shall see that these small values do not propagate elsewhere but are confined to the columns and rows where they were introduced. This procedure is analogous to ‘‘singularity confinement’’ [17, 18] used as an integrability criterion for difference equations.

We first note that we have

$$\lim_{u_{m,n+1} \rightarrow 0} A_{m,n+1;m+1,n} = \lim_{u_{m+1,n} \rightarrow 0} A_{m,n+1;m+1,n} = 1. \quad (2.9)$$

Three cases will be considered separately:

1. A column of zeroes. The boundary conditions are

$$u_{m_0,0} = \mu, \quad u_{m,0} \neq 0 \quad \text{for } m \neq m_0.$$

Using (2.7), (2.9) we obtain expressions for $u_{m_0,n}$, namely

$$u_{m_0,n} = \frac{u_{m_0-1,n}}{u_{m_0-1,0}} \mu, \quad n \geq 0.$$

In the column to the right of the zeroes we obtain two equivalent expressions:

$$u_{m_0+1,n} = u_{m_0+1,n-1} \frac{u_{m_0-1,n}}{u_{m_0-1,n-1}}, \quad (2.10)$$

$$u_{m_0+1,n} = u_{m_0-1,n} \frac{u_{m_0+1,0}}{u_{m_0-1,0}}. \quad (2.11)$$

Thus the zero quantity μ cancels out and $u_{m_0+1,n}$ is finite and nonzero for all $n \geq 0$. Moreover $u_{m_0+1,n}$ is expressed in terms of the given initial values and values calculated at previous nonzero values.

2. A row of zeroes can be treated completely analogously. The boundary conditions are replaced by

$$u_{0,n_0} = \nu, \quad u_{0,n} \neq 0 \quad \text{for } n \neq n_0,$$

and we obtain

$$u_{m,n_0} = \frac{u_{m,n_0-1}}{u_{0,n_0-1}} \nu, \quad m \geq 0,$$

i.e., a row of zeroes for $\nu = 0$. The row above the zeroes satisfies

$$u_{m,n_0+1} = u_{m-1,n_0+1} \frac{u_{m,n_0-1}}{u_{m-1,n_0-1}}, \quad (2.12)$$

$$u_{m,n_0+1} = u_{m,n_0-1} \frac{u_{0,n_0+1}}{u_{0,n_0-1}}. \quad (2.13)$$

3. Two intersecting lines of zeroes. The boundary conditions are

$$u_{0,n_0} = \nu, \quad u_{m_0,0} = \mu; \quad u_{m,0} \neq 0 \quad \text{for } m \neq m_0, \quad u_{0,n} \neq 0 \quad \text{for } n \neq n_0.$$

Using the same considerations as above we find a column and a row of zeroes satisfying

$$u_{m_0,n} = \frac{u_{m_0-1,n}}{u_{m_0-1,0}} \mu, \quad n \neq n_0, \quad u_{m,n_0} = \frac{u_{m,n_0-1}}{u_{0,n_0-1}} \nu, \quad m \neq m_0,$$

$$u_{m_0,n_0} = \frac{u_{m_0-1,n_0-1}}{u_{0,n_0-1} u_{m_0-1,0}} \mu \nu.$$

Thus, for $\mu = 0$, $\nu = 0$ the solutions $u_{m,n}$ have zeroes precisely where they should. Now let us use (2.7), (2.9) to calculate the values of $u_{m_0+1,n}$ and u_{m,n_0+1} , i.e., the column at the right and the row above the zeroes. The final result is that (2.10) is valid for all $n \neq n_0, n_0 + 1$ and (2.12) for all $m \neq m_0, m_0 + 1$ with

$$u_{m_0+1,n_0+1} = u_{m_0-1,n_0+1} \frac{u_{m_0+1,n_0-1}}{u_{m_0-1,n_0-1}},$$

while (2.11) is valid for all $n \neq n_0$ and (2.13) for all $m \neq m_0$.

Finally we see that the zeroes are confined to the rows and columns determined by a zero in the boundary condition and that the values of $u_{m,n}$ everywhere else are finite, non zero and determined by the equations (2.7), (2.8) and the boundary conditions. In other words the rows and columns of zeroes do not interfere with the integration algorithm. This will be confirmed by numerical calculations in Section 5.

3 Invariant discretization of the algebraic Liouville equation using a larger number of points

There are several reasons to increase the number of points on the stencil that we use.

1. To determine whether the entire $\text{VIR}(x) \otimes \text{VIR}(y)$ symmetry group can be preserved on a larger lattice.

2. To determine whether the only other $\text{SL}_x(2, \mathbb{R}) \otimes \text{SL}_y(2, \mathbb{R})$ differential invariant [23], namely

$$I_2 = \frac{1}{u^6} (2uu_{xx} - 3u_x^2)(2uu_{yy} - 3u_y^2) \quad (3.1)$$

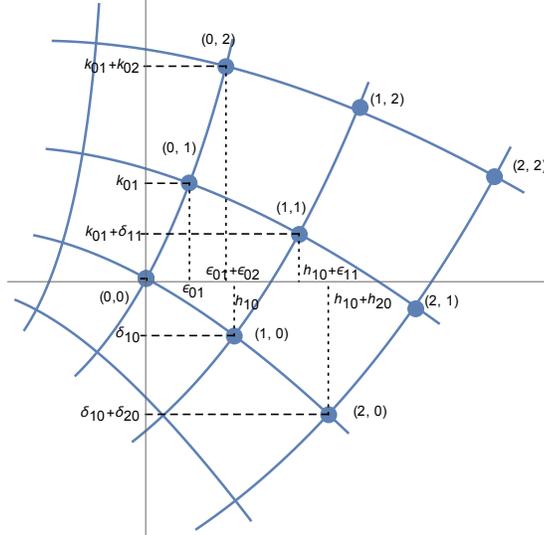


Figure 1. Points on a general lattice, e.g., $x_{00} = x$, $x_{10} = x + h_{10}$, $x_{01} = x + \epsilon_{01}$, $x_{11} = x + h_{10} + \epsilon_{11}$, $x_{20} = x + h_{10} + h_{20}$, $x_{02} = x + \epsilon_{01} + \epsilon_{02}$, $x_{12} = x + h_{10} + \epsilon_{11} + \epsilon_{12}$, $x_{21} = x + h_{10} + h_{20} + \epsilon_{21}$, $x_{22} = x + h_{10} + h_{20} + \epsilon_{21} + \epsilon_{22}$, $y_{00} = y$, $y_{01} = y + k_{01}$, $y_{10} = y + \delta_{10}$, $y_{11} = y + k_{01} + \delta_{11}$, $y_{02} = y + k_{01} + k_{02}$, $y_{20} = y + \delta_{10} + \delta_{20}$, $y_{12} = y + k_{01} + k_{02} + \delta_{12}$, $y_{21} = y + k_{01} + \delta_{10} + \delta_{20}$, $y_{22} = y + k_{01} + k_{02} + \delta_{12} + \delta_{22}$.

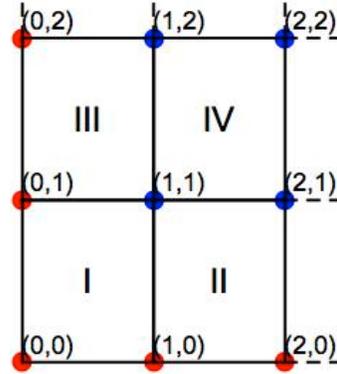


Figure 2. A stencil for the 9-points scheme.

can be invariantly discretized on a larger lattice. Four points are clearly not sufficient to approximate two first- and three second-order derivatives.

3. To approximate the algebraic Liouville equation with a higher degree of accuracy in h and k and thus possibly improve the numerical calculations.

In Section 2 and in [23] we have shown that the Liouville equation can be approximated on 4 points. To approximate an arbitrary second-order PDE for a function $u(x, y)$ we need at least 6 points. An invariant discretization may need more than six.

Equation (2.6) satisfies

$$\begin{aligned} \lim_{h,k \rightarrow 0} \frac{1}{h^3 k^3} \{ J_2 - J_1 - [a|J_1|^{3/2} + bJ_1|J_2|^{1/2} + c|J_1|^{1/2}J_2 + d|J_2|^{3/2}] \} \\ = [uu_{xy} - u_x u_y - u^3][1 + \mathcal{O}(h, k)], \end{aligned}$$

and thus provides a first-order approximation of the algebraic Liouville equation. In this section we will explore a second-order approximation (order $\mathcal{O}(h^2, k^2, hk)$) of the equation (1.2). To do this we shall use a 9-point stencil as shown on Figs. 1 and 2. The 4 well-behaved invariants (2.2), (2.3) make use of the four vertices of rectangle I on Fig. 2. Instead of the vertices of

rectangle I we could use any other 4 points, and we shall use the vertices of the rectangles II, III and IV. The invariants involving the independent variables ξ_a and η_a ($a = 1, \dots, 4$) all vanish on the orthogonal lattice (2.5). The invariants depending on the dependent variables u_{ij} that are finite and nonzero on this lattice are

$$\begin{aligned} J_1 &= u_{01}u_{10}h^2k^2, & J_2 &= u_{00}u_{11}h^2k^2, \\ J_3 &= u_{11}u_{20}h^2k^2, & J_4 &= u_{10}u_{21}h^2k^2, \\ J_5 &= u_{11}u_{02}h^2k^2, & J_6 &= u_{01}u_{12}h^2k^2, \\ J_7 &= u_{12}u_{21}h^2k^2, & J_8 &= u_{11}u_{22}h^2k^2. \end{aligned}$$

The quantities J_1, \dots, J_8 are *linearly* independent but one polynomial relation exists between them, namely

$$J_4J_6 = J_1J_7.$$

The continuous limit is obtained by expanding the invariants into Taylor series and then taking $h \rightarrow 0$, $k \rightarrow 0$. We shall assume that they tend to zero at the same rate, i.e., $k = \alpha h$, $\alpha \sim 1$. We have

$$\begin{aligned} J_1 &= h^2k^2 \left[u^2 + hu u_x + ku u_y + hku_x u_y + \frac{1}{2}h^2 u u_{xx} + \frac{1}{2}k^2 u u_{yy} + \dots \right], \\ J_2 &= h^2k^2 \left[u^2 + hu u_x + ku u_y + hku u_{xy} + \frac{1}{2}h^2 u u_{xx} + \frac{1}{2}k^2 u u_{yy} + \dots \right], \\ J_3 &= h^2k^2 \left[u^2 + 3hu u_x + ku u_y + \frac{1}{2}k^2 u u_{yy} + hk(u u_{xy} + 2u_y u_x) + h^2 \left(\frac{5}{2} u u_{xx} + 2u_x^2 \right) + \dots \right], \\ J_4 &= h^2k^2 \left[u^2 + 3hu u_x + ku u_y + \frac{1}{2}k^2 u u_{yy} + hk(2u u_{xy} + u_y u_x) + h^2 \left(\frac{5}{2} u u_{xx} + 2u_x^2 \right) + \dots \right], \\ J_5 &= h^2k^2 \left[u^2 + hu u_x + 3ku u_y + k^2 \left(\frac{5}{2} u u_{yy} + 2u_y^2 \right) + hk(u u_{xy} + 2u_y u_x) + \frac{1}{2}h^2 u u_{xx} + \dots \right], \\ J_6 &= h^2k^2 \left[u^2 + hu u_x + 3ku u_y + k^2 \left(\frac{5}{2} u u_{yy} + 2u_y^2 \right) + hk(2u u_{xy} + u_y u_x) + \frac{1}{2}h^2 u u_{xx} + \dots \right], \\ J_7 &= h^2k^2 \left[u^2 + 3hu u_x + 3ku u_y + k^2 \left(2u_y^2 + \frac{5}{2} u u_{yy} \right) + hk(4u u_{xy} + 5u_y u_x) \right. \\ &\quad \left. + h^2 \left(2u_x^2 + \frac{5}{2} u u_{xx} \right) + \dots \right], \\ J_8 &= h^2k^2 \left[u^2 + 3hu u_x + 3ku u_y + k^2 \left(\frac{5}{2} u u_{yy} + 2u_y^2 \right) + hk(5u u_{xy} + 4u_y u_x) \right. \\ &\quad \left. + h^2 \left(\frac{5}{2} u u_{xx} + 2u_x^2 \right) + \dots \right]. \end{aligned} \tag{3.2}$$

We see that u_{22} , u_{02} , u_{20} and u_{00} figure only once each in the invariants, namely in J_8 , J_5 , J_3 and J_2 , respectively. On the other hand u_{01} , u_{10} , u_{12} , and u_{21} figure twice each, respectively in (J_1, J_6) , (J_1, J_4) , (J_6, J_7) and (J_4, J_7) . The value u_{11} figures in all four of J_2 , J_3 , J_5 and J_8 .

To lowest-order we have

$$J_2 - J_1 = J_4 - J_3 = J_6 - J_5 = J_8 - J_7 = h^3 k^3 (u u_{xy} - u_x u_y) [1 + \mathcal{O}(h, k)].$$

To obtain the left hand side of the algebraic Liouville equation (1.2) up to order $\mathcal{O}(h^2, hk, k^2)$ we need the differences $J_{2a} - J_{2a-1}$ to a higher-order than in (3.2), namely

$$\begin{aligned} J_2 - J_1 &= h^3 k^3 \left\{ (u u_{xy} - u_x u_y) + \frac{h}{2} (u u_{xxy} - u_y u_{xx}) + \frac{k}{2} (u u_{xyy} - u_x u_{yy}) \right\}, \\ J_4 - J_3 &= h^3 k^3 \left\{ (u u_{xy} - u_x u_y) + \frac{3h}{2} (u u_{xxy} - u_y u_{xx}) + \frac{k}{2} (u u_{xyy} - u_x u_{yy}) \right\}, \\ J_6 - J_5 &= h^3 k^3 \left\{ (u u_{xy} - u_x u_y) + \frac{h}{2} (u u_{xxy} - u_y u_{xx}) + \frac{3k}{2} (u u_{xyy} - u_x u_{yy}) \right\}, \\ J_8 - J_7 &= h^3 k^3 \left\{ (u u_{xy} - u_x u_y) + \frac{3h}{2} (u u_{xxy} - u_y u_{xx}) + \frac{3k}{2} (u u_{xyy} - u_x u_{yy}) \right\}. \end{aligned} \tag{3.3}$$

In [23] equation (1.2) was approximated to order $\mathcal{O}(h, k)$. To approximate it to $\mathcal{O}(h^2, hk, k^2)$ we must get rid of the terms of order $\mathcal{O}(h, k)$ in (3.3).

The left hand side is approximated to the needed order by

$$\begin{aligned} & \alpha[4(J_2 - J_1) - (J_6 - J_5 + J_4 - J_3)] + \beta[4(J_8 - J_7) - 3(J_6 - J_5 + J_4 - J_3)] \\ & = 2h^3k^3(uu_{xy} - u_xu_y)(\alpha - \beta)[1 + \mathcal{O}(h^2, hk, k^2)], \end{aligned} \quad (3.4)$$

where α and β are arbitrary real constants.

To express the right-hand side of (1.2) we use the basis

$$\begin{aligned} B_1 &= \frac{1}{2}(3J_1 - J_8) = h^2k^2u^2(1 + R_1), & B_2 &= J_4 + J_6 - J_8 - J_1 = h^2k^2R_2, \\ B_3 &= J_2 - J_1 = h^2k^2R_3, & B_4 &= J_4 - J_3 = h^2k^2R_4, & B_5 &= J_6 - J_5 = h^2k^2R_5, \\ B_6 &= J_8 - J_7 = h^2k^2R_6, \end{aligned} \quad (3.5)$$

where R_1, \dots, R_6 are all of the order $\mathcal{O}(h^2, hk, k^2)$.

The left hand side of (1.2) is already expressed in this basis (see (3.4) using B_3, \dots, B_6).

From the basis elements (3.5) we can calculate u^2 as

$$B_1 + \sum_{i=2}^6 c_i B_i = h^2k^2u^2[1 + \mathcal{O}(h^2, hk, k^2)],$$

with 5 free real parameters c_i . To obtain u^3 we have several possibilities. One is to take

$$\left(B_1 + \sum_{i=2}^6 c_i B_i \right)^{3/2} = h^3k^3u^3[1 + \mathcal{O}(h^2, hk, k^2)]. \quad (3.6)$$

The corresponding discrete Liouville equation is then

$$\alpha[4B_3 - (B_4 + B_5)] + \beta[4B_6 - 3(B_4 + B_5)] = \left(B_1 + \sum_{i=2}^6 c_i B_i \right) \left\{ \left| B_1 + \sum_{i=2}^6 c_i B_i \right| \right\}^{1/2}, \quad (3.7)$$

with

$$2(\alpha - \beta) = 1. \quad (3.8)$$

Another possibility is to replace the basis (3.5) by

$$A_1 = B_1, \quad A_a = B_1 + \frac{1}{2}B_a, \quad a = 2, \dots, 6.$$

We can then approximate the right-hand side of the discrete algebraic Liouville equation by

$$\sum_{a,b=1}^6 \gamma_{a,b} A_a \sqrt{|A_b|} = h^3k^3u^3 \sum_{a,b=1}^6 \gamma_{a,b} [1 + \mathcal{O}(h^2, hk, k^2)].$$

Then the discrete algebraic Liouville equation reads

$$2\alpha[4A_3 - 2A_1 - (A_4 + A_5)] + 2\beta[4A_6 + 2A_1 - 3(A_4 + A_5)] = \sum_{a,b=1}^6 \gamma_{a,b} A_a \sqrt{|A_b|}, \quad (3.9)$$

$$\sum_{a,b=1}^6 \gamma_{a,b} = 2(\alpha - \beta).$$

The general invariant equations (3.9), (3.6) use all 9 points on the stencil and contain a lot of free parameters. The parameters can be chosen to simplify calculations, though the choice of $(\alpha, \beta, c_1, \dots, c_6)$ or $(\alpha, \beta, \gamma_{a,b})$ is restricted by the type of boundary conditions we wish to impose.

The quantity u_{22} figures in J_8 only. An explicit scheme is obtained if J_8 figures linearly in the corresponding invariant discrete Liouville equation.

One possibility is to choose $\alpha = -3\beta$ and $c_2 = c_3 = c_4 = c_5 = 0$, $c_6 = 1/2$ in (3.7), (3.8). Then $\beta = -1/8$ and the invariant Liouville equation reduces to

$$J_8 = J_7 + 3(J_2 - J_1) - \frac{1}{\sqrt{2}}(3J_1 - J_7)\sqrt{|3J_1 - J_7|}. \quad (3.10)$$

In terms of the field u_{ij} (3.10) reads

$$\begin{aligned} u_{22} = & \frac{1}{u_{11}}[u_{12}u_{21} + 3(u_{11}u_{00} - u_{10}u_{01}) \\ & - \frac{1}{\sqrt{2}}hk(3u_{01}u_{10} - u_{12}u_{21})\sqrt{|3u_{01}u_{10} - u_{12}u_{21}|}]. \end{aligned} \quad (3.11)$$

so that u_{22} is expressed in terms of u_{00} , u_{01} , u_{10} , u_{11} , u_{12} and u_{21} , i.e., only 7 points are involved.

Another simple possibility is to choose $\alpha = -3\beta$ and $\gamma_{ab} = \delta_{a1}\delta_{b6}$ in (3.9). Then we have $\beta = -1/8$ and we obtain

$$J_8 = \frac{J_7 + 3(J_2 - J_1) - \frac{3}{\sqrt{2}}J_1 \operatorname{sign}(3J_1 - J_7)\sqrt{|3J_1 - J_7|}}{1 - \frac{1}{\sqrt{2}}\operatorname{sign}(3J_1 - J_7)\sqrt{|3J_1 - J_7|}} \quad (3.12)$$

In terms of the field u_{ij} (3.12) reads

$$\begin{aligned} u_{22} = & \left\{ u_{12}u_{21} + 3(u_{11}u_{00} - u_{10}u_{01}) \right. \\ & \left. - \frac{3}{\sqrt{2}}hku_{01}u_{10} \operatorname{sign}(3u_{01}u_{10} - u_{12}u_{21})\sqrt{|3u_{01}u_{10} - u_{12}u_{21}|} \right\} \\ & \times \left\{ u_{11} \left[1 - \frac{hk}{\sqrt{2}} \operatorname{sign}(3u_{01}u_{10} - u_{12}u_{21})\sqrt{|3u_{01}u_{10} - u_{12}u_{21}|} \right] \right\}^{-1}. \end{aligned} \quad (3.13)$$

Again only 7 of the 9 points on a stencil are used.

Equations (3.11) and (3.13) are to be viewed as recursion relations, expressing $u_{2,2}$ in terms of 6 points on a rectangle of which the point (2,2) is the top right vertex (see Fig. 2).

By construction (3.11) and (3.13) are better approximations of the equation (1.1) than is (2.6). This does not mean that they will provide better numerical results and some comments are in order.

1. Boundary conditions for a numerical solution on a 4-point lattice require the knowledge of $u(x, y)$ on two lines, e.g., $u_{m,0}$ and $u_{0,n}$, i.e., $u(x, 0)$ and $u(0, y)$. On the 9-point lattice we must start with 2 sets of parallel lines, e.g., $u_{m,0}$, $u_{m,1}$ and $u_{0,n}$, $u_{1,n}$. This amounts to giving $u(x, 0)$, $u(0, y)$ and the first term of $u_y(x, 0)$, $u_x(0, y)$. This is more information than is needed in standard (non invariant) discretizations and indeed more information than is needed in theory to determine a solution completely. Hence once $u(x, 0)$ and $u(0, y)$ are given $u_y(x, 0)$ and $u_x(0, y)$ *cannot* be chosen arbitrarily. In our numerical solutions we calculated the conditions using an exact solution so $u(x, 0)$, $u(0, y)$ and $u_x(0, y)$, $u_y(x, 0)$ are consistent.

2. Contrary to the case of a 4-point lattice, instabilities close to zero lines of solutions cannot be avoided on 7- or 9-point lattices. Indeed let us give initial conditions on the first square satisfying $u_{00} \neq 0$, $u_{01} \neq 0$, $u_{10} \neq 0$, $u_{11} \neq 0$, $u_{12} \neq 0$, $u_{20} = \epsilon_1$, $u_{21} = \epsilon_2$. From the known

solution of the PDE (1.1) we expect the solution to satisfy $u_{2,m} = 0$ for $m \geq 2$. Equation (3.11) implies

$$u_{22} = \frac{1}{u_{11}} \left[u_{12}\epsilon_2 + 3(u_{11}u_{00} - u_{10}u_{01}) - \frac{1}{\sqrt{2}}hk(3u_{01}u_{10} - u_{12}\epsilon_2)\sqrt{|3u_{01}u_{10} - u_{12}\epsilon_2|} \right].$$

Thus u_{22} is not strictly zero for $\epsilon_1 = \epsilon_2 = 0$, it does however satisfy $u_{22} \sim \mathcal{O}(h^2, k^2, hk)$. This is acceptable, however the problem arises when we shift the stencil and calculate u_{32} which is supposed to be finite and nonzero if we assume $u_{30} \neq 0$, $u_{31} \neq 0$. What we obtain from (3.11) is

$$u_{32} = \frac{1}{\epsilon_2} \left[\mathcal{O}(h^2, k^2, hk)u_{31} + 3(\epsilon_2u_{10} - \epsilon_1u_{11}) - \frac{1}{\sqrt{2}}hk(3\epsilon_1u_{11} - \mathcal{O}(h^2, k^2, hk)u_{31})\sqrt{|3\epsilon_1u_{11} - \mathcal{O}(h^2, k^2, hk)u_{31}|} \right].$$

Thus, u_{32} is singular for $\epsilon_2 = 0$ and becomes finite only in the continuous limit $h = k = 0$. This will quite obviously create numerical instabilities. They are avoided only for very special initial conditions, such that $u_{22} = 0$ for all h and k . Using (3.13) leads to the same kind of problems.

Sadly (for the 9 points scheme) the answers to all three questions posed in the beginning of this section are negative.

1. The only $\text{VIR}(x) \otimes \text{VIR}(y)$ invariant lattice is given by requiring conditions (2.2) in all four quadrangles of Fig. 2. Indeed we have

$$\begin{aligned} \text{pr } X(x^3)\xi_a &= (x_{11} - x_{00})(x_{10} - x_{01})\xi_a, \\ \text{pr } Y(y^3)\eta_a &= (y_{11} - y_{00})(y_{10} - y_{01})\eta_a, \quad a = 1, 2, 3, 4 \end{aligned}$$

so (weak) invariance requires $\xi_a = \eta_a = 0$ and adding further points does not help. Moreover, no function of J_1, \dots, J_8 has the correct continuous limit and is invariant under $\text{VIR}(x) \otimes \text{VIR}(y)$.

2. We have determined that the invariant (3.1) cannot be discretized in an $\text{SL}_x(2, \mathbb{R}) \otimes \text{SL}_y(2, \mathbb{R})$ invariant manner using J_1, \dots, J_8 but we do not present the details here since this question is not related to the Liouville equation.

3. Numerical results for several exact solutions showed that serious instabilities occur for the 7-point scheme. As one can see in Fig. 3 representing the solution f_1 given in (5.9), instabilities in the 7-point case occur almost immediately. The same is true for other solutions of equation (1.2).

Our conclusion is that the 9-point (or 7-point) scheme is too unstable to be useful. We present it here because we think that this negative result is not a priori obvious and that this discussion may be useful.

4 The Adler–Startsev linearizable discrete Liouville equation

Adler and Startsev [1] have presented a discretization of the algebraic Liouville equation (1.2) on a four-point lattice, namely

$$a_{m+1,n+1} \left(1 + \frac{1}{a_{m+1,n}} \right) \left(1 + \frac{1}{a_{m,n+1}} \right) a_{m,n} = 1. \quad (4.1)$$

This equation is linearizable by the substitution

$$a_{m,n} = -\frac{(b_{m+1,n} - b_{m,n})(b_{m,n+1} - b_{m,n})}{b_{m+1,n}b_{m,n+1}},$$

where $b_{m,n}$ satisfies the linear equation

$$b_{m+1,n+1} - b_{m+1,n} - b_{m,n+1} + b_{m,n} = 0.$$

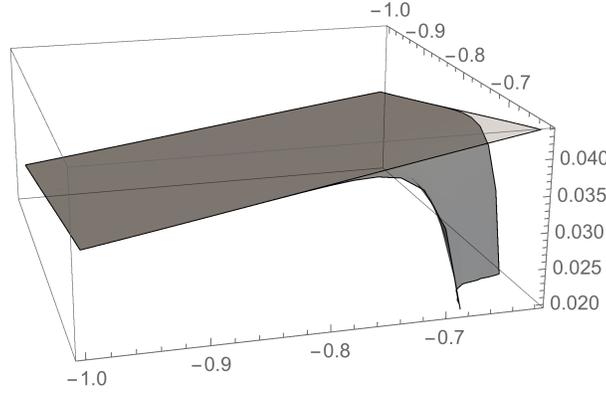


Figure 3. Comparison between the 4-point and 7-point approximation for the solution f_1 given in (5.9). The dark grey graph represents the results of the application of (3.12), (3.13), while the light grey one is obtained by using the 4-point formula (2.7), (2.8). The integration is carried out starting from the point $(-1.0, -1.0)$, the recursion formula is applied for steps $h = k = 0.02$ on a grid 20×20 .

Hence the exact general solution of (4.1) is

$$a_{m,n} = -\frac{(c_{m+1} - c_m)(k_{n+1} - k_n)}{(c_{m+1} + k_n)(c_m + k_{n+1})},$$

where c_m, k_n are arbitrary functions of one index each.

In [23] we showed, following [1] that the continuous limit of (4.1), for $a_{m,n} = -\frac{hk}{2}u_{m,n}$ when h and k go to zero, gives (1.2) and that it has no continuous point symmetries but must have generalized symmetries. Moreover by defining $c_m = \phi_1(x_{m,n})$, $k_n = \phi_2(y_{m,n})$ with $x_{m,n}$ and $y_{m,n}$ defined in (2.5) we have

$$c_{m+1} = \phi_1(x) + h \frac{d\phi_1}{dx} + \mathcal{O}(h^2), \quad k_{n+1} = \phi_2(y) + k \frac{d\phi_2}{dy} + \mathcal{O}(k^2),$$

and thus $a_{m,n} = -hk \frac{\phi_{1,x}\phi_{2,y}}{(\phi_1 + \phi_2)^2} + \mathcal{O}(h^3, h^2k, hk^2, k^3)$ a first-order approximation of the general solution of (1.2) given by (1.3).

5 Numerical tests of the 4-point scheme

In this section we shall apply the invariant recursion formula (2.7), (2.8) to solve a set of boundary value problems on a quadrant in the xy -plane. Boundary conditions will be given on two orthogonal lines parallel to the x and y axes, respectively, and numerical solutions will be constructed above and to the right of these lines. The numerical solutions will be compared with exact solutions of the continuous equation for the same boundary conditions. In practice we will start from exact solutions given by choosing $\phi(x, y) = \phi_1(x) + \phi_2(y)$ in (1.3) and calculate the values of these functions on the boundaries. The global estimator which we use is the discrete analog of relative distance in $L^2_{\mathcal{D}}$. We compute the quantity

$$\chi_\alpha(F) = \sqrt{\frac{\sum_{ij} (F_{ij}^\alpha - F_{ij})^2}{\sum_{ij} F_{ij}^2}}, \quad (5.1)$$

where F_{ij} are the values of the exact solution F on the lattice sites and F_{ij}^α , with $\alpha = \text{Inv, AS, RV}$, or stand are the values computed numerically for the invariant, Adler–Startsev, Rebelo–Valiquette or standard discretization, respectively. A similar analysis is performed for the other

recursion formulae. The summation will be over all points of the lattice for which the calculation was performed.

The χ_α quantity (5.1) provides information about the overall averaged behavior of numerical solutions, rather than about their point-by-point behavior. Geometrical features of solutions are better reflected by plots of individual solutions and by a relative error function such as

$$R_{ij} = \left| \frac{F_{ij}^\alpha - F_{ij}}{F_{ij}} \right|. \quad (5.2)$$

A characteristic property of all solutions of the algebraic Liouville equation concern the zeroes. They either have no zero in any finite domain \mathcal{D} , or the zeroes are not isolated, but occur along continuous lines parallel to the x or y axes.

We will compare results using four different discretization methods and thus four different recursion formulae, expressing $u_{m+1,n+1}$ in terms of $u_{m,n}$, $u_{m+1,n}$ and $u_{m,n+1}$. For comparison we present the four formulae for the first position of the stencil, i.e., $m = n = 0$. In all cases the left hand side of (1.2) is approximated by

$$u_{11}u_{00} - u_{10}u_{01} = h^2k^2(uu_{xy} - u_xu_y),$$

where h and k are the lengths of the steps in the x and y directions, respectively. The right-hand side of (1.2) is approximated differently in each case. The corresponding recursion formulae and their continuous limits up to one order beyond the leading one are:

1. The *invariant* method (2.7), (2.8) (preserving the $\mathrm{SL}_x(2, \mathbb{R}) \otimes \mathrm{SL}_y(2, \mathbb{R})$ symmetry group as point symmetries)

$$u_{11}u_{00} - u_{10}u_{01} = hk[au_{01}u_{10} + (1-a)u_{00}u_{11}] \mathrm{sign}(u_{01}u_{10})\sqrt{|u_{01}u_{10}|}, \quad (5.3)$$

$$uu_{xy} - u_xu_y = u^3 + (3u^2u_y + u_{yy}u_x + uu_{xyy})\frac{k}{2} + (3u^2u_x + u_yu_{xx} - uu_{xxy})\frac{h}{2}. \quad (5.4)$$

2. The Rebelo and Valiquette method (preserving the entire infinite-dimensional symmetry algebra as generalized symmetries)

$$u_{11}u_{00} - u_{10}u_{01} = hku_{00}u_{01}u_{10}, \quad (5.5)$$

$$uu_{xy} - u_xu_y = u^3 + (2u^2u_y + u_xu_{yy} - uu_{xyy})\frac{k}{2} + (2u^2u_x + u_{xx}u_y - uu_{xxy})\frac{h}{2}.$$

3. The Adler–Startsev method (preserving linearizability of the Liouville equation)

$$u_{11}u_{00} - u_{10}u_{01} = hku_{00}u_{11} \left[\frac{u_{01} + u_{10}}{2} - hku_{01}u_{10} \right], \quad (5.6)$$

$$uu_{xy} - u_xu_y = u^3 + (2u^2u_y + u_xu_{yy} - uu_{xyy})\frac{k}{2} + (2u^2u_x + u_{xx}u_y - uu_{xxy})\frac{h}{2}. \quad (5.7)$$

4. The *standard* method (not preserving any specific structure) defined on the 4 points of a square lattice is

$$u_{11}u_{00} - u_{10}u_{01} = hku_{00}^3, \quad (5.8)$$

$$uu_{xy} - u_xu_y = u^3 + (u_{xx}u_y - uu_{xxy})\frac{h}{2} + (u_xu_{yy} - uu_{xyy})\frac{k}{2}.$$

Each of these formulae gives a different explicit expression for u_{11} in terms of the already known values of u_{00} , u_{01} and u_{10} .

Several comments are in order:

Table 1. Relative mean square distance (5.1) between the numerical solutions and the analytic one in the domain \mathcal{D}_0 .

	χ_{Inv}	χ_{AS}	χ_{RV}	χ_{stand}
f_1	5.2×10^{-6}	2.7×10^{-6}	3.1×10^{-4}	9.2×10^{-4}
f_2	3.4×10^{-4}	1.5×10^{-4}	7.6×10^{-3}	2.2×10^{-2}
f_3	4.7×10^{-5}	1.5×10^{-5}	3.0×10^{-3}	9.2×10^{-3}
f_4	4.3×10^{-5}	7.9×10^{-5}	5.2×10^{-3}	2.0×10^{-2}
f_5	3.8×10^{-2}	3.0×10^{-2}	2.8×10^{-1}	4.3×10^{-1}

1. In (5.4) there is no dependence on the parameter a . It will only appear at the order hk (not h^2 or k^2). That is the reason why the dependence of the numerical results depends weakly on the choice of a (see Fig. 7 below to confirm this).

2. Formulas (5.4) (invariant method) and (5.7) (Adler–Startsev method) coincide. They differ in the higher-order terms. Table 1, as could be expected, confirms that the results obtained by these two methods are similar.

We consider 5 different solutions of the continuous algebraic Liouville equation (1.2), namely

$$\begin{aligned}
 f_1 &= \frac{2}{(x^2 + 1)(y^2 + 1)(\tan^{-1}(x) + \tan^{-1}(y) + 6)^2}, \\
 f_2 &= \frac{8(1 - 4(x + \frac{1}{2}))(1 - 4y)\exp(-2x(1 + 2x) - 2y(y + 2))}{(e^{-2x(1+2x)} + e^{2y(1-2y)} + 1)^2}, \\
 f_3 &= -\frac{3.38 \sin(1.3(x + 10^{-2})) \cos(1.3(y + 10^{-2}))}{(\cos(1.3(x + 10^{-2})) + \sin(1.3(y + 10^{-2})) + 3)^2}, \\
 f_4 &= \frac{8xy}{(x^2 + y^2 + 2)^2}, \\
 f_5 &= \frac{383.1e^{3.862(2.5(x-0.5)+0.4y+2.5)}}{(e^{9.655(x+0.5)} + 12.83e^{1.545y})^2}.
 \end{aligned} \tag{5.9}$$

The functions f_1 and f_5 do not contain any zeroes in any finite domain. The functions f_2 and f_4 have one row and one column of zeroes each. The function f_3 contains infinitely many orthogonal lines of zeroes, since it is a periodic function. Finally, f_5 is a wall like function, with no zeroes. We mention that for f_1 , f_3 and f_4 the first-order corrections in (5.4) and (5.7) vanish. Plots of the exact solutions f_1, \dots, f_4 are given in Fig. 4, f_5 on Fig. 6a below. The right-hand sides of (5.5), (5.6) and (5.8) are polynomials whereas the invariant case (5.3) involves square roots.

The numerical computations were performed on the square domain $\mathcal{D}_0 = [-1.5, 1.1] \times [-1.0, 1.6]$, with steps of equal length $h = k = 0.02$, for a lattice of 130×130 points. Somewhat arbitrarily we choose the parameter a in the symmetry invariant recursion formula to be $a = 1.0$. The boundary conditions are given on the bottom and left side of the square.

In Table 1 we give the χ quantity (5.1) for all five solutions using 4 different methods. We see from Table 1 that χ_{Inv} and χ_{AS} are in general of the same order, as are χ_{RV} and χ_{stand} . The values of χ_{Inv} and χ_{AS} are better than those of χ_{RV} and χ_{stand} by at least one order of magnitude, usually by 2 orders, with χ_{RV} always better than χ_{stand} . The faster the solution changes, the worse is the result (for all methods), specially for the solutions f_2 and f_5 .

In order to test the stability of the algorithms with respect to the size of the adopted meshes, we made another series of calculations involving the above test functions over a fixed domain $\mathcal{D}_1 = [-1.905, 1.895] \times [-1.905, 1.895]$, larger than \mathcal{D}_0 , and spanned it using different lattice scales with $h = k$. A general flavor of such calculations can be extracted from Table 2 for

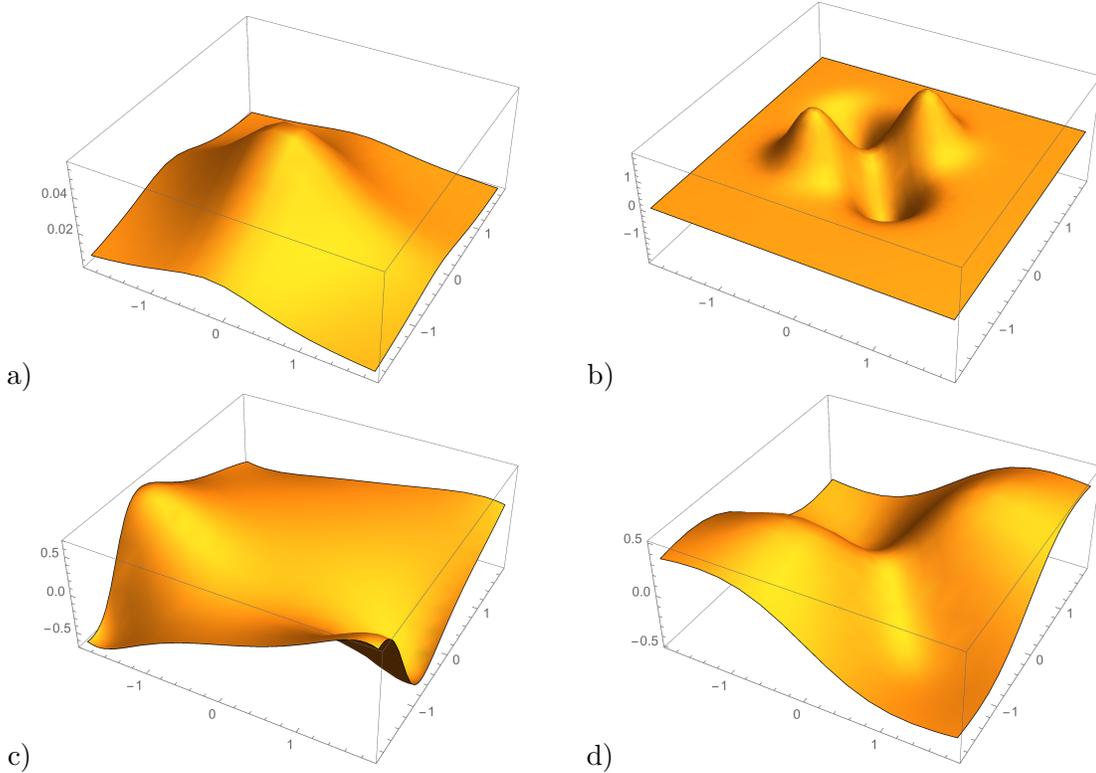


Figure 4. Plots of the exact solutions (a) f_1 , (b) f_2 , (c) f_3 and (d) f_4 in the domain \mathcal{D}_1 used in the numerical integrations.

Table 2. $\log_{10}\chi$ in the domain \mathcal{D}_1 for each discretization procedure we considered changing the size of the lattice meshes. We report only the results for f_1 with the parameter $a = 1$.

$h = k$	χ_{Inv}	χ_{AS}	χ_{RV}	χ_{stand}
$8. \times 10^{-1}$	-3.98219	-4.30917	-2.57467	-2.13488
$4. \times 10^{-1}$	-4.5782	-4.91072	-2.87464	-2.41647
$2. \times 10^{-1}$	-5.17707	-5.51254	-3.17517	-2.7076
$1. \times 10^{-1}$	-5.77759	-6.11449	-3.47595	-3.00363

the function f_1 For the solution f_1 (with no zeroes) the value of χ_{Inv} and χ_{AS} are comparable and at least two orders of magnitude lower than the other two. The values of χ_{RV} are always lower than χ_{stand} but of the same order. Generally speaking, decreasing the scale of the mesh by a factor of 0.3 implies decreasing the value of χ by a factor of 0.6 for the invariant and Adler–Startsev discretization and by a factor of 0.3 for the other two discretization (quadratic as opposed to linear convergence).

Let us now turn to the point-by-point behavior of the solutions. As discussed in Section 2 a characteristic property of all solutions of the Liouville equation is that zeroes are not isolated but occur in straight lines parallel to the axes. To see how well this is reflected in numerical solutions, let us first concentrate on solution f_2 which has zeroes on horizontal and vertical lines passing through the saddle point $x = -0.25$, $y = +0.25$. These points are not on the lattice due to the definition of the domain \mathcal{D}_1 with $h = k = 0.01$. On Table 3 we give the values of the solution f_2 at the four lattice points nearest to the saddle point. The AS solution has the first 4 digits coinciding with the exact one, the invariant one has 3. The RV and the standard method have 2. Similar results are also valid for the other solutions with zeroes (f_3 and f_4).

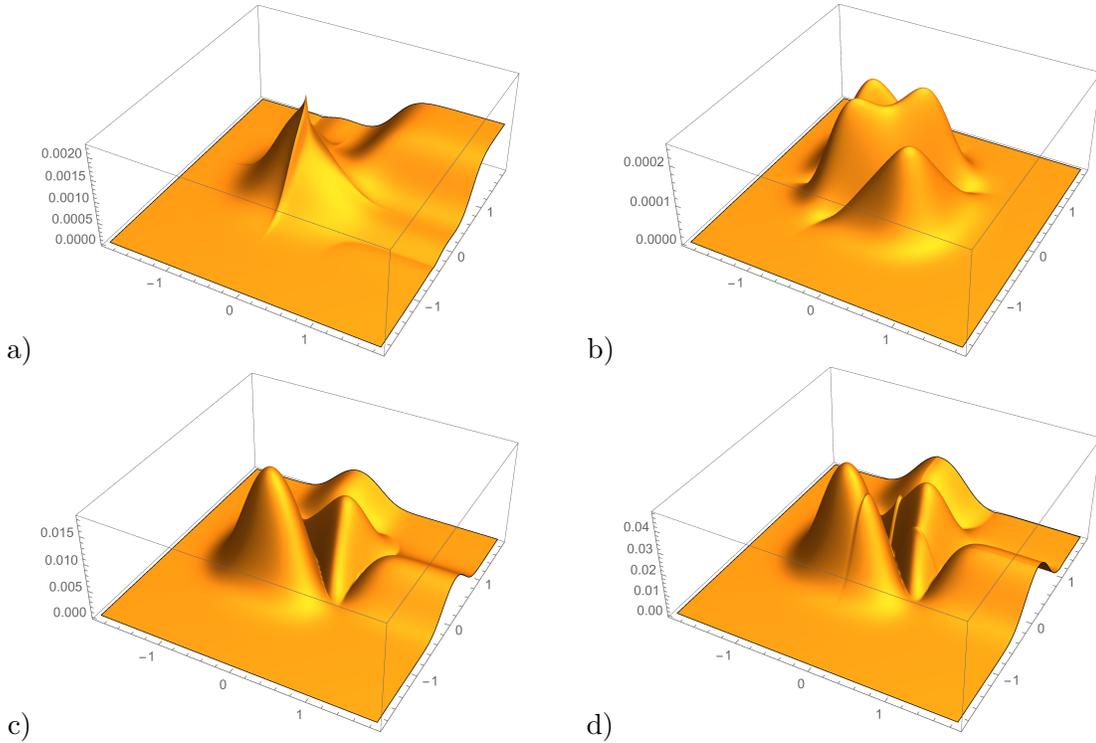


Figure 5. Plots of the relative error $R_{i,j}$ defined in (5.2) for the numerical approximation of the solution f_2 in the domain \mathcal{D}_1 using the individual discretizations: (a) Invariant, (b) Adler–Startsev, (c) Rebelo–Valiquette and (d) standard. The graphs are not in the same scale. The maximal value of the error is approximately 2×10^{-3} , 2×10^{-4} , 1.5×10^{-2} , 4×10^{-2} for the discretizations (a), (b), (c) and (d), respectively.

Table 3. Value of f_2 at the four lattice points nearest to the saddle point $(-0.25, +0.25)$ for the exact solution and the four numerical approximations.

point	Exact	Inv	AS	RV	stand
$(-0.255, 0.255)$	$-4.14391 \cdot 10^{-4}$	$-4.1419 \cdot 10^{-4}$	$-4.14369 \cdot 10^{-4}$	$-4.1281 \cdot 10^{-4}$	$-4.11579 \cdot 10^{-4}$
$(-0.245, 0.255)$	$4.14391 \cdot 10^{-4}$	$4.14257 \cdot 10^{-4}$	$4.14369 \cdot 10^{-4}$	$4.12877 \cdot 10^{-4}$	$4.11513 \cdot 10^{-4}$
$(-0.255, 0.245)$	$4.14391 \cdot 10^{-4}$	$4.14123 \cdot 10^{-4}$	$4.14369 \cdot 10^{-4}$	$4.12744 \cdot 10^{-4}$	$4.11645 \cdot 10^{-4}$
$(-0.245, 0.245)$	$-4.14391 \cdot 10^{-4}$	$-4.1419 \cdot 10^{-4}$	$-4.14369 \cdot 10^{-4}$	$-4.1281 \cdot 10^{-4}$	$-4.11579 \cdot 10^{-4}$

The behavior of the solution f_2 is plotted on Fig. 5 for the entire region \mathcal{D}_1 where we show the values of the error function $R_{i,j}$ of (5.2). The maximal value of $R_{i,j}$ is 2×10^{-4} in Fig. 5b for the AS method, 10 times higher for the invariant method, 75 higher for RV and 200 times higher for the standard one. For the invariant method the error is concentrated at the saddle point (see Fig. 5a) with a tail behind it. The AS method has maximal error at the four extrema (see Fig. 5b) with no tail. The other two methods have maximal errors on the maxima of the solutions (not the minima) with tails in both directions.

On Fig. 6 we analyze the solution f_5 in detail in a point-by-point manner in the domain \mathcal{D}_1 . Fig. 6a represents the exact solution (a wall of constant height). It has no zeroes anywhere in the finite real plane. The height of the wall gradually decreases for solutions b, d and e, but much more slowly for the invariant method b. For the AS method the height increases so we present the height of the solution on Fig. 6c in a different scale. The increase in Fig. 6c is comparable with the decrease in Fig. 6b (a factor of about 2.5).

Finally we analyze the role of the parameter a in the formula (2.7), (2.8). The continuous limit (5.4) shows that a appears for the first time in terms of order $O(hk)$. Numerical calculations

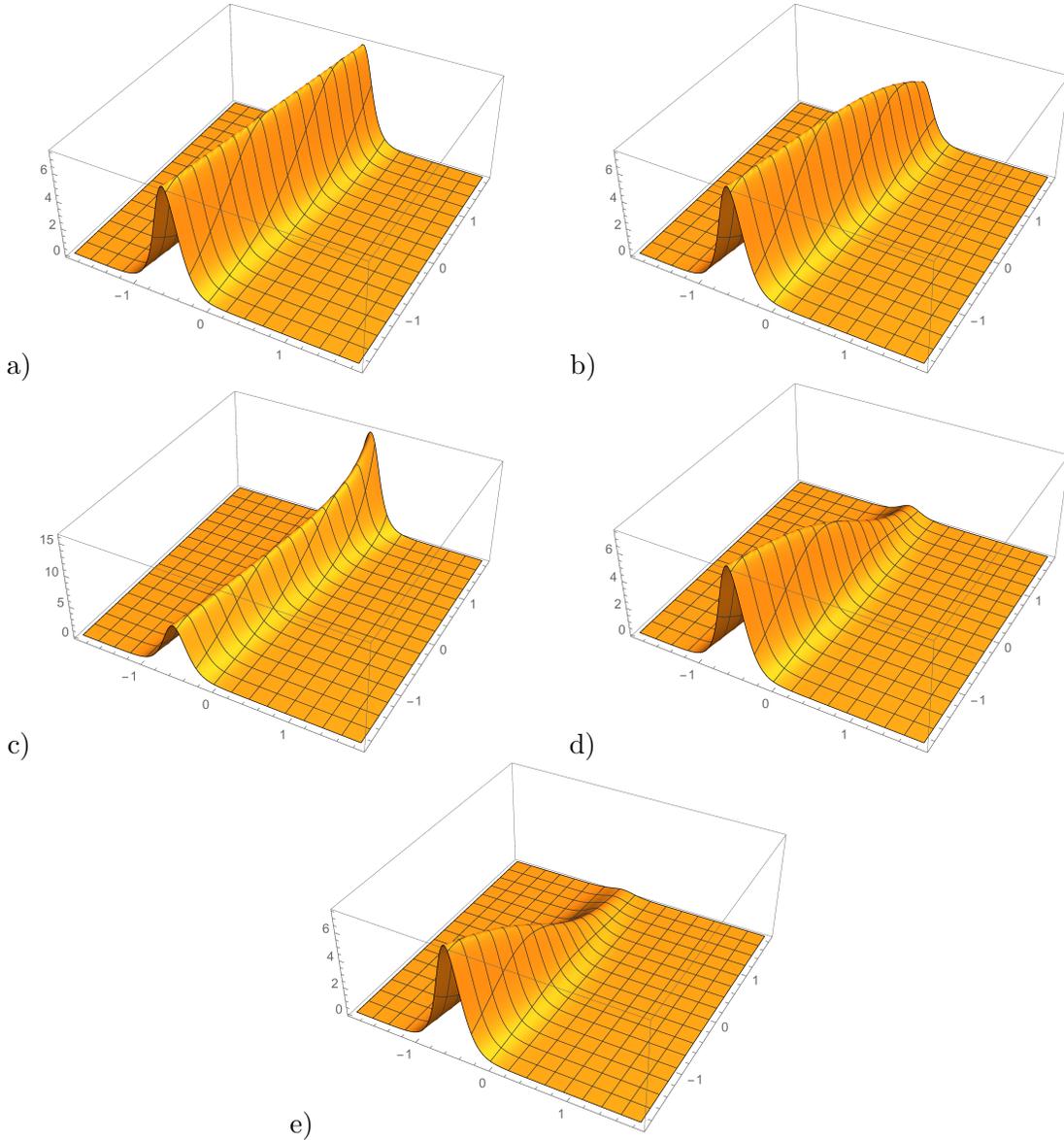


Figure 6. Pictures of the function f_5 in the domain \mathcal{D}_1 for the exact solution (a) and for the individual discretizations: (b) Invariant, (c) Adler–Startsev, (d) Rebelo–Valiquette and (e) standard. It is clear by inspection that in this case all proposed discretizations are numerically unstable, leading to decaying or blow up of the computed solutions by recursive formulae.

of χ_{Inv} for the function f_2 shows a variation of about a factor 2, when $a \in [-0.5, 1.5]$ (see Fig. 7). Furthermore, this function takes a minimum for $a = 0.17$. However, this value is strongly dependent on the test function considered.

6 Conclusions

Both from the point of physics and from the point of view of geometric integration we see that for discretizing the Liouville equation we have to choose which characteristic feature of the equation we wish to preserve. Adler and Startsev [1] have shown how to preserve linearizability and the existence of a class of exact solutions depending on two arbitrary functions of one variable. We have shown that for a wide class of solutions a recurrence formula based on their method provides

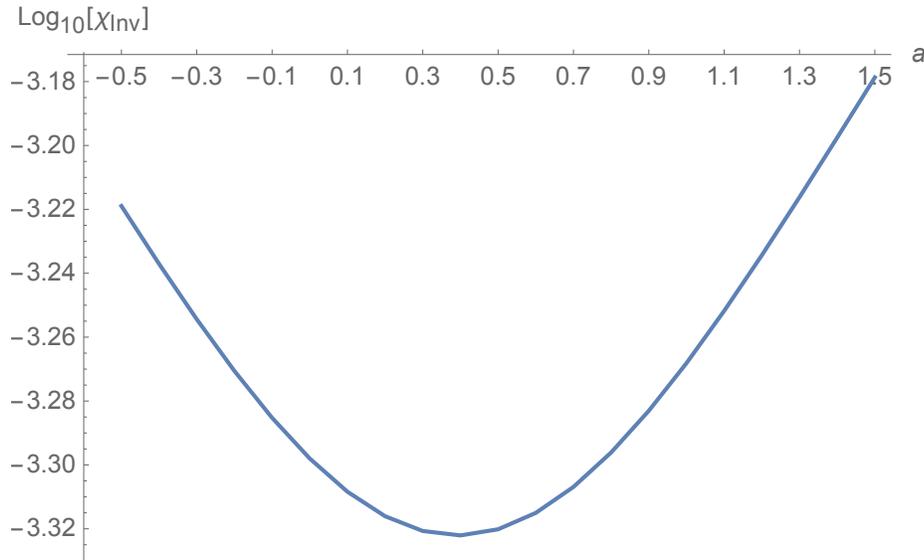


Figure 7. Plot of χ_{Inv} for the function f_2 in the parameter range $a \in [-0.5, 1.5]$.

the most accurate results, both using the global χ^2 criterion and comparing local point by point convergence using the R_{ij} criterion. On the other hand, linearizability, just like integrability is a property of a very restricted class of nonlinear PDEs.

The existence of a nontrivial Lie point symmetry group is a much more generic property, specially for PDEs coming from fundamental physical theories. From this point of view the Liouville equation is again special: its Lie point symmetry group is infinite-dimensional. Rebelo and Valiquette [38] have presented a discretization that preserves this entire infinite-dimensional symmetry group as a special type of generalized symmetries. As opposed to more general higher symmetries, their symmetries have a global group action and are very interesting from the theoretical point of view. From the numerical point of view of we have shown that that the precision of the RV solutions is systematically better than that of those obtained by the standard method (though of the same order of magnitude). The measure of the validity is given by the quantity χ of (5.1).

Finally, the method proposed in [23] and further developed in this article preserves point invariance under the maximal finite subgroup of the infinite-dimensional symmetry group. Numerical methods based on this *partial* preservation of symmetries perform very well for all solutions in some case even better than the Adler–Startsev case.

In future work we plan to study symmetry preserving discretizations of other equations with infinite-dimensional symmetry groups, such as the Kadomtsev–Petviashvili equation, and the three-wave interaction equation. A symmetry preserving discretization of the Korteweg–de Vries equation has provided encouraging results [3].

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References

- [1] Adler V.E., Startsev S.Ya., On discrete analogues of the Liouville equation, *Theoret. and Math. Phys.* **121** (1999), 1484–1495, [arXiv:solv-int/9902016](#).
- [2] Bihlo A., Invariant meshless discretization schemes, *J. Phys. A: Math. Theor.* **46** (2013), 062001, 12 pages, [arXiv:1210.2762](#).
- [3] Bihlo A., Coiteux-Roy X., Winternitz P., The Korteweg–de Vries equation and its symmetry-preserving discretization, *J. Phys. A: Math. Theor.* **48** (2015), 055201, 25 pages, [arXiv:1409.4340](#).
- [4] Bihlo A., Nave J.-C., Invariant discretization schemes using evolution-projection techniques, *SIGMA* **9** (2013), 052, 23 pages, [arXiv:1209.5028](#).
- [5] Bourlioux A., Cyr-Gagnon C., Winternitz P., Difference schemes with point symmetries and their numerical tests, *J. Phys. A: Math. Gen.* **39** (2006), 6877–6896, [math-ph/0602057](#).
- [6] Budd C., Dorodnitsyn V., Symmetry-adapted moving mesh schemes for the nonlinear Schrödinger equation, *J. Phys. A: Math. Gen.* **34** (2001), 10387–10400.
- [7] Dorodnitsyn V., Transformation groups in difference spaces, *J. Sov. Math.* **55** (1991), 1490–1517.
- [8] Dorodnitsyn V., Applications of Lie groups to difference equations, *Differential and Integral Equations and Their Applications*, Vol. 8, CRC Press, Boca Raton, FL, 2011.
- [9] Dorodnitsyn V., Kaptsov E., Kozlov R., Winternitz P., The adjoint equation method for constructing first integrals of difference equations, *J. Phys. A: Math. Theor.* **48** (2015), 055202, 32 pages, [arXiv:1311.1597](#).
- [10] Dorodnitsyn V., Kozlov R., A heat transfer with a source: the complete set of invariant difference schemes, *J. Nonlinear Math. Phys.* **10** (2003), 16–50, [math.AP/0309139](#).
- [11] Dorodnitsyn V., Kozlov R., Winternitz P., Lie group classification of second-order ordinary difference equations, *J. Math. Phys.* **41** (2000), 480–504.
- [12] Dorodnitsyn V., Kozlov R., Winternitz P., Continuous symmetries of Lagrangians and exact solutions of discrete equations, *J. Math. Phys.* **45** (2004), 336–359, [nlin.SI/0307042](#).
- [13] Dubrovin B.A., Fomenko A.T., Novikov S.P., Modern geometry – methods and applications. Part I. The geometry of surfaces, transformation groups, and fields, *Graduate Texts in Mathematics*, Vol. 93, 2nd ed., Springer-Verlag, New York, 1992.
- [14] Floreanini R., Negro J., Nieto L.M., Vinet L., Symmetries of the heat equation on the lattice, *Lett. Math. Phys.* **36** (1996), 351–355.
- [15] Floreanini R., Vinet L., Lie symmetries of finite-difference equations, *J. Math. Phys.* **36** (1995), 7024–7042.
- [16] Floreanini R., Vinet L., Quantum symmetries of q -difference equations, *J. Math. Phys.* **36** (1995), 3134–3156.
- [17] Grammaticos B., Ramani A., Painlevé equations, continuous, discrete and ultradiscrete, in Symmetries and Integrability of Difference Equations (Université de Montréal, Montréal, QC, June 8–21, 2008), *London Mathematical Society Lecture Note Series*, Vol. 381, Editors D. Levi, P.J. Olver, Z. Thomova, P. Winternitz, Cambridge University Press, Cambridge, 2011, 50–82.
- [18] Grammaticos B., Ramani A., Papageorgiou V., Do integrable mappings have the Painlevé property?, *Phys. Rev. Lett.* **67** (1991), 1825–1828.
- [19] Grant T.J., Bespoke finite difference schemes that preserve multiple conservation laws, *LMS J. Comput. Math.* **18** (2015), 372–403.
- [20] Hairer E., Lubich C., Wanner G., Geometric numerical integration. Structure-preserving algorithms for ordinary differential equations, *Springer Series in Computational Mathematics*, Vol. 31, 2nd ed., Springer-Verlag, Berlin, 2006.
- [21] Hydon P.E., Difference equations by differential equation methods, *Cambridge Monographs on Applied and Computational Mathematics*, Cambridge University Press, Cambridge, 2014.

- [22] Iserles A., A first course in the numerical analysis of differential equations, 2nd ed., *Cambridge Texts in Applied Mathematics*, Cambridge University Press, Cambridge, 2009.
- [23] Levi D., Martina L., Winternitz P., Lie-point symmetries of the discrete Liouville equation, *J. Phys. A: Math. Theor.* **48** (2015), 025204, 18 pages, [arXiv:1407.4043](#).
- [24] Levi D., Olver P.J., Thomova Z., Winternitz P. (Editors), Symmetries and integrability of difference equations, *London Mathematical Society Lecture Note Series*, Vol. 381, Cambridge University Press, Cambridge, 2011.
- [25] Levi D., Rodríguez M.A., Construction of partial difference schemes: I. The Clairaut, Schwarz, Young theorem on the lattice, *J. Phys. A: Math. Theor.* **46** (2013), 295203, 19 pages.
- [26] Levi D., Rodríguez M.A., On the construction of partial difference schemes: II. Discrete variables and invariant schemes, [arXiv:1407.0838](#).
- [27] Levi D., Vinet L., Winternitz P., Lie group formalism for difference equations, *J. Phys. A: Math. Gen.* **30** (1997), 633–649.
- [28] Levi D., Winternitz P., Continuous symmetries of discrete equations, *Phys. Lett. A* **152** (1991), 335–338.
- [29] Levi D., Winternitz P., Continuous symmetries of difference equations, *J. Phys. A: Math. Gen.* **39** (2006), R1–R63, [nlin.SI/0502004](#).
- [30] Levi D., Winternitz P., Yamilov R.I., Lie point symmetries of differential-difference equations, *J. Phys. A: Math. Theor.* **43** (2010), 292002, 14 pages, [arXiv:1004.5311](#).
- [31] Lie S., Theorie der Transformationsgruppen, Bds. 1–3, B.G. Teubner, Leipzig, 1888, 1890, 1893.
- [32] Liouville J., Sur l'équation aux différences partielles $\frac{d^2 \log \lambda}{du dv} \pm \frac{\lambda}{2a^2} = 0$, *J. Math. Pures Appl.* **18** (1853), 71–72.
- [33] Marsden J.E., West M., Discrete mechanics and variational integrators, *Acta Numer.* **10** (2001), 357–514.
- [34] McLachlan R.I., Quispel G.R.W., Geometric integrators for ODEs, *J. Phys. A: Math. Gen.* **39** (2006), 5251–5285.
- [35] Medolaghi P., Classificazione delle equazioni alle derivate parziali del secondo ordine, che ammettono un gruppo infinito di trasformazioni puntuali, *Ann. Mat. Pura Appl.* **1** (1898), 229–263.
- [36] Olver P.J., Applications of Lie groups to differential equations, *Graduate Texts in Mathematics*, Vol. 107, 2nd ed., Springer-Verlag, New York, 1993.
- [37] Rebelo R., Valiquette F., Symmetry preserving numerical schemes for partial differential equations and their numerical tests, *J. Difference Equ. Appl.* **19** (2013), 738–757, [arXiv:1110.5921](#).
- [38] Rebelo R., Valiquette F., Invariant discretization of partial differential equations admitting infinite-dimensional symmetry groups, *J. Difference Equ. Appl.* **21** (2015), 285–318, [arXiv:1401.4380](#).
- [39] Rebelo R., Winternitz P., Invariant difference schemes and their application to $\mathfrak{sl}(2, \mathbb{R})$ invariant ordinary differential equations, *J. Phys. A: Math. Theor.* **42** (2009), 454016, 10 pages, [arXiv:0906.2980](#).
- [40] Rodríguez M.A., Winternitz P., Lie symmetries and exact solutions of first-order difference schemes, *J. Phys. A: Math. Gen.* **37** (2004), 6129–6142, [nlin.SI/0402047](#).
- [41] Valiquette F., Winternitz P., Discretization of partial differential equations preserving their physical symmetries, *J. Phys. A: Math. Gen.* **38** (2005), 9765–9783, [math-ph/0507061](#).
- [42] Winternitz P., Symmetry preserving discretization of differential equations and Lie point symmetries of differential-difference equations, in Symmetries and Integrability of Difference Equations (Université de Montréal, Montréal, QC, June 8–21, 2008), *London Mathematical Society Lecture Note Series*, Vol. 381, Editors D. Levi, P.J. Olver, Z. Thomova, P. Winternitz, Cambridge University Press, Cambridge, 2011, 292–341.