

# Topological Monodromy of an Integrable Heisenberg Spin Chain<sup>\*</sup>

Jeremy LANE

Department of Mathematics, University of Toronto,  
40 St. George Street, Toronto, Ontario, Canada M5S 2E4

E-mail: [jeremy.lane@mail.utoronto.ca](mailto:jeremy.lane@mail.utoronto.ca)

URL: <https://www.math.toronto.edu/cms/lane-jeremy/>

Received November 27, 2014, in final form July 29, 2015; Published online July 31, 2015

<http://dx.doi.org/10.3842/SIGMA.2015.062>

**Abstract.** We investigate topological properties of a completely integrable system on  $S^2 \times S^2 \times S^2$  which was recently shown to have a Lagrangian fiber diffeomorphic to  $\mathbb{R}P^3$  not displaceable by a Hamiltonian isotopy [Oakley J., Ph.D. Thesis, University of Georgia, 2014]. This system can be viewed as integrating the determinant, or alternatively, as integrating a classical Heisenberg spin chain. We show that the system has non-trivial topological monodromy and relate this to the geometric interpretation of its integrals.

*Key words:* integrable system; monodromy; Lagrangian fibration; Heisenberg spin chain

*2010 Mathematics Subject Classification:* 37J35; 53D12

## 1 Introduction

The Hamiltonian

$$H(X, Y, Z) = \sqrt{3 + \langle X, Y \rangle + \langle Y, Z \rangle + \langle Z, X \rangle}$$

models pairwise interaction of three identical spin vectors  $X, Y, Z \in S^2$  which are fixed to the vertices of an equilateral triangle. Systems of this type – called Heisenberg spin chains – are of interest to physicists as they provide a classical model for quantum spin in a fixed lattice. Together with the Hamiltonians

$$I(X, Y, Z) = \langle X + Y + Z, e_3 \rangle \quad \text{and} \quad J(X, Y, Z) = \det(X, Y, Z),$$

this Heisenberg spin chain becomes a completely integrable system on  $(S^2 \times S^2 \times S^2, \omega_{\text{STD}} \oplus \omega_{\text{STD}} \oplus \omega_{\text{STD}})$ . Lagrangian fibers of this system were recently studied in [16], and it was shown that the system has a Lagrangian fiber  $L \cong \mathbb{R}P^3$  which is not displaceable by Hamiltonian diffeomorphisms.

There is an analogous coupled spin system given by the Hamiltonians

$$H_1(X, Y) = \sqrt{1 - \langle X, Y \rangle} \quad \text{and} \quad H_2(X, Y) = \langle X + Y, e_3 \rangle,$$

which has provided interesting examples of non-displaceable Lagrangian tori in  $(S^2 \times S^2, \omega_{\text{STD}} \oplus \omega_{\text{STD}})$  [7, 8, 17, 20]. In fact,  $(H_1, H_2)$  is the moment map of a Hamiltonian  $T^2$ -action on the complement of the Lagrangian sphere  $\tilde{\Delta}$  of anti-diagonal elements  $(X, -X)$ , where  $H_1$  fails to be smooth.

---

<sup>\*</sup>This paper is a contribution to the Special Issue on Algebraic Methods in Dynamical Systems. The full collection is available at <http://www.emis.de/journals/SIGMA/AMDS2014.html>

In this note we study the integrable system  $\mathcal{H} = (H, I, J)$  from the perspective of Lagrangian torus fibrations and global action-angle coordinates, as introduced by Duistermaat in [5]. Unlike the system on  $S^2 \times S^2 \setminus \widetilde{\Delta}$ , the system  $\mathcal{H}$  is not toric on the set where it is smooth,  $S^2 \times S^2 \times S^2 \setminus L$ , and cannot be made so: there is a global obstruction to the existence of a diffeomorphism  $f$  such that the Hamiltonian flow of  $f \circ J$  is periodic. We compute this obstruction, called topological monodromy, explicitly

**Theorem 1.1.** *The topological monodromy of the system  $\mathcal{H}$  is generated by the matrix*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}.$$

*Thus the system does not admit global action coordinates (since  $H$  and  $I$  are already periodic, this implies that there does not exist a diffeomorphism  $f: \text{Im}(J) \rightarrow \mathbb{R}$  such that the Hamiltonian  $f \circ J$  is periodic).*

Although topological monodromy of a given integrable system may be computed directly using elliptic integrals – as has been done in [2, 5] and many other places – we derive this result from general theorems of Zung and Izosimov about the structure of integrable systems near singularities [12, 21]. Section 2 reviews the theory of singularities for integrable systems developed in [21] and the relation of this theory to the topological monodromy obstruction. Section 3 establishes basic facts and notation for adjoint orbits in Lie algebras which we use throughout the paper. Section 4 introduces the system  $\mathcal{H}$  and provides an algebraic proof that the Hamiltonians  $H, I, J$  Poisson commute. In order to apply the theorems of Zung and Isozimov to this system, we must describe the topology of the critical fibers of the map  $\mathcal{H}$ . In Section 5 we use the underlying Euclidean geometry of the system and the structure of the Lie algebra  $\mathfrak{so}(3)$  to completely describe the critical set, critical values and image of  $\mathcal{H}$ ,

**Theorem 1.2.** *The image of the moment map  $\mathcal{H}$  is the set of points  $(r, s, t) \in \mathbb{R}^3$  that satisfy the equations*

$$|t| \leq \sqrt{1 + 2 \left( \frac{r^2 - 3}{6} \right)^3 - 3 \left( \frac{r^2 - 3}{6} \right)^2}, \quad |s| \leq r, \quad \text{and} \quad 0 \leq r \leq 3 \quad (1.1)$$

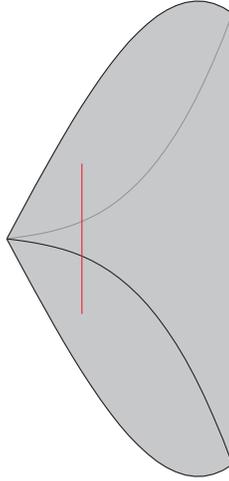
(see Figs. 1 and 2). *The set of critical values consists of the boundary of this set, together with the line segment  $\{(1, s, 0) : -1 < s < 1\}$ .*

We show in Section 7 that the critical fibers above the line segment  $\{(1, s, 0) : -1 < s < 1\}$ , which we call the ‘critical line’, are all topologically stable, rank 1, focus-focus (see Section 2 for definitions). Theorem 1.1 follows from this by the theorems of Zung and Izosimov.

Although it is not necessary in order to prove Theorem 1.1, we give a hands-on proof that the regular fibers of  $\mathcal{H}$  are all connected in Section 6. Combining our description of the moment map image, and results of [5], the Lagrangian torus fibration over the set of regular values of  $\mathcal{H}$  is determined up to fiber-preserving symplectomorphism by Theorem 1.1 (see Remark 7.3).

## 2 Singularities and topological monodromy of integrable systems

In this section we review the theory of singularities for integrable systems developed in [21] and its relation to topological monodromy. We make several simplifying assumptions (such as fiber connectedness) and the reader should note that Theorem 2.10 is not stated in full generality. Historical background and further details can be found in [3, 21].



**Figure 1.** The moment map image is a solid with one ‘orbifold’ corner, four ‘toric’ faces, and a ‘critical line’ through the interior (red).

**Definition 2.1.** An *integrable system* is a triple  $(M, \omega, \mathcal{H})$  where  $(M, \omega)$  is a symplectic manifold of dimension  $2n$ , and  $\mathcal{H} = (H_1, \dots, H_n): M \rightarrow \mathbb{R}^n$  is a smooth map such that

- 1) the Poisson bracket  $\{H_i, H_j\} = 0$  for all  $i, j$ , and
- 2) there is an open, dense subset of  $M$  where the functions  $H_1, \dots, H_n$  are all functionally independent (i.e.,  $dH_1 \wedge \dots \wedge dH_n \neq 0$ ).

We denote the image of  $\mathcal{H}$  by  $B$ , the set of regular values by  $B_{\text{reg}}$ , and preimage  $\mathcal{H}^{-1}(B_{\text{reg}})$  by  $M_{\text{reg}}$ .

**Definition 2.2.** An integrable system  $(M, \omega, \mathcal{H})$  admits *global action coordinates* if there is a diffeomorphism  $a = (a_1, \dots, a_n): B \rightarrow \mathbb{R}^n$  such that the Hamiltonian flow of  $a_i \circ H_i$  is periodic for each  $1 \leq i \leq n$ . The map  $a$  is often referred to as *global action coordinates* for  $\mathcal{H}$ .

If an integrable system with compact fibers admits global action coordinates, then  $(M, \omega, a \circ \mathcal{H})$  is a symplectic toric manifold. Thus, an important question in the study of integrable systems is

**Question 2.3.** *Does a given integrable system  $(M, \omega, \mathcal{H})$  admit global action coordinates?*

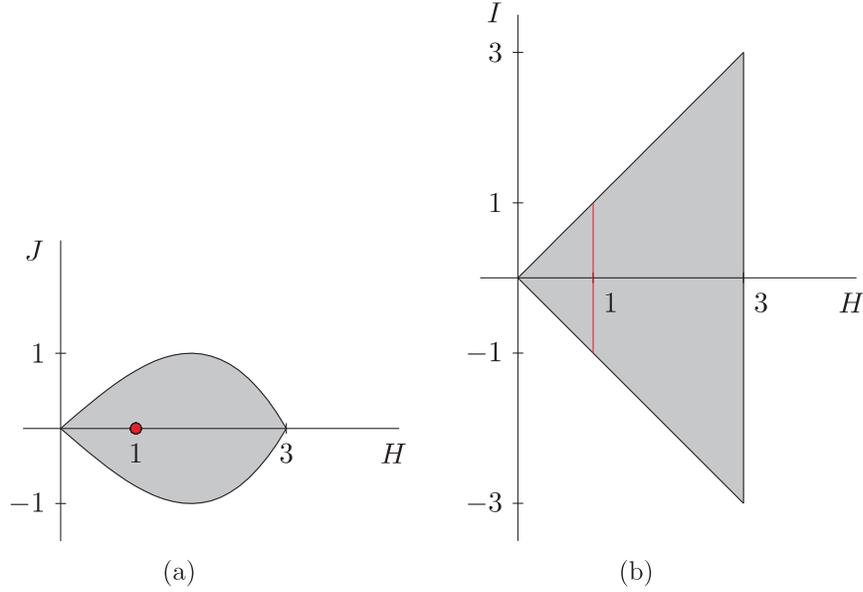
Suppose the fibers of  $\mathcal{H}$  are compact and connected; then the regular fibers are Lagrangian  $n$ -tori and the Arnold–Liouville theorem implies that the restricted map  $\mathcal{H}: M_{\text{reg}} \rightarrow B_{\text{reg}}$  is a fiber bundle of compact, connected Lagrangian tori  $T^n$  that is locally symplectically equivalent to the trivial system

$$\mathcal{H} = (x_1, \dots, x_n): (D^n \times T^n, dx \wedge d\theta) \rightarrow D^n.$$

Regular integrable systems such as this are called *Lagrangian torus fibrations*. Given a Lagrangian torus fibration, there is a natural fiber-wise action

$$T^*B \times_B M \rightarrow M$$

defined by  $(\alpha_b, m) \mapsto \phi_{\alpha_b}^1(m)$  where  $\phi_{\alpha_b}^1$  is the time-1 flow of the vector-field  $\omega^{-1}\mathcal{H}^*\alpha$  for some 1-form  $\alpha \in \Omega^1(B)$  such that  $\alpha(b) = \alpha_b$ . The stabilizer of this action is a smooth submanifold  $\Lambda \subset T^*B$  that is a full rank lattice in each fiber, called the *period lattice* of the fibration. Restricting



**Figure 2.** Projections of the moment map image.

the projection map  $\pi: T^*B \rightarrow B$  to this lattice, we obtain a covering space  $\pi: \Lambda \rightarrow B$  whose topological monodromy is a group homomorphism  $m_b: \pi_1(B, b) \rightarrow GL(n, \mathbb{Z})$  after identifying the fiber  $\Lambda_b$  with  $\mathbb{Z}^n$  by a choice of basis (see, e.g., [15] or [5] for full details of this construction). The *topological monodromy* of a Lagrangian torus fibration is the topological monodromy of its period lattice, and its non-triviality is the obstruction to the torus fiber bundles

$$T^*B/\Lambda \rightarrow B \quad \text{and} \quad M \rightarrow B$$

being principal, and thus to  $\mathcal{H}$  admitting global action coordinates. In his seminal 1980 paper *On global action-angle coordinates*, Duistermaat proved

**Theorem 2.4** ([5]). *A Lagrangian torus fibration  $\mathcal{H}: M \rightarrow B$  admits global action coordinates if and only if the topological monodromy of the period lattice is trivial.*

An integrable system with compact, connected fibers admits global action coordinates only if the corresponding Lagrangian torus fibration  $M_{\text{reg}} \rightarrow B_{\text{reg}}$  admits global action coordinates, so to answer Question 2.3 one may begin by studying the topological monodromy of this fibration. On the other hand, there is a subtle interplay between the topological monodromy of  $M_{\text{reg}} \rightarrow B_{\text{reg}}$  and the critical fibers of  $\mathcal{H}$ . To describe this interplay we need to recall several definitions and theorems from [12, 21].

Let  $p$  be a critical point of rank  $n - k$  of an integrable system  $\mathcal{H} = (H_1, \dots, H_n)$  on  $(M, \omega)$ . Without loss of generality, we may assume that  $dH_1, \dots, dH_k = 0$ . The operators  $\omega^{-1}d^2H_1, \dots, \omega^{-1}d^2H_k$  form a commutative subalgebra  $\mathfrak{h}$  of  $\mathfrak{sp}(L^\perp/L) \cong \mathfrak{sp}(\mathbb{R}, 2k)$  where the subspace  $L \subset T_pM$  is the span of the vector fields  $X_{H_{k+1}}, \dots, X_{H_n}$  and  $L^\perp$  is its symplectic orthocomplement. A generic critical point of  $\mathcal{H}$  will satisfy the following Morse–Bott type condition for integrable systems:

**Definition 2.5.** A rank  $n - k$  critical point  $p$  of an integrable system  $\mathcal{H}$  is *non-degenerate* if the subalgebra  $\mathfrak{h}$  defined above is a Cartan subalgebra. Similarly, a critical fiber of an integrable system is *non-degenerate* if all of its critical points are non-degenerate.

Equivalently,  $p$  is non-degenerate if some linear combination of the operators  $\omega^{-1}d^2H_1, \dots, \omega^{-1}d^2H_k$  has  $2k$  distinct eigenvalues (and the operators are independent). It was shown in [19]

that a Cartan subalgebra  $\mathfrak{h} \subset \mathfrak{sp}(\mathbb{R}, 2k)$  decomposes as a direct sum of three types of Lie subalgebra, called *elliptic*, *hyperbolic*, and *focus-focus*, and the conjugacy class of a given Cartan subalgebra  $\mathfrak{h}$  can be described by the three positive integers  $h_e$ ,  $h_h$ , and  $h_f$ , which denote the number of elliptic, hyperbolic, and focus-focus blocks respectively in this decomposition<sup>1</sup>.

**Definition 2.6.** The *Williamson type* of a non-degenerate critical point  $p$  for an integrable system  $\mathcal{H}$  is the 3-tuple  $(h_e, h_h, h_f)$  corresponding to the Cartan subalgebra  $\mathfrak{h} \subset \mathfrak{sp}(\mathbb{R}, 2k)$ .

The remainder of this paper is primarily concerned with critical points for which the Williamson type is  $(h_e, 0, h_f)$ ; such critical points are sometimes called *almost toric*. A fiber is called *almost toric* if all the critical points it contains are almost toric.

It was shown in [21] that the Williamson type of a critical point of lowest rank in a non-degenerate critical fiber is an invariant of the fiber, which allows the definition,

**Definition 2.7.** The *Williamson type* of a non-degenerate critical fiber  $N$  in an integrable system  $(M, \omega, \mathcal{H})$  with compact and connected fibers is the Williamson type of any critical point of lowest rank in the fiber. The *rank* of  $N$  is the rank of these critical points.

Assuming once again that all the fibers of an integrable system are compact and connected, we are interested in the foliation of a neighbourhood  $\mathcal{U}(N)$  of a critical fiber  $N$  by the fibers of  $\mathcal{H}$  (this is the *Liouville foliation*, although it is defined differently in more general settings, see [21]). We consider neighbourhoods of the form  $\mathcal{U}(N) = \mathcal{H}^{-1}(D)$  where  $D$  is a small disc centred at  $\mathcal{H}(N)$ . Two foliations are said to be *topologically equivalent* if they are related by a foliation preserving homeomorphism. A *singularity* of an integrable system is an equivalence class of a Liouville foliation in a neighbourhood of  $N$ .

Let  $(H_1, H_2)$  be an integrable system with compact, connected fibers on a symplectic 4-manifold  $M$ . The (topological equivalence class of a) Liouville foliation in a neighbourhood of critical fiber  $N$  whose only critical points are non-degenerate, rank 0, focus-focus is called a *stable focus-focus singularity*. The critical fiber  $N$  of such a singularity is a disjoint union of  $c \geq 1$  critical points  $x_1, \dots, x_c$  and  $c$  open annuli  $A_1, \dots, A_c$ ,  $A_i \cong \mathbb{R} \times S^1$ , such that each annulus  $A_i$  has  $\{x_i, x_{i+1}\}$  as its boundary.

**Theorem 2.8** ([22]). *If  $N$  is a stable focus-focus singularity of an integrable system which contains  $c \geq 1$  critical points then there is a neighbourhood  $\mathcal{U}(N) = \mathcal{H}^{-1}(D^2)$  such that  $\mathcal{H}(N) = 0$  is the only critical value in  $D^2$ , and the topological monodromy of the torus fibration  $\mathcal{U}(N) \setminus N \rightarrow D^2 \setminus \{0\}$  is generated by*

$$A = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$$

(i.e., up to a choice of basis for  $\mathbb{Z}^n$ ,  $m_b(\gamma) = A$  for a representative  $\gamma$  of a generator of  $\pi_b(D^2 \setminus \{0\})$ ). Furthermore, every two such Liouville foliations are topologically equivalent.

Note that since  $m_b$  is a group homomorphism,

$$m_b(\gamma^{-1}) = A^{-1} = \begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix},$$

but  $A^{-1}$  and  $A$  are in the same  $GL(n, \mathbb{Z})$  conjugacy class.

In particular, a model Liouville foliation  $\mathcal{U}(N_c^f) \rightarrow D^2$  can be constructed for any  $c \geq 1$  [22]. It has been shown that this prototypical model can be used to understand almost toric Liouville foliations in higher degrees of freedom up to topological equivalence, under an additional assumption:

<sup>1</sup>See Appendix A for a more explicit description of these subalgebras.

**Definition 2.9** ([21, Definition 6.3]). A non-degenerate singularity  $\mathcal{H}: \mathcal{U}(N) \rightarrow D^n$  of an integrable system is called *topologically stable* if the local critical value set of the moment map restricted to  $\mathcal{U}(N)$  coincides with the critical value set of the moment map restricted to a small neighbourhood of a singular point of minimal rank in  $N$ .

The following theorem was proven for more general singularities, but we only state it for almost toric fibers.

**Theorem 2.10** ([21]). *Suppose  $N$  is a rank  $n - k$ , topologically stable, almost toric critical fiber of an integrable system  $\mathcal{H}$  with compact and connected fibers. Then the Liouville foliation of  $\mathcal{U}(N)$  is topologically equivalent to a finite quotient of the product foliation:*

$$((D^{n-k} \times T^{n-k}) \times \mathcal{U}(N_{h_e}^e) \times \mathcal{U}(N_{c_1}^f) \times \cdots \times \mathcal{U}(N_{c_f}^f))/G,$$

where  $G$  is a finite group which acts freely and component-wise on each factor in a foliation preserving manner<sup>2</sup>. Moreover,  $G$  acts trivially on the elliptic component.

This result was later strengthened by Izosimov,

**Theorem 2.11** ([12]). *The Liouville foliation of  $\mathcal{U}(N)$  in the preceding theorem is topologically equivalent to the product foliation of*

$$(D^{n-k} \times T^{n-k}) \times \mathcal{U}(N_{h_e}^e) \times (\mathcal{U}(N_{c_1}^f) \times \cdots \times \mathcal{U}(N_{c_f}^f))/G.$$

Given an almost toric critical fiber, this theorem implies that the associated Lagrangian torus fibration is topologically equivalent to the associated Lagrangian torus fibration of a model product system. In particular, both fibrations will have the same topological monodromy, since it is a topological invariant of torus fibrations. The topological monodromy of a product of Lagrangian torus fibrations  $\mathcal{H} \times \mathcal{H}': M \times M' \rightarrow B \times B'$  is the topological monodromy of the lattice  $\Lambda_{\mathcal{H} \times \mathcal{H}'} \subset T^*B \times T^*B'$  which decomposes as the product covering space  $\Lambda_{\mathcal{H}} \times \Lambda_{\mathcal{H}'} \rightarrow B \times B'$ , and the topological monodromy of this covering space decomposes as

$$m_b(\gamma) = \begin{pmatrix} m_{b, \mathcal{H}}(\gamma) & 0 \\ 0 & m_{b, \mathcal{H}'}(\gamma) \end{pmatrix}.$$

**Remark 2.12.** In Section 7 we will use Theorem 2.11 to conclude that (for the Heisenberg spin system) the Liouville foliation above the critical line is homeomorphic to a product, and thus the topological monodromy around the critical line decomposes as a product, as in the preceding discussion. In fact, one can prove this more directly by using the description in [21] of the group  $G$  appearing in Theorem 2.10, and our explicit identification of the singular fibers with the product  $S^1 \times N_3^f$  (Remark 7.1) to show that the group  $G$  will vanish, and hence the foliation is homeomorphic to a product.

### 3 Symplectic geometry of coadjoint orbits

Let  $G$  be a compact, connected Lie group with Lie algebra  $\mathfrak{g}$  endowed with an Ad-invariant inner product  $\langle \cdot, \cdot \rangle$ . After  $G$ -equivariant identification of  $\mathfrak{g}$  with  $\mathfrak{g}^*$  via the inner product, the Kostant–Kirillov–Souriau symplectic structure on an adjoint orbit  $\mathcal{O}_Z$  of  $Z \in \mathfrak{g}$  is given by

$$\omega_Z([X, Z], [Y, Z]) = \langle Z, [X, Y] \rangle,$$

<sup>2</sup>Here  $\mathcal{U}(N_{h_e}^e)$  is the model Liouville foliation  $D^{2h_e} \rightarrow \mathbb{R}^{h_e}$  given by  $(x_1, y_1, \dots, x_{h_e}, y_{h_e}) \mapsto (x_1^2 + y_1^2, \dots, x_{h_e}^2 + y_{h_e}^2)$  as proven by [6].

where  $X, Y \in \mathfrak{g}$ . Hamilton's equation for a function  $H: \mathfrak{g} \rightarrow \mathbb{R}$  can be written as

$$\frac{dZ}{dt} = X_H(Z) = [\nabla H(Z), Z],$$

where  $\nabla H$  is the gradient vector field defined by the equation  $\langle \nabla H(Z), Y \rangle = dH_Z(Y)$  for all  $Y \in T_Z \mathcal{O}$ . Hence, the Poisson bracket of two functions  $H, F$  can be conveniently written as

$$\{H, F\}_Z = \omega_Z(X_H, X_F) = \omega_Z([\nabla H, Z], [\nabla F, Z]) = \langle Z, [\nabla H, \nabla F] \rangle.$$

A direct sum of semisimple Lie algebras  $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_N$  is endowed with direct sum Lie brackets and Killing forms. An adjoint orbit in  $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_N$  is a product of adjoint orbits  $\mathcal{O}_{Z_1} \times \cdots \times \mathcal{O}_{Z_N}$  and the symplectic structure coincides with the direct sum of their respective symplectic structures,  $\omega = \omega_1 \oplus \cdots \oplus \omega_N$ .

The moment map for the adjoint action of  $G$  on an orbit in  $\mathfrak{g}$  is inclusion of the orbit into  $\mathfrak{g}^*$  via the Ad-equivariant identification. The moment map for the diagonal adjoint action of  $G$  on  $\mathcal{O}_{Z_1} \times \cdots \times \mathcal{O}_{Z_N}$  is hence the map  $(X_1, \dots, X_N) \mapsto \sum X_i$ .

**Example 3.1.** Let  $G = \text{SO}(3)$  be the group of rotations of  $\mathbb{R}^3$  equipped with the standard basis and inner product. Its Lie algebra is

$$\mathfrak{so}(3) = \left\{ \begin{pmatrix} 0 & x_1 & x_2 \\ -x_1 & 0 & x_3 \\ -x_2 & -x_3 & 0 \end{pmatrix} : x_1, x_2, x_3 \in \mathbb{R} \right\},$$

which has the Ad-invariant inner product  $\langle X, Y \rangle = -\frac{1}{2} \text{tr}(XY)$  and Lie bracket  $[X, Y] = XY - YX$ . The map  $\Psi: \mathfrak{so}(3) \rightarrow (\mathbb{R}^3, \times)$  given by

$$\begin{pmatrix} 0 & x_1 & x_2 \\ -x_1 & 0 & x_3 \\ -x_2 & -x_3 & 0 \end{pmatrix} \mapsto (x_1, x_2, x_3) \in \mathbb{R}^3$$

is an isomorphism of Lie algebras, where  $(\mathbb{R}^3, \times)$  is the cross-product Lie algebra. Under this identification, the adjoint action of  $\text{SO}(3)$  on  $\mathfrak{so}(3)$  is identified with the standard action on  $\mathbb{R}^3$ , and the Ad-invariant inner product is simply the standard Euclidean inner product:

$$\langle X, Y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 = -\frac{1}{2} \text{tr}(XY).$$

Thus, the adjoint orbits are identified with concentric spheres in  $\mathbb{R}^3$  and the symplectic structure on an adjoint orbit through the point  $Z = (z_1, z_2, z_3)$  is

$$\omega_Z([X, Z], [Y, Z]) = \langle Z, [X, Y] \rangle = \det \begin{pmatrix} z_1 & z_2 & z_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix},$$

which is precisely the standard symplectic structure on a sphere with radius  $|Z|$ . In what follows, we will consider the adjoint orbit in  $\mathfrak{so}(3) \oplus \mathfrak{so}(3) \oplus \mathfrak{so}(3)$  that is a product of three spheres with radius 1,  $(S^2 \times S^2 \times S^2, \omega_{\text{STD}} \oplus \omega_{\text{STD}} \oplus \omega_{\text{STD}})$ .

## 4 An integrable Heisenberg spin chain

As in Example 3.1, consider a product of three spheres of radius 1 whose elements are triples  $(X, Y, Z) \in M = S^2 \times S^2 \times S^2$ . Define the Hamiltonians

$$\begin{aligned} H(X, Y, Z) &= |X + Y + Z| = \sqrt{3 + \langle X, Y \rangle + \langle Y, Z \rangle + \langle Z, X \rangle}, \\ I(X, Y, Z) &= \langle X + Y + Z, e_3 \rangle, \quad \text{and} \quad J(X, Y, Z) = \langle X, [Y, Z] \rangle = \det(X, Y, Z). \end{aligned}$$

By ad-invariance of the inner product, the Hamiltonian vector fields of these functions are

$$\begin{aligned} X_H &= [\nabla H, (X, Y, Z)] = \frac{1}{H(X, Y, Z)}([Y + Z, X], [X + Z, Y], [X + Y, Z]), \\ X_I &= [\nabla I, (X, Y, Z)] = ([e_3, X], [e_3, Y], [e_3, Z]), \quad \text{and} \\ X_J &= [\nabla J, (X, Y, Z)] = ([[Y, Z], X], [[Z, X], Y], [[X, Y], Z]). \end{aligned}$$

The flow  $\varphi_{[v, X]}^t$  of a vector field  $[v, X]$  acts by rotation of the vector  $X$  around the axis  $v$  with period  $2\pi/|v|$ , which we denote as  $R_t^v$ ,

$$\varphi_{[v, X]}^t X = R_t^v X.$$

Thus, the Hamiltonian flow of  $I$  acts by rotating each sphere around the  $e_3$ -axis with period  $2\pi$ ,

$$\varphi_{X_I}^t(X, Y, Z) = (R_t^{e_3} X, R_t^{e_3} Y, R_t^{e_3} Z).$$

Where defined, the Hamiltonian flow of  $H$  rotates each sphere around the axis  $X + Y + Z$  with period  $2\pi$ ,

$$\varphi_{X_H}^t(X, Y, Z) = (R_t^v X, R_t^v Y, R_t^v Z),$$

where  $v = (X + Y + Z)/|X + Y + Z|$ . This is perhaps best visualized as rotating the polygon with edges  $X, Y, Z, -X - Y - Z$  around the edge  $-X - Y - Z$ .

**Proposition 4.1.**  $\{H, J\} = \{H, I\} = \{J, I\} = 0$ .

**Proof.** It is a nice exercise to see that this is true based on the geometric description of the Hamiltonians and their flows given above. More algebraically, one can see this using the Lie algebra structure that is present. For example,

$$\begin{aligned} \{J, I\}_{(X, Y, Z)} &= \langle (X, Y, Z), ([Y, Z], [Z, X], [X, Y]), (e_3, e_3, e_3) \rangle \\ &= \langle [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]], e_3 \rangle = 0 \end{aligned}$$

by the Jacobi identity and ad-invariance of our inner product. A similar calculation shows that  $\{H, I\} = \{H, J\} = 0$ . ■

**Remark 4.2.** The Hamiltonian flow of  $J$  is less straightforward to describe, but we can say something about how it acts on the submanifold

$$L = \{(X, Y, Z) \in M : X + Y + Z = 0\} = H^{-1}(0).$$

There the vectors  $X, Y$ , and  $Z$  are coplanar and the vector field

$$X_J(X, Y, Z) = ([[X, Y], X], [[Y, Z], Y], [[Z, X], Z]).$$

The flow of this vector field acts on  $L$  by rotation of each vector  $X, Y$  and  $Z$  around the axis  $\bar{n} = [X, Y] = [Y, Z] = [Z, X]$  with constant period: since each component of  $X_J$  is tangent to the plane spanned by  $X, Y$ , and  $Z$ , and the flow of  $X_J$  preserves  $L$ , the vector  $[X, Y]$  is preserved by  $X_J$ , so

$$X_J = ([\bar{n}, X], [\bar{n}, Y], [\bar{n}, Z]).$$

In [16] it was shown that the fiber  $L$  is a non-displaceable Lagrangian submanifold of  $(S^2 \times S^2 \times S^2, \omega_{\text{STD}} \oplus \omega_{\text{STD}} \oplus \omega_{\text{STD}})$ . To see that it is an embedded Lagrangian  $\mathbb{R}P^3$ , observe that it is the zero level set for the moment map of the diagonal  $\text{SO}(3)$ -action,  $(X, Y, Z) \mapsto X + Y + Z$ , and the diagonal action of  $\text{SO}(3)$  is free and transitive.

**Remark 4.3.** The Hamiltonian  $H$  has several important symmetries. First, note that  $H$  is invariant under the diagonal Hamiltonian  $\text{SO}(3)$ -action, whose moment map is

$$S^2 \times S^2 \times S^2 \xrightarrow{(X,Y,Z) \mapsto X+Y+Z} \mathfrak{so}(3) \cong \mathfrak{so}(3)^*.$$

Thus  $H$  is non-commutatively integrable, or super-integrable. Indeed, there are three independent choices for our second integral

$$I_v(X, Y, Z) = \langle X + Y + Z, v \rangle,$$

which do not pairwise Poisson commute.

In addition, all three Hamiltonians,  $H, I, J$  have a symplectic  $\mathbb{Z}_3$ -symmetry by cyclic permutations  $(X, Y, Z) \mapsto (Z, X, Y)$ . As we will see, this symmetry of the system by a group of order 3 is also a symmetry of the rank 1 focus-focus critical fibers, and thus the system's topological monodromy contains the number 3 (see Theorem 2.8).

## 5 Image of the moment map

In this section we give a complete description the image, critical set, and critical values for the moment map  $\mathcal{H} = (H, I, J)$ .

The image of the moment map has two reflective symmetries coming from the fact that  $J(X, Y, Z) = -J(X, Z, Y)$  and  $I(-X, -Y, -Z) = -I(X, Y, Z)$ . It is obvious that  $|I| \leq H$ , with equality when  $X + Y + Z \in \text{span}(e_3)$ , and that  $0 \leq H \leq 3$  with equality when  $X = Y = Z$ .

Observe that

$$H = \sqrt{3 + 2(a + b + c)} \quad \text{and} \quad |J| = \sqrt{1 + 2abc - (a^2 + b^2 + c^2)},$$

where  $a = \langle X, Y \rangle$ ,  $b = \langle Y, Z \rangle$  and  $c = \langle Z, X \rangle$  (the second formula is the volume of a parallelepiped). If we maximize  $|J|$  with the constraint  $H = \text{const}$ , then we must have  $a = b = c$  (the interior angles between the three vectors are the same). Using this we can deduce that

$$|J| \leq \sqrt{1 + 2 \left( \frac{H^2 - 3}{6} \right)^3 - 3 \left( \frac{H^2 - 3}{6} \right)^2},$$

and equality is achieved when  $a = b = c$ .

**Proposition 5.1.** *The image of the moment map  $\mathcal{H}$  is the set of points  $(r, s, t) \in \mathbb{R}^3$  that satisfy the equations*

$$|t| \leq \sqrt{1 + 2 \left( \frac{r^2 - 3}{6} \right)^3 - 3 \left( \frac{r^2 - 3}{6} \right)^2}, \quad |s| \leq r, \quad \text{and} \quad 0 \leq r \leq 3.$$

**Proof.** Observe that the level sets of  $H$  are all connected (see the proof of Theorem 6.1 in Section 6). For a given tuple  $(r, s, t) \in \mathbb{R}^3$  that satisfies the inequalities of (1.1), consider the restriction of the map  $J$  to the connected set  $H^{-1}(r)$ . Since  $0 \leq r \leq 3$ , we can construct a tuple  $(X, Y, Z) \in H^{-1}(r)$  such that

$$\langle X, Y \rangle = \langle Y, Z \rangle = \langle Z, X \rangle = \frac{r^2 - 3}{6}.$$

$J(X, Y, Z)$  is maximal or minimal depending on the orientation of the basis  $\{X, Y, Z\}$  (if  $r = 0$  or  $3$  then  $\{X, Y, Z\}$  does not form a basis and  $J(X, Y, Z) = 0$  is both maximal and minimal). It follows by the Intermediate Value Theorem that there is a tuple  $(X, Y, Z) \in S^2 \times S^2 \times S^2$  such that  $H(X, Y, Z) = r$  and  $J(X, Y, Z) = t$ . Using the Intermediate Value Theorem again, it is easy to see that there exists a  $\theta \in [0, 2\pi]$  such that  $I(R_\theta^{e_3} X, R_\theta^{e_3} Y, R_\theta^{e_3} Z) = s$ . Since  $H$  and  $J$  are invariant under diagonal rotations, we are done.  $\blacksquare$

Next, we turn our attention to the critical set for the system. The critical set consists of several subsets:

1. The sets where  $H$  is critical:
  - (a) the embedded  $\text{SO}(3) \cong H^{-1}(0)$ , which lies over the vertex  $(0, 0, 0)$ ,
  - (b) the three embedded spheres  $S_1 = \{(-X, X, X)\}$ ,  $S_2 = \{(X, -X, X)\}$ , and  $S_3 = \{(X, X, -X)\}$ , which lie over the critical line ( $H = 1$  and  $J = 0$ ), and
  - (c) the diagonally embedded sphere  $S_4 = \{(X, X, X)\}$ , which lies over the edge  $H = 3$ .
2. The set  $C_1 = \{(X, Y, Z): X + Y + Z \in \text{Span}(e_3)\}$  where  $dI$  is proportional to  $dH$ . This contains the critical set of  $I$  and is mapped to two opposite faces of the moment map image.
3. The set  $C_2 = \{(X, Y, Z): \langle X, Y \rangle = \langle Y, Z \rangle = \langle Z, X \rangle\}$  where  $dJ$  is proportional to  $dH$ . Note that  $S_4, H^{-1}(0) \subset C_2$ . This set maps to the other two opposite faces of the moment map image. Points in  $C_1 \cap C_2$  map to edges of the moment map image.

**Proposition 5.2.** *The critical set for  $\mathcal{H}$  is*

$$C = C_1 \cup C_2 \cup S_1 \cup S_2 \cup S_3.$$

*The set  $\mathcal{H}(C_1 \cup C_2)$  is the boundary of the image  $\mathcal{H}(M)$  and the set  $\mathcal{H}(S_1) = \mathcal{H}(S_2) = \mathcal{H}(S_3)$  is the line segment  $\mathcal{H}(M) \cap \{H = 1, J = 0\}$  (see Fig. 1).*

In the terminology of integrable systems, the set of critical values is the system's 'bifurcation diagram'. This description of the bifurcation diagram will be of use to us in Section 7. Note that the three critical spheres  $S_1, S_2, S_3$  are permuted by the system's  $\mathbb{Z}_3$ -symmetry (cf. Remark 4.3).

**Proof.** Throughout we use the fact that  $df = 0$  if and only if  $X_f = 0$  (when  $f$  is smooth).

1. Since 0 is the global minimum for  $H$ , the set  $H^{-1}(0)$  is critical. To find the other critical sets of  $H$ , observe that  $X_H = 0$  if and only if  $[X + Y + Z, X] = 0$ ,  $[X + Y + Z, Y] = 0$ , and  $[X + Y + Z, Z] = 0$ . If  $X + Y + Z \neq 0$ , this occurs if and only if  $X + Y + Z$  is contained in the lines spanned by  $X, Y$ , and  $Z$ , so this happens if and only if  $X, Y$ , and  $Z$  are collinear. This entails cases (1b) and (1c).
2. Observe that  $X_I = \alpha X_H$  for some  $\alpha \neq 0$  if and only if

$$\begin{aligned} [e_3, X] &= \alpha[X + Y + Z, X], & [e_3, Y] &= \alpha[X + Y + Z, Y], & \text{and} \\ [e_3, Z] &= \alpha[X + Y + Z, Z], \end{aligned}$$

which is true if and only if  $X + Y + Z \in \text{span}(e_3)$ .

3. Observe that  $X_J = \alpha X_H$  if and only if

$$\begin{aligned} [[Y, Z], X] &= \alpha[Y + Z, X], & [[Z, X], Y] &= \alpha[X + Z, Y], & \text{and} \\ [[X, Y], Z] &= \alpha[X + Y, Z] \end{aligned}$$

for some  $\alpha$ . If  $\alpha = 0$  then the 3-tuple  $X, Y, Z$  forms an oriented or anti-oriented orthonormal frame, or  $X, Y$  and  $Z$  are collinear. If  $\alpha \neq 0$  then

- (a)  $[Y, Z], Y + Z$ , and  $X$  are coplanar,
- (b)  $[Z, X], X + Z$ , and  $Y$  are coplanar, and
- (c)  $[X, Y], X + Y$ , and  $Z$  are coplanar.

Since  $X$ ,  $Y$  and  $Z$  all have the same length, the three items listed above are true if and only if  $X$ ,  $Y$ ,  $Z$  are collinear, or  $\langle X, Y \rangle = \langle Y, Z \rangle = \langle Z, X \rangle$  (this fact is a straightforward exercise in Euclidean geometry).

Finally, suppose that  $\alpha X_I + \beta X_H + \gamma X_J = 0$  for  $\alpha, \beta, \gamma \in \mathbb{R}$  not all zero. Then

$$\begin{aligned} \alpha[e_3, X] + \beta[Y + Z, X] + \gamma[[Y, Z], X] &= 0, \\ \alpha[e_3, Y] + \beta[X + Z, Y] + \gamma[[Z, X], Y] &= 0, \quad \text{and} \\ \alpha[e_3, Z] + \beta[X + Y, Z] + \gamma[[X, Y], Z] &= 0. \end{aligned}$$

Adding these equations together we obtain

$$\alpha[e_3, X + Y + Z] + \gamma([X, Y], Z) + \gamma([Y, Z], X) + \gamma([Z, X], Y) = 0,$$

which by the Jacobi identity reduces to  $\alpha[e_3, X + Y + Z] = 0$ . If  $[e_3, X + Y + Z] = 0$ , then  $(X, Y, Z) \in C_1$ . If  $\alpha = 0$ , then  $(X, Y, Z)$  is in the set  $S_1 \cup S_2 \cup S_3 \cup C_2$ .  $\blacksquare$

Combining Propositions 5.1 and 5.2, we have Theorem 1.2. Since we now know that our Hamiltonians are independent on an open dense subset of  $M$ , we can conclude:

**Corollary 5.3.**  *$\mathcal{H}$  is a completely integrable system. In particular, the regular level sets of  $\mathcal{H}$  are homeomorphic to a disjoint union of finitely many 3-tori.*

**Corollary 5.4.** *The set of regular values is homotopy-equivalent to  $S^1$ .*

In the next section, we will see that the regular level sets are connected and in Section 7 we will describe the structure of the associated Lagrangian torus fibration.

## 6 Connectedness of regular level sets

In this section we prove

**Theorem 6.1.** *The regular fibers of the map  $\mathcal{H} = (H, I, J)$  are all connected.*

**Proof.** The proof will have two parts. For  $H \neq 1$  we can make a general argument and for  $H = 1$  we will apply Ehresmann's theorem.

Pick a regular value  $(r, s, t)$  with  $r \neq 1$ . Since  $H$  generates a Hamiltonian  $S^1$ -action on  $M \setminus H^{-1}(0)$ , it is a Morse–Bott function such that all critical sets have even index [1]. Thus, the level sets of  $H$  are connected. Since the regular level sets of  $H$  are compact and connected, the symplectic reductions  $M_r \equiv H^{-1}(r)/S^1$  are all compact, connected symplectic manifolds. Since  $s$  is a regular value of the reduced Hamiltonian  $\tilde{I}$ , and  $\tilde{I}$  generates a free  $S^1$ -action, we can reduce once more to obtain the compact and connected manifold  $M_{r,s} = \tilde{I}^{-1}(s)/S^1$ .

The image of the twice-reduced Hamiltonian  $\tilde{J}$  on  $M_{r,s}$  is a line segment with the only critical values being the maximum and minimum. Recall from Proposition 5.2 that in the unreduced manifold,  $dJ$  is dependent on  $dH$  and  $dI$  at a point  $(X, Y, Z) \in H^{-1}(r) \cap I^{-1}(s)$  if and only if  $dJ$  is proportional to  $dH$ , and this occurs if and only if  $\langle X, Y \rangle = \langle Y, Z \rangle = \langle Z, X \rangle$ . It is easy to see geometrically that the set of all 3-tuples of unit length vectors which satisfy the conditions

- 1)  $\langle X, Y \rangle = \langle Y, Z \rangle = \langle Z, X \rangle$ ,
- 2)  $|X + Y + Z| = r$ , and
- 3)  $\langle X + Y + Z, e_3 \rangle = s$ .

is two orbits of the Hamiltonian  $T^2$ -action generated by  $(H, I)$ , which are distinguished by whether the basis  $\{X, Y, Z\}$  is positively or negatively oriented, corresponding to being the maximum or minimum set for  $J$  on  $H^{-1}(r) \cap I^{-1}(s)$ . Thus there are two critical points of  $\tilde{J}$  on  $M_{r,s}$  and the regular fibers  $\tilde{J}^{-1}(s)$  are all connected. This implies that the regular fiber  $\mathcal{H}^{-1}(r, s, t) \subset M$  is connected.

Now consider a regular value  $(1, s, t)$ . Since the map  $\mathcal{H}$  is proper, Ehresmann's theorem implies that the fiber  $\mathcal{H}^{-1}(1, s, t)$  is diffeomorphic to a nearby fiber  $\mathcal{H}^{-1}(r, s, t)$  with  $r \neq 1$ , which we have just shown is connected.  $\blacksquare$

**Remark 6.2.** The image of the invariant Lagrangian  $L = H^{-1}(0)$  in the reduction at 0 by  $I$  is a Lagrangian  $S^2$ .

**Remark 6.3.** In [18] it is shown that if  $(M^4, \omega, (H_1, H_2))$  is a non-degenerate completely integrable system (cf. Definition 2.5) with two degrees of freedom, whose set of critical values has no vertical tangent lines, then the system has connected fibers. After rotating the moment map image, and checking that the boundary of  $\mathcal{H}(M)$  consists generically (almost all values of  $H$  or  $I$ ) of non-degenerate elliptic critical values one could deduce connectedness of almost all the fibers of  $\mathcal{H}$  or  $I$  by applying this theorem to the reduced systems, then applying Ehresmann to the remaining fibers. For example, the systems obtained by reducing at  $H = 0$  or  $\pm 1$  will have degenerate critical points (the Lagrangian  $S^2$  of the previous remark and image of intersections  $S_i \cap C_1$ , respectively), so the theorem of [18] cannot be applied directly.

## 7 Topological monodromy around the critical line

As we have seen, the regular fibers of the moment map  $\mathcal{H} = (H, I, J)$  for the Heisenberg spin chain are connected and, because of the critical line, the set of regular values  $B_{\text{reg}}$  is homotopy equivalent to  $S^1$ . Thus, the associated Lagrangian torus fibration  $M_{\text{reg}} \rightarrow B_{\text{reg}}$  of the Heisenberg spin chain may have non-trivial topological monodromy. This section uses our understanding of the critical set and critical values of  $\mathcal{H}$  from Section 5, together with a non-degeneracy computation that is relegated to Appendix A, to deduce the topological monodromy directly from Theorem 2.11. Recall

**Theorem 1.1.** *The topological monodromy of the Lagrangian torus fibration associated to the integrable system  $(H, I, J)$  is generated by the matrix*

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}.$$

*Thus the system does not admit global action coordinates.*

**Proof.** Consider critical the fibers  $N(s) = \mathcal{H}^{-1}(1, s, 0)$  for  $-1 < s < 1$ . The condition  $J = 0$  implies that the unit-length vectors  $X, Y, Z$  are coplanar and the condition  $H = 1$  implies that they form three sides of the parallelogram  $X, Y, Z, -X - Y - Z$ . The set of all such vectors is compact and connected (see the remark following the proof). By Proposition 5.2, the set of critical points in the fiber  $N(s)$  contains three connected components:

$$\begin{aligned} N(s) \cap S_1 &= \{(-X, X, X) \in S^2 \times S^2 \times S^2: \langle X, e_3 \rangle = s\}, \\ N(s) \cap S_2 &= \{(X, -X, X) \in S^2 \times S^2 \times S^2: \langle X, e_3 \rangle = s\}, \\ N(s) \cap S_3 &= \{(X, X, -X) \in S^2 \times S^2 \times S^2: \langle X, e_3 \rangle = s\}, \end{aligned}$$

each homeomorphic to  $S^1$ , and they are permuted by the system's  $\mathbb{Z}_3$  symmetry (cf. Remark 4.3). Each critical fiber is topologically stable (Definition 2.9) since the system's  $\mathbb{Z}_3 \times S^1$ -symmetry acts transitively on the critical set in each fiber  $N(s)$ , and preserves the map  $\mathcal{H}$ .

By Proposition A.1 the critical points in  $N(s)$  are rank 1 non-degenerate, with Williamson type  $(0, 0, 1)$ . Thus by Theorem 2.11, the Liouville foliation of  $\mathcal{U}(N(s))$  is topologically equivalent to the product foliation of  $(D^1 \times S^1) \times \mathcal{U}(N_3^f)$ . As noted in Section 2, this implies that the topological monodromy of our system is the same as the product foliation of  $(D^1 \times S^1) \times \mathcal{U}(N_3^f)$ , which decomposes into blocks for each component of the direct sum,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}.$$

By Proposition 2.8, the bottom right minor is

$$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}. \quad \blacksquare$$

Since the Hamiltonian flows of  $H$  and  $I$  are periodic, the 1-forms  $dH$  and  $dI$  are global sections of the period lattice  $\Lambda \rightarrow B_{\text{reg}}$ . Thus if  $\{dH(b), dI(b), \alpha_b\}$  is a  $\mathbb{Z}$ -basis for  $\Lambda_b$ , and  $\gamma$  is a choice of generator for  $\pi_1(B_{\text{reg}}, b)$ , then we must have

$$m_b(\gamma) = \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & \pm 1 \end{pmatrix}.$$

Theorem 1.1 tells us that

$$m_b(\gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

up to a choice of basis fixing  $dH(b)$  and  $dI(b)$ .

**Remark 7.1.** It is easy to see how the fiber  $N(s)$  is homeomorphic to  $S^1 \times N_3^f$ , where  $N_3^f$  is the focus-focus fiber with three critical points as described in Section 2. First note that the set

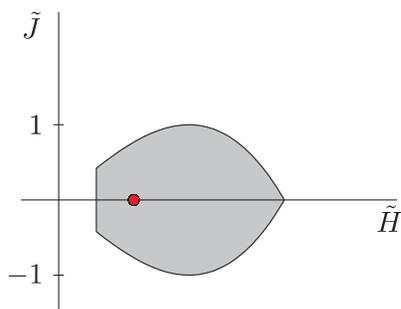
$$S = \{(X, Y, Z) \in S^2 \times S^2 \times S^2 : X + Y + Z = \beta e_1 + s e_3, \beta \in \mathbb{R}^+\}$$

is a slice for the Hamiltonian  $S^1$ -action generated by  $I$  in a neighbourhood of  $N(s)$ . Since this  $S^1$ -action is free in a neighbourhood of  $N(s)$ , the fiber  $N(s)$  is homeomorphic to  $S^1 \times (N(s) \cap S)$ . The set  $N(s) \cap S$  consists of: the three annuli,

$$\begin{aligned} &\{(X, \beta e_1 + s e_3, -X) \in S \mid X \in S^2 \setminus \{\pm(\beta e_1 + s e_3)\}\}, \\ &\{(\beta e_1 + s e_3, X, -X) \in S \mid X \in S^2 \setminus \{\pm(\beta e_1 + s e_3)\}\}, \\ &\{(X, -X, \beta e_1 + s e_3) \in S \mid X \in S^2 \setminus \{\pm(\beta e_1 + s e_3)\}\} \end{aligned}$$

together with the three points,

$$\begin{aligned} &(-\beta e_1 - s e_3, \beta e_1 + s e_3, \beta e_1 + s e_3), & (\beta e_1 + s e_3, -\beta e_1 - s e_3, \beta e_1 + s e_3), \\ &(\beta e_1 + s e_3, \beta e_1 + s e_3, -\beta e_1 - s e_3). \end{aligned}$$



**Figure 3.** Reduced system on  $M_s$  for  $-1 < s < 1$ ,  $t \neq 0$ .

**Remark 7.2.** The fact that this system has non-trivial monodromy should be unsurprising for the following reason: the topology of the  $H$ -level sets changes as you pass through the critical value 1. This can be seen directly with Morse theory, but there is also a natural interpretation in terms of the topology of polygon spaces (as introduced by [13]). There is a natural diffeomorphism of the level set  $H^{-1}(r)$  with the manifold  $M(1, 1, 1, r)$  of closed 4-gons in  $\mathbb{R}^3$  with side lengths 1, 1, 1, and  $r$ . When  $r \neq 1$ , it has been observed by Knutson, Hausmann [10], and Kapovitch and Millson [13] that the quotient  $M(1, 1, 1, r)/\mathrm{SO}(3)$  is homeomorphic to  $S^2$ , and that the quotient map  $\pi: M(1, 1, 1, r) \rightarrow S^2$  is a principal  $\mathrm{SO}(3)$ -bundle. Further, the characteristic classes of this principal  $\mathrm{SO}(3)$ -bundle were described by Knutson and Hausmann in their paper [11]. Their result says that for  $0 < r < 1$  the bundle is trivial, whereas for  $1 < r < 3$  the bundle is non-trivial. It is a fun exercise to check that  $\mathrm{SO}(3) \times S^2$  and the total space of the non-trivial principal  $\mathrm{SO}(3)$ -bundle over  $S^2$  are not homeomorphic<sup>3</sup>.

It was observed in [4] that such a change in the topology of the level set  $H^{-1}(r)$  as  $r$  passes through an interior critical value indicates that there *must* be non-trivial monodromy around the associated critical fibres, since this forces the pullback of the torus bundle to any circle around the critical line to be non-trivial.

**Remark 7.3.** Since  $B_{\mathrm{reg}}$  is homotopy equivalent to  $S^1$ , the Chern class of the torus fibration  $M_{\mathrm{reg}} \rightarrow B_{\mathrm{reg}}$  is trivial, so there exists a global section  $\sigma: B_{\mathrm{reg}} \rightarrow M_{\mathrm{reg}}$ . Since the Lagrangian Chern class [5] also vanishes,  $\sigma$  can be chosen to be Lagrangian, so the map

$$\Psi: T^*(B_{\mathrm{reg}})/\Lambda \times_{B_{\mathrm{reg}}} M_{\mathrm{reg}} \rightarrow M_{\mathrm{reg}}$$

gives a symplectomorphism  $\alpha \mapsto \Psi(\alpha, \sigma(\pi(\alpha)))$  which is an isomorphism of the Lagrangian torus fibrations  $(T^*(B_{\mathrm{reg}})/\Lambda, d\lambda) \rightarrow B_{\mathrm{reg}}$  and  $(M_{\mathrm{reg}}, \omega) \rightarrow B_{\mathrm{reg}}$  (see [5, 15] for details).

**Remark 7.4.** For  $-1 < s < 1$ , the reduced system on  $M_s = I^{-1}(s)/S^1$  has a stable focus-focus critical fiber with three critical points. The manifold  $M_s$  is the blow-up  $Bl_3\mathbb{C}P^2$ , and the reduced moment map image has 3 vertices (see Fig. 3). A quick comparison with the list of almost toric systems in [14] shows that this checks out.

## 8 Further directions

In quantum mechanics, the spin- $S$  isotropic Heisenberg spin chain on the lattice  $L = \mathbb{Z}$  is an operator  $\mathcal{K}$  on the tensor product  $V = \bigotimes_L \mathbb{C}^{2S+1}$  of irreducible  $\mathfrak{su}(2)$  representations, defined as

$$\mathcal{K} = \sum_{i \in L} \langle X_i, X_{i+1} \rangle = \sum_{i \in L} (X_i^1 \circ X_{i+1}^1 + X_i^2 \circ X_{i+1}^2 + X_i^3 \circ X_{i+1}^3),$$

<sup>3</sup>Hint: compute their fundamental groups.

where  $X_i^j$  are the Pauli matrices

$$X_i^1 = \frac{1}{S} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad X_i^2 = \frac{1}{S} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad X_i^3 = \frac{1}{S} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

acting on  $V$  by the irreducible representation of  $\mathfrak{su}(2)$  in the  $i$ th factor. This operator models ‘nearest-neighbour’ spin coupling on the lattice. One may consider the infinite chain  $L = \mathbb{Z}$  or a chain with boundary conditions,  $L = \mathbb{Z}_N$ . For a chain with boundary conditions, the large- $S$  limit of this system is the Hamiltonian

$$K = \sum_{i \in L} \langle X_i, X_{i+1} \rangle$$

on the symplectic manifold  $M_N = (S^2 \times \cdots \times S^2, \omega_{\text{STD}} \oplus \cdots \oplus \omega_{\text{STD}})$  with elements  $(X_1, \dots, X_N)$ . In the case  $N = 3$ , this is related to the Hamiltonian  $H$  studied in this paper by the identity

$$3 - H^2 = K$$

(of course, the analogous identity fails for  $N > 3$ ). In [9], the algebraic structure of the spin- $\frac{1}{2}$  representation was used to find a recursive formula for the operators commuting with  $\mathcal{K}$  for any lattice  $L$  and  $S = \frac{1}{2}$ . Inspired by this, one might ask if there is a similar combinatorial pattern generating integrals of the classical Hamiltonians  $K$  or  $H$  on  $M_N$ , where we might define  $H_N$  for  $N > 3$  as the Hamiltonian describing ‘complete graph’ coupling

$$H_N = \sum_{i, j \in L, i \neq j} \langle X_i, X_j \rangle.$$

**Question 8.1.** *How is the structure of the Lie algebra  $\mathfrak{so}(3) \cong \mathfrak{su}(2)$  reflected in the conservation laws of the Hamiltonians  $H_N$  on  $M_N$ .*

For example, for arbitrary  $N$  one can check using the Jacobi identity and ad-invariance of  $\langle \cdot, \cdot \rangle$ , as in the proof of Proposition 4.1 that  $H_N$  has integrals

$$I_v = \sum_{i \in L} \langle X_i, v \rangle, \quad v \in \mathbb{R}^3, \quad J = \sum_{i \in L} \langle X_i, [X_{i+1}, X_{i+2}] \rangle$$

with  $\{H_N, J\} = \{H_N, I_v\} = \{J, I_v\} = 0$ , and one could ask if further independent integrals exist. As in Remark 7.2, the level sets  $H_N^{-1}(h)$  are diffeomorphic to the space  $M(1, \dots, 1, h)$  of  $N$ -gons in  $\mathbb{R}^3$  with corresponding edge lengths, and come with a quotient map to the associated polygon space  $M(1, \dots, 1, h)/\text{SO}(3)$ . As  $h$  passes through critical values (which are even or odd integers depending on the parity of  $N$ ) the topology of this bundle will change, as described in [11], so if  $H_N$  is completely integrable we anticipate non-trivial topological monodromy, provided these critical values lie in the interior of the moment map image.

## A Non-degeneracy computation

In this appendix we prove

**Proposition A.1.** *The critical points of  $\mathcal{H} = (H, I, J)$  which lie above the critical line  $\{(1, s, 0) : -1 < s < 1\}$  are non-degenerate and have Williamson type  $(0, 0, 1)$  (see Definitions 2.5 and 2.7).*

As we have seen, this set of critical points is precisely

$$\{(X, Y, Z) \in S_1 \cup S_2 \cup S_3 : I(X, Y, Z) \neq \pm 1\}.$$

On this set  $dJ = dH = 0$  and  $dI \neq 0$ , so these points are all rank 1. To show the non-degeneracy of these critical points, we need to show that for each such  $p$ , the operators  $A_J(p) = \omega^{-1}d^2J(p)$ , and  $A_H(p) = \omega^{-1}d^2H(p)$  span a Cartan subalgebra (see Section 2). Equivalently, we must find a linear combination of the operators  $A_J(p)$  and  $A_H(p)$  which has 4 distinct eigenvalues. Note that it is sufficient to check non-degeneracy of a single critical point in each fiber  $\mathcal{H}^{-1}(1, s, 0)$  because each connected component of the critical set is an orbit of the Hamiltonian flow of  $X_J$ , and the symplectic  $\mathbb{Z}_3$ -action generated by the permutation  $\sigma(X, Y, Z) = (Z, X, Y)$  preserves  $\mathcal{H}$  and permutes these connected components transitively.

According to the classification of [19], the Williamson type of  $p$  will then be determined by the form of these eigenvalues: in  $\mathfrak{sp}(\mathbb{R}, 4)$  there are four conjugacy classes of Cartan subalgebras corresponding to four possible combinations of eigenvalues for a generic element:

- 1) elliptic-elliptic:  $\pm iA, \pm iB$ ,
- 2) elliptic-hyperbolic:  $\pm A, \pm iB$ ,
- 3) hyperbolic-hyperbolic:  $\pm A, \pm B$ , and
- 4) focus-focus:  $A \pm iB, -A \pm iB$ .

In Darboux coordinates the operator  $A_H(p)$  is equal to the linearization of the Hamiltonian vector field  $X_H$  at  $p$ , since

$$\frac{\partial X_H^i}{\partial x^j} = \frac{\partial}{\partial x^j} \left( \omega^{ik} \frac{\partial f}{\partial x^k} \right) = \omega^{ik} \frac{\partial^2 f}{\partial x^j \partial x^k} = (\omega^{-1}d^2H)_j^i.$$

Consider the cylindrical coordinates  $(\theta, z) \in (-\pi/2, 3\pi/2) \times (-1, 1)$  with symplectic form  $d\theta_i \wedge dz_i$ . The map  $\phi: (-\pi/2, 3\pi/2) \times (-1, 1) \rightarrow S^2$  given by

$$\phi(\theta, z) = ((1 - z^2)^{1/2} \cos(\theta), (1 - z^2)^{1/2} \sin(\theta), z)$$

is a symplectomorphism. In cylindrical coordinates  $(\theta_1, z_1, \theta_2, z_2, \theta_3, z_3)$ , the Hamiltonians are

$$\begin{aligned} \hat{H} &= \left( \sum_j (1 - z_j^2)^{1/2} \cos(\theta_j) \right)^2 + \left( \sum_j (1 - z_j^2)^{1/2} \sin(\theta_j) \right)^2 + \left( \sum_j z_j \right)^2, \\ \hat{J} &= \sum_{j=1,2,3} z_j (1 - z_{j+1}^2)^{1/2} (1 - z_{j-1}^2)^{1/2} \sin(\theta_{j-1} - \theta_{j+1}), \quad \hat{I} = z_1 + z_2 + z_3 \end{aligned}$$

(where we have pulled back  $(H^2 - 3)/2$  instead of  $H$  for computational convenience). Hamilton's equations tell us that

$$X_f = \frac{\partial f}{\partial z_i} \frac{\partial}{\partial \theta_i} - \frac{\partial f}{\partial \theta_i} \frac{\partial}{\partial z_i}.$$

The linearization of  $X_f$  at a fixed point of the flow of  $f$  is then

$$A_f = \begin{pmatrix} \left( \frac{-\partial^2 f}{\partial z_k \partial \theta_i} \right)_{ik} & \left( \frac{-\partial^2 f}{\partial \theta_k \partial \theta_i} \right)_{ik} \\ \left( \frac{\partial^2 f}{\partial z_k \partial z_i} \right)_{ik} & \left( \frac{\partial^2 f}{\partial \theta_k \partial z_i} \right)_{ik} \end{pmatrix}.$$

In order to check non-degeneracy, one therefore computes the partial derivatives:

$$\frac{\partial^2 \hat{H}}{\partial z_k \partial z_i} = \begin{cases} -2(1 - z_i^2)^{-3/2} \left( \sum_{j \neq i} (1 - z_j^2)^{1/2} \cos(\theta_i - \theta_j) \right), & k = i, \\ 2z_i z_k (1 - z_i^2)^{-1/2} (1 - z_k^2)^{-1/2} \cos(\theta_i - \theta_k) + 2, & k \neq i, \end{cases}$$

$$\begin{aligned}
\frac{\partial^2 \hat{H}}{\partial \theta_k \partial z_i} &= \begin{cases} 2z_i(1-z_i^2)^{-1/2} \left( \sum_j (1-z_j^2)^{1/2} \sin(\theta_i - \theta_j) \right), & k = i, \\ -2z_i(1-z_i^2)^{-1/2} (1-z_k^2)^{1/2} \sin(\theta_i - \theta_k), & k \neq i, \end{cases} \\
\frac{\partial^2 \hat{H}}{\partial \theta_k \partial \theta_i} &= \begin{cases} -2(1-z_i^2)^{1/2} \left( \sum_{j \neq i} (1-z_j^2)^{1/2} \cos(\theta_i - \theta_j) \right), & k = i, \\ 2(1-z_i^2)^{1/2} (1-z_k^2)^{1/2} \cos(\theta_i - \theta_k), & k \neq i, \end{cases} \\
\frac{\partial^2 \hat{J}}{\partial z_k \partial z_i} &= \begin{cases} -\frac{(1-z_i^2)^{-3/2} (z_{i-1}(1-z_{i+1}^2)^{1/2} \sin(\theta_{i+1} - \theta_i) + z_{i+1}(1-z_{i-1}^2)^{1/2} \sin(\theta_i - \theta_{i-1}))}{(1-z_i^2)^{1/2}} \left( (1-z_{i+1}^2)^{1/2} \sin(\theta_{i+1} - \theta_i) - \frac{z_{i+1}z_{i-1}}{(1-z_{i-1}^2)^{1/2}} \sin(\theta_i - \theta_{i-1}) \right) & k = i, \\ -\frac{z_{i-1}(1-z_{i+1}^2)^{1/2}}{(1-z_{i-1}^2)^{1/2}} \sin(\theta_{i-1} - \theta_{i+1}), & k = i-1, \\ \frac{-z_i}{(1-z_i^2)^{1/2}} \left( \frac{-z_{i-1}z_{i+1}}{(1-z_{i+1}^2)^{1/2}} \sin(\theta_{i+1} - \theta_i) + (1-z_{i-1}^2)^{1/2} \sin(\theta_i - \theta_{i-1}) \right) & \\ -\frac{z_{i+1}(1-z_{i-1}^2)^{1/2}}{(1-z_{i+1}^2)^{1/2}} \sin(\theta_{i-1} - \theta_{i+1}), & k = i+1, \end{cases} \\
\frac{\partial^2 \hat{J}}{\partial \theta_k \partial z_i} &= \begin{cases} -z_i(1-z_i^2)^{-1/2} (z_{i+1}(1-z_{i-1}^2)^{1/2} \cos(\theta_i - \theta_{i-1}) - z_{i-1}(1-z_{i+1}^2)^{1/2} \cos(\theta_{i+1} - \theta_i)), & k = i, \\ z_i z_{i+1} (1-z_{i-1}^2)^{1/2} (1-z_i^2)^{-1/2} \cos(\theta_i - \theta_{i-1}) + (1-z_{i+1}^2)^{1/2} (1-z_{i-1}^2)^{1/2} \cos(\theta_{i-1} - \theta_{i+1}), & k = i-1, \\ -z_i z_{i-1} (1-z_{i+1}^2)^{1/2} (1-z_i^2)^{-1/2} \cos(\theta_{i+1} - \theta_i) - (1-z_{i+1}^2)^{1/2} (1-z_{i-1}^2)^{1/2} \cos(\theta_{i-1} - \theta_{i+1}), & k = i+1, \end{cases} \\
\frac{\partial^2 \hat{J}}{\partial \theta_k \partial \theta_i} &= \begin{cases} (1-z_i)^{1/2} (-z_{i+1}(1-z_{i-1}^2)^{1/2} \sin(\theta_i - \theta_{i-1}) - z_{i-1}(1-z_{i+1}^2)^{1/2} \sin(\theta_{i+1} - \theta_i)), & k = i, \\ z_{i+1}(1-z_i)^{1/2} (1-z_{i-1}^2)^{1/2} \sin(\theta_i - \theta_{i-1}), & k = i-1, \\ z_{i-1}(1-z_i)^{1/2} (1-z_{i+1}^2)^{1/2} \sin(\theta_{i+1} - \theta_i), & k = i+1. \end{cases}
\end{aligned}$$

Let  $p = (0, s, 0, s, \pi, -s)$  in cylindrical coordinates. The linearization of  $X_j$  at  $p$  is

$$A_j(p) = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{pmatrix}.$$

The linearization of  $X_{\hat{H}}$  at  $p$  is

$$A_{\hat{H}}(p) = 2 \begin{pmatrix} 0 & 0 & 0 & 0 & -b^2 & b^2 \\ 0 & 0 & 0 & -b^2 & 0 & b^2 \\ 0 & 0 & 0 & b^2 & b^2 & -2b^2 \\ 0 & b^{-2} & b^{-2} & 0 & 0 & 0 \\ b^{-2} & 0 & b^{-2} & 0 & 0 & 0 \\ b^{-2} & b^{-2} & 2b^{-2} & 0 & 0 & 0 \end{pmatrix},$$

where  $b^2 = 1 - s^2$ . These operators are independent and a quick computation shows that for any  $-1 < s < 1$  the operator on  $L^\perp/L$  induced by  $A_j + A_{\hat{H}}$  has four distinct complex eigenvalues of the form  $A \pm iB$ ,  $-A \pm iB$ . Hence the critical point is rank 1 non-degenerate focus-focus.

## Acknowledgements

The author would like to thank his advisor Yael Karshon for her guidance and support and Leonid Polterovich for suggesting the study of integrable spin chains. Special thanks also goes to Joel Oakley for discussing his recent work on displaceability, Anton Izosimov for explaining his results on non-degenerate singularities, Peter Crooks for his editorial assistance, and the referees for their detailed feedback. The author was supported by NSERC PGS-D and OGS scholarships during the preparation of this work.

## References

- [1] Atiyah M.F., Convexity and commuting Hamiltonians, *Bull. London Math. Soc.* **14** (1982), 1–15.
- [2] Bates L.M., Monodromy in the champagne bottle, *Z. Angew. Math. Phys.* **42** (1991), 837–847.
- [3] Bolsinov A.V., Fomenko A.T., Integrable Hamiltonian systems. Geometry, topology, classification, *Chapman & Hall/CRC*, Boca Raton, FL, 2004.
- [4] Cushman R., Geometry of the energy momentum mapping of the spherical pendulum, *CWI Newslett.* (1983), 4–18.
- [5] Duistermaat J.J., On global action-angle coordinates, *Comm. Pure Appl. Math.* **33** (1980), 687–706.
- [6] Eliasson L.H., Normal forms for Hamiltonian systems with Poisson commuting integrals – elliptic case, *Comment. Math. Helv.* **65** (1990), 4–35.
- [7] Entov M., Polterovich L., Rigid subsets of symplectic manifolds, *Compos. Math.* **145** (2009), 773–826, [arXiv:0704.0105](https://arxiv.org/abs/0704.0105).
- [8] Fukaya K., Oh Y.-G., Ohta H., Ono K., Toric degeneration and nondisplaceable Lagrangian tori in  $S^2 \times S^2$ , *Int. Math. Res. Not.* **2012** (2012), 2942–2993, [arXiv:1002.1660](https://arxiv.org/abs/1002.1660).
- [9] Grabowski M.P., Mathieu P., Quantum integrals of motion for the Heisenberg spin chain, *Modern Phys. Lett. A* **9** (1994), 2197–2206, [hep-th/9403149](https://arxiv.org/abs/hep-th/9403149).
- [10] Hausmann J.-C., Knutson A., Polygon spaces and Grassmannians, *Enseign. Math.* **43** (1997), 173–198, [dg-ga/9602012](https://arxiv.org/abs/dg-ga/9602012).
- [11] Hausmann J.-C., Knutson A., The cohomology ring of polygon spaces, *Ann. Inst. Fourier (Grenoble)* **48** (1998), 281–321, [dg-ga/9706003](https://arxiv.org/abs/dg-ga/9706003).
- [12] Izosimov A.M., Classification of almost toric singularities of Lagrangian foliations, *Sb. Math.* **202** (2011), 1021–1042.
- [13] Kapovich M., Millson J.J., The symplectic geometry of polygons in Euclidean space, *J. Differential Geom.* **44** (1996), 479–513.
- [14] Leung N.C., Symington M., Almost toric symplectic four-manifolds, *J. Symplectic Geom.* **8** (2010), 143–187, [math.SG/0312165](https://arxiv.org/abs/math.SG/0312165).
- [15] Meinrenken E., Symplectic geometry, course notes, Unpublished lecture notes, 2000, available at <http://www.math.toronto.edu/mein/teaching/syimpl.pdf>.
- [16] Oakley J., Lagrangian submanifolds of products of spheres, Ph.D. Thesis, University of Georgia, 2014.
- [17] Oakley J., Usher M., On certain Lagrangian submanifolds of  $S^2 \times S^2$  and  $\mathbb{C}P^n$ , [arXiv:1311.5152](https://arxiv.org/abs/1311.5152).
- [18] Pelayo A., Ratiu T.S., Vu Ngoc S., Symplectic bifurcation theory for integrable systems, [arXiv:1108.0328](https://arxiv.org/abs/1108.0328).
- [19] Williamson J., On the algebraic problem concerning the normal forms of linear dynamical systems, *Amer. J. Math.* **58** (1936), 141–163.
- [20] Wu W., On an exotic Lagrangian torus in  $\mathbb{C}P^2$ , *Compos. Math.* **151** (2015), 1372–1394, [arXiv:1201.2446](https://arxiv.org/abs/1201.2446).
- [21] Zung N.T., Symplectic topology of integrable Hamiltonian systems. I. Arnold–Liouville with singularities, *Compositio Math.* **101** (1996), 179–215, [math.DS/0106013](https://arxiv.org/abs/math.DS/0106013).
- [22] Zung N.T., A note on focus-focus singularities, *Differential Geom. Appl.* **7** (1997), 123–130, [math.DS/0110147](https://arxiv.org/abs/math.DS/0110147).