

On Certain Wronskians of Multiple Orthogonal Polynomials

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Abstract. We consider determinants of Wronskian type whose entries are multiple orthogonal polynomials associated with a path connecting two multi-indices. By assuming that the weight functions form an algebraic Chebyshev (AT) system, we show that the polynomials represented by the Wronskians keep a constant sign in some cases, while in some other cases oscillatory behavior appears, which generalizes classical results for orthogonal polynomials due to Karlin and Szegő. There are two applications of our results. The first application arises from the observation that the m -th moment of the average characteristic polynomials for multiple orthogonal polynomial ensembles can be expressed as a Wronskian of the type II multiple orthogonal polynomials. Hence, it is straightforward to obtain the distinct behavior of the moments for odd and even m in a special multiple orthogonal ensemble – the AT ensemble. As the second application, we derive some Turán type inequalities for multiple Hermite and multiple Laguerre polynomials (of two kinds). Finally, we study numerically the geometric configuration of zeros for the Wronskians of these multiple orthogonal polynomials. We observe that the zeros have regular configurations in the complex plane, which might be of independent interest.

Key words: Wronskians; algebraic Chebyshev systems; multiple orthogonal polynomials; moments of the average characteristic polynomials; multiple orthogonal polynomial ensembles; Turán inequalities; zeros

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1 Introduction and statement of the main results

1.1 Determinants whose entries are orthogonal polynomials

In a classical paper [39], Karlin and Szegő developed an interesting and general theory regarding the determinants whose entries are orthogonal polynomials [18, 31, 54]. They showed that the polynomials represented by certain determinants whose elements are orthogonal polynomials keep a constant sign in some cases, while in some other cases the polynomials are oscillatory. More precisely, let

$$Q_n(x) = k_n(-x)^n + \dots, \quad k_n > 0, \quad n \in \mathbb{N} = \{0, 1, 2, 3, \dots\},$$

be a sequence of orthogonal polynomials with respect to an arbitrary measure whose distribution function has an infinite number of increasing points. The Wronskian of these polynomials is

then defined by

$$\begin{aligned} W(n, l; x) &:= W(Q_n(x), Q_{n+1}(x), \dots, Q_{n+l-1}(x)) \\ &= \det \begin{pmatrix} Q_n(x) & Q_{n+1}(x) & \cdots & Q_{n+l-1}(x) \\ Q'_n(x) & Q'_{n+1}(x) & \cdots & Q'_{n+l-1}(x) \\ \vdots & \vdots & \ddots & \vdots \\ Q_n^{(l-1)}(x) & Q_{n+1}^{(l-1)}(x) & \cdots & Q_{n+l-1}^{(l-1)}(x) \end{pmatrix}. \end{aligned}$$

By [39, Theorems 1 and 2], it is known that, for l even,

$$(-1)^{l/2} W(n, l; x) > 0, \quad x \in \mathbb{R},$$

i.e., the Wronskian keeps a constant sign for all real x ; if l is odd, then $W(n, l; x)$ has exactly n simple real zeros and the zeros of $W(n, l; x)$ and $W(n+1, l; x)$ strictly interlace.

Another important class of determinants considered in [39] is the Hankel determinant

$$\begin{aligned} T(n, l; x) &:= T(Q_n(x), Q_{n+1}(x), \dots, Q_{n+l-1}(x)) \\ &= \det \begin{pmatrix} Q_n(x) & Q_{n+1}(x) & \cdots & Q_{n+l-1}(x) \\ Q_{n+1}(x) & Q_{n+2}(x) & \cdots & Q_{n+l}(x) \\ \vdots & \vdots & \ddots & \vdots \\ Q_{n+l-1}(x) & Q_{n+l}(x) & \cdots & Q_{n+2l-2}(x) \end{pmatrix}, \end{aligned}$$

which is called the Turánian. Karlin and Szegő showed that, if l is even, $T(n, l; x)$ has the sign $(-1)^{l/2}$ on the interval I for the following three classical systems of orthogonal polynomials [54]:

- $Q_n(x) = P_n^{(\lambda)}(x)/P_n^{(\lambda)}(1)$, $\lambda > -1/2$ and $I = (-1, 1)$, where $P_n^{(\lambda)}(x)$ are the ultraspherical polynomials,
- $Q_n(x) = L_n^{(\alpha)}(x)/L_n^{(\alpha)}(0)$, $\alpha > -1$ and $I = (0, +\infty)$, where $L_n^{(\alpha)}(x)$ are the Laguerre polynomials,
- $Q_n(x) = H_n(x)$ and $I = (-\infty, +\infty)$, where $H_n(x)$ are the Hermite polynomials.

The strategy of proofs is to represent the Hankel determinants in terms of the Wronskian of orthogonal polynomials of another type. Note that, if $l = 2$, one has

$$T(n, 2; x) = Q_{n+1}^2(x) - Q_n(x)Q_{n+2}(x) > 0.$$

This inequality is called the Turán inequality, which was first proved for the Legendre polynomials $P_n(x) = P_n^{(1/2)}(x)$ [53, 55] and inspired the work of Karlin and Szegő. The analogous results for determinants involving orthogonal polynomials associated with discrete weights are also presented in [39].

Nowadays, determinants whose entries are orthogonal polynomials still attract much attention. For instance, the Wronskians of orthogonal polynomials also appear in random matrix theory; cf. [15, 47] and Section 3 below. The relationship between the Wronskian of orthogonal polynomials and the Hankel determinant of polynomials is clarified by Leclerc in [45], and further generalized by Durán [22]. In addition, it comes out that Turán inequality holds not only for a large class of orthogonal polynomials including the most classical orthogonal polynomials (cf. [10, 17, 23, 24, 29, 30, 41, 51, 52]), but also for many special functions and their q -analogues with important applications; we refer to [1, 6, 7, 8, 9, 20, 37, 44, 46, 49, 50] and the references therein for the development of that aspect. Other studies of the Wronskians of orthogonal polynomials can be found in [36, 38, 59].

for every choice of n different points $x_1, \dots, x_n \in [a, b]$. To see this, suppose x_1, \dots, x_n are such that the determinant is zero, then there is a linear combination of the rows $\sum_{k=1}^n b_k \varphi_k$ that gives a zero row. We then obtain a linear combination of functions φ_k admitting n zeros at x_1, \dots, x_n , which is a contradiction.

A system of r weights (w_1, \dots, w_r) is an AT system for the multi-index \vec{n} if each w_j is defined on a fixed interval $[a, b] \subseteq \mathbb{R}$ such that

$$\{w_1, xw_1, \dots, x^{n_1-1}w_1, w_2, xw_2, \dots, x^{n_2-1}w_2, \dots, w_r, xw_r, \dots, x^{n_r-1}w_r\}$$

is a Chebyshev system on $[a, b]$. If \vec{n} is a multi-index such that the weights (w_1, \dots, w_r) form an AT system for every index \vec{m} satisfying $\vec{m} \leq \vec{n}$ (in the componentwise sense, that is, $m_j \leq n_j$, $j = 1, \dots, r$), by [35, Theorem 23.2]. We have that \vec{n} is a normal index, which implies the existence and uniqueness of the polynomials $P_{\vec{n}}$. In particular, the weights for many classical multiple orthogonal polynomials (including multiple Hermite polynomials, multiple Laguerre polynomials, Jacobi–Piñeiro polynomials) belong to the AT systems.

For more information about multiple orthogonal polynomials, we refer to Aptekarev et al. [3, 5], Coussement and Van Assche [58], Nikishin and Sorokin [48, Chapter 4, § 3], Ismail [35, Chapter 23] and Filipuk, Van Assche and Zhang [26].

1.3 Statement of the main results

To state our main results, we need to define the Wronskian of multiple orthogonal polynomials. Let us consider a sequence of multi-indices $(\vec{n}_0, \vec{n}_1, \dots, \vec{n}_{l-1})$, $l \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$, such that

- $\vec{n}_0 = \vec{n}$ for a given initial multi-index \vec{n} ,
- $|\vec{n}_j| = |\vec{n}| + j$ for $j = 1, \dots, l-1$,
- $\vec{n}_0 \leq \vec{n}_1 \leq \dots \leq \vec{n}_{l-2} \leq \vec{n}_{l-1}$ (componentwise).

Therefore, $(\vec{n}_0, \vec{n}_1, \dots, \vec{n}_{l-1})$ defines a path connecting \vec{n} to \vec{n}_{l-1} , where in each step the multi-index \vec{n}_k is increased by one in exactly one direction.

For every such kind of a fixed path, we define the associated Wronskian of multiple orthogonal polynomials by

$$\begin{aligned} W(\vec{n}, l; x) &:= W(P_{\vec{n}_0}(x), P_{\vec{n}_1}(x), \dots, P_{\vec{n}_{l-1}}(x)) \\ &= \det \begin{pmatrix} P_{\vec{n}_0}(x) & P_{\vec{n}_1}(x) & \cdots & P_{\vec{n}_{l-1}}(x) \\ P'_{\vec{n}_0}(x) & P'_{\vec{n}_1}(x) & \cdots & P'_{\vec{n}_{l-1}}(x) \\ \vdots & \vdots & \ddots & \vdots \\ P_{\vec{n}_0}^{(l-1)}(x) & P_{\vec{n}_1}^{(l-1)}(x) & \cdots & P_{\vec{n}_{l-1}}^{(l-1)}(x) \end{pmatrix}, \end{aligned} \quad (1.5)$$

where $P_{\vec{n}}$ is the type II multiple orthogonal polynomial given in (1.4). Clearly, $W(\vec{n}, l; x)$ is a polynomial in x depending on the parameters \vec{n} , l and the path starting from \vec{n} . We shall also use the notation $W(\vec{a}, \vec{b}, \dots, \vec{e}; x)$ to emphasize the dependence on a specific path consisting of the multi-indices $(\vec{a}, \vec{b}, \dots, \vec{e})$ if necessary.

Our main results are stated as follows.

Theorem 1.1. *Suppose that the weights (w_1, w_2, \dots, w_r) form an AT system on $[a, b]$ for all the multi-indices $\vec{n} \in \mathbb{N}^r$, then we have*

$$W(\vec{n}, l; x) > 0, \quad x \in \mathbb{R}, \quad (1.6)$$

if l is even, where $W(\vec{n}, l; x)$ is defined in (1.5).

Note that our assumption on the weights ensures the existence of multiple orthogonal polynomials (see the discussion at the end of the previous section), thus the Wronskian is well-defined. If l is odd, then we have the following result.

Theorem 1.2. *Let w_1, w_2, \dots, w_r be r weights as in Theorem 1.1 and let l be odd. For each fixed multi-index \vec{n} the polynomials $W(\vec{n}, l; x)$ have exactly $|\vec{n}|$ simple zeros on the real axis. Furthermore, given two paths consisting of l multi-indices such that the last $l-1$ multi-indices of one path starting from \vec{n} coincide with the first $l-1$ multi-indices of the other path ending at \vec{m} (which also means $|\vec{m}| = |\vec{n}| + l$ and $\vec{n} \leq \vec{m}$), then the real zeros of two associated Wronskians strictly interlace.*

In case $l = 1$, this theorem states that the type II multiple orthogonal polynomial $P_{\vec{n}}$ whose weights form an AT system has $|\vec{n}|$ zeros and the zeros of $P_{\vec{n}}$ and $P_{\vec{n} + \vec{e}_k}$ ($k = 1, \dots, r$) interlace, where $\vec{e}_k = (0, \dots, 0, 1, 0, \dots, 0)$ denotes the k -th standard unit vector with 1 on the k -th entry. These facts are already known; cf. [35, Theorem 23.2] and [33]. Moreover, if $r = 1$, the type II multiple orthogonal polynomials reduce to the usual orthogonal polynomials, hence, Theorems 1.1 and 1.2 generalize classical results of Karlin and Szegő mentioned at the beginning.

Finally, one can also consider the Wronskians involving type I multiple orthogonal polynomials by replacing $P_{\vec{n}}$ in (1.5) by $Q_{\vec{n}}$ defined in (1.1). In this case, the Wronskians are not polynomials in general.

Theorem 1.3. *Let w_1, w_2, \dots, w_r be r sufficiently many times differentiable weights as in Theorem 1.1 and assume that l is even. For each path starting from the multi-index \vec{n} , we have that the function $W(Q_{\vec{n}}(x), Q_{\vec{n}_1}(x), \dots, Q_{\vec{n}_{l-1}}(x))$ given in (1.5) keeps a constant sign on (a, b) , if the Wronskian is well defined.*

1.4 Outline of the paper

The rest of this paper is organized as follows. In Section 2, we prove Theorems 1.1–1.3. We next give two applications of our main results. In Section 3, we show that the m -th moment of the average characteristic polynomials for multiple orthogonal polynomial ensembles can be expressed as a Wronskian of the type II multiple orthogonal polynomials. It is then straightforward to obtain the distinct behavior of the moments for odd and even m in a special multiple orthogonal ensemble – the AT ensemble. In Section 4 we derive the inequalities of Turán type for some classical multiple orthogonal polynomials, namely, for multiple Hermite polynomials and multiple Laguerre polynomials. We conclude this paper with numerical study of the zero configurations of the Wronskians for multiple Hermite polynomials and multiple Laguerre polynomials in Section 5. It comes out that the zeros have fascinating and regular configurations in the complex plane, which might be of independent interest.

2 Proofs of Theorems 1.1–1.3

We shall prove our main theorems by extending the arguments in [39]. Roughly speaking, the proofs of Theorems 1.1 and 1.3 use the properties of an AT system, while for the proof of Theorem 1.2 one needs Theorem 1.1 and Sylvester’s theorem concerning the relation between the determinants of a square matrix and its minors.

2.1 Proof of Theorem 1.1

We first show that the Wronskian (1.6) keeps a constant sign on the real axis for l even. If this is not true, we may find a real number x_0 such that $W(\vec{n}, l; x_0) = 0$. This in turn implies the

existence of the constants $\lambda_0, \lambda_1, \dots, \lambda_{l-1}$ depending on the path such that the function

$$f(x) := \sum_{i=0}^{l-1} \lambda_i P_{\vec{n}_i}(x) \quad (2.1)$$

satisfies

$$f^{(k)}(x_0) = 0, \quad k = 0, 1, \dots, l-1.$$

Thus, x_0 is a zero of $f(x)$ of multiplicity at least l .

We further claim that f has at least $|\vec{n}|$ zeros on (a, b) where f changes sign. Such zeros are also called *nodal zeros*. To see this, we first observe from (1.4) and (2.1) that the equality

$$\int f(x) \sum_{i=1}^r q_i(x) w_i(x) dx = 0 \quad (2.2)$$

holds for any polynomials $q_i(x)$ of degree less than or equal to $n_i - 1$, where we also make use of the fact that $\vec{n} \leq \vec{n}_1 \leq \dots \leq \vec{n}_{l-2} \leq \vec{n}_{l-1}$. In order to obtain a contradiction, suppose that f has at most $k \leq |\vec{n}| - 1$ nodal zeros on (a, b) , say, x_1, x_2, \dots, x_k , then

$$f(x) = \prod_{i=1}^k (x - x_i) Q(x), \quad (2.3)$$

where Q does not change the sign on the the interval (a, b) . Since $k \leq |\vec{n}| - 1$, it is always possible to construct a multi-index $\vec{m} = (m_1, m_2, \dots, m_r) \in \mathbb{N}^r$ such that $\vec{m} \leq \vec{n}$, and $|\vec{m}| = k + 1 \leq |\vec{n}|$ for any given initial multi-index \vec{n} . Then there exist polynomials \tilde{q}_i , $1 \leq i \leq r$, with degrees less than or equal to $m_i - 1$ satisfying

$$\sum_{i=1}^r \tilde{q}_i(x) w_i(x) = \begin{cases} 0, & \text{if } x = x_1, x_2, \dots, x_k, \\ 1, & \text{if } x = x_{k+1}, \end{cases}$$

where x_1, x_2, \dots, x_k are the nodal zeros as in (2.3) and x_{k+1} is an arbitrary point on (a, b) but different from those k nodal zeros. Indeed, this is equivalent to solving a linear system of $|\vec{m}| = k + 1$ equations for the unknown coefficients of \tilde{q}_i . This system is uniquely solvable if the matrix

$$\begin{pmatrix} w_1(x_1) & \cdots & x_1^{m_1-1} w_1(x_1) & & w_r(x_1) & \cdots & x_1^{m_r-1} w_r(x_1) \\ w_1(x_2) & \cdots & x_2^{m_1-1} w_1(x_2) & & w_r(x_2) & \cdots & x_2^{m_r-1} w_r(x_2) \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ w_1(x_{k+1}) & \cdots & x_{k+1}^{m_1-1} w_1(x_{k+1}) & & w_r(x_{k+1}) & \cdots & x_{k+1}^{m_r-1} w_r(x_{k+1}) \end{pmatrix}$$

is not singular, which is immediate on account of the fact that our weights (w_1, w_2, \dots, w_r) form an AT system on $[a, b]$ for all the multi-indices. The Chebyshev property further indicates that the function $\prod_{i=1}^k (x - x_i) (\sum_{i=1}^r \tilde{q}_i(x) w_i(x))$ does not change the sign on (a, b) (cf. [40]). This, together with (2.3), implies that

$$\int f(x) \sum_{i=1}^r \tilde{q}_i(x) w_i(x) dx \neq 0,$$

which is a contradiction with (2.2). Thus, we have proved f has at least $|\vec{n}|$ nodal zeros on (a, b) .

Note that our definition of f in (2.1) shows that f is a polynomial of degree less than or equal to $|\vec{n}| + l - 1$. If x_0 is different from all these nodal zeros, then f will have at least $|\vec{n}| + l$ zeros, which is a contradiction. On the other hand, if x_0 coincides with one of the nodal zeros, multiplicity of x_0 must be at least $l + 1$ (since the zero of even multiplicity l cannot be nodal), hence, the number of total zeros is at least $|\vec{n}| - 1 + l + 1 = |\vec{n}| + l$, again a contradiction. In summary, we have proved that $W(\vec{n}, l; x)$ keeps a constant sign on the real axis if l is even.

Finally, the inequality (1.6) follows from the observation that the leading term of the $W(\vec{n}, l; x)$ is $x^{|\vec{n}|}$, up to a positive constant. This can be seen by first multiplying the rows of (1.5) by $1, x, \dots, x^{l-1}$ and the columns by $x^{-|\vec{n}|}, x^{-|\vec{n}|-1}, \dots, x^{-|\vec{n}|-l+1}$, respectively, and then letting $x \rightarrow \infty$.

2.2 Proof of Theorem 1.2

The proof of this theorem is similar to that for the case of orthogonal polynomials in [39]. We start with a special form of Sylvester's identity which states that for any square matrix A of size n and $n \geq m_1 \geq m_2 \geq 1$ and $n \geq n_1 > n_2 \geq 1$, the following identity (cf. [28]) holds:

$$\det A \cdot \det A[m_1, m_2; n_1, n_2] = \det \begin{pmatrix} \det A[m_1; n_1] & \det A[m_1; n_2] \\ \det A[m_2; n_1] & \det A[m_2; n_2] \end{pmatrix}, \quad (2.4)$$

where $A[m_1, m_2; n_1, n_2]$ denotes the submatrix obtained from A by deleting rows m_1, m_2 and columns n_1, n_2 , and a similar definition holds for $A[m_i; n_j]$, $i, j = 1, 2$.

Given any path, say, $(\vec{n}_0, \vec{n}_1, \dots, \vec{n}_{l-1})$, we can find another path $(\vec{n}_1, \vec{n}_2, \dots, \vec{n}_l)$ whose first $l - 1$ multi-indices coincide with the last $l - 1$ multi-indices of the original path. Applying (2.4) to $\det A = W(\vec{n}_0, \vec{n}_1, \dots, \vec{n}_{l-1}, \vec{n}_l; x)$ with $m_1 = l + 1, m_2 = l$ and $n_1 = l + 1, n_2 = 1$, we have

$$W(\vec{n}_0, \dots, \vec{n}_l; x) \cdot W(\vec{n}_1, \dots, \vec{n}_{l-1}; x) = \det \begin{pmatrix} W(\vec{n}_0, \dots, \vec{n}_{l-1}; x) & W(\vec{n}_1, \dots, \vec{n}_l; x) \\ W'(\vec{n}_0, \dots, \vec{n}_{l-1}; x) & W'(\vec{n}_1, \dots, \vec{n}_l; x) \end{pmatrix},$$

where the derivative $'$ is with respect to x . Recall that l is odd, hence, Theorem 1.1 gives us

$$W(\vec{n}_0, \dots, \vec{n}_{l-1}; x)W'(\vec{n}_1, \dots, \vec{n}_l; x) - W'(\vec{n}_0, \dots, \vec{n}_{l-1}; x)W(\vec{n}_1, \dots, \vec{n}_l; x) > 0,$$

for all $x \in \mathbb{R}$. This inequality shows that all zeros of $W(\vec{n}_0, \dots, \vec{n}_{l-1}; x)$ must be simple. Moreover, we have

$$W(\vec{n}_0, \dots, \vec{n}_{l-1}; x)W'(\vec{n}_1, \dots, \vec{n}_l; x) > 0, \quad \text{if} \quad W(\vec{n}_1, \dots, \vec{n}_l; x) = 0, \quad (2.5)$$

and

$$W'(\vec{n}_0, \dots, \vec{n}_{l-1}; y)W(\vec{n}_1, \dots, \vec{n}_l; y) < 0, \quad \text{if} \quad W(\vec{n}_0, \dots, \vec{n}_{l-1}; y) = 0. \quad (2.6)$$

If x_0 and \tilde{x}_0 are two consecutive zeros of $W(\vec{n}_1, \dots, \vec{n}_l; x)$, then

$$W'(\vec{n}_1, \dots, \vec{n}_l; x_0)W'(\vec{n}_1, \dots, \vec{n}_l; \tilde{x}_0) < 0.$$

This, together with (2.5), implies that

$$W(\vec{n}_0, \dots, \vec{n}_{l-1}; x_0)W(\vec{n}_0, \dots, \vec{n}_{l-1}; \tilde{x}_0) < 0.$$

Hence, there must be a zero of $W(\vec{n}_0, \dots, \vec{n}_{l-1}; x)$ between (x_0, \tilde{x}_0) . By (2.6) and the same argument, we conclude that there exists at least one zero of $W(\vec{n}_1, \dots, \vec{n}_l; x)$ between any two consecutive zeros of $W(\vec{n}_0, \dots, \vec{n}_{l-1}; x)$. This completes the proof of the simplicity and the interlacing property of real zeros stated in Theorem 1.2.

Finally, it remains to calculate the number of real zeros of the Wronskian. This can be achieved by induction argument on $|\vec{n}|$. If $|\vec{n}| = 0$ (i.e., $\vec{n} = \vec{0}$), then $W(\vec{0}, l; x) > 0$, since the Wronskian matrix reduces to the upper diagonal matrix with positive diagonal entries. Hence, there is no real zero in this case. If $|\vec{n}| = 1$, we have $W(\vec{n}, l; x) \sim x^{|\vec{n}|l} = x^l$ as $x \rightarrow \pm\infty$ and l is odd, thus, $W(\vec{n}, l; x)$ has at least one real zero. If it has more than one real zero, by interlacing property this will lead to the existence of a real zero for certain Wronskian associated with a path starting from $\vec{0}$, hence, a contradiction. Suppose now that $W(\vec{n}, l; x)$ has exactly $k > 0$ simple real zeros if $|\vec{n}| = k$. When $|\vec{n}| = k + 1$, for any path starting from \vec{n} , we can find another path starting from \vec{m} such that $|\vec{m}| = k$ and the zeros of $W(\vec{n}, l; x)$ and $W(\vec{m}, l; x)$ interlace. Then, $W(\vec{n}, l; x)$ will have at least $k - 1$ simple zeros. If x_k is the largest zero of $W(\vec{m}, l; x)$, then $W'(\vec{m}, l; x_k) > 0$ since $W(\vec{m}, l; x)$ is positive for x large. From (2.6), we have that $W(\vec{n}, l; x_k) < 0$, hence there will be at least one zero of $W(\vec{n}, l; x)$ on the right hand side of x_k . There would be only one such zero, again by interlacing property. Similar argument implies that there will be exactly one zero on the left of the smallest real zero of $W(\vec{m}, l; x)$. Thus, $W(\vec{n}, l; x)$ will have exactly $k + 1 = |\vec{n}|$ real simple zeros.

2.3 Proof of Theorem 1.3

The proof is similar to that of Theorem 1.1. Suppose that there exists a point $x_0 \in (a, b)$ such that $W(Q_{\vec{n}_1}(x_0), Q_{\vec{n}_2}(x_0), \dots, Q_{\vec{n}_{l-1}}(x_0)) = 0$. Then we can find constants λ_i , $i = 0, 1, \dots, l - 1$, such that the function

$$f(x) := \sum_{i=0}^{r-1} \lambda_i Q_{\vec{n}_i}(x), \quad \vec{n}_0 = \vec{n},$$

has a zero at x_0 of multiplicity at least l . Since

$$\int f(x)p(x) dx = 0$$

for any polynomial p of degree less than or equal to $|\vec{n}| - 2$, we have that f has at least $|\vec{n}| - 1$ nodal zeros. Thus, as in the proof of Theorem 1.1, we conclude that f will have at least $|\vec{n}| + l - 1$ zeros. This is a contradiction to the fact that f is a linear combination of the Chebyshev system for the multi-index \vec{n}_{l-1} , which has at most $|\vec{n}_{l-1}| - 1 = |\vec{n}| + l - 2$ zeros on $[a, b]$.

3 Moments of the average characteristic polynomials for multiple orthogonal polynomial ensembles

In this section, we shall apply our results to the moments of the average characteristic polynomials for multiple orthogonal polynomial ensembles.

It is well-known that, besides the interest from the approximation theory, multiple orthogonal polynomials have also arisen recently in a natural way in certain models of mathematical physics, including random matrix theory, non-intersecting paths, etc; cf. [42, 43] and the references therein. Indeed, let us consider $|\vec{n}| = \sum_{i=1}^r n_i = n$ random points on the real line whose joint probability density function (p.d.f.) can be written as a product of two determinants:

$$\frac{1}{Z_n} \det (f_i(x_j))_{i,j=1}^n \det (g_i(x_j))_{i,j=1}^n, \quad (3.1)$$

where Z_n is the normalizing constant to make the total probability on \mathbb{R}^n equal to one, and the two sequences of functions f_i, g_i are given by

$$f_i(x) = x^{i-1}, \quad i = 1, \dots, n,$$

where W is defined in (1.5). Note that when $g_i(x) = x^{i-1}w(x)$ in (3.1) (i.e., in the case of orthogonal polynomial ensembles), the formula above was first shown by Brézin and Hikami [15]; see also [47].

Combining (3.4) and Theorems 1.1–1.2, we obtain

Corollary 3.1. *Assume that the weights (w_1, w_2, \dots, w_r) in (3.1) form an AT system on $[a, b]$ for all the multi-indices in \mathbb{N}^r (i.e., an AT ensemble in the sense of [42]). Then we have that the moments of the average characteristic polynomials with respect to (3.1)*

$$\mathbb{E} \left[\prod_{k=1}^n (z - x_k)^m \right]$$

are strictly positive on the real axis if m is even; while for odd m , the moments admit oscillatory behavior as stated in Theorem 1.2.

4 Some inequalities for multiple orthogonal polynomials

In this section, we shall use our results to derive the inequalities of Turán type for some classical multiple orthogonal polynomials, namely, for multiple Hermite polynomials and multiple Laguerre polynomials. It is known that the weights for these polynomials form an AT system for any multi-index $\vec{n} \in \mathbb{N}^r$.

4.1 Turán inequalities for multiple Hermite polynomials

Multiple Hermite polynomials are defined by

$$\int_{-\infty}^{\infty} x^k H_{\vec{n}}(x) e^{-x^2 + c_j x} dx = 0, \quad k = 0, 1, \dots, n_j - 1,$$

for $j = 1, \dots, r$, where $c_i \neq c_j$ if $i \neq j$; cf. [12], [35, § 23.5] and [58, § 3.4]. An explicit formula for multiple Hermite polynomials is

$$H_{\vec{n}} = \frac{(-1)^{|\vec{n}|}}{2^{|\vec{n}|}} \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} \binom{n_1}{k_1} \cdots \binom{n_r}{k_r} \prod_{j=1}^r c_j^{n_j - k_j} (-1)^{|\vec{k}|} H_{|\vec{k}|}(x),$$

where $\vec{k} = (k_1, \dots, k_r)$ and $H_{|\vec{k}|}$ is the usual Hermite polynomial of degree $|\vec{k}|$ with the leading coefficient $2^{|\vec{k}|}$. The following statement holds for multiple Hermite polynomials.

Theorem 4.1. *The multiple Hermite polynomials satisfy the following inequalities:*

$$H_{\vec{n} + \vec{e}_j}(x) H_{\vec{n} + \vec{e}_k}(x) - H_{\vec{n}}(x) H_{\vec{n} + \vec{e}_j + \vec{e}_k}(x) > 0, \quad x \in \mathbb{R}, \quad (4.1)$$

for $j, k = 1, \dots, r$, where $\vec{e}_i = (0, \dots, 0, 1, 0, \dots, 0)$ denotes the i -th standard unit vector with 1 on the i -th entry. In particular, by taking $j = k$, we have

$$H_{\vec{n} + 2\vec{e}_j}^2(x) - H_{\vec{n}}(x) H_{\vec{n} + 2\vec{e}_j}(x) > 0, \quad x \in \mathbb{R}. \quad (4.2)$$

Remark 4.2. It is readily seen that the formula (4.2) extends the Turán inequality for the Hermite polynomials [39].

Proof. From Theorem 1.1 with $l = 2$, it follows that

$$\det \begin{pmatrix} H_{\vec{n}}(x) & H_{\vec{n}+\vec{e}_j}(x) \\ H'_{\vec{n}}(x) & H'_{\vec{n}+\vec{e}_j}(x) \end{pmatrix} > 0, \quad j = 1, \dots, r, \quad (4.3)$$

for any $x \in \mathbb{R}$. Note that $H_{\vec{n}}$ satisfies the following raising operations (cf. [35, § 23.8.2])

$$\frac{d}{dx} (e^{-x^2+c_k x} H_{\vec{n}}(x)) = -2e^{-x^2+c_k x} H_{\vec{n}+\vec{e}_k}(x), \quad k = 1, \dots, r,$$

or, equivalently,

$$\frac{d}{dx} H_{\vec{n}}(x) = -2H_{\vec{n}+\vec{e}_k}(x) + (2x - c_k)H_{\vec{n}}(x), \quad k = 1, \dots, r. \quad (4.4)$$

Inserting this formula into (4.3) gives (4.1). ■

It is worthwhile to point out that the inequality (4.1) is independent of the parameters c_j appearing in the weight functions. Furthermore, by choosing the path in the Wronskian matrix to be $(\vec{n}, \vec{n} + \vec{e}_j, \dots, \vec{n} + (l - 1)\vec{e}_j)$ for fixed $j = 1, \dots, r$ and using (4.4) with $k = j$, it is readily seen that

$$\begin{aligned} & W(H_{\vec{n}}(x), H_{\vec{n}+\vec{e}_j}(x), \dots, H_{\vec{n}+(l-1)\vec{e}_j}(x)) \\ &= (-2)^{\frac{l(l-1)}{2}} \det \begin{pmatrix} H_{\vec{n}}(x) & H_{\vec{n}+\vec{e}_j}(x) & \dots & H_{\vec{n}+(l-1)\vec{e}_j}(x) \\ H_{\vec{n}+\vec{e}_j}(x) & H_{\vec{n}+2\vec{e}_j}(x) & \dots & H_{\vec{n}+l\vec{e}_j}(x) \\ \vdots & \vdots & \ddots & \vdots \\ H_{\vec{n}+(l-1)\vec{e}_j}(x) & H_{\vec{n}+l\vec{e}_j}(x) & \dots & H_{\vec{n}+2(l-1)\vec{e}_j}(x) \end{pmatrix}, \end{aligned} \quad (4.5)$$

that is, we pass from the determinant of the Wronskian type to the Hankel determinant. This, together with Theorem 1.1, implies

Corollary 4.3. *Let $T(H_{\vec{n}}(x), H_{\vec{n}+\vec{e}_j}(x), \dots, H_{\vec{n}+(l-1)\vec{e}_j}(x))$ be the Hankel determinant of multiple Hermite polynomials on the right hand side of (4.5), then*

$$(-1)^{\frac{l(l-1)}{2}} T(H_{\vec{n}}(x), H_{\vec{n}+\vec{e}_j}(x), \dots, H_{\vec{n}+(l-1)\vec{e}_j}(x)) > 0, \quad x \in \mathbb{R}, \quad j = 1, \dots, r,$$

if l is even.

4.2 Two-parameter Turán inequalities for multiple Laguerre polynomials

There are two kinds of multiple Laguerre polynomials. Multiple Laguerre polynomials of the first kind are defined by the orthogonality conditions

$$\int_0^\infty x^k L_{\vec{n}}^{\vec{\alpha}}(x) x^{\alpha_j} e^{-x} dx = 0, \quad k = 0, 1, \dots, n_j - 1,$$

for $j = 1, \dots, r$, where $\vec{\alpha} = (\alpha_1, \dots, \alpha_r)$ with $\alpha_j > -1$ and $\alpha_i - \alpha_j \notin \mathbb{Z}$ whenever $i \neq j$; cf. [12], [35, § 23.4.1] and [58, § 3.2]. An explicit formula for the multiple Laguerre polynomials of the first kind is

$$\begin{aligned} L_{\vec{n}}^{\vec{\alpha}}(x) &= \sum_{k_1=0}^{n_1} \dots \sum_{k_r=0}^{n_r} \binom{n_1}{k_1} \dots \binom{n_r}{k_r} \binom{n_r + \alpha_r}{k_r} \dots \\ &\quad \times \binom{|\vec{n}| - |\vec{k}| + k_1 + \alpha_1}{k_1} \prod_{i=1}^r k_i! (-1)^{|\vec{k}|} |x^{|\vec{n}|-|\vec{k}|}. \end{aligned} \quad (4.6)$$

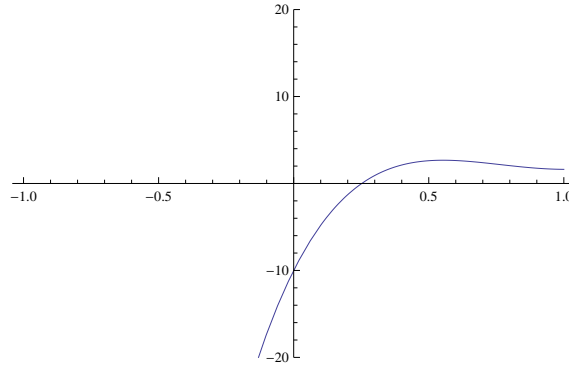


Figure 1. Plot of (4.7) with $\vec{n} = (n, m) = (1, 1)$ and $\vec{\alpha} = (1/2, 1/3)$.

Multiple Laguerre polynomials of the second kind are defined by the orthogonality conditions

$$\int_0^\infty x^k L_{\vec{n}}^{(\alpha, \vec{c})}(x) x^\alpha e^{-c_j x} dx = 0, \quad k = 0, 1, \dots, n_j - 1,$$

for $j = 1, \dots, r$, where $\vec{c} = (c_1, \dots, c_r)$ and we assume that $\alpha > -1$, $c_j > 0$ and $c_i \neq c_j$ whenever $i \neq j$; cf. [12], [35, § 23.4.2], [48, Remark 5 on p. 160] and [58, § 3.3]. An explicit formula for these polynomials is

$$L_{\vec{n}}^{(\alpha, \vec{c})}(x) = \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} \binom{n_1}{k_1} \cdots \binom{n_r}{k_r} \binom{|\vec{n}| + \alpha}{|\vec{k}|} |\vec{k}|! \frac{(-1)^{|\vec{k}|}}{\prod_{j=1}^r c_j^{k_j}} x^{|\vec{n}| - |\vec{k}|}.$$

It turns out that the Turán inequalities of the form (4.1) do not hold for multiple Laguerre polynomials in general. We can easily calculate from (4.6) that, for instance, for multiple Laguerre polynomials of the first kind with $r = 2$, $\vec{n} = (n, m) = (1, 1)$ and $\vec{\alpha} = (1/2, 1/3)$ the expression

$$(L_{n+1, m}^{\vec{\alpha}}(x))^2 - L_{n, m}^{\vec{\alpha}}(x) L_{n+2, m}^{\vec{\alpha}}(x) \tag{4.7}$$

is reduced to

$$2x^5 - \frac{119}{6}x^4 + \frac{647}{9}x^3 - \frac{7495}{72}x^2 + \frac{185}{3}x - 10,$$

which can be both positive and negative; see Fig. 1. We can also find similar counterexamples for the other choices of indices and also for multiple Laguerre polynomials of the second kind. However, multiple Laguerre polynomials satisfy the following two-parameter Turán inequalities in the sense of [17].

Theorem 4.4. *For multiple Laguerre polynomials of the first kind we have*

$$L_{\vec{n}+\vec{e}_k}^{\vec{\alpha}}(x) L_{\vec{n}+\vec{e}_j}^{\vec{\alpha}-\vec{e}_j}(x) - L_{\vec{n}}^{\vec{\alpha}}(x) L_{\vec{n}+\vec{e}_j+\vec{e}_k}^{\vec{\alpha}-\vec{e}_j}(x) > 0, \quad x > 0, \tag{4.8}$$

for $\vec{\alpha} > \vec{0}$ and $j, k = 1, \dots, r$.

Similarly, for multiple Laguerre polynomials of the second kind, we have

$$L_{\vec{n}+\vec{e}_k}^{(\alpha, \vec{c})}(x) L_{\vec{n}+\vec{e}_j}^{(\alpha-1, \vec{c})}(x) - L_{\vec{n}}^{(\alpha, \vec{c})}(x) L_{\vec{n}+\vec{e}_j+\vec{e}_k}^{(\alpha-1, \vec{c})}(x) > 0, \quad x > 0, \tag{4.9}$$

for $\alpha > 0$ and $j, k = 1, \dots, r$.

Proof. From Theorem 1.1 with $l = 2$, we see that

$$\det \begin{pmatrix} L_{\vec{n}}(x) & L_{\vec{n}+\vec{e}_k}(x) \\ xL'_{\vec{n}}(x) & xL'_{\vec{n}+\vec{e}_k}(x) \end{pmatrix} > 0, \quad x > 0, \quad (4.10)$$

for $k = 1, \dots, r$, where $L_{\vec{n}}$ is the multiple Laguerre polynomial of the first or second kind. Note that $L_{\vec{n}}^{\vec{\alpha}}$ satisfies the following raising operator (cf. [35, § 23.4.2]):

$$\frac{d}{dx} (x^{\alpha_j} e^{-x} L_{\vec{n}}^{\vec{\alpha}}(x)) = -x^{\alpha_j-1} e^{-x} L_{\vec{n}+\vec{e}_j}^{\vec{\alpha}-\vec{e}_j}(x), \quad j = 1, \dots, r,$$

or, equivalently,

$$x \frac{d}{dx} L_{\vec{n}}^{\vec{\alpha}}(x) = (x - \alpha_j) L_{\vec{n}}^{\vec{\alpha}}(x) - L_{\vec{n}+\vec{e}_j}^{\vec{\alpha}-\vec{e}_j}(x), \quad j = 1, \dots, r,$$

Substituting this formula into (4.10) gives us (4.8).

Similarly, (4.9) follows from (4.10) and the relation (cf. [35, § 23.4.4])

$$\frac{d}{dx} (x^{\alpha} e^{-c_j x} L_{\vec{n}}^{(\alpha, \vec{c})}(x)) = -c_j x^{\alpha-1} e^{-c_j x} L_{\vec{n}+\vec{e}_j}^{(\alpha-1, \vec{c})}(x), \quad j = 1, \dots, r,$$

or, equivalently,

$$x \frac{d}{dx} L_{\vec{n}}^{(\alpha, \vec{c})}(x) = (c_j x - \alpha) L_{\vec{n}}^{(\alpha, \vec{c})}(x) - c_j L_{\vec{n}+\vec{e}_j}^{(\alpha-1, \vec{c})}(x), \quad j = 1, \dots, r.$$

This completes the proof of Theorem 4.4. ■

5 Configurations of zeros for the Wronskians of multiple Hermite and Laguerre polynomials

We conclude this paper by studying numerically the geometric configuration of zeros of the Wronskians of multiple Hermite and Laguerre (of both kinds) polynomials using *Mathematica*¹. Our motivation arises from the fact that the structure of zeros of certain Wronskians of orthogonal polynomials or special functions has recently been studied numerically (cf. [19, 25, 27] and the references therein), where it is shown that they have highly regular configurations in the complex plane. It turns out that the zeros of Wronskians for certain multiple orthogonal polynomials produce intriguing pictures as well, which might be of independent interest.

Throughout this section, we take $r = 2$ and denote the multi-index \vec{n} by $(n, m) \in \mathbb{N}^2$. Unless otherwise stated, the path associated with the Wronskian (1.5) is chosen in such a way that in each step it is increased by one in the horizontal direction, i.e.,

$$(n, m) \rightarrow (n+1, m) \rightarrow (n+2, m) \rightarrow \dots$$

Other choices of the paths show similar behavior of the zeros. Clearly, the structure of the roots in the complex plane depends on l , \vec{n} , the path chosen and the values of the parameters, therefore, it is difficult to be described completely. We thus have chosen a few illustrative examples from numerical experiments for multiple Hermite and Laguerre polynomials to show the fascinating configurations.

¹<http://www.wolfram.com>

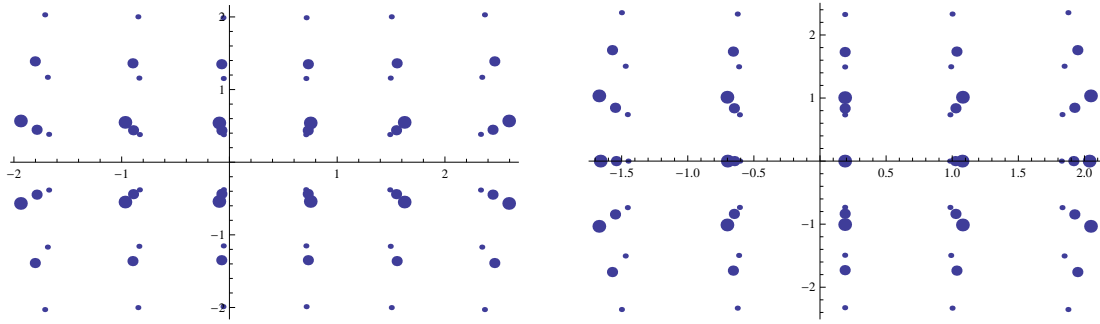


Figure 2. Zeros of the Wronskians for multiple Hermite polynomials with $\vec{n} = (3, 3)$, $\vec{c} = (1/3, 34/35)$ for $l = 2, 4, 6$ (left) and $\vec{n} = (2, 3)$, $\vec{c} = (1/3, 2/5)$ $l = 3, 5, 7$ (right). The size of points decreases as l increases.

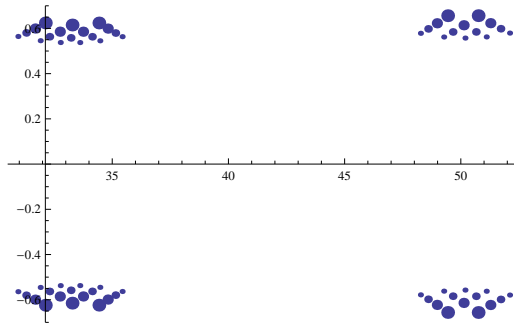


Figure 3. Zeros of the Wronskians for multiple Hermite polynomials with $\vec{n} = (2, 3), (3, 4), (4, 5), (5, 6)$, $\vec{c} = (100, 200/3)$ for $l = 2$. The size of points decreases as $|\vec{n}|$ increases.

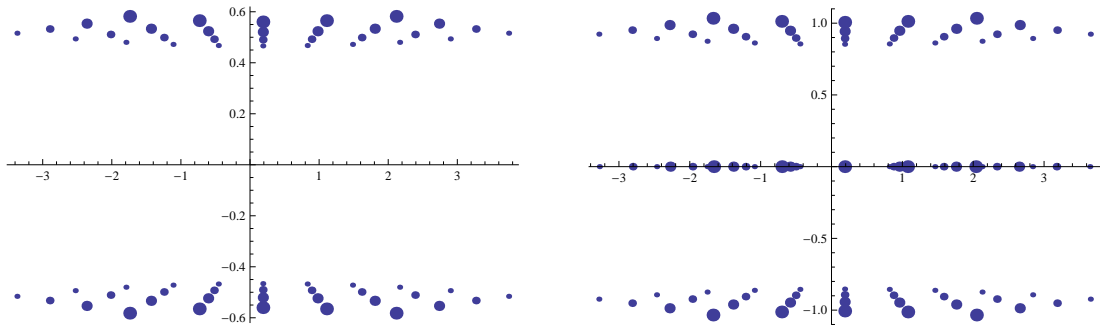


Figure 4. Zeros of the Wronskians for multiple Hermite polynomials with $\vec{n} = (2, 3), (3, 4), (4, 5), (5, 6)$ and $\vec{c} = (1/3, 2/5)$ for $l = 2$ (left) and $l = 3$ (right). The size of points decreases as $|\vec{n}|$ increases.

5.1 Zeros of the Wronskians for multiple Hermite polynomials

The zeros of Wronskians for multiple Hermite polynomials numerically have roughly rectangular-like structure in the complex plane. In Fig. 2 we plot zeros of the Wronskians for these polynomials by fixing \vec{n} , the parameter \vec{c} and increasing the length of the path (the size of points decreases as l increases). We can see additional row of zeros on the real axis for l odd, as indicated by the first part of Theorem 1.2.

If the two values in the parameter \vec{c} differ too much, it seems that the zeros may separate into several rectangles, which is shown in Fig. 3 (the size of points decreases as $|\vec{n}|$ increase). When l is odd, we have additional groups of zeros on the real line as projections of complex groups of roots.

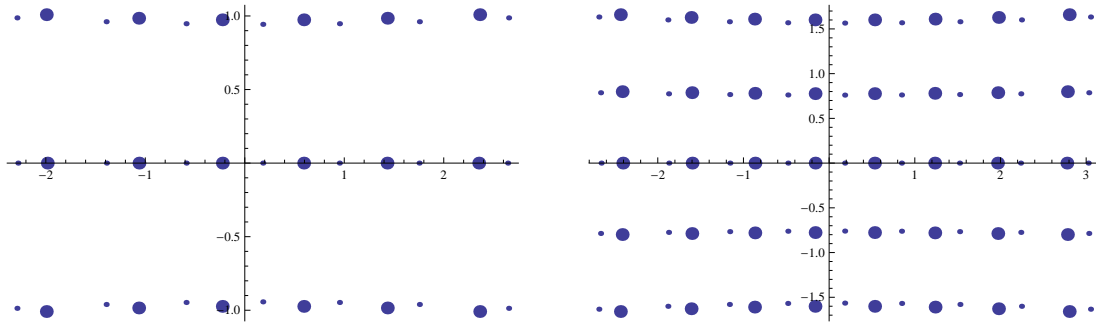


Figure 5. Theorem 1.2 for multiple Hermite polynomials with $\vec{c} = (1/3, 2/5)$, $\vec{n} = (3, 3)$ for $l = 3$ (left) and $\vec{n} = (4, 4)$ for $l = 5$ (right). The size of points decreases as $|\vec{n}|$ increases.

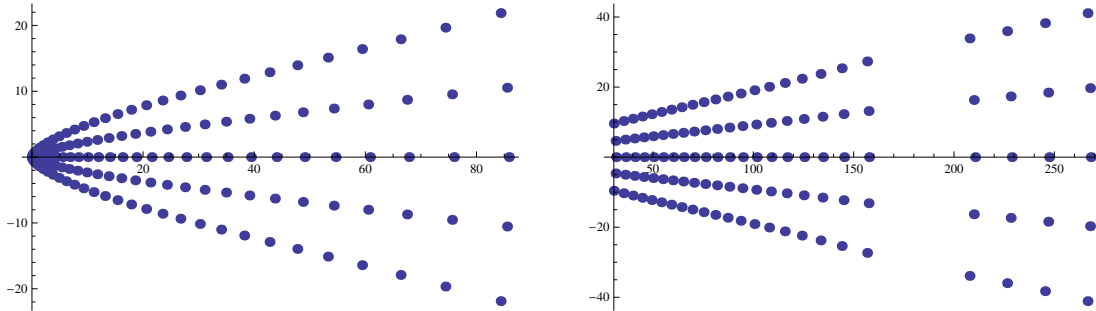


Figure 6. Zeros of the Wronskians for multiple Laguerre polynomials of the first kind with $\vec{\alpha} = (1/2, 1/3)$, $\vec{n} = (10, 20)$ (left) and $\vec{\alpha} = (200, 200/3)$, $\vec{n} = (4, 20)$ (right) for $l = 5$.

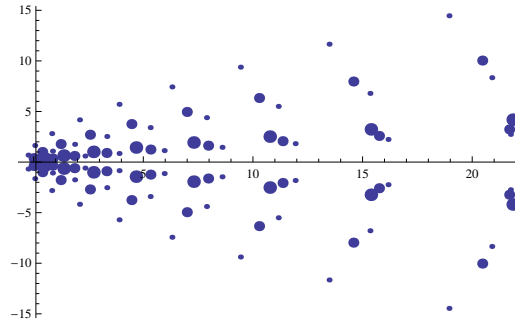


Figure 7. Zeros of the Wronskians for multiple Laguerre polynomials of the first kind with $\vec{n} = (4, 5)$, $\vec{\alpha} = (1/2, 1/3)$ for $l = 2, 4, 6$. The size of points decreases as l increases.

The effect of increasing $|\vec{n}|$ is illustrated in Fig. 4 for l even and odd respectively. As $|\vec{n}|$ increases, the zeros are distributed in a wider range. Furthermore, if l increases, we can see more horizontal lines.

We illustrate Theorem 1.2 in Fig. 5. We clearly see the interlacing of real zeros and regular configurations of zeros in the complex plane. It seems that the interlacing property also appears on the other lines parallel to the real axis. For even l the structure of complex roots is similar (there are no real roots in this case).

5.2 Zeros of the Wronskians for multiple Laguerre polynomials

Since the observed structure of roots of the Wronskians for multiple Laguerre polynomials of the first and second kind is numerically quite similar, we shall concentrate more on the multiple

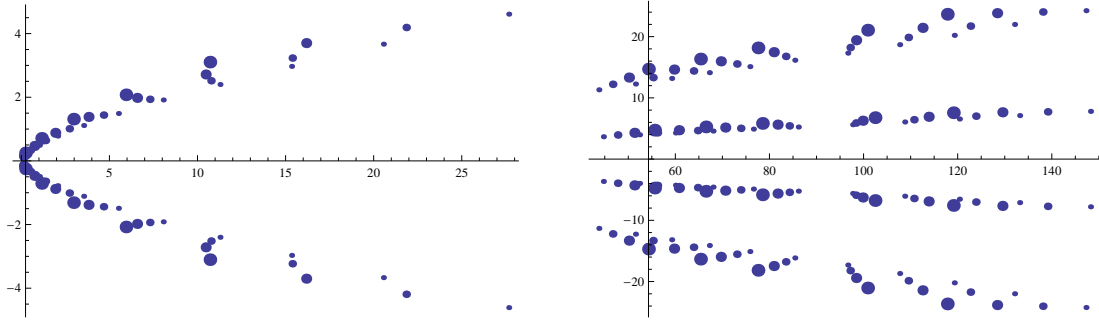


Figure 8. Zeros of the Wronskians for multiple Laguerre polynomials of the first kind with $\vec{n} = (2, 3), (3, 4), (4, 5), (5, 6)$, $\vec{\alpha} = (1/2, 1/3)$ for $l = 2$ (left) and $\vec{\alpha} = (100, 200/3)$ for $l = 4$ (right). The size of points decreases as $|\vec{n}|$ increases.

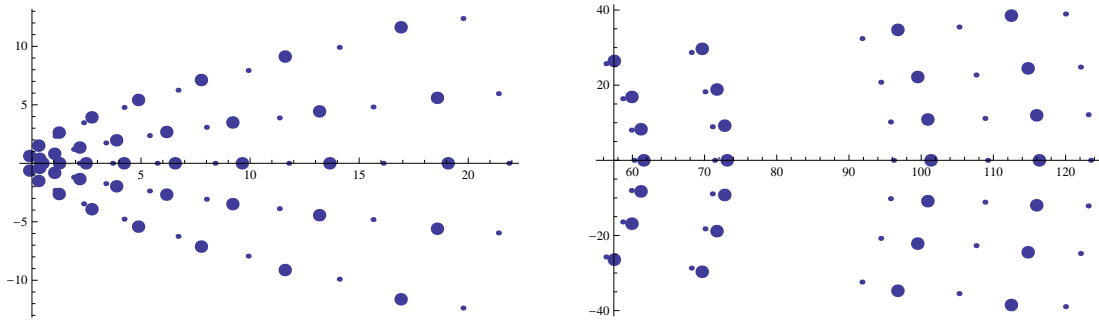


Figure 9. Theorem 1.2 for multiple Laguerre polynomials of the first kind with $\vec{n} = (4, 4)$, $\vec{\alpha} = (1/2, 1/3)$ for $l = 5$ (left) and $\vec{n} = (2, 2)$, $\vec{\alpha} = (100, 200/3)$ for $l = 7$ (right). The size of points decreases as $|\vec{n}|$ increases.

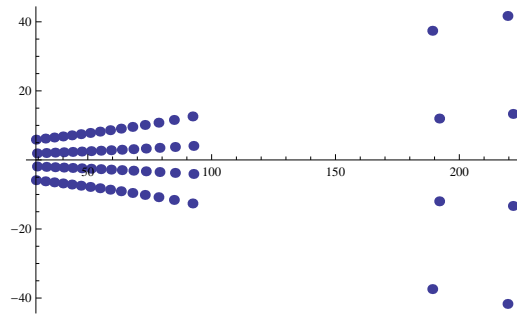


Figure 10. Zeros of the Wronskians for multiple Laguerre polynomials of the second kind with $\vec{n} = (15, 2)$, $\alpha = 100$, $\vec{c} = (2, 3/5)$ for $l = 4$.

Laguerre polynomials of the first kind. The configurations of zeros of Wronskians for multiple Laguerre polynomials of the first kind resemble (several) parabolas (with additional zeros on the real line in case l is odd). Sometimes it looks like zeros lie on arcs of circles with increasing radius. As we change the parameters, we can observe that the zeros on the left can do not accumulate and, as in the case of multiple Hermite polynomials, they can be grouped into several clusters with a certain gap between them. Fig. 6 illustrates these observations.

The zeros of several Wronskians, if plotted together, also present nice interlacing properties. Following the same strategy in previous section, we fix \vec{n} , the parameter $\vec{\alpha}$ and increase l in Fig. 7 (compare with Fig. 2), while in Fig. 8 we increase \vec{n} and fix other parameters (compare with Fig. 4). Theorem 1.2 in this case is illustrated in Fig. 9.

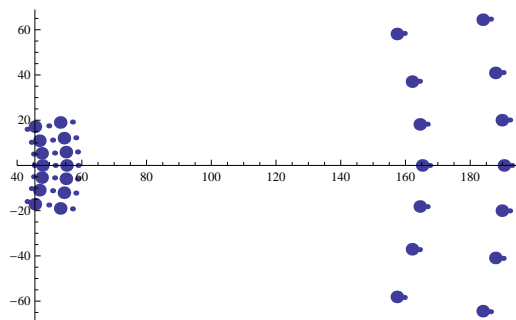


Figure 11. Theorem 1.2 for multiple Laguerre polynomials of the second kind with $\vec{n} = (2, 2)$, $\alpha = 100$, $\vec{c} = (2, 3/5)$ for $l = 7$. The size of points decreases as $|\vec{n}|$ increases.

Finally, the roots of the Wronskians for multiple Laguerre polynomials of the second kind are depicted in Fig. 10 and Theorem 1.2 is illustrated in Fig. 11.

More pictures with different configurations of roots can be found in Mathematica files on the web-pages of the authors (or available on request). We believe the nice and regular geometric configurations generated from the zeros of Wronskians deserve further analytic investigations.

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