Center of Twisted Graded Hecke Algebras for Homocyclic Groups*

Wee Liang GAN † and *Matthew HIGHFIELD* ‡

† *University of California, Riverside, CA 92521, USA*
E-mail: *wlgan@math.ucr.edu*

‡ *Pepperdine University, Malibu, CA 90263, USA*
E-mail: *matthew.highfield@pepperdine.edu*

Received March 31, 2014, in final form October 10, 2014; Published online October 15, 2014
http://dx.doi.org/10.3842/SIGMA.2014.098

Abstract. We determine explicitly the center of the twisted graded Hecke algebras associated to homocyclic groups. Our results are a generalization of formulas by M. Douglas and B. Fiol in [*J. High Energy Phys.* **2005** (2005), no. 9, 053, 22 pages].

Key words: twisted graded Hecke algebra; homocyclic group

2010 Mathematics Subject Classification: 20C08

1 Main results

The notion of twisted graded Hecke algebras was introduced by S. Witherspoon in [10]; they are variants of the graded Hecke algebras of V. Drinfel'd [4] and G. Lusztig [6] (see also [7]) and twisted symplectic reflection algebras of T. Chmutova [2]. To a finite dimensional complex vector space V, a finite subgroup G of $\mathrm{GL}(V)$, and a 2-cocycle α of G, the associated twisted graded Hecke algebra H is, by definition, a Poincaré–Birkhoff–Witt deformation of the crossed-product algebra $SV\#_\alpha G$, where SV denotes the symmetric algebra of V. The center of $SV\#_\alpha G$ is $(SV)^G$, and it is a natural question to determine the center of H. In the non-twisted case, the center of the graded Hecke algebra associated to a finite real reflection group was determined by G. Lusztig in [5, Theorem 6.5]. In this paper, we determine the center of H for the twisted graded Hecke algebra in [10, Example 2.16], where $V=\mathbb{C}^n$ and G is isomorphic to a homocyclic group $(\mathbb{Z}/\ell\mathbb{Z})^{n-1}$. (By a homocyclic group, we mean a direct product of cyclic groups of the same order.) In this example, the algebra H is finitely generated as a module over its center; the center of H therefore plays an important role in the representation theory of H. We show that the center of H is generated by $n+1$ elements subject to one relation, which we determine explicitly. Our results are a generalization of formulas by M. Douglas and B. Fiol who considered the special case when $n=3$ in their paper [3] on $\mathbb{C}^3/(\mathbb{Z}/\ell\mathbb{Z})^2$ orbifolds with discrete torsion.

We state our main results in this section and give the proofs in Section 2. We shall work over $\mathbb{C}$. Let n be an integer ≥ 3, and ℓ an integer ≥ 2. Let $V=\mathbb{C}^n$ and let $x_1,\dots,x_n$ be the standard basis of V. Let G be the subgroup of $SL_n(\mathbb{C})$ consisting of all diagonal matrices g satisfying $g^\ell=1$. Let ζ be a primitive ℓ-th root of unity.

Notation 1.1. All subscripts are taken modulo n. For example, $x_{n+1}=x_1$.

*This paper is a contribution to the Special Issue on New Directions in Lie Theory. The full collection is available at http://www.emis.de/journals/SIGMA/LieTheory2014.html

For $i = 1, \dots, n$, let g_i be the element of G such that

$$g_i(x_j) = \begin{cases} \zeta x_j, & \text{if } \ j = i, \\ \zeta^{-1} x_j, & \text{if } \ j = i+1, \\ x_j, & \text{else.} \end{cases}$$

Observe that $g_n = g_1^{-1} \cdots g_{n-1}^{-1}$. We have an isomorphism $(\mathbb{Z}/\ell\mathbb{Z})^{n-1} \xrightarrow{\sim} G$ defined by sending $(1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$ to $g_1, \dots, g_{n-1}$, respectively.

Define the 2-cocycle $\alpha : G \times G \to \mathbb{C}^\times$ of G by

$$\alpha\big(g_1^{i_1} \cdots g_{n-1}^{i_{n-1}}, g_1^{j_1} \cdots g_{n-1}^{j_{n-1}}\big) = \zeta^{-i_1 j_2 - i_2 j_3 - \cdots - i_{n-2} j_{n-1}}.$$

If E is an algebra, an action of G on E is a homomorphism $G \to \mathrm{Aut}(E)$. Recall that for any algebra E and an action of G on E, one has the crossed product algebra $E \#_\alpha G$. As a vector space, $E \#_\alpha G$ is $E \otimes \mathbb{C}G$; the product is defined by

$$(r \otimes g)(s \otimes h) = \alpha(g, h) r (g \cdot s) \otimes gh$$

for all $r, s \in E$ and $g, h \in G$. If $g, h \in G$, then we shall denote their product in $E \#_\alpha G$ by $g * h$; thus,

$$g * h = \alpha(g, h) gh.$$

One has, for any $i, j \in \{1, \dots, n\}$ with $|i - j| \notin \{1, n-1\}$,

$$g_{i+1} * g_i = \zeta g_i * g_{i+1}, \qquad g_i * g_j = g_j * g_i.$$

Let $t = (t_1, \dots, t_n) \in \mathbb{C}^n$, and write TV for the tensor algebra of V. Following [10, Example 2.16], we make the following definition.

Definition 1.2. Let H be the associative algebra defined as the quotient of $TV \#_\alpha G$ by the relations:

$$x_i x_{i+1} - x_{i+1} x_i = t_i g_i, \qquad x_i x_j - x_j x_i = 0$$

for all $i, j \in \{1, \dots, n\}$ with $|i - j| \notin \{1, n-1\}$.

Remark 1.3. By [10, Theorem 2.10] and [10, Example 2.16], the algebra H in Definition 1.2 is a twisted graded Hecke algebra for G. (However, when $n > 3$ and $\ell = 2$, this is not the most general twisted graded Hecke algebra for G; see [10, Example 2.16] and [9, Example 5.1].)

Let $\mathbb{C}[y_1^\pm, \dots, y_n^\pm]$ be the algebra of Laurent polynomials in the variables $y_1, \dots, y_n$. The group G acts on $\mathbb{C}[y_1^\pm, \dots, y_n^\pm]$ by

$$g_i y_1^{p_1} \cdots y_n^{p_n} = \zeta^{p_i - p_{i+1}} y_1^{p_1} \cdots y_n^{p_n}$$

for all $i \in \{1, \dots, n-1\}$ and $p_1, \dots, p_n \in \mathbb{Z}$.

Proposition 1.4. *There is an injective homomorphism*

$$\Theta : \ \mathsf{H} \longrightarrow \mathbb{C}[y_1^\pm, \dots, y_n^\pm] \#_\alpha G$$

such that

$$\Theta(x_i) = y_i - \left(\frac{\zeta t_i}{\zeta - 1}\right) y_{i+1}^{-1} g_i, \tag{1.1}$$

$$\Theta(g_i) = g_i \tag{1.2}$$

for all $i \in \{1, \dots, n\}$.

Let

$$I = \{\{i_1 < \cdots < i_k\} \mid k \geq 0;\ i_1, \ldots, i_k \in \{1, \ldots, n\}\},$$
$$J = \{\{i_1 < \cdots < i_k\} \in I \mid |i_r - i_s| \notin \{1, n-1\} \text{ for all } r, s\}.$$

Define the elements $\delta, \varepsilon_1, \ldots, \varepsilon_n$ of $\mathbb{Z}^n$ by

$$\delta = (1, 1, \ldots, 1), \quad \varepsilon_1 = (1, 1, 0, \ldots, 0), \quad \varepsilon_2 = (0, 1, 1, 0, \ldots), \quad \ldots, \quad \varepsilon_n = (1, 0, \ldots, 0, 1).$$

Notation 1.5. For any variables $\omega_1, \ldots, \omega_n$ and $p = (p_1, \ldots, p_n) \in \mathbb{Z}^n$, we denote by ω^p the expression $\omega_1^{p_1} \cdots \omega_n^{p_n}$.

We shall set

$$\tau_i = \frac{t_i}{\zeta - 1} \quad \text{for} \quad i = 1, \ldots, n-1, \qquad \tau_n = \frac{\zeta t_n}{\zeta - 1}.$$

Define the element $w \in \mathsf{H}$ by

$$w = \sum_{\{i_1 < \cdots < i_k\} \in J} \tau_{i_1} \cdots \tau_{i_k} x^{\delta - \varepsilon_{i_1} - \cdots - \varepsilon_{i_k}} g_{i_1} * \cdots * g_{i_k}.$$

Example 1.6. If $n = 3$, then

$$w = x_1 x_2 x_3 + \tau_1 x_3 g_1 + \tau_2 x_1 g_2 + \tau_3 x_2 g_3 = x_1 x_2 x_3 + \frac{1}{\zeta - 1}\left(t_1 x_3 g_1 + t_2 x_1 g_2 + \zeta t_3 x_2 g_3\right).$$

In particular, if $n = 3$ and $\ell = 2$, the formula for w is in [1, Lemma 7.1].

Theorem 1.7. *The center of* H *is generated as an algebra by* $x_1^\ell, \ldots, x_n^\ell$, *and* w.

Let Z be the center of H. For $r = 0, \ldots, \lfloor \ell/2 \rfloor$, set

$$\nu_r = (-1)^r \frac{\ell}{\ell - r}\binom{\ell - r}{r},$$

and set

$$\widetilde{\tau}_i = \tau_i^\ell \quad \text{for} \quad i = 1, \ldots, n-1, \qquad \widetilde{\tau}_n = (-1)^{n(\ell-1)} \tau_n^\ell.$$

We define a polynomial F in the $n+1$ variables $a_1, \ldots, a_n$ and b by

$$F = \sum_{\{i_1 < \cdots < i_k\} \in J} \widetilde{\tau}_{i_1} \cdots \widetilde{\tau}_{i_k} a^{\delta - \varepsilon_{i_1} - \cdots - \varepsilon_{i_k}} - \sum_{r=0}^{\lfloor \ell/2 \rfloor} (-1)^{nr} \zeta^{(n-2)r} \nu_r (\tau_1 \cdots \tau_n)^r b^{\ell - 2r}. \tag{1.3}$$

Corollary 1.8. *The assignment*

$$a_i \mapsto x_i^\ell \quad \textit{for} \quad i = 1, \ldots, n, \qquad b \mapsto w \tag{1.4}$$

defines an isomorphism

$$\mathbb{C}[a_1, \ldots, a_n, b]/(F) \xrightarrow{\sim} \mathsf{Z}. \tag{1.5}$$

In the undeformed case, when $t_1 = \cdots = t_n = 0$, the polynomial F is equal to $a_1 \cdots a_n - b^\ell$.

2 Proof of main results

Proof of Proposition 1.4. For $i = 1, \dots, n-1$, we define $\Theta(x_i)$, $\Theta(x_n)$, and $\Theta(g_i)$ by (1.1) and (1.2). It follows from a straightforward verification that Θ is a well-defined homomorphism.

It remains to see that Θ is injective. Observe that H is spanned by the monomials $x^p g$ for $p = (p_1, \dots, p_n) \in \mathbb{Z}^n$ and $g \in G$, where $p_1, \dots, p_n \geq 0$. We call $p_1 + \dots + p_n$ the total degree of the monomial $x^p g$. The image of $x^p g$ under Θ is the sum of $y^p g$ with terms of strictly smaller total degrees. Therefore, if $\alpha \in \mathsf{H}$ is nonzero, we can write it as a sum $\alpha_0 + \alpha_1 + \cdots$, where α_k is a linear combination of monomials $x^p g$ with total degree k. If k is the maximal integer with α_k nonzero, then $\Theta(\alpha_k)$ is nonzero, and hence $\Theta(\alpha)$ is also nonzero. ■

Remark 2.1. It follows from Proposition 1.4 that the monomials $x_1^{p_1} \cdots x_n^{p_n} g$ for non-negative integers $p_1, \dots, p_n$ and $g \in G$ form a basis for H (called the PBW basis of H). This was first proved in [10, Example 2.16] using [10, Theorem 2.10].

We have an increasing filtration on H defined by setting $\deg(x_i) = 1$ and $\deg(g) = 0$ for all $i \in \{1, \dots, n\}$, $g \in G$. It is immediate from Remark 2.1 that the natural homomorphism $SV\#_\alpha G \to \mathrm{gr}\mathsf{H}$ is an isomorphism, where $\mathrm{gr}\mathsf{H}$ denotes the associated graded algebra of H.

The proof of (2.3) in the following lemma is the key calculation in this paper.

Lemma 2.2.

(i) *One has:*

$$\Theta\big(x_i^\ell\big) = y_i^\ell - \tau_i^\ell y_{i+1}^{-\ell}, \tag{2.1}$$
$$\Theta\big(x_n^\ell\big) = y_n^\ell - (-1)^{n(\ell-1)} \tau_n^\ell y_1^{-\ell}, \tag{2.2}$$

for all $i \in \{1, \dots, n-1\}$.

(ii) *One has:*

$$\Theta(w) = y_1 \cdots y_n + (-1)^n \zeta^{n-2} \tau_1 \cdots \tau_n y_1^{-1} \cdots y_n^{-1}. \tag{2.3}$$

Proof. (i) To prove (2.1), we need to show that

$$\underbrace{\big(y_i - \zeta\tau_i y_{i+1}^{-1} g_i\big) \cdots \big(y_i - \zeta\tau_i y_{i+1}^{-1} g_i\big)}_{\ell} = y_i^\ell - \tau_i^\ell y_{i+1}^{-\ell}. \tag{2.4}$$

Since $g_i y_i = \zeta y_i g_i$ and $g_i y_{i+1}^{-1} = \zeta y_{i+1}^{-1} g_i$, the product on the left hand side of (2.4) is a linear combination of $y_i^k y_{i+1}^{k-\ell} g_i^{\ell-k}$ for $k = 0, 1, \dots, \ell$. Moreover, the coefficient of $y_i^k y_{i+1}^{k-\ell} g_i^{\ell-k}$ in this linear combination is the same as the coefficient of u^k when we expand the product

$$\big(u - \zeta^\ell \tau_i\big)\big(u - \zeta^{\ell-1}\tau_i\big) \cdots \big(u - \zeta\tau_i\big) \tag{2.5}$$

in the polynomial ring $\mathbb{C}[u]$. Since the polynomial in (2.5) is equal to $u^\ell - \tau_i^\ell$, the identity (2.1) follows. The proof of (2.2) is similar except that

$$\underbrace{g_n * \cdots * g_n}_{\ell} = (-1)^{n(\ell-1)}.$$

(ii) For any $h_* = \{h_1 < \cdots < h_j\} \in I$, we let

$$h'_* = \{h_r \in h_* \mid h_s - h_r \in \{1, 1-n\} \text{ for some } s\},$$
$$\chi(h_*) = |\{h_r \in h'_* \mid h_r \neq n\}| - |\{h_r \in h'_* \mid h_r = n\}|,$$

$$E(h_*) = \zeta^{\chi(h_*)}\tau_{h_1}\cdots\tau_{h_j}y^{\delta-\varepsilon_{h_1}-\cdots-\varepsilon_{h_j}}g_{h_1}*\cdots*g_{h_j}.$$

Now suppose $i_* = \{i_1 < \cdots < i_k\} \in J$. Let D be the subset of $\{1,\ldots,n\}$ consisting of all d such that $d \not\equiv i_r,\ i_r+1 \pmod n$ for all r. We denote by $d_1 < \cdots < d_p$ the elements of D. Then

$$\begin{aligned}
&\Theta\big(\tau_{i_1}\cdots\tau_{i_k}x^{\delta-\varepsilon_{i_1}-\cdots-\varepsilon_{i_k}}g_{i_1}*\cdots*g_{i_k}\big)\\
&\quad= \tau_{i_1}\cdots\tau_{i_k}\left(y_{d_1} - \frac{\zeta t_{d_1}}{\zeta-1}y_{d_1+1}^{-1}g_{d_1}\right)\cdots\left(y_{d_p} - \frac{\zeta t_{d_p}}{\zeta-1}y_{d_p+1}^{-1}g_{d_p}\right)g_{i_1}*\cdots*g_{i_k}\\
&\quad= \tau_{i_1}\cdots\tau_{i_k}\sum_{S\subset D}Y_{d_1}(S)\cdots Y_{d_p}(S)g_{i_1}*\cdots*g_{i_k},
\end{aligned}$$

where, for $r = 1,\ldots,p$,

$$Y_{d_r}(S) = \begin{cases} y_{d_r}, & \text{if} \quad d_r \notin S,\\ -\zeta(\zeta-1)^{-1}t_{d_r}y_{d_r+1}^{-1}g_{d_r}, & \text{if} \quad d_r \in S.\end{cases}$$

Setting $h_* = i_* \cup S$, we obtain[1]

$$\Theta\big(\tau_{i_1}\cdots\tau_{i_k}x^{\delta-\varepsilon_{i_1}-\cdots-\varepsilon_{i_k}}g_{i_1}*\cdots*g_{i_k}\big) = \sum_{\{h_*\in I\,|\,i_*\subset h_*-h'_*\}}(-1)^{|h_*|-|i_*|}E(h_*).$$

Hence,

$$\begin{aligned}
\Theta(w) &= \sum_{\{i_1<\cdots<i_k\}\in J}\Theta\big(\tau_{i_1}\cdots\tau_{i_k}x^{\delta-\varepsilon_{i_1}-\cdots-\varepsilon_{i_k}}g_{i_1}*\cdots*g_{i_k}\big)\\
&= \sum_{i_*\in J}\left(\sum_{\{h_*\in I\,|\,i_*\subset h_*-h'_*\}}(-1)^{|h_*|-|i_*|}E(h_*)\right) = \sum_{h_*\in I}\left(E(h_*)\sum_{i_*\subset h_*-h'_*}(-1)^{|h_*|-|i_*|}\right).
\end{aligned}$$

If $|h_*| = n$, then $h'_* = h_*$. If $|h_*| \notin \{0,n\}$, then $h'_* \neq h_*$. Therefore,

$$E(h_*)\sum_{i_*\subset h_*-h'_*}(-1)^{|h_*|-|i_*|} = \begin{cases} y_1\cdots y_n & \text{if} \quad |h_*| = 0,\\ (-1)^n\zeta^{n-2}\tau_1\cdots\tau_n y_1^{-1}\cdots y_n^{-1} & \text{if} \quad |h_*| = n,\\ 0 & \text{else.}\end{cases} \qquad \blacksquare$$

Proof of Theorem 1.7. It is easy to see that the center of $SV\#_\alpha G$ is the algebra of G-invariant elements $(SV)^G$ of SV, and moreover, the algebra $(SV)^G$ is generated by x_i^ℓ $(i = 1,\ldots,n)$ and $x_1\cdots x_n$.

Using Lemma 2.2, we see that

$$\Theta\big(x_i^\ell\big) \quad \text{for} \quad i = 1,\ldots,n, \qquad \text{and} \qquad \Theta(w)$$

are in the center of $\mathbb{C}[y_1^\pm,\ldots,y_n^\pm]\#_\alpha G$. Since the homomorphism Θ is injective, the elements x_i^ℓ $(i = 1,\ldots,n)$ and w are in the center of H. Since the principal symbols of $x_1^\ell,\ldots,x_n^\ell$ and w in $SV\#_\alpha G$ are, respectively, $x_1^\ell,\ldots,x_n^\ell$ and $x_1\cdots x_n$, the theorem follows from a standard argument. $\blacksquare$

[1] Note that if $d_r \in S$ but $d_r+1 \in D-S$, then the term g_{d_r} in $Y_{d_r}(S)$ appears on the left of the term y_{d_r+1} of $Y_{d_r+1}(S)$ and one has $g_{d_r}y_{d_r+1} = \zeta^{-1}y_{d_r+1}g_{d_r}$. However, if $n \in S$ but $1 \in D-S$, then the term g_n in $Y_n(S)$ already appears to the right of the term y_1 of $Y_1(S)$. This is the reason why the definition of τ_n differs from the corresponding definitions of $\tau_1,\ldots,\tau_{n-1}$ by a factor of ζ.

Proof of Corollary 1.8. Let $\widetilde{a}_1 = \Theta(x_1^\ell)$, …, $\widetilde{a}_n = \Theta(x_n^\ell)$, and $\widetilde{b} = \Theta(w)$. By Lemma 2.2,

$$\widetilde{a}_i = y_i^\ell - \widetilde{\tau}_i y_{i+1}^{-\ell} \quad \text{for} \quad i = 1, \dots, n,$$
$$\widetilde{b} = y_1 \cdots y_n + (-1)^n \zeta^{n-2} \tau_1 \cdots \tau_n y_1^{-1} \cdots y_n^{-1}.$$

By a calculation completely similar to the proof of (2.3), one has

$$\sum_{\{i_1 < \cdots < i_k\} \in J} \widetilde{\tau}_{i_1} \cdots \widetilde{\tau}_{i_k} \widetilde{a}^{\delta - \varepsilon_{i_1} - \cdots - \varepsilon_{i_k}} = (y_1 \cdots y_n)^\ell + (-1)^{n\ell} (\tau_1 \cdots \tau_n)^\ell (y_1 \cdots y_n)^{-\ell}. \tag{2.6}$$

We claim that we also have

$$\sum_{r=0}^{\lfloor \ell/2 \rfloor} (-1)^{nr} \zeta^{(n-2)r} \nu_r (\tau_1 \cdots \tau_n)^r \widetilde{b}^{\ell - 2r} = (y_1 \cdots y_n)^\ell + (-1)^{n\ell} (\tau_1 \cdots \tau_n)^\ell (y_1 \cdots y_n)^{-\ell}. \tag{2.7}$$

To see this, recall that the Chebyshev polynomials of the first kind are defined recursively by $T_0(\xi) = 1$, $T_1(\xi) = \xi$, and

$$T_m(\xi) = 2\xi T_{m-1}(\xi) - T_{m-2}(\xi) \quad \text{for} \quad m = 2, 3, \dots.$$

It is well known (and can be easily proved by induction) that

$$2T_\ell\left(\frac{\xi}{2}\right) = \sum_{r=0}^{\lfloor \ell/2 \rfloor} \nu_r \xi^{\ell - 2r}, \tag{2.8}$$
$$2T_\ell\left(\frac{\xi + \xi^{-1}}{2}\right) = \xi^\ell + \xi^{-\ell}. \tag{2.9}$$

By (2.8) and (2.9), one has the identity

$$\xi^\ell + \xi^{-\ell} = \sum_{r=0}^{\lfloor \ell/2 \rfloor} \nu_r \left(\xi + \xi^{-1}\right)^{\ell - 2r},$$

and hence the identity

$$\xi^\ell + \varrho^{2\ell} \xi^{-\ell} = \sum_{r=0}^{\lfloor \ell/2 \rfloor} \nu_r \varrho^{2r} \left(\xi + \varrho^2 \xi^{-1}\right)^{\ell - 2r}$$

where ξ and ϱ are formal variables. By setting $\xi = y_1 \cdots y_n$ and choosing ϱ to be a square-root of $(-1)^n \zeta^{n-2} \tau_1 \cdots \tau_n$, we obtain (2.7).

By Proposition 1.4, Theorem 1.7, and the equations (2.6) and (2.7), the assignment (1.4) defines a surjective homomorphism

$$\Phi : \mathbb{C}[a_1, \dots, a_n, b] \to \mathsf{Z}$$

such that $\Phi(F) = 0$. Suppose $D \in \mathbb{C}[a_1, \dots, a_n, b]$ and $\Phi(D) = 0$. We can write

$$D = \sum_{r=0}^{\ell-1} D_r(a_1, \dots, a_n) b^r + R,$$

where $D_r(a_1, \dots, a_n) \in \mathbb{C}[a_1, \dots, a_n]$ for $r = 0, \dots, \ell - 1$, and $R \in (F)$. Thus,

$$\sum_{r=0}^{\ell-1} D_r\left(x_1^\ell, \dots, x_n^\ell\right) w^r = 0. \tag{2.10}$$

We claim that $D_r(a_1,\dots,a_n)=0$ for all r. Suppose not; then let m be the maximal integer such that $D_m(a_1,\dots,a_n)\neq 0$. Let $x_1^{\ell p_1}\cdots x_n^{\ell p_n}$ be a monomial in $D_m(x_1^\ell,\dots,x_n^\ell)$ with nonzero coefficient. Since $0\le m<\ell$, when we write the left hand side of (2.10) in terms of the PBW basis, the coefficient of $x_1^{\ell p_1+m}\cdots x_n^{\ell p_n+m}$ is nonzero, a contradiction. Hence, the kernel of Φ is (F). This proves (1.5). ■

Remark 2.3. When $n=3$, the algebra H is Morita equivalent to a deformed Sklyanin algebra $S_{\rm def}$ defined by C. Walton in [8, Definition IV.2]. More precisely, if $n=3$ and

$$e=\frac{1}{\ell}\sum_{r=0}^{\ell-1}g_1^r,$$

one has $\mathsf{H}e\mathsf{H}=\mathsf{H}$ and $e\mathsf{H}e\cong S_{\rm def}$ where the parameters for $S_{\rm def}$ (following the notations in [8, Definition IV.2]) are $a=1$, $b=\zeta$, $c=d_i=0$, and $e_i=-\zeta t_i$ for $i=1,2,3$. This follows from the observation that, for $n=3$, setting $\phi_i=x_ig_{i+1}$, one has $\phi_i\phi_{i+1}-\zeta\phi_{i+1}\phi_i=\zeta t_i$ for all i. The algebra $S_{\rm def}$ (with above parameters) was first studied by M. Douglas and B. Fiol, see [3, (3.10)]. Our formulas (1.1)–(1.2) are a generalization of [3, (4.6)], and our equation (1.3) is a generalization of [3, (4.7)]. The formulas in (2.1)–(2.3) are generalizations of [3, (4.8)].

Acknowledgements

We thank the referees for their many helpful comments.

References

[1] Căldăraru A., Giaquinto A., Witherspoon S., Algebraic deformations arising from orbifolds with discrete torsion, *J. Pure Appl. Algebra* **187** (2004), 51–70, math.KT/0210027.

[2] Chmutova T., Twisted symplectic reflection algebras, math.RT/0505653.

[3] Douglas M.R., Fiol B., D-branes and discrete torsion. II, *J. High Energy Phys.* **2005** (2005), no. 9, 053, 22 pages, hep-th/9903031.

[4] Drinfel'd V.G., Degenerate affine Hecke algebras and Yangians, *Funct. Anal. Appl.* **20** (1986), 58–60.

[5] Lusztig G., Cuspidal local systems and graded Hecke algebras. I, *Inst. Hautes Études Sci. Publ. Math.* (1988), 145–202.

[6] Lusztig G., Affine Hecke algebras and their graded version, *J. Amer. Math. Soc.* **2** (1989), 599–635.

[7] Ram A., Shepler A.V., Classification of graded Hecke algebras for complex reflection groups, *Comment. Math. Helv.* **78** (2003), 308–334, math.GR/0209135.

[8] Walton C.M., On degenerations and deformations of Sklyanin algebras, Ph.D. Thesis, University of Michigan, 2011.

[9] Witherspoon S., Skew derivations and deformations of a family of group crossed products, *Comm. Algebra* **34** (2006), 4187–4206, math.RA/0506154.

[10] Witherspoon S., Twisted graded Hecke algebras, *J. Algebra* **317** (2007), 30–42, math.RT/0506152.