

# The Soccer-Ball Problem<sup>\*</sup>

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**Abstract.** The idea that Lorentz-symmetry in momentum space could be modified but still remain observer-independent has received quite some attention in the recent years. This modified Lorentz-symmetry, which has been argued to arise in Loop Quantum Gravity, is being used as a phenomenological model to test possibly observable effects of quantum gravity. The most pressing problem in these models is the treatment of multi-particle states, known as the ‘soccer-ball problem’. This article briefly reviews the problem and the status of existing solution attempts.

*Key words:* Lorentz-invariance; quantum gravity; quantum gravity phenomenology; deformed special relativity

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## 1 Introduction

The Lorentz-group is non-compact and boosts can result in arbitrarily large blue shifts. Regardless of how long a wavelength appears to you, there always exists a restframe in relative motion where this wavelength is just a Planck length or even below that. This means in return that discarding short wavelengths, or high frequencies respectively, beyond any finite cutoff necessarily violates Lorentz-invariance.

Since the Planck energy is often expected to act as a natural regulator at high frequencies, this brings up the question how it can be given an observer-independent meaning. One way to do this is to regard it as a Lorentz-scalar, much like the mass of the W-boson in the coupling of Fermi’s theory of beta decay. Another way to do it is to modify the action of Lorentz-transformation on momentum space so that the Planck energy is observer-independent in the same way the speed of light is observer-independent. This is the idea underlying deformations of special relativity (DSR). First proposed in [2, 3, 29, 35, 36], these modifications of special relativity have received much attention both because they have observable consequences and because of their potential to illuminate the role of the Planck scale as a regulator. DSR was originally conceived as a modification of the Lorentz-transformations that keeps the Planck energy as a component of a four-vector invariant. It has later been motivated by Loop Quantum Gravity (for references see [24, Section 5.1]).

However, while it is straight-forward to construct modified Lorentz-transformations for momentum space by rescaling the momentum [17], this procedure cannot work in position space. The Lorentz-group is the unique group that leaves the Minkowski-metric invariant with the origin as fixed point, ie the only point that is mapped unto itself. The deformed transformation in momentum space is a non-linear correction to the normally linear Lorentz-transformation and

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singles out the zero-vector as a preferred point that remains invariant. That is unproblematic in momentum space because a vanishing momentum is indeed a special case. However, a point marked zero in space-time is as good as any other point, and thus accepting the special role of the zero in position space violates observer-independence. The group of translations in this case is no longer a normal subgroup of the Poincaré group, which means that boosting followed by a spatial shift does not give the same result as first shifting and then boosting. This is problematic because the spatial shift may be passive, i.e. be merely a relabeling of coordinates. One is left with the conclusion that points labeled as being distant from zero have an ambiguous transformation behavior that prevents one from precisely localizing the point, resulting in a non-locality that can become macroscopically large (much larger than the Planck length).

How to formulate DSR in position space is thus an important question and one that has so far not been satisfactorily addressed. It was pointed out already in [4, 19, 43], that the early versions of DSR violate locality at large scales. In [21, 22] it was shown that this is the case for any theory with an energy-dependent speed of light that also maintains observer-independence. Partly in response to this highly problematic non-locality, recent attempts have been to reformulate DSR as a theory of ‘relative locality’ [5, 6, 9, 11, 25, 44, 45].

In the approach of relative locality the momentum space is not constructed from the tensor bundle over the space-time manifold. Instead, one starts with phase space and aims to reconstruct space-time from it. Locality, then, can be interpreted as being observer-dependent, and relative rather than absolute. It remains to be seen how this approach circumvents the issue pointed out in [21, 22] in any other way than just making the speed of light energy-independent and thus reproducing ordinary Special Relativity. The big problem in the relative locality approach is that, when starting from phase space rather than space-time, it is unclear in the end how to identify a physically meaningful coordinate system on space-time, for the very reason spelled out above, that the behavior of coordinates now depends on unphysical labels. However, this is an original new idea and still young, and so it can be hoped that with more study it will result in a satisfactory theory of DSR in position space in the soon future.

In the following, I want to focus on a different, though of course related, problem that occurs when modifying the action of the Lorentz-group on momentum space, that is the addition of momenta. We will use the unit convention  $c = \hbar = 1$  and the Planck mass  $m_{\text{Pl}}$  is the inverse of the Planck length  $l_{\text{Pl}} = 1/m_{\text{Pl}}$ .

## 2 The problem

It is presently not known how to define the sum of momenta in approaches that modify Lorentz-symmetry in momentum space and maintain observer-independence. In a nutshell, the problem is that the modified Lorentz-transformation,  $\tilde{\Lambda}$ , that acts on momenta is nonlinear, and thus the transformation of the sum of momenta is not the same as sum of the transformations of the momenta

$$\tilde{\Lambda}(\mathbf{p}_1 + \mathbf{p}_2) \neq \tilde{\Lambda}(\mathbf{p}_1) + \tilde{\Lambda}(\mathbf{p}_2).$$

This then ruins observer-independence which was one of the main motivations to introduce deformed Lorentz-symmetry: If  $\tilde{\Lambda}$  is the unit element of the group, then the equation is fulfilled, and thus the restframe in which it is fulfilled singles out a preferred frame.

The first way to address this problem that comes to mind is to note that the momenta of different particles are elements of different subspaces of the phase-space. One could thus in principle just define the transformation acting on the sum of momenta to be the sum of the transformations. However, this procedure will fail for momentum conservation in interactions, in which the terms contributing to the sum can be different before and after the interaction.

One thus concludes that it is necessary to define a new, non-linear, addition law  $\oplus$  for momenta that has the property that it remains invariant under Lorentz-transformations and that can be rightfully interpreted as a conserved quantity.

Note that this problem is about the sum of momenta, and not necessarily about bound states or even quantum particles. This problem appears already in the attempt to formulate a classical theory which obeys the modified transformation behavior. The question is not about interacting particles or quantum superpositions, the question is what is the total momentum of any collection of particles. It is apparent that this is an essential point to address for any model that should reproduce known physics.

Luckily, it is possible to construct a Lorentz-invariant new addition law  $\oplus$  without too much trouble. To see how this works, we note that given a modified nonlinear Lorentz-transformation,  $\tilde{\Lambda}$ , that acts on elements  $\mathbf{p}$  of momentum space, it is always possible to construct a vector  $\mathbf{k}$  that is a function of  $\mathbf{p}$ , so that  $\mathbf{k} = f(\mathbf{p})$  transforms under the normal Lorentz-transformation  $\Lambda$ . In the literature,  $\mathbf{k}$  is often referred to as the ‘pseudo-momentum’ [26]. Here and in the following bold-faced quantities denote vectors, and so the function  $f$  is a map from  $\mathbb{R}^4 \rightarrow \mathbb{R}^4$ . It can be shown that the opposite is also true: given the function  $f$  it is possible to construct the non-linear Lorentz-transformation  $\tilde{\Lambda}$ , so these two formulations of DSR are actually equivalent to each other [17].

It is then straightforward to construct the modified addition law by use of the pseudo-momentum  $\mathbf{k}$  that by assumption transforms under the normal Lorentz transformation. To each momentum  $\mathbf{p}$  we have an associated pseudo-momentum  $\mathbf{k}_1 = f(\mathbf{p}_1)$ ,  $\mathbf{k}_2 = f(\mathbf{p}_2)$ . The sum  $\mathbf{k}_1 + \mathbf{k}_2$  is invariant under normal Lorentz transformations, and so we construct the sum of the  $\mathbf{p}$ ’s as

$$\mathbf{p}_1 \oplus \mathbf{p}_2 = f^{-1}(\mathbf{k}_1 + \mathbf{k}_2) = f^{-1}(f(\mathbf{p}_1) + f(\mathbf{p}_2)).$$

This is one way to arrive at the modified addition law for momenta. It requires one to first construct the pseudo-momentum. In some cases it is easier to extract the modified addition law from an algebraic approach that starts with the modified commutation relations in the Poincaré-algebra; it is the coproduct of the  $\kappa$ -Poincaré algebra [32].

The new sum is in general not associative. If one does not symmetrize the new addition rule, the result of the new sum may also depend on the order in which momenta are added, ie  $(\mathbf{p}_1 \oplus \mathbf{p}_2) \oplus \mathbf{p}_3 \neq \mathbf{p}_1 \oplus (\mathbf{p}_2 \oplus \mathbf{p}_3)$ . This means in particular the sum of two momenta can depend on a third term that may describe a completely unrelated (and arbitrarily far away) part of the universe, which has been dubbed the ‘spectator problem’ [16, 31]. We will not further address this problem here.

This new definition for a sum is now nicely observer-independent by construction, but brings with it a new problem. The non-linear contributions in  $f$  by construction – going back to the motivation for DSR – become relevant when the momenta (or some of its components respectively) come close by the Planck energy. For a single elementary particle these nonlinear contributions will be small. But the (usual) total momentum of a collection of particles can easily exceed the Planck energy. The Planck mass is a large energy as far as particle physics is concerned, but in everyday units it is about  $10^{-5}$  gram, a value that is easily exceeded by some large molecules. This problem of reproducing a sensible multi-particle limit when one chooses the physical momentum to transform under modified Lorentz transformations has become known as the ‘soccer-ball problem’, where the soccer-ball stands in for any object exceeding the Planck mass.

The soccer-ball problem can be sharpened as follows. If one makes an expansion of the function  $f$  to include the first correction terms in  $\mathbf{p}/m_{\text{Pl}}$ , and from that derives the sum  $\oplus$ , then it remains to be shown that the correction terms stay smaller than the linear terms if one calculates sums over a large number of momenta. One expects that the sum then has approximately the form  $\mathbf{p}_1 \oplus \mathbf{p}_2 \approx \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_1 \mathbf{p}_2 \Gamma / m_{\text{Pl}}$ , where  $\Gamma$  are some coefficients of order one. If one iterates this sum for  $N$  terms, the normal sum grows with  $N$  but the number of

additional terms scales with  $\sim N^2$  for  $N \gg 1$ . To solve the soccer-ball problem, the non-linear contributions have to grow slower than  $N$  in spite of this, and are still small compared to the linear term even when the total momentum exceeds the Planck scale.

One may slightly weaken this requirement because the universe does not contain an infinite amount of particles, so strictly speaking the non-linear terms only have to remain negligible for collections of particles that we have observed. However, we note that once one ventures into the realm of quantum field theory – which is necessary eventually to reproduce the physics we know – the number of constituent particles becomes ill-defined. It can plausibly be said that the proton does in fact contain an infinite amount of particles, and is composed of a whole sea of virtual quarks rather than just three valence quarks. If one only solves the soccer-ball problem for finite  $N$ , it is thus bound to come back once one attempts to treat quantized interactions.

### 3 Proposed solutions

There have been various attempts to address the soccer-ball problem, but so far none has been generally accepted.

For example, it has been suggested that with the addition of  $N$  particles, the Planck scale that appears in the non-linear Lorentz transformation and in the modified addition law should be rescaled to  $m_{\text{Pl}}N$  [26, 34, 36]. It is however difficult to see how this ad-hoc solution could follow from the theory. In [33] it has been proposed to reinterpret DSR as a modified theory of measurement, thereby also addressing the soccer-ball problem.

Alternatively, it has been proposed that the scaling of modifications should go with the density [20, 38] rather than with the total momentum or energy respectively. While the energy of macroscopic objects is larger than that of microscopic ones, the energy density decreases instead. This seems a natural solution to the issue but would necessitate a completely different ansatz to implement.

The so far most promising approach has been put forward in [7] in the context of relative locality. It was claimed in this paper that the soccer ball problem is absent for a particular choice of connection  $\Gamma_{\mu}^{\alpha\beta}$  and basis on the manifold that is momentum space. This basis are normal coordinates in which the connection is totally antisymmetric  $\Gamma_{\mu}^{(\alpha\beta)} = 0$  with the consequence that collinear momenta add under the normal addition law:

$$\mathbf{p}_1 \oplus \mathbf{p}_2 = \mathbf{p}_1 + \mathbf{p}_2 \quad \text{if} \quad \mathbf{p}_1 \parallel \mathbf{p}_2.$$

Now in the first conceptions of relative locality, the connection  $\Gamma$  was a metric connection on the momentum space manifold. In that case the requirement of the existence of global normal coordinates means that the curvature of momentum space actually has to vanish and only the torsion is non-trivial. This is in stark contrast to the momentum space geometries that have previously been considered, typically Anti-de Sitter spaces with vanishing torsion. Alternatively, the connection  $\Gamma$  is not a metric connection but a connection deriving from the structure of the group and its addition law. This interpretation is natural when dealing with a group manifold. Alas, when dealing with a general manifold like one does in relative locality it is not a priori clear that normal coordinates exist globally.

However, having mentioned this caveat we will in the following assume the coordinates exist and interpret  $\Gamma$  as a general, not necessarily a Riemannian, connection on the manifold. Note however, that in the relative locality approach the Lagrangian is constructed from the geodesic distance of momenta from the origin, and for that one needs the metric connection.

Using the normal coordinate basis then, it was shown in [7] that the momentum exchange between two macroscopic bodies (‘soccer-balls’) does not pose a problem provided the total momentum of the bodies themselves is well-defined. However, it was argued in [23] that the

total momentum of the macroscopic bodies is in fact not well-defined: Assuming a body of a certain temperature  $T$ , the fluctuations in the constituents' momenta are not aligned and random in direction. The fluctuations do thus contribute to the modified sum in the normal coordinates even though the average momenta don't. It was estimated in [23] on fairly general grounds that the non-linear terms scale with  $N^{3/2}$  and thus the sum does not converge.

In the reply [8] it was pointed out that this estimate neglected some second-order terms that mix into the first order-terms. The more detailed analysis leads to a constraint on the connection coefficients that has to be fulfilled in order for the terms that scale with powers larger than  $N$  to be absent. This analysis is a big step forward. However, it remains to be shown that there is any non-trivial manifold that fulfils these requirements and that not taking into account subsequently higher orders will leave only the linear addition law as option. It must be kept in mind that by the construction of DSR one expects all the additional terms in the non-linear sum to asymptotically scale with  $N$  and not any faster, so that they do exactly cancel the growth of the linear term. This makes it quite implausible that there is a maximum power of  $N$  in the expansion. Alas, as previously mentioned, as long as one does not deal with quantized interaction, taking  $N \rightarrow \infty$  might not be required.

That having been said, the approach to the soccer-ball problem pursued in [8] is technically rigorous and offers the possibility of settling the issue at least for the normal coordinates. What is left to show is that the coordinate system exists globally and that the requirements for absence of the divergent terms can be fulfilled by any non-trivial addition laws.

## 4 Composite systems and statistical mechanics

Another way to deal with the soccer-ball problem that has become very popular is to just ignore it. Many approaches to the description of composite systems or many particle states have been made based on the modified commutation relations either without subscribing to the deformed Lorentz transformations, and thereby generically breaking Lorentz invariance, or by employing an ad hoc solution by rescaling the bound on the energy with the number of constituents, or by not rescaling the bound and thus finding – not so surprisingly – that Planck scale effects become noticeable in composite objects with masses approaching the Planck mass.

For example, it was proposed recently that a massive quantum mechanical oscillator might allow one to test Planck-scale physics [41] in a parameter range close to the Planck scale. This conclusion was reached not taking into account that, to circumvent the soccer-ball problem, the Planck-mass has to be rescaled with the number of constituents. If one does not do this rescaling the model does not even reproduce classical Newtonian mechanics because the momenta of macroscopic objects do no longer add linearly, and the violations of linearity are dominant. The experiment proposed in [41] thus tests a model we already know is wrong. This is also so if one does not subscribe to the deformed Lorentz-transformation and instead considers the modified addition law to just violate observer-independence because Lorentz-invariance violation is extremely strongly constrained already [28], far beyond the Planck scale.

That having been said, one can of course investigate the statistical mechanics and thermodynamics on a purely mathematical footing and just leave the deformation scale unspecified, so that its relation to the Planck scale and the number of constituents can be taken into account later, once the soccer-ball problem has been solved.

In that spirit, the statistical mechanics from the  $\kappa$ -Poincaré algebra was investigated in general in [15, 30]. The partition functions of deformed quantized statistical mechanics have been derived in [39], in [1] the consequences for the Liouville theorem were investigated, and in [13] the modification of the density of states and the arising consequences for black-hole thermodynamics were studied. In [42] corrections to the effective Hamiltonian of macroscopic bodies have been considered and in [27] statistical mechanics with a generalized uncertainty

and possible applications for cosmology have been looked at. [37] studied the thermodynamics of ultra-relativistic particles in the early universe and relativistic thermodynamics has been investigated in [14]. [46] studied the equation of state for ultra-relativistic Fermi gases in compact stars, the ideal gas was addressed in [12] and photon gas thermodynamics in [47].

As previously mentioned, the physical consequences of these investigations should be interpreted with caution whenever energies have not been rescaled with the number of constituents.

## 5 Summary

The idea that Lorentz-symmetry might be modified so that the Planck energy becomes observer-independent is interesting and leads to testable phenomenological consequences. But consistently modifying Special Relativity is hard. We have seen here that such a deformation of Special Relativity necessitates a non-linear addition law for momenta and it is presently not known whether a sum of a large number of momenta converges suitably. If it does not converge, these models cannot reproduce even classical mechanics. This so-called ‘soccer-ball problem’ is thus a pressing one, for without solving it models with deformed Lorentz-symmetry fail to reproduce well-tested physics.

A recent approach within the scenario of relative locality seems promising. It has been shown that there exist addition rules which do not suffer from the problem and concrete requirements for the absence of the soccer-ball problem have been derived in this context. It yet remains to be seen however whether in the class of these addition rules which are immune to the soccer-ball problem are any which are in fact non-linear and do not just reproduce Special Relativity.

Finally, we note that the soccer-ball problem does not occur in the case in which the modification of the Lorentz-transformation only occurs for off-shell momenta which seems to be suggested in some interpretations [18]. Then, if one identifies the momenta of particles as those of the asymptotically free states, the addition of their momenta is linear as usual. For the same reason, the problem also does not appear in the interpretation of such modifications of conservation laws as being caused by a running Planck’s constant, put forward [10, 40], and discussed in [24]. In this case, the relevant energy is the momentum transfer, and for bound states this remains small even if the total mass increases beyond the Planck mass.

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