The GraviGUT Algebra Is not a Subalgebra of E_8, but E_8 Does Contain an Extended GraviGUT Algebra

Andrew DOUGLAS † *and Joe REPKA* ‡

† *CUNY Graduate Center and New York City College of Technology, City University of New York, USA*
E-mail: *adouglas2@gc.cuny.edu*

‡ *Department of Mathematics, University of Toronto, Canada*
E-mail: *repka@math.toronto.edu*

Received April 04, 2014, in final form July 03, 2014; Published online July 08, 2014
http://dx.doi.org/10.3842/SIGMA.2014.072

Abstract. The (real) GraviGUT algebra is an extension of the $\mathfrak{spin}(11,3)$ algebra by a 64-dimensional Lie algebra, but there is some ambiguity in the literature about its definition. Recently, Lisi constructed an embedding of the GraviGUT algebra into the quaternionic real form of E_8. We clarify the definition, showing that there is only one possibility, and then prove that the GraviGUT algebra cannot be embedded into any real form of E_8. We then modify Lisi's construction to create true Lie algebra embeddings of the extended GraviGUT algebra into E_8. We classify these embeddings up to inner automorphism.

Key words: exceptional Lie algebra E_8; GraviGUT algebra; extended GraviGUT algebra; Lie algebra embeddings

2010 Mathematics Subject Classification: 17B05; 17B10; 17B20; 17B25; 17B81

1 Introduction

The Standard Model of particle physics, with gauge group $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$, attempts to describe all particles and all forces, except gravity. Grand Unified Theories (GUT) attempt to unify the forces and particles of the Standard Model. The three main GUTs are Georgi and Glashow's SU(5) theory, Georgi's Spin(10) theory, and the Pati–Salam model based on the Lie group $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4)$ [2].

In [8], Lisi attempts to construct a unification which includes gravity. In this construction, Lisi first embeds gravity and the standard model into $\mathfrak{spin}(11,3)$. He then embeds $\mathfrak{spin}(11,3)$ together with the positive chirality 64-dimensional $\mathfrak{spin}(11,3)$ irrep into the quaternionic real form of E_8. Lisi refers to the embedded Lie algebra as the GraviGUT algebra. For a description of Lisi's theory see [8] or [7]. For a critique of Lisi's theory involving the GraviGUT algebra see [3]. We note that the GraviGUT algebra was first introduced by Nesti and Percacci [12].

In Section 4 of [11], it is observed that one of the Lie algebras associated with a point in Vogel's plane (cf. [13]) has the same dimension as the GraviGUT algebra, and it is hypothesized there that these two algebras are in fact isomorphic. It would certainly be interesting to identify the algebra in question.

Because the exposition of [8] reflects the process of exploring the possible realizations of the GraviGUT algebra, there is some potential for confusion about its definition. Lisi first describes the algebraic structure of $\mathfrak{spin}(11,3)$ and the action of $\mathfrak{spin}(11,3)$ on its 64-dimensional irrep V in equations (3.9) and (3.10) from [8], respectively. He then notes that the structure of the GraviGUT algebra *could* be completed by defining a trivial Lie bracket on V.

However, the actual algebraic structure of his explicit realization of V is not established until equations (4.3) and (4.4) from [8], when Lisi describes his embedding; it is here that we see that V is not abelian relative to the Lie bracket inherited from E_8. In fact, with this definition of the Lie bracket, the subspace of E_8 spanned by the embedded copies of $\mathfrak{spin}(11,3)$ and V is not a Lie algebra at all: it is not closed under the bracket. Lisi acknowledges this in his remark near the end of § 3 that "The word 'algebra' is used here in a generalized sense".

Theorem 2 of the present paper actually shows that the *only* way to extend the usual bracket on $\mathfrak{so}(14)_{\mathbb{C}}$ and its action on V to make $\mathfrak{so}(14)_{\mathbb{C}} \Subset V$ into a Lie algebra is to require V to be abelian. In particular, the only possible definition of the (complexified) GraviGUT algebra as a Lie algebra is $\mathfrak{so}(14)_{\mathbb{C}} \Subset V$, where V is a 64-dimensional abelian ideal which is irreducible under the action of $\mathfrak{so}(14)_{\mathbb{C}}$.

Once the structure of the complexified GraviGUT algebra is specified, it is not difficult to show that it cannot be embedded into the complex algebra E_8; cf. Corollary 1. Hence, the (real) GraviGUT algebra cannot be embedded into the quaternionic real form of E_8, or any other real form of E_8.

However, the operators in E_8 described by Lisi do generate a larger Lie algebra, which contains an additional 14-dimensional ideal. We call this larger algebra the extended GraviGUT algebra.

We modify Lisi's construction to create true Lie algebra embeddings of the extended GraviGUT algebra into E_8. The (complex) *extended GraviGUT algebra* is a nonabelian, nilpotent extension of $\mathfrak{so}(14)_{\mathbb{C}}$ by a 78-dimensional $\mathfrak{so}(14)_{\mathbb{C}}$-representation. This 78-dimensional representation is composed of a 64-dimensional irrep and the standard 14-dimensional $\mathfrak{so}(14)_{\mathbb{C}}$-irrep. Its precise structure is described in Section 6, but we do note here that the (complexified) GraviGUT algebra is a quotient of the extended GraviGUT algebra. We classify these embeddings up to inner automorphism.

The article is organized as follows. Section 2 contains relevant background on Lie algebras and their representations: in particular, it deals with the complex, simple Lie algebras $\mathfrak{so}(14)_{\mathbb{C}}$ and E_8. Section 3 presents additional notation and terminology. In Section 4 we describe the classification of embeddings of $\mathfrak{so}(14)_{\mathbb{C}}$ into E_8, which will be used in the following section. In Section 5 we determine the only possible definition of the GraviGUT algebra and also establish that the complexified GraviGUT algebra cannot be embedded into the complex algebra E_8. Finally, in Section 6 we classify the embeddings of the extended GraviGUT algebra into E_8.

2 The complex Lie algebras $\mathfrak{so}(14)_{\mathbb{C}}$ and E_8, and their representations

The special orthogonal algebra $\mathfrak{so}(14)_{\mathbb{C}}$ is the complexification of $\mathfrak{spin}(11,3)$. It is the Lie algebra of complex 14×14 matrices N satisfying $N^{\mathrm{tr}} = -N$. The dimension of $\mathfrak{so}(14)_{\mathbb{C}}$ is 91 and its rank is 7. The Lie group corresponding to $\mathfrak{so}(14)_{\mathbb{C}}$ arises naturally as the symmetry group of a projective space over $\mathbb{R}$ [1].

E_8 is the complex, exceptional Lie algebra of rank 8. It is 248-dimensional. Like $\mathfrak{so}(14)_{\mathbb{C}}$, E_8 has a close connection to the Riemannian geometry of projective spaces (for details, we refer the reader to [1]).

Let $\mathfrak{g}$ denote $\mathfrak{so}(14)_{\mathbb{C}}$ or E_8. Let $k = 7$ or 8 when $\mathfrak{g} = \mathfrak{so}(14)_{\mathbb{C}}$ or E_8, respectively. We may define $\mathfrak{g}$ by a set of generators $\{H_i, X_i, Y_i\}_{1 \le i \le k}$ together with the Chevalley–Serre relations [6]:

$$[H_i, H_j] = 0, \qquad [H_i, X_j] = M^{\mathfrak{g}}_{ji} X_j,$$
$$[H_i, Y_j] = -M^{\mathfrak{g}}_{ji} Y_j, \qquad [X_i, Y_j] = \delta_{ij} H_i,$$
$$(\operatorname{ad} X_i)^{1-M^{\mathfrak{g}}_{ji}}(X_j) = 0, \qquad (\operatorname{ad} Y_i)^{1-M^{\mathfrak{g}}_{ji}}(Y_j) = 0, \quad \text{when} \quad i \neq j.$$

Here $1 \le i, j \le k$, and $M^{\mathfrak{g}}$ is the Cartan matrix of $\mathfrak{g}$. The X_i, for $1 \le i \le k$, correspond to the simple roots. We write H for the Cartan subalgebra spanned by $\{H_i\}$.

For future reference, the Dynkin diagrams of $\mathfrak{so}(14)_{\mathbb{C}}$ and E_8, indicating the numbering of simple roots, are given in Fig. 1.

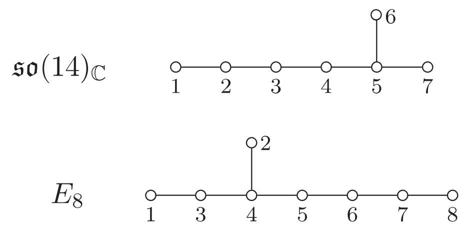

Figure 1. Dynkin diagrams of $\mathfrak{so}(14)_{\mathbb{C}}$ and E_8.

We now briefly describe the finite-dimensional, irreducible representations (irreps) of $\mathfrak{so}(14)_{\mathbb{C}}$ and E_8, with $\mathfrak{g}$ and k defined as above. For $i = 1, \ldots, k$, define $\alpha_i, \lambda_i \in H^*$ by $\alpha_i(H_j) = M^{\mathfrak{g}}_{ji}$, and $\lambda_i(H_j) = \delta_{ij}$, where $M^{\mathfrak{g}}$ is the Cartan matrix of $\mathfrak{g}$. The λ_i are the *fundamental weights*, and their indexing corresponds with that of the Dynkin diagram of type $\mathfrak{so}(14)_{\mathbb{C}}$ or E_8 in Fig. 1.

For each $\lambda = m_1\lambda_1 + \cdots + m_k\lambda_k \in H^*$ with nonnegative integers $m_1, \ldots, m_k$, there exists an irrep of $\mathfrak{g}$ with highest weight λ, denoted $V_{\mathfrak{g}}(\lambda)$. The irreps $V_{\mathfrak{g}}(\lambda_i)$ for $1 \leq i \leq k$ are the *fundamental representations*. Each irrep of $\mathfrak{g}$ is equivalent to $V_{\mathfrak{g}}(\lambda)$, where $\lambda = m_1\lambda_1 + \cdots + m_k\lambda_k$ for some nonnegative integers $m_1, \ldots, m_k$.

3 Additional definitions and notation

The following definitions and notation will be used in this article:

- For $1 \leq a_i \leq 8$, let X_{a_i} correspond to the a_ith simple root of E_8. We then define

  $$X_{a_1,a_2,a_3,\ldots,a_m} \equiv [[\ldots[[X_{a_1}, X_{a_2}], X_{a_3}], \ldots], X_{a_m}].$$

  $Y_{a_1,a_2,a_3,\ldots,a_m}$ is defined analogously.

- Let $\varphi : \mathfrak{so}(14)_{\mathbb{C}} \hookrightarrow E_8$ be an embedding. Further, let W be an element of E_8. Then, $[W]_{\varphi(\mathfrak{so}(14)_{\mathbb{C}})}$ is the $\mathfrak{so}(14)_{\mathbb{C}}$ representation generated by W with respect to the adjoint action of $\varphi(\mathfrak{so}(14)_{\mathbb{C}})$. When the embedding φ is clear, as will be the case below, we simply write $[W]_{\mathfrak{so}(14)_{\mathbb{C}}}$.

- Let φ and ϱ be Lie algebra embeddings of $\mathfrak{g}'$ into $\mathfrak{g}$. Then φ and ϱ are *equivalent* if there is an inner automorphism $\rho : \mathfrak{g} \to \mathfrak{g}$ such that $\varphi = \rho \circ \varrho$, and we write

  $$\varphi \sim \varrho.$$

- Two embeddings φ and ϱ of $\mathfrak{g}'$ into $\mathfrak{g}$ are *linearly equivalent* if for each representation $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ the induced $\mathfrak{g}'$ representations $\pi \circ \varphi$, $\pi \circ \varrho$ are equivalent, and we write

  $$\varphi \sim_L \varrho.$$

  Clearly equivalence of embeddings implies linear equivalence, but the converse is not in general true.

We define equivalence and linear equivalence of subalgebras much as we did for embeddings:

- Two subalgebras $\mathfrak{g}'$ and $\mathfrak{g}''$ of $\mathfrak{g}$ are equivalent if there is an inner automorphism ρ of $\mathfrak{g}$ such that $\rho(\mathfrak{g}') = \mathfrak{g}''$.

- Two subalgebras $\mathfrak{g}'$ and $\mathfrak{g}''$ of $\mathfrak{g}$ are linearly equivalent if for every representation $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ the subalgebras $\pi(\mathfrak{g}')$, $\pi(\mathfrak{g}'')$ of $\mathfrak{gl}(V)$ are conjugate under $\mathrm{GL}(V)$.

4 Embedding $\mathfrak{so}(14)_{\mathbb{C}}$ into E_8

In [4], the authors presented the following well-known "natural" embedding of $\mathfrak{so}(14)_{\mathbb{C}}$ into E_8:

$$\begin{aligned}\varphi:\ \mathfrak{so}(14)_{\mathbb{C}} &\hookrightarrow E_8,\\ H_{8-i} &\mapsto H_{i+1},\\ X_{8-i} &\mapsto X_{i+1},\\ Y_{8-i} &\mapsto Y_{i+1},\end{aligned} \tag{1}$$

where $1 \leq i \leq 7$. This embedding may be visualized as a "natural" subgraph of the Dynkin diagram of E_8 which is isomorphic to the Dynkin diagram of $\mathfrak{so}(14)_{\mathbb{C}}$ (see Fig. 1).

In [10], Minchenko showed that there is a unique subalgebra isomorphic to $\mathfrak{so}(14)_{\mathbb{C}}$ in E_8, up to inner automorphism. Hence, the only way to get new embeddings of $\mathfrak{so}(14)_{\mathbb{C}}$ into E_8 other than the φ described in equation (1) is to compose φ with an outer automorphism of $\mathfrak{so}(14)_{\mathbb{C}}$. However, it was shown by the authors in [4] that outer automorphisms of $\mathfrak{so}(14)_{\mathbb{C}}$ do not produce new embeddings of $\mathfrak{so}(14)_{\mathbb{C}}$ into E_8. We thus have the following theorem [4].

Theorem 1. *The map $\varphi : \mathfrak{so}(14)_{\mathbb{C}} \hookrightarrow E_8$ defined in equation (1) is the unique embedding of $\mathfrak{so}(14)_{\mathbb{C}}$ into E_8, up to inner automorphism.*

5 The GraviGUT algebra is not a subalgebra of E_8

Theorem 2. *Consider a sum of complex vector spaces $\mathfrak{so}(14)_{\mathbb{C}} \oplus V$, where V is a 64-dimensional space. Suppose a Lie bracket is defined which gives the usual structure to the $\mathfrak{so}(14)_{\mathbb{C}}$ subspace and in such a way that the brackets $[X, v]$, for $X \in \mathfrak{so}(14)_{\mathbb{C}}$ and $v \in V$, define an action of $\mathfrak{so}(14)_{\mathbb{C}}$ on V under which V becomes an irreducible $\mathfrak{so}(14)_{\mathbb{C}}$-module. Then the only way to extend this bracket to make $\mathfrak{so}(14)_{\mathbb{C}} \Subset V$ into a Lie algebra is to put the abelian structure on V, i.e., $[v, v'] = 0$, for all $v, v' \in V$.*

In particular, the only possible definition of the (complexified) GraviGUT algebra as a Lie algebra is $\mathfrak{so}(14)_{\mathbb{C}} \Subset V$, where V is a 64-dimensional abelian ideal which is irreducible under the action of $\mathfrak{so}(14)_{\mathbb{C}}$.

Proof. Let V be a 64-dimensional $\mathfrak{so}(14)_{\mathbb{C}}$-irrep. Then $V \cong V(\lambda_6)$ or $V \cong V(\lambda_7)$. Consider the tensor product decompositions:

$$\begin{aligned}V(\lambda_6) \otimes V(\lambda_6) &\cong V(2\lambda_6) \oplus V(\lambda_5) \oplus V(\lambda_3) \oplus V(\lambda_1),\\ V(\lambda_7) \otimes V(\lambda_7) &\cong V(2\lambda_7) \oplus V(\lambda_5) \oplus V(\lambda_3) \oplus V(\lambda_1).\end{aligned}$$

Since $V(\lambda_6)$ does not occur in the tensor product decomposition of $V(\lambda_6) \otimes V(\lambda_6)$, $V(\lambda_7)$ does not occur in the tensor product decomposition of $V(\lambda_7) \otimes V(\lambda_7)$, and neither decomposition contains a 91-dimensional irrep, we cannot have a nontrivial product $V \otimes V \to V$ or $V \otimes V \to \mathfrak{so}(14)_{\mathbb{C}}$. Hence, we cannot have a nontrivial product $V \times V \to V$ or $V \times V \to \mathfrak{so}(14)_{\mathbb{C}}$ or $V \times V \to (\mathfrak{so}(14)_{\mathbb{C}} \Subset V)$. The only possible definition of a Lie algebra structure is to make V an abelian subalgebra. ■

Corollary 1. *The GraviGUT algebra cannot be embedded into the quaternionic real form of E_8, or any other real form of E_8.*

Proof. The maximal dimension of an abelian subalgebra of E_8 is 36 [9]. This implies that V cannot be a subalgebra of E_8, and that the complexified GraviGUT algebra cannot be embedded into E_8. The result follows. ■

Remarks.

1. We also note that if we have a 64-dimensional representation V that is not irreducible, then embedding $\mathfrak{so}(14)_{\mathbb{C}} \Subset V$ into E_8 is still not possible. The summands in the direct sum decomposition of E_8 as an $\varphi(\mathfrak{so}(14)_{\mathbb{C}})$-module have dimensions 1, 14, 14, 64, 64, 91, as we will see below in equation (2). Hence, the only 64-dimensional $\mathfrak{so}(14)_{\mathbb{C}}$ submodules of E_8 are irreducible.
2. The subspace of E_8 defined by Lisi in his equations (4.3), (4.4) (see [8]) is not closed under the Lie bracket it inherits from E_8.

6 The extended GraviGUT algebra in E_8

In [4], the authors computed the following decomposition of E_8 with respect to the adjoint action of $\varphi(\mathfrak{so}(14)_{\mathbb{C}})$:

$$\begin{aligned} E_8 &\cong_{\mathfrak{so}(14)_{\mathbb{C}}} V(\lambda_2) \oplus V(\lambda_1) \oplus V(\lambda_6) \oplus V(\lambda_7) \oplus V(\lambda_1) \oplus V(0) \\ &\cong_{\mathfrak{so}(14)_{\mathbb{C}}} [X_{74}]_{\mathfrak{so}(14)_{\mathbb{C}}} \oplus [X_{120}]_{\mathfrak{so}(14)_{\mathbb{C}}} \oplus [Y_1]_{\mathfrak{so}(14)_{\mathbb{C}}} \oplus [X_{112}]_{\mathfrak{so}(14)_{\mathbb{C}}} \oplus [Y_{97}]_{\mathfrak{so}(14)_{\mathbb{C}}} \oplus [H]_{\mathfrak{so}(14)_{\mathbb{C}}}, \end{aligned} \tag{2}$$

where

$$\begin{aligned} X_{74} &= X_{4,5,6,7,8,2,3,4,5,6,7}, \\ X_{112} &= -X_{3,4,2,1,5,4,3,6,5,4,7,2,6,5,8,7,6,4,5,3,4,2}, \\ X_{120} &= X_{8,7,6,5,4,3,2,1,4,5,6,7,3,4,5,6,2,4,5,3,4,2,1,3,4,5,6,7,8}, \\ Y_{97} &= -Y_{5,4,2,3,6,4,1,3,5,4,7,2,6,5,4,3,1}, \\ H &= 4H_1 + 5H_2 + 7H_3 + 10H_4 + 8H_5 + 6H_6 + 4H_7 + 2H_8. \end{aligned}$$

Lemma 1. *The following are 78-dimensional, nonabelian nilpotent subalgebras of E_8:*

$$[Y_1]_{\mathfrak{so}(14)_{\mathbb{C}}} \oplus [Y_{97}]_{\mathfrak{so}(14)_{\mathbb{C}}}, \qquad [X_{112}]_{\mathfrak{so}(14)_{\mathbb{C}}} \oplus [X_{120}]_{\mathfrak{so}(14)_{\mathbb{C}}}.$$

Note that the sums are direct as $\mathfrak{so}(14)_{\mathbb{C}}$ irreps, but not as subalgebras of E_8. Further, the subalgebras $[Y_{97}]_{\mathfrak{so}(14)_{\mathbb{C}}}$ and $[X_{120}]_{\mathfrak{so}(14)_{\mathbb{C}}}$ of E_8 are abelian.

Proof. The authors showed in [4] that $[Y_{97}]_{\mathfrak{so}(14)_{\mathbb{C}}}$ and $[X_{120}]_{\mathfrak{so}(14)_{\mathbb{C}}}$ are abelian subalgebras of E_8.

The positive roots of E_8, as explicitly described in Appendix A, give us a triangular decomposition of E_8: $E_{8,+} \oplus E_{8,0} \oplus E_{8,-}$. In Appendix A we also explicitly describe bases for the representations $[X_{120}]_{\mathfrak{so}(14)_{\mathbb{C}}}$, $[X_{112}]_{\mathfrak{so}(14)_{\mathbb{C}}}$, $[Y_{97}]_{\mathfrak{so}(14)_{\mathbb{C}}}$, and $[Y_1]_{\mathfrak{so}(14)_{\mathbb{C}}}$. Since $\varphi(\mathfrak{so}(14)_{\mathbb{C}}) = [X_{74}]_{\mathfrak{so}(14)_{\mathbb{C}}}$, we also give bases for $\varphi(\mathfrak{so}(14)_{\mathbb{C}+})$ and $\varphi(\mathfrak{so}(14)_{\mathbb{C}-})$. Each of these bases consists of all positive root vectors, or all negative root vectors.

The bases given in Appendix A imply

$$\begin{aligned} E_{8,+} &= \varphi(\mathfrak{so}(14)_{\mathbb{C}+}) \oplus [X_{112}]_{\mathfrak{so}(14)_{\mathbb{C}}} \oplus [X_{120}]_{\mathfrak{so}(14)_{\mathbb{C}}}, \\ E_{8,-} &= \varphi(\mathfrak{so}(14)_{\mathbb{C}-}) \oplus [Y_1]_{\mathfrak{so}(14)_{\mathbb{C}}} \oplus [Y_{97}]_{\mathfrak{so}(14)_{\mathbb{C}}}. \end{aligned}$$

And of course

$$\begin{aligned} &[[X_{112}]_{\mathfrak{so}(14)_{\mathbb{C}}} \oplus [X_{120}]_{\mathfrak{so}(14)_{\mathbb{C}}}, [X_{112}]_{\mathfrak{so}(14)_{\mathbb{C}}} \oplus [X_{120}]_{\mathfrak{so}(14)_{\mathbb{C}}}], \\ &[[Y_1]_{\mathfrak{so}(14)_{\mathbb{C}}} \oplus [Y_{97}]_{\mathfrak{so}(14)_{\mathbb{C}}}, [Y_1]_{\mathfrak{so}(14)_{\mathbb{C}}} \oplus [Y_{97}]_{\mathfrak{so}(14)_{\mathbb{C}}}] \end{aligned}$$

are subsets of $E_{8,+}$ and $E_{8,-}$, respectively.

In Appendix A, we see that the positive root vector X_α is in the basis of $[X_{112}]_{\mathfrak{so}(14)_\mathbb{C}}$ or $[X_{120}]_{\mathfrak{so}(14)_\mathbb{C}}$ if $\alpha^1 \neq 0$, where α^1 is the first entry of α. If $\alpha^1 = 0$, then X_α is in the basis of $\varphi(\mathfrak{so}(14)_{\mathbb{C}+})$.

Thus, if X_α and $X_{\alpha'}$ are positive root vectors in the basis of $[X_{112}]_{\mathfrak{so}(14)_\mathbb{C}}$ or $[X_{120}]_{\mathfrak{so}(14)_\mathbb{C}}$ such that $[X_\alpha, X_{\alpha'}] \neq 0$, then this product is a nonzero scalar multiple of $X_{\alpha+\alpha'}$, where $(\alpha+\alpha')^1 \neq 0$, so that $X_{\alpha+\alpha'}$ is an element of $[X_{112}]_{\mathfrak{so}(14)_\mathbb{C}} \oplus [X_{120}]_{\mathfrak{so}(14)_\mathbb{C}}$. Therefore,

$$[[X_{112}]_{\mathfrak{so}(14)_\mathbb{C}} \oplus [X_{120}]_{\mathfrak{so}(14)_\mathbb{C}}, [X_{112}]_{\mathfrak{so}(14)_\mathbb{C}} \oplus [X_{120}]_{\mathfrak{so}(14)_\mathbb{C}}] \subseteq [X_{112}]_{\mathfrak{so}(14)_\mathbb{C}} \oplus [X_{120}]_{\mathfrak{so}(14)_\mathbb{C}}.$$

In a similar manner we show

$$[[Y_1]_{\mathfrak{so}(14)_\mathbb{C}} \oplus [Y_{97}]_{\mathfrak{so}(14)_\mathbb{C}}, [Y_1]_{\mathfrak{so}(14)_\mathbb{C}} \oplus [Y_{97}]_{\mathfrak{so}(14)_\mathbb{C}}] \subseteq [Y_1]_{\mathfrak{so}(14)_\mathbb{C}} \oplus [Y_{97}]_{\mathfrak{so}(14)_\mathbb{C}}.$$

Thus $[Y_1]_{\mathfrak{so}(14)_\mathbb{C}} \oplus [Y_{97}]_{\mathfrak{so}(14)_\mathbb{C}}$ and $[X_{112}]_{\mathfrak{so}(14)_\mathbb{C}} \oplus [X_{120}]_{\mathfrak{so}(14)_\mathbb{C}}$ are subalgebras of E_8. Further, they are nilpotent since they are contained in $E_{8,-}$ or $E_{8,+}$, respectively. ■

Lemma 2. *The following are not subalgebras of* E_8*:*

$$[Y_{97}]_{\mathfrak{so}(14)_\mathbb{C}} \oplus [X_{112}]_{\mathfrak{so}(14)_\mathbb{C}}, \qquad [X_{120}]_{\mathfrak{so}(14)_\mathbb{C}} \oplus [Y_1]_{\mathfrak{so}(14)_\mathbb{C}}.$$

Proof. Referring to the bases of $[X_{120}]_{\mathfrak{so}(14)_\mathbb{C}}$ and $[Y_1]_{\mathfrak{so}(14)_\mathbb{C}}$ described in Appendix A, we have $Y_{112} \in [Y_1]_{\mathfrak{so}(14)_\mathbb{C}}$, and of course $X_{120} \in [X_{120}]_{\mathfrak{so}(14)_\mathbb{C}}$. However, $[Y_{112}, X_{120}]$ is a nonzero multiple of X_{47}, which is not in $[X_{120}]_{\mathfrak{so}(14)_\mathbb{C}} \oplus [Y_1]_{\mathfrak{so}(14)_\mathbb{C}}$. Hence $[X_{120}]_{\mathfrak{so}(14)_\mathbb{C}} \oplus [Y_1]_{\mathfrak{so}(14)_\mathbb{C}}$ is not a subalgebra. Similarly $[Y_{97}]_{\mathfrak{so}(14)_\mathbb{C}} \oplus [X_{112}]_{\mathfrak{so}(14)_\mathbb{C}}$ is not a subalgebra. ■

We may now explicitly define the *extended GraviGUT algebra* as follows. As a vector space, it is

$$\mathfrak{so}(14)_\mathbb{C} \ltimes (V(\lambda_6) \oplus V(\lambda_1)) \cong \mathfrak{so}(14)_\mathbb{C} \ltimes (V(\lambda_7) \oplus V(\lambda_1)).$$

The Lie algebra structure is inherited from that of E_8. In particular, the following subalgebras are *not* direct sums as algebras, though the sums are direct as vector spaces:

$$V(\lambda_6) + V(\lambda_1) \cong [Y_1]_{\mathfrak{so}(14)_\mathbb{C}} + [Y_{97}]_{\mathfrak{so}(14)_\mathbb{C}},$$
$$V(\lambda_7) + V(\lambda_1) \cong [X_{112}]_{\mathfrak{so}(14)_\mathbb{C}} + [X_{120}]_{\mathfrak{so}(14)_\mathbb{C}}.$$

We note further that not only are $\mathfrak{so}(14)_\mathbb{C} \ltimes (V(\lambda_6) + V(\lambda_1))$ and $\mathfrak{so}(14)_\mathbb{C} \ltimes (V(\lambda_7) + V(\lambda_1))$ isomorphic subalgebras, but they are equivalent subalgebas of E_8, related by the Chevalley involution of E_8. Hence, we shall only consider $\mathfrak{so}(14)_\mathbb{C} \ltimes (V(\lambda_7) + V(\lambda_1))$. It is significant to observe that the (complexified) GraviGUT algebra is a quotient of the extended GraviGUT algebra. The only distinction that can be made is that these two subalgebras of E_8 are *not* the same as $\mathfrak{so}(14)_\mathbb{C}$-modules.

We now proceed to the classification of embeddings of the extended GraviGUT algebra into E_8. A lift of $\varphi : \mathfrak{so}(14)_\mathbb{C} \hookrightarrow E_8$ to $\mathfrak{so}(14)_\mathbb{C} \ltimes (V(\lambda_7)+V(\lambda_1))$ is completely determined by its definition on highest weight vectors of $V(\lambda_7)$ and $V(\lambda_1)$. Call these vectors u and v, respectively. Hence, for any $\alpha, \beta \in \mathbb{C}^*$, the following is a lift of $\varphi : \mathfrak{so}(14)_\mathbb{C} \hookrightarrow E_8$ to $\mathfrak{so}(14)_\mathbb{C} \ltimes (V(\lambda_7)+V(\lambda_1))$:

$$\begin{aligned} \widetilde{\varphi}_{\alpha,\beta} :\ & \mathfrak{so}(14)_\mathbb{C} \ltimes (V(\lambda_7) + V(\lambda_1)) \hookrightarrow E_8, \\ & u \mapsto \alpha X_{112}, \\ & v \mapsto \beta X_{120}. \end{aligned} \tag{3}$$

Theorem 3. *All embeddings of the extended GraviGUT algebra into* E_8 *are given by* $\widetilde{\varphi}_{\alpha,\beta}$*, for all* $\alpha, \beta \in \mathbb{C}^*$*. These embeddings are classified according to the rule*

$$\widetilde{\varphi}_{\alpha,\beta} \sim \widetilde{\varphi}_{\alpha',\beta'} \ \Leftrightarrow\ \alpha'^2\beta = \alpha^2\beta'.$$

Proof. First note that by Theorem 1, all embeddings of the extended GraviGUT algebras must come from lifts of φ, and hence, considering Lemmas 1 and 2, equation (3) defines all embeddings of the extended GraviGUT algebra into E_8.

Define inner automorphisms of E_8 as follows:

$$\begin{aligned} \rho: \quad & X_1 \mapsto \alpha X_1, && Y_1 \mapsto \frac{1}{\alpha} Y_1, \\ & X_i \mapsto X_i, && Y_i \mapsto Y_i, \\ \rho': \quad & X_1 \mapsto \alpha' X_1, && Y_1 \mapsto \frac{1}{\alpha'} Y_1, \\ & X_i \mapsto X_i, && Y_i \mapsto Y_i, \end{aligned}$$

for $2 \le i \le 8$. Then $\widetilde{\varphi}_{\alpha,\beta} = \rho \circ \widetilde{\varphi}_{1,\frac{\beta}{\alpha^2}}$, and $\widetilde{\varphi}_{\alpha',\beta'} = \rho' \circ \widetilde{\varphi}_{1,\frac{\beta'}{\alpha'^2}}$. Hence $\widetilde{\varphi}_{\alpha,\beta} \sim \widetilde{\varphi}_{1,\frac{\beta}{\alpha^2}}$, and $\widetilde{\varphi}_{\alpha',\beta'} \sim \widetilde{\varphi}_{1,\frac{\beta'}{\alpha'^2}}$.

If ϑ is an inner automorphism of E_8 such that $\vartheta \circ \widetilde{\varphi}_{1,\frac{\beta}{\alpha^2}} = \widetilde{\varphi}_{1,\frac{\beta'}{\alpha'^2}}$, then ϑ fixes X_i and Y_i for $2 \le i \le 8$, and also X_{112}. We have

$$\begin{aligned} &[\dots[X_{112}, Y_2], Y_4], Y_3], Y_5], Y_6], Y_7], Y_8], Y_4], Y_5], \\ &\qquad Y_2], Y_6], Y_4], Y_5], Y_3], Y_4], Y_7], Y_6], Y_5], Y_2], Y_4], Y_3] = X_1, \end{aligned}$$

so that ϑ fixes X_1. Hence $\vartheta(X_{120}) = X_{120}$, so that $\frac{\beta}{\alpha^2} = \frac{\beta'}{\alpha'^2}$. The opposite implication is obvious. Hence we have established

$$\widetilde{\varphi}_{\alpha,\beta} \sim \widetilde{\varphi}_{\alpha,\beta} \Leftrightarrow \alpha'^2 \beta = \alpha^2 \beta'. \qquad \blacksquare$$

Remark. Theorem 3 implies, of course, that there are an infinite number of embeddings of the extended GraviGUT algebra into E_8, up to inner automorphism. However, it is interesting to note that there is a unique subalgebra of E_8 which is isomorphic to the extended GraviGUT algebra up to inner automorphism.

7 Conclusions

In [8], Lisi identified a copy of $\mathfrak{spin}(11,3)$ in the quaternionic real form of E_8 and a 64-dimensional subspace on which $\mathfrak{spin}(11,3)$ acts irreducibly. His hope was to embed the GraviGUT algebra. However, the subspace spanned by these spaces is not closed under the Lie bracket.

We proved that the only possible Lie algebra structure on the GraviGUT algebra has a trivial (abelian) bracket on the 64-dimensional subspace. In particular, the GraviGUT algebra cannot be embedded into any real form of E_8. We then modified Lisi's construction to create true Lie algebra embeddings of the extended GraviGUT algebra into E_8. We classified these embeddings up to inner automorphism.

A The representations $[X_{74}]_{\mathfrak{so}(14)_\mathbb{C}}$, $[X_{120}]_{\mathfrak{so}(14)_\mathbb{C}}$, $[X_{112}]_{\mathfrak{so}(14)_\mathbb{C}}$, $[Y_{97}]_{\mathfrak{so}(14)_\mathbb{C}}$, and $[Y_1]_{\mathfrak{so}(14)_\mathbb{C}}$

In this appendix we describe the representations $[X_{74}]_{\mathfrak{so}(14)_\mathbb{C}}$, $[X_{120}]_{\mathfrak{so}(14)_\mathbb{C}}$, $[X_{112}]_{\mathfrak{so}(14)_\mathbb{C}}$, $[Y_{97}]_{\mathfrak{so}(14)_\mathbb{C}}$ and $[Y_1]_{\mathfrak{so}(14)_\mathbb{C}}$ from equation (2).

Let $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_8$ be a set of simple roots for E_8. To any positive root $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + \cdots + a_8\alpha_8$ we may associate a vector $[a_1, a_2, a_3, \dots, a_8] \in \mathbb{Z}^8_{\ge 0}$. With this convention, the positive roots of E_8, as computed with GAP [5], are as follows:

$$\alpha_1 = [1,0,0,0,0,0,0,0], \qquad \alpha_2 = [0,1,0,0,0,0,0,0],$$

$$\alpha_3 = [0,0,1,0,0,0,0,0], \quad \alpha_4 = [0,0,0,1,0,0,0,0],$$
$$\alpha_5 = [0,0,0,0,1,0,0,0], \quad \alpha_6 = [0,0,0,0,0,1,0,0],$$
$$\alpha_7 = [0,0,0,0,0,0,1,0], \quad \alpha_8 = [0,0,0,0,0,0,0,1],$$
$$\alpha_9 = [1,0,1,0,0,0,0,0], \quad \alpha_{10} = [0,1,0,1,0,0,0,0],$$
$$\alpha_{11} = [0,0,1,1,0,0,0,0], \quad \alpha_{12} = [0,0,0,1,1,0,0,0],$$
$$\alpha_{13} = [0,0,0,0,1,1,0,0], \quad \alpha_{14} = [0,0,0,0,0,1,1,0],$$
$$\alpha_{15} = [0,0,0,0,0,0,1,1], \quad \alpha_{16} = [1,0,1,1,0,0,0,0],$$
$$\alpha_{17} = [0,1,1,1,0,0,0,0], \quad \alpha_{18} = [0,1,0,1,1,0,0,0],$$
$$\alpha_{19} = [0,0,1,1,1,0,0,0], \quad \alpha_{20} = [0,0,0,1,1,1,0,0],$$
$$\alpha_{21} = [0,0,0,0,1,1,1,0], \quad \alpha_{22} = [0,0,0,0,0,1,1,1],$$
$$\alpha_{23} = [1,1,1,1,0,0,0,0], \quad \alpha_{24} = [1,0,1,1,1,0,0,0],$$
$$\alpha_{25} = [0,1,1,1,1,0,0,0], \quad \alpha_{26} = [0,1,0,1,1,1,0,0],$$
$$\alpha_{27} = [0,0,1,1,1,1,0,0], \quad \alpha_{28} = [0,0,0,1,1,1,1,0],$$
$$\alpha_{29} = [0,0,0,0,1,1,1,1], \quad \alpha_{30} = [1,1,1,1,1,0,0,0],$$
$$\alpha_{31} = [1,0,1,1,1,1,0,0], \quad \alpha_{32} = [0,1,1,2,1,0,0,0],$$
$$\alpha_{33} = [0,1,1,1,1,1,0,0], \quad \alpha_{34} = [0,1,0,1,1,1,1,0],$$
$$\alpha_{35} = [0,0,1,1,1,1,1,0], \quad \alpha_{36} = [0,0,0,1,1,1,1,1],$$
$$\alpha_{37} = [1,1,1,2,1,0,0,0], \quad \alpha_{38} = [1,1,1,1,1,1,0,0],$$
$$\alpha_{39} = [1,0,1,1,1,1,1,0], \quad \alpha_{40} = [0,1,1,2,1,1,0,0],$$
$$\alpha_{41} = [0,1,1,1,1,1,1,0], \quad \alpha_{42} = [0,1,0,1,1,1,1,1],$$
$$\alpha_{43} = [0,0,1,1,1,1,1,1], \quad \alpha_{44} = [1,1,2,2,1,0,0,0],$$
$$\alpha_{45} = [1,1,1,2,1,1,0,0], \quad \alpha_{46} = [1,1,1,1,1,1,1,0],$$
$$\alpha_{47} = [1,0,1,1,1,1,1,1], \quad \alpha_{48} = [0,1,1,2,2,1,0,0],$$
$$\alpha_{49} = [0,1,1,2,1,1,1,0], \quad \alpha_{50} = [0,1,1,1,1,1,1,1],$$
$$\alpha_{51} = [1,1,2,2,1,1,0,0], \quad \alpha_{52} = [1,1,1,2,2,1,0,0],$$
$$\alpha_{53} = [1,1,1,2,1,1,1,0], \quad \alpha_{54} = [1,1,1,1,1,1,1,1],$$
$$\alpha_{55} = [0,1,1,2,2,1,1,0], \quad \alpha_{56} = [0,1,1,2,1,1,1,1],$$
$$\alpha_{57} = [1,1,2,2,2,1,0,0], \quad \alpha_{58} = [1,1,2,2,1,1,1,0],$$
$$\alpha_{59} = [1,1,1,2,2,1,1,0], \quad \alpha_{60} = [1,1,1,2,1,1,1,1],$$
$$\alpha_{61} = [0,1,1,2,2,2,1,0], \quad \alpha_{62} = [0,1,1,2,2,1,1,1],$$
$$\alpha_{63} = [1,1,2,3,2,1,0,0], \quad \alpha_{64} = [1,1,2,2,2,1,1,0],$$
$$\alpha_{65} = [1,1,2,2,1,1,1,1], \quad \alpha_{66} = [1,1,1,2,2,2,1,0],$$
$$\alpha_{67} = [1,1,1,2,2,1,1,1], \quad \alpha_{68} = [0,1,1,2,2,2,1,1],$$
$$\alpha_{69} = [1,2,2,3,2,1,0,0], \quad \alpha_{70} = [1,1,2,3,2,1,1,0],$$
$$\alpha_{71} = [1,1,2,2,2,2,1,0], \quad \alpha_{72} = [1,1,2,2,2,1,1,1],$$
$$\alpha_{73} = [1,1,1,2,2,2,1,1], \quad \alpha_{74} = [0,1,1,2,2,2,2,1],$$
$$\alpha_{75} = [1,2,2,3,2,1,1,0], \quad \alpha_{76} = [1,1,2,3,2,2,1,0],$$
$$\alpha_{77} = [1,1,2,3,2,1,1,1], \quad \alpha_{78} = [1,1,2,2,2,2,1,1],$$
$$\alpha_{79} = [1,1,1,2,2,2,2,1], \quad \alpha_{80} = [1,2,2,3,2,2,1,0],$$
$$\alpha_{81} = [1,2,2,3,2,1,1,1], \quad \alpha_{82} = [1,1,2,3,3,2,1,0],$$
$$\alpha_{83} = [1,1,2,3,2,2,1,1], \quad \alpha_{84} = [1,1,2,2,2,2,2,1],$$

$$\begin{aligned}
&\alpha_{85} = [1,2,2,3,3,2,1,0], &&\alpha_{86} = [1,2,2,3,2,2,1,1],\\
&\alpha_{87} = [1,1,2,3,3,2,1,1], &&\alpha_{88} = [1,1,2,3,2,2,2,1],\\
&\alpha_{89} = [1,2,2,4,3,2,1,0], &&\alpha_{90} = [1,2,2,3,3,2,1,1],\\
&\alpha_{91} = [1,2,2,3,2,2,2,1], &&\alpha_{92} = [1,1,2,3,3,2,2,1],\\
&\alpha_{93} = [1,2,3,4,3,2,1,0], &&\alpha_{94} = [1,2,2,4,3,2,1,1],\\
&\alpha_{95} = [1,2,2,3,3,2,2,1], &&\alpha_{96} = [1,1,2,3,3,3,2,1],\\
&\alpha_{97} = [2,2,3,4,3,2,1,0], &&\alpha_{98} = [1,2,3,4,3,2,1,1],\\
&\alpha_{99} = [1,2,2,4,3,2,2,1], &&\alpha_{100} = [1,2,2,3,3,3,2,1],\\
&\alpha_{101} = [2,2,3,4,3,2,1,1], &&\alpha_{102} = [1,2,3,4,3,2,2,1],\\
&\alpha_{103} = [1,2,2,4,3,3,2,1], &&\alpha_{104} = [2,2,3,4,3,2,2,1],\\
&\alpha_{105} = [1,2,3,4,3,3,2,1], &&\alpha_{106} = [1,2,2,4,4,3,2,1],\\
&\alpha_{107} = [2,2,3,4,3,3,2,1], &&\alpha_{108} = [1,2,3,4,4,3,2,1],\\
&\alpha_{109} = [2,2,3,4,4,3,2,1], &&\alpha_{110} = [1,2,3,5,4,3,2,1],\\
&\alpha_{111} = [2,2,3,5,4,3,2,1], &&\alpha_{112} = [1,3,3,5,4,3,2,1],\\
&\alpha_{113} = [2,3,3,5,4,3,2,1], &&\alpha_{114} = [2,2,4,5,4,3,2,1],\\
&\alpha_{115} = [2,3,4,5,4,3,2,1], &&\alpha_{116} = [2,3,4,6,4,3,2,1],\\
&\alpha_{117} = [2,3,4,6,5,3,2,1], &&\alpha_{118} = [2,3,4,6,5,4,2,1],\\
&\alpha_{119} = [2,3,4,6,5,4,3,1], &&\alpha_{120} = [2,3,4,6,5,4,3,2].
\end{aligned}$$

Let $X_{\alpha_i} = X_i$, and $Y_{\alpha_i} = Y_i$ be a choice of positive (resp. negative) root vector corresponding to the root α_i.

A basis of $V(\lambda_1) = [X_{120}]_{\mathfrak{so}(14)_\mathbb{C}}$ is given by the 14 positive root vectors:

$$\begin{aligned}
&X_{97}, \quad X_{101}, \quad X_{104}, \quad X_{107}, \quad X_{109}, \quad X_{111}, \quad X_{113},\\
&X_{114}, \quad X_{115}, \quad X_{116}, \quad X_{117}, \quad X_{118}, \quad X_{119}, \quad X_{120}.
\end{aligned}$$

A basis of $V(\lambda_1) = [Y_{97}]_{\mathfrak{so}(14)_\mathbb{C}}$ is given by the 14 negative root vectors:

$$\begin{aligned}
&Y_{97}, \quad Y_{101}, \quad Y_{104}, \quad Y_{107}, \quad Y_{109}, \quad Y_{111}, \quad Y_{113},\\
&Y_{114}, \quad Y_{115}, \quad Y_{116}, \quad Y_{117}, \quad Y_{118}, \quad Y_{119}, \quad Y_{120}.
\end{aligned}$$

A basis of $V(\lambda_7) = [X_{112}]_{\mathfrak{so}(14)_\mathbb{C}}$ is given by the 64 positive root vectors:

$$\begin{aligned}
&X_1, \quad X_9, \quad X_{16}, \quad X_{23}, \quad X_{24}, \quad X_{30}, \quad X_{31}, \quad X_{37}, \quad X_{38},\\
&X_{39}, \quad X_{44}, \quad X_{45}, \quad X_{46}, \quad X_{47}, \quad X_{51}, \quad X_{52}, \quad X_{53}, \quad X_{54},\\
&X_{57}, \quad X_{58}, \quad X_{59}, \quad X_{60}, \quad X_{63}, \quad X_{64}, \quad X_{65}, \quad X_{66}, \quad X_{67},\\
&X_{69}, \quad X_{70}, \quad X_{71}, \quad X_{72}, \quad X_{73}, \quad X_{75}, \quad X_{76}, \quad X_{77}, \quad X_{78},\\
&X_{79}, \quad X_{80}, \quad X_{81}, \quad X_{82}, \quad X_{83}, \quad X_{84}, \quad X_{85}, \quad X_{86}, \quad X_{87},\\
&X_{88}, \quad X_{89}, \quad X_{90}, \quad X_{91}, \quad X_{92}, \quad X_{93}, \quad X_{94}, \quad X_{95}, \quad X_{96},\\
&X_{98}, \quad X_{99}, \quad X_{100}, \quad X_{102}, \quad X_{103}, \quad X_{105}, \quad X_{106}, \quad X_{108}, \quad X_{110},\\
&X_{112}.
\end{aligned}$$

A basis of $V(\lambda_6) = [Y_1]_{\mathfrak{so}(14)_\mathbb{C}}$ is given by the 64 negative root vectors:

$$\begin{aligned}
&Y_1, \quad Y_9, \quad Y_{16}, \quad Y_{23}, \quad Y_{24}, \quad Y_{30}, \quad Y_{31}, \quad Y_{37}, \quad Y_{38},\\
&Y_{39}, \quad Y_{44}, \quad Y_{45}, \quad Y_{46}, \quad Y_{47}, \quad Y_{51}, \quad Y_{52}, \quad Y_{53}, \quad Y_{54},
\end{aligned}$$

$$\begin{array}{lllllllll} Y_{57}, & Y_{58}, & Y_{59}, & Y_{60}, & Y_{63}, & Y_{64}, & Y_{65}, & Y_{66}, & Y_{67}, \\ Y_{69}, & Y_{70}, & Y_{71}, & Y_{72}, & Y_{73}, & Y_{75}, & Y_{76}, & Y_{77}, & Y_{78}, \\ Y_{79}, & Y_{80}, & Y_{81}, & Y_{82}, & Y_{83}, & Y_{84}, & Y_{85}, & Y_{86}, & Y_{87}, \\ Y_{88}, & Y_{89}, & Y_{90}, & Y_{91}, & Y_{92}, & Y_{93}, & Y_{94}, & Y_{95}, & Y_{96}, \\ Y_{98}, & Y_{99}, & Y_{100}, & Y_{102}, & Y_{103}, & Y_{105}, & Y_{106}, & Y_{108}, & Y_{110}, \\ Y_{112}. \end{array}$$

Note that $\varphi(\mathfrak{so}(14)_{\mathbb{C}}) = [X_{74}]_{\mathfrak{so}(14)_{\mathbb{C}}}$. We describe bases of $\varphi(\mathfrak{so}(14)_{\mathbb{C}+})$ and $\varphi(\mathfrak{so}(14)_{\mathbb{C}-})$, respectively:

$$\begin{array}{lllllllll} X_{2}, & X_{3}, & X_{4}, & X_{5}, & X_{6}, & X_{7}, & X_{8}, & X_{10}, & X_{11}, \\ X_{12}, & X_{13}, & X_{14}, & X_{15}, & X_{17}, & X_{18}, & X_{19}, & X_{20}, & X_{21}, \\ X_{22}, & X_{25}, & X_{26}, & X_{27}, & X_{28}, & X_{29}, & X_{32}, & X_{33}, & X_{34}, \\ X_{35}, & X_{36}, & X_{40}, & X_{41}, & X_{42}, & X_{43}, & X_{48}, & X_{49}, & X_{50}, \\ X_{55}, & X_{56}, & X_{61}, & X_{62}, & X_{68}, & X_{74}; \\ Y_{2}, & Y_{3}, & Y_{4}, & Y_{5}, & Y_{6}, & Y_{7}, & Y_{8}, & Y_{10}, & Y_{11}, \\ Y_{12}, & Y_{13}, & Y_{14}, & Y_{15}, & Y_{17}, & Y_{18}, & Y_{19}, & Y_{20}, & Y_{21}, \\ Y_{22}, & Y_{25}, & Y_{26}, & Y_{27}, & Y_{28}, & Y_{29}, & Y_{32}, & Y_{33}, & Y_{34}, \\ Y_{35}, & Y_{36}, & Y_{40}, & Y_{41}, & Y_{42}, & Y_{43}, & Y_{48}, & Y_{49}, & Y_{50}, \\ Y_{55}, & Y_{56}, & Y_{61}, & Y_{62}, & Y_{68}, & Y_{74}. \end{array}$$

Acknowledgements. The work of A.D. is partially supported by a research grant from the Professional Staff Congress/ City University of New York (PSC/CUNY). The work of J.R. is partially supported by the Natural Sciences and Engineering Research Council (NSERC). The authors would also like to thank the anonymous referees for valuable comments.

References

[1] Baez J., The octonions, *Bull. Amer. Math. Soc.* **39** (2002), 145–205, math.RA/0105155.

[2] Baez J., Huerta J., The algebra of grand unified theories, *Bull. Amer. Math. Soc.* **47** (2010), 483–552, arXiv:0904.1556.

[3] Distler J., Garibaldi S., There is no "theory of everything" inside E_8, *Comm. Math. Phys.* **298** (2010), 419–436, arXiv:0905.2658.

[4] Douglas A., Kahrobaei D., Repka J., Classification of embeddings of abelian extensions of D_n into E_{n+1}, *J. Pure Appl. Algebra* **217** (2013), 1942–1954.

[5] GAP-Groups, Algorithms, and programming, Version 4.2, 2000, `http://www-gap.dcs.st-and.ac.uk/~gap`.

[6] Humphreys J.E., Introduction to Lie algebras and representation theory, *Graduate Texts in Mathematics*, Vol. 9, Springer-Verlag, New York – Berlin, 1972.

[7] Lisi A.G., An exceptionally simple theory of everything, arXiv:0711.0770.

[8] Lisi A.G., An explicit embedding of gravity and the Standard Model in E_8, in Representation Theory and Mathematical Physics, *Contemp. Math.*, Vol. 557, Amer. Math. Soc., Providence, RI, 2011, 231–244, arXiv:1006.4908.

[9] Malcev A.I., Commutative subalgebras of semi-simple Lie algebras, *Amer. Math. Soc. Translation* **1951** (1951), no. 40, 15 pages.

[10] Minchenko A.N., Semisimple subalgebras of exceptional Lie algebras, *Trans. Moscow Math. Soc.* **67** (2006), 225–259.

[11] Mkrtchyan R.L., On the map of Vogel's plane, arXiv:1209.5709.

[12] Nesti F., Percacci R., Chirality in unified theories of gravity, *Phys. Rev. D* **81** (2010), 025010, 7 pages, arXiv:0909.4537.

[13] Vogel P., Algebraic structures on modules of diagrams, *J. Pure Appl. Algebra* **215** (2011), 1292–1339.