

Gravity in Twistor Space and its Grassmannian Formulation^{*}

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Abstract. We prove the formula for the complete tree-level S -matrix of $\mathcal{N} = 8$ supergravity recently conjectured by two of the authors. The proof proceeds by showing that the new formula satisfies the same BCFW recursion relations that physical amplitudes are known to satisfy, with the same initial conditions. As part of the proof, the behavior of the new formula under large BCFW deformations is studied. An unexpected bonus of the analysis is a very straightforward proof of the enigmatic $1/z^2$ behavior of gravity. In addition, we provide a description of gravity amplitudes as a multidimensional contour integral over a Grassmannian. The Grassmannian formulation has a very simple structure; in the \mathcal{N}^{k-2} MHV sector the integrand is essentially the product of that of an MHV and an $\overline{\text{MHV}}$ amplitude, with $k + 1$ and $n - k - 1$ particles respectively.

Key words: twistor theory; scattering amplitudes; gravity

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1 Introduction

In a recent paper [15], two of us conjectured that the complete classical S -matrix of maximal supergravity in four dimensions can be described by a certain integral over the space of rational maps to twistor space. The main aim of this paper is to prove that conjecture.

In [5, 6, 7, 8, 16, 17] it was shown that gravitational tree amplitudes obey the BCFW recursion relations [11, 12]. Our method here is to show that the formula presented in [15] obeys these same relations, and produces the correct three-particle MHV and $\overline{\text{MHV}}$ amplitudes to start the recursion.

In the analogous formulation of tree amplitudes in $\mathcal{N} = 4$ super Yang–Mills [9, 30, 36], BCFW decomposition is closely related to performing a contour integral in the moduli space of holomorphic maps so as to localize on the boundary where the worldsheet degenerates to a nodal curve [10, 18, 20, 21, 31, 32, 34]. The various summands on the right hand side of the recursion relation correspond to the various ways the vertex operators and map degree may be distributed among the two curve components.

The relation between factorization channels of amplitudes and shrinking cycles on the worldsheet that separate some vertex operators from others is of course a fairly general property of

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string theory. In the present case, it is also necessary to prove that the rest of the integrand behaves well under this degeneration. In particular, the formula of [15] involves a product of two determinants that generalize Hodges' construction of gravitational MHV amplitudes [23, 24] to arbitrary external helicities. One of the striking properties of these determinants is that they each depend in a simple way on the 'infinity twistor', and thereby the breaking of conformal invariance inherent in gravitational amplitudes becomes completely explicit. More specifically, the determinants are each monomials in the infinity twistor, to a power that depends only on the number of external states and the MHV degree of the scattering process. Furthermore, as explained in [15], the way arbitrary gravitational amplitudes depend on the infinity twistor can easily be traced through BCFW recursion. This strongly suggests that the determinants behave simply under factorization. We shall see that this is indeed the case.

Along the way, we show that the $1/z^2$ decay [6, 17] of gravitational tree amplitudes at large values of the BCFW shift parameter z is also simple to see from the formula of [15]. This behaviour is responsible for many remarkable properties of these amplitudes (see, e.g., [33] for some applications).

In addition, in the second part of the paper we reformulate the construction of [15] as an integral over the Grassmannian $G(k, n)$, written in terms of the 'link' coordinates of [4]. As preparation, we show how the two determinants, which in twistor space look very different from one another, are naturally conjugate under parity. The formulation of gravitational tree amplitudes as an integral over $G(k, n)$ is strikingly simple: the integrand is the product of that of an MHV and an $\overline{\text{MHV}}$ amplitude, with $k + 1$ and $n - k - 1$ particles respectively.

2 Gravity from rational curves

We begin by briefly reviewing the conjecture of [15] (see Appendix C for notation). All n -particle tree amplitudes in $\mathcal{N} = 8$ supergravity are given by the sum

$$\mathcal{M}_n = \sum_{d=0}^{\infty} \mathcal{M}_{n,d}$$

over N^{d-1} MHV partial amplitudes. The main claim of [15] is that these N^{d-1} MHV amplitudes may be represented by the integral

$$\mathcal{M}_{n,d}(\{\lambda_i, \tilde{\lambda}_i, \eta_i\}) = \int \frac{\prod_{a=0}^d d^{4|8} \mathcal{Z}_a}{\text{vol GL}(2; \mathbb{C})} |\Phi|' |\tilde{\Phi}|' \prod_{i=1}^n (\sigma_i d\sigma_i) h_i(\mathcal{Z}(\sigma_i)). \quad (2.1)$$

Here, \mathcal{Z} is a holomorphic map from a rational curve Σ with homogeneous coordinates σ^α to $\mathcal{N} = 8$ supertwistor space with homogeneous coordinates $\mathcal{Z}^I = (Z^a, \chi^A) = (\mu^{\dot{\alpha}}, \lambda_\alpha, \chi^A)$. The external states are $\mathcal{N} = 8$ linearized supergravitons, and are described on twistor space by classes $h_i \in H^{0,1}(\mathbb{P}\mathbb{T}, \mathcal{O}(2))$. These wavefunctions are pulled back to points σ_i on the curve via the map \mathcal{Z} . We usually take

$$h_i(\mathcal{Z}(\sigma_i)) := \int \frac{dt_i}{t_i^3} \delta^2(\lambda_i - t_i \lambda(\sigma_i)) \exp(t_i \llbracket \mu(\sigma_i) \tilde{\lambda}_i \rrbracket) \quad (2.2)$$

so as to represent momentum eigenstates¹. It is easy to check that such an $h_i(\mathcal{Z}(\sigma_i))$ is homogeneous of degree -4 in the external data λ_i , as required for a positive helicity graviton supermultiplet in on-shell momentum space.

¹In [15] we used a $G(2, n)$ notation for the worldsheet variables, whereas here we are working projectively on the worldsheet. The two pictures are related by taking $\sigma_i^{\text{there}} = t_i^{1/d} \sigma_i^{\text{here}}$ when $d \geq 1$, so that $d^2 \sigma_i^{\text{there}} = \frac{1}{d} (\sigma_i d\sigma_i)^{\text{here}} dt_i / t_i$. When $d = 0$ the relation is simply $\sigma_i^{\text{there}} = \sigma_i^{\text{here}}$ and $d^2 \sigma_i^{\text{there}} = (\sigma_i d\sigma_i)^{\text{here}} dt_i / t_i$. This rescaling also accounts for the factors of t_i that appear in the matrices Φ and $\tilde{\Phi}$ in (2.4) and (2.3), respectively.

The main content of (2.1) is the operators $|\Phi|'$ and $|\tilde{\Phi}|'$. These arise as a generalization of Hodges' formulation of MHV amplitudes [23, 24] and were defined in [15] as follows. We first let $\tilde{\Phi}$ be the $n \times n$ matrix operator with elements

$$\begin{aligned}\tilde{\Phi}_{ij} &= \frac{1}{(ij)} \left[\frac{\partial}{\partial \mu(\sigma_i)} \frac{\partial}{\partial \mu(\sigma_j)} \right] = \frac{[ij]}{(ij)} t_i t_j \quad \text{for } i \neq j, \\ \tilde{\Phi}_{ii} &= - \sum_{j \neq i} \tilde{\Phi}_{ij} \prod_{a=0}^d \frac{(jp_a)}{(ip_a)},\end{aligned}\tag{2.3}$$

where the second equality in the first line follows when $\tilde{\Phi}$ acts on the momentum eigenstates (2.2). It was shown in [15] that $\tilde{\Phi}$ has rank $n - d - 2$, with the $(d + 2)$ -dimensional kernel spanned by vectors whose j^{th} component is $\sigma_j^{\alpha_0} \cdots \sigma_j^{\alpha_d}$, i.e.

$$\sum_{j=1}^n \tilde{\Phi}_{ij} \sigma_j^{\alpha_0} \cdots \sigma_j^{\alpha_d} = 0.$$

This equation holds on the support of the δ -functions $\prod_{a=0}^d \delta^{2|8} \left(\sum_i t_i \tilde{\lambda}_i (\sigma_i^1)^a (\sigma_i^2)^{d-a} \right)$ obtained by integrating out the map coefficients μ_a from (2.6).

Φ is similarly defined as the symmetric $n \times n$ matrix with elements

$$\begin{aligned}\Phi_{ij} &= \frac{\langle \lambda(\sigma_i) \lambda(\sigma_j) \rangle}{(ij)} = \frac{\langle ij \rangle}{(ij)} \frac{1}{t_i t_j} \quad \text{for } i \neq j, \\ \Phi_{ii} &= - \sum_{j \neq i} \Phi_{ij} \prod_{r=0}^{\tilde{d}} \frac{(jp_r)}{(ip_r)} \frac{\prod_{k \neq i} (ik)}{\prod_{l \neq j} (jl)},\end{aligned}\tag{2.4}$$

where

$$\tilde{d} := n - d - 2$$

is introduced for later convenience, and again the second equality in the first line of (2.4) follows when acting on (2.2). Φ has rank d , with kernel defined by the equation

$$\sum_{j=1}^n \Phi_{ij} \frac{\sigma_j^{\alpha_0} \cdots \sigma_j^{\alpha_{\tilde{d}}}}{\prod_{k \neq j} (jk)} = 0.\tag{2.5}$$

This equation holds because for any degree d polynomial $\lambda(\sigma)$, the residue theorem gives

$$\oint_{|\langle i\sigma \rangle|=\epsilon} (\sigma d\sigma) \frac{\lambda(\sigma) \sigma^{\alpha_1} \cdots \sigma^{\alpha_{\tilde{d}}}}{\prod_{j=1}^n (\sigma j)} = \frac{\lambda(\sigma_i) \sigma_i^{\alpha_1} \cdots \sigma_i^{\alpha_{\tilde{d}}}}{\prod_{j \neq i} (ij)}.$$

The sum of this residue over all $i \in \{1, \dots, n\}$ vanishes because the resulting contour is homologically trivial.

The operators $|\Phi|'$ and $|\tilde{\Phi}|'$ are defined as follows. Remove any $d + 2$ rows and any $d + 2$ columns from $\tilde{\Phi}$ to produce a non-singular matrix $\tilde{\Phi}_{\text{red}}$. Then

$$|\tilde{\Phi}|' := \frac{\det(\tilde{\Phi}_{\text{red}})}{|\tilde{r}_1 \cdots \tilde{r}_{d+2}| |\tilde{c}_1 \cdots \tilde{c}_{d+2}|}.$$

In this ratio, $|\tilde{r}_1 \dots \tilde{r}_{d+2}|$ denotes the Vandermonde determinant

$$|\tilde{r}_1 \dots \tilde{r}_{d+2}| = \prod_{\substack{i < j \\ i, j \in \{\text{removed}\}}} (ij)$$

made from all possible combinations of the worldsheet coordinates corresponding to the deleted rows; $|\tilde{c}_1 \dots \tilde{c}_{d+2}|$ is the same Vandermonde determinant, but for the deleted columns.

In [15], $|\Phi|'$ was also defined in terms of the determinant of a non-singular matrix Φ_{red} obtained by similarly removing rows and columns from Φ , now $n - d$ of each. However, $|\Phi|'$ itself was constructed as

$$|\Phi|' := \frac{\det(\Phi_{\text{red}})}{|r_1 \dots r_d| |c_1 \dots c_d|}$$

using the Vandermonde determinants $|r_1 \dots r_d|$ and $|c_1 \dots c_d|$ of the rows and columns that *remain* in Φ_{red} .

2.1 Definition of \det'

The above definitions of $|\tilde{\Phi}|'$ and $|\Phi|'$ are actually quite different. This motivates us to find a more canonical way to define these determinants. It turns out that the most natural definition has to do with the null vectors of each matrix. Once the null space of any symmetric $n \times n$ matrix K of rank m is determined, one can compute any two maximal minors of the $n \times (n - m)$ matrix with the null vectors of K as its columns. Denote the two maximal minors chosen by $|R|$ and $|C|$. Then

$$\det'(K) = \frac{\det(K_{\text{red}})}{|R||C|},$$

where K_{red} is the reduced matrix obtained after removing $n - m$ rows and columns whose row and column label coincide with the ones removed from the $n \times (n - m)$ matrix of null vectors to obtain $|R|$ and $|C|$.

Appendix A contains a formal motivation for this definition and explains how the old and new definitions are related. At this point it suffices to say that

$$|\tilde{\Phi}|' = \det'(\tilde{\Phi}) \quad \text{while} \quad |\Phi|' = \frac{\det'(\Phi)}{|1 \dots n|^2},$$

so that an alternative presentation of the gravity formula (2.1) is

$$\mathcal{M}_{n,d}(\{\lambda_i, \tilde{\lambda}_i, \eta_i\}) = \int \frac{\prod_{a=0}^d d^{4|8} \mathcal{Z}_a}{\text{vol GL}(2; \mathbb{C})} \frac{\det'(\Phi) \det'(\tilde{\Phi})}{|1 \dots n|^2} \prod_{i=1}^n (\sigma_i d\sigma_i) h_i(\mathcal{Z}(\sigma_i)). \quad (2.6)$$

In the rest of the paper we will use whichever form is more convenient for the argument at hand.

3 BCFW recursion

In this section, we will prove the conjecture of [15] by showing that the $\mathcal{M}_{n,d}$ defined by equation (2.6) correctly obeys BCFW recursion. There are four aspects to the proof. Firstly, we must

show that the formula correctly reproduces the 3-particle amplitudes that seed the recursion. This step is straightforward. Next, we must show that under the BCFW shift²

$$\lambda_1 \rightarrow \lambda_1 + z\lambda_n \quad \tilde{\lambda}_n \rightarrow \tilde{\lambda}_n - z\tilde{\lambda}_1, \quad \chi_n \rightarrow \chi_n - z\chi_1 \quad (3.1)$$

the integral in (2.6) decays at least as fast as $1/z$ in the limit that $z \rightarrow \infty$. Thirdly, we must show that $\mathcal{M}_{n,d}$ has a pole whenever the sum of momenta of any two or more particles becomes null, with the residue of this pole being the product of two subamplitudes. Finally, we complete the argument by showing that $\mathcal{M}_{n,d}$ has no poles other than the physical ones. This being the case, the usual BCFW contour argument [11] may be applied to construct $\mathcal{M}_{n,d}$ recursively from smaller amplitudes. Equation (2.6) will then agree with the tree amplitudes in $\mathcal{N} = 8$ supergravity since it satisfies the same recursion relation with the same initial conditions [5, 6, 7, 8, 16, 17].

In fact, it is known that gravitational scattering amplitudes decay as $1/z^2$ under the BCFW shift [6, 17]. We will see that $\mathcal{M}_{n,d}$ has precisely this behaviour quite transparently. Although this fact can be shown using Lagrangian techniques [6, 17], the proof is rather opaque from the view point of the S -matrix.

3.1 3-particle seed amplitudes

We first check that (2.6) yields the correct 3-point amplitudes. For the $\overline{\text{MHV}}$ we have $n = 3$ and $d = 0$, so that the map \mathcal{Z} is constant, $\mathcal{Z}(\sigma) = \mathcal{Z}$. We can remove all three rows and columns of Φ and it is simple to show that $\det'(\Phi)$ cancels³ the factor of $|\underline{123}|^2$ in the denominator of (2.6). We can also remove two of the three rows and columns from Φ . Choosing these to be the first and second rows and the first and third columns, the reduced determinant becomes

$$\det'(\tilde{\Phi}) = \frac{1}{(12)(23)(31)} \left[\frac{\partial}{\partial \mu(\sigma_2)} \frac{\partial}{\partial \mu(\sigma_3)} \right].$$

The denominator (12)(23)(31) is exactly compensated by the Jacobian from fixing worldsheet $\text{SL}(2; \mathbb{C})$ invariance, so (2.6) reduces to

$$\begin{aligned} \mathcal{M}_{3,0} &= [23] \int \frac{d^{4|8} \mathcal{Z}}{\text{vol}(\mathbb{C}^*)} t_2 t_3 \prod_{i=1}^3 \frac{dt_i}{t_i^3} \bar{\delta}^2(\lambda_i - t_i \lambda) \exp(t_i \llbracket \mu \tilde{\lambda}_i \rrbracket) \\ &= [23] \int \prod_{i=2}^3 \frac{dt_i}{t_i^2} \bar{\delta}^2(\lambda_i - t_i \lambda_1) \bar{\delta}^{2|8}(\tilde{\lambda}_1 + t_2 \tilde{\lambda}_2 + t_3 \tilde{\lambda}_3), \end{aligned}$$

where in going the second line we fixed the \mathbb{C}^* scaling by setting $t_1 = 1$ and then performed the $d^{4|8} \mathcal{Z}$ integral. Using the two bosonic δ -functions involving the $\tilde{\lambda}$'s to fix t_2 and t_3 shows that

$$\mathcal{M}_{3,0} = \frac{\delta^4 \left(\sum_{i=1}^3 p_i \right) \delta^{0|8} (\eta_1[23] + \eta_2[31] + \eta_3[12])}{([\underline{12}][\underline{23}][\underline{31}])^2},$$

which is exactly the required 3-point $\overline{\text{MHV}}$ amplitude.

For general wavefunctions h_i , $\mathcal{M}_{0,3}$ is simply the cubic vertex of the twistor action for self-dual $\mathcal{N} = 8$ supergravity [27]

$$S_{\text{sdG}} = \int_{\mathbb{PT}} D^{3|8} \mathcal{Z} \wedge \left(h \wedge \bar{\partial} h + \frac{2}{3} h \wedge \{h, h\} \right), \quad (3.2)$$

²In Appendix A we prove that $\mathcal{M}_{n,d}$ is completely permutation symmetric in the external states. Hence there is no loss of generality in considering this BCFW shift.

³Here and below we take $\det'(\Phi)$ to be defined as in (A.1); see the discussion after equation (A.6).

evaluated on on-shell states. In this action,

$$\{f, g\} := I^{ab} \frac{\partial f}{\partial Z^a} \frac{\partial g}{\partial Z^b} = \left[\frac{\partial f}{\partial \mu} \frac{\partial g}{\partial \mu} \right]$$

is the Poisson bracket associated to the infinity twistor. Notice that the negative homogeneity of the Poisson bracket ensures the interaction term scales the same way as the kinetic term, and that each balances the scaling of the $\mathcal{N} = 8$ measure.

The linearized field equations of (3.2) state that $h(Z)$ represents a class in $H^{0,1}(\mathbb{P}\mathbb{T}, \mathcal{O}(2))$, as required by the Penrose transform for massless free fields of helicity +2. At the non-linear level, the field equations state that $\bar{\partial} + \{h, \cdot\}$ defines an integrable almost complex structure on $\mathbb{P}\mathbb{T}$ that is compatible with the Poisson structure. This is exactly the content of Penrose's non-linear graviton construction [29]. A twistor space with an integrable almost complex structure corresponds to a conformal equivalence class of space-times with self-dual Weyl tensor. The additional information that the complex structure is compatible with the Poisson structure picks a distinguished metric in the conformal class that satisfies the vacuum Einstein equations. The presence of the infinity twistor in $\mathcal{M}_{\overline{\text{MHV}}}$ is thus a direct consequence of its presence in the self-dual action, reflecting the very nature of the non-linear graviton construction.

For the 3-point MHV we have $(n, d) = (3, 1)$, so that $\mathcal{Z}(\sigma) = A\sigma^0 + B\sigma^1$. We can now remove two rows and columns from Φ . Choosing these to be the first and second rows and the first and third columns, equation (A.2) gives

$$\frac{\det'(\Phi)}{|\mathbf{123}|^2} = \frac{\langle 23 \rangle}{(23)} \frac{1}{t_2 t_3} = \langle AB \rangle,$$

where the second equality holds on the support of the δ -functions for λ_i . We can remove all three rows and columns from $\tilde{\Phi}$, so

$$\det'(\tilde{\Phi}) = \frac{1}{(12)^2(23)^2(31)^2} = \frac{\langle AB \rangle^6}{\langle 12 \rangle^2 \langle 23 \rangle^2 \langle 31 \rangle^2} \prod_{i=1}^3 t_i^4.$$

The integral $\mathcal{M}_{3,1}$ then becomes

$$\begin{aligned} \mathcal{M}_{3,1} &= \frac{1}{\langle 12 \rangle^2 \langle 23 \rangle^2 \langle 31 \rangle^2} \\ &\times \int \frac{d^{4|8} A d^{4|8} B}{\text{vol GL}(2)} \langle AB \rangle^7 \prod_{i=1}^3 (\sigma_i d\sigma_i) dt_i t_i \delta^2(\lambda_i - t_i \lambda(\sigma_i)) \exp(t_i \llbracket \mu(\sigma_i) \tilde{\lambda}_i \rrbracket). \end{aligned}$$

The integrals over t_i and σ_i integrals may be fixed by the δ -functions. The integrals over $(\mu_{A,B}, \chi_{A,B})$ then provide a super-momentum conserving δ -function, while the four remaining integrals over $|A\rangle$ and $|B\rangle$ are compensated by the $\text{GL}(2)$. Overall, we have

$$\mathcal{M}_{3,1} = \frac{\delta^{4|16} \left(\sum_i p_i \right)}{\langle 12 \rangle^2 \langle 23 \rangle^2 \langle 31 \rangle^2},$$

exactly the 3-particle MHV amplitude in $\mathcal{N} = 8$ supergravity, as expected.

The 3-particle amplitudes that seed BCFW recursion are thus correctly reproduced by the integral (2.6).

3.2 Decay as $z \rightarrow \infty$

We now investigate the behaviour of $\mathcal{M}_{n,d}$ under the BCFW shift in the limit that the shift parameter $z \rightarrow \infty$. We shall see that the highly non-trivial fact that the gravitational amplitudes decay as $1/z^2$ in this limit is made manifest by (2.6).

At degree d , we can remove $d+2$ rows and columns from $\tilde{\Phi}$ and $n-d$ rows and columns from Φ . Hence, since we are only interested in BCFW recursion for $1 \leq d \leq n-3$, we can always remove at least two rows and columns from each. With the shift (3.1) that affects only $|1\rangle$ and $|n\rangle$, we choose the removed rows and columns to include 1 and n in both cases. In addition, we choose one of the arbitrary points $p_r \in \Sigma$ in (2.4) to be σ_1 and another to be σ_n so that the terms $j=1, n$ drop out of the sum over j the diagonal elements Φ_{ii} . Similarly, we choose the arbitrary points $p_a \in \Sigma$ in (2.3) to include σ_1 and σ_n . With these choices, the external data $|1\rangle, |n\rangle$ and $[1], [n]$ does not appear in the determinants $\det'(\Phi)\det'(\tilde{\Phi})$. Thus, after integrating out the (μ, χ) components of the map, the shift (3.1) affects (2.6) only by changing the arguments of the δ -functions involving λ_1 and $\tilde{\lambda}_n$. On the coordinate patch $\sigma^\alpha = (1, u)$ of the worldsheet, these shifted δ -functions become

$$\bar{\delta}^2 \left(\lambda_1 + z\lambda_n - t_1 \sum_{a=0}^d \rho_a u_1^a \right) \prod_{a=0}^d \bar{\delta}^{2|8} \left(\sum_{j=1}^n t_j \tilde{\lambda}_j u_j^a - z t_n \tilde{\lambda}_1 u_n^a \right),$$

where ρ_a is the λ part of the map \mathcal{Z} .

To absorb these shifts, we introduce new worldsheet variables (\hat{u}, \hat{t}) for particle 1, defined by

$$\hat{t}_1 := t_1 - z t_n, \quad \hat{t}_1 (\hat{u}_1)^d := t_1 (u_1)^d - z t_n (u_n)^d, \quad (3.3)$$

where z is the BCFW shift parameter. This definition absorbs the shifts in the δ -functions, up to terms that vanish as $z \rightarrow \infty$. Specifically, the argument of the shifted λ_1 δ -function becomes

$$\lambda_1 - \hat{t}_1 \sum_{a=0}^d \rho_a \left(\frac{d-a}{d} u_n^a + \frac{a}{d} \hat{u}_1^d u_n^{a-d} \right) + \mathcal{O}(1/z)$$

while the arguments of the δ -functions involving the $\tilde{\lambda}$'s become

$$\sum_{j=2}^n t_j \tilde{\lambda}_j u_j^a + \hat{t}_1 \tilde{\lambda}_1 \left(\frac{d-a}{d} u_n^a + \frac{a}{d} \hat{u}_1^d u_n^{a-d} \right) + \mathcal{O}(1/z).$$

The important point is that the new variables (\hat{u}_1, \hat{t}_1) remain finite as $z \rightarrow \infty$. Therefore, to study the behaviour of (2.6) in this limit, we should express its integrand in terms of these variables.

Begin with the measure for particle 1. It follows from (3.3) that

$$\begin{aligned} (\sigma_1 d\sigma_1) &= du_1 = \left(\frac{z t_n u_n^d + \hat{t}_1 \hat{u}_1^d}{z t_n + \hat{t}_1} \right)^{\frac{1}{d}-1} \xrightarrow{z \rightarrow \infty} \frac{\hat{t}_1}{z t_n} u_n^{\frac{d-1}{d}} \hat{u}_1^{d-1} d\hat{u}_1, \\ \frac{dt_1}{t_1^3} &= \frac{d\hat{t}_1}{(z t_n + \hat{t}_1)^3} \xrightarrow{z \rightarrow \infty} \frac{1}{z^3} \frac{d\hat{t}_1}{t_n^3}, \end{aligned}$$

where we have dropped terms that wedge to zero against the measure for particle n . Thus the integration measure of (2.6) falls as z^{-4} as the shift parameter tends to infinity. Similarly, we find that

$$(1j) = u_1 - u_j = \left(\frac{z t_n u_n^d + \hat{u}_1^d \hat{t}_1}{z t_n + \hat{t}_1} \right)^{\frac{1}{d}} - u_j \xrightarrow{z \rightarrow \infty} (nj)$$

whenever $(nj) \neq 0$, but that

$$(1n) = \left(\frac{zt_n u_n^d + \hat{u}_1^d \hat{t}_1}{zt_n + \hat{t}_1} \right)^{\frac{1}{d}} - u_j \xrightarrow{z \rightarrow \infty} \frac{u_n \hat{t}_1}{zt_n} \left(\frac{\hat{u}_1^d}{u_n^d} + 1 \right),$$

so that the special case of $(1n)$ decays as z^{-1} , as the order z^0 term cancels exactly.

We now investigate the occurrence of $(1n)$ and t_1 in the integrand of $\mathcal{M}_{n,d}$, since these are the only terms that have non-trivial large z behaviour. We can always choose to remove row and columns 1 and n from both Φ and $\tilde{\Phi}$. This does not quite suffice to remove $(1n)$ and t_1 from the matrices, because they still appear in the diagonal terms

$$\Phi_{ii} = - \sum_{j \neq i} \frac{\langle ij \rangle}{(ij)} \frac{1}{t_i t_j} \prod_{r=0}^{\tilde{d}} \frac{(jp_r)}{(ip_r)} \frac{\prod_{k \neq i} (ik)}{\prod_{l \neq j} (jl)}, \quad \tilde{\Phi}_{ii} = - \sum_{j \neq i} \frac{[ij]}{(ij)} t_i t_j \prod_{a=0}^d \frac{(jp_a)}{(ip_a)}.$$

By further choosing one of the p_r and one of the p_a to be σ_1 the summand with $j = 1$ vanishes in each of these matrices, and since $i \neq 1, n$ the matrices themselves approach a constant value as $z \rightarrow \infty$, obtained by simply replacing $\sigma_1 \rightarrow \sigma_n$.

Aside from the measure then, the only pieces of the integrand which affect the large z behaviour are the Vandermonde determinants in the definition of the reduced determinants. Since we have removed rows and columns 1 and n , the Vandermonde determinants associated with Φ are independent of $(1n)$. However, we find that the definition of $\det'(\tilde{\Phi})$ involves the denominator

$$|1nr_1 \dots r_d|^2 \propto (1n)^2,$$

where r_1, \dots, r_d are the other rows and columns that were removed from $\tilde{\Phi}$. This factor, appearing in the denominator of $\det'(\tilde{\Phi})$, behaves as $1/z^2$ in the large shift limit. Combined with the $1/z^4$ behaviour of the integration measure, we see that $\mathcal{M}_{n,d}(z) \propto 1/z^2$ as $z \rightarrow \infty$. This ensures that the BCFW integrand $\mathcal{M}_{n,d}(z) dz/z$ has no pole at infinity, allowing the BCFW residue theorem to proceed. It is quite remarkable that the formula (2.6) for $\mathcal{M}_{n,d}$ reproduces the correct $1/z^2$ behaviour of gravity so transparently. We repeat that this behaviour is highly non-trivial to prove by any other means, and yet is a key property of gravitational scattering amplitudes [33].

Incidentally, exactly the same argument as above may be applied to the Witten-RSV formula [30, 36]

$$\mathcal{A}_{n,d} = \int \frac{\prod_{a=0}^d d^{4|4} \mathcal{Z}_a}{\text{vol GL}(2; \mathbb{C})} \prod_{i=1}^n \frac{(\sigma_i d\sigma_i)}{(\sigma_i \sigma_{i+1})} \frac{dt_i}{t_i} \delta^2(\lambda_i - t_i \lambda(\sigma_i)) \exp(t_i \llbracket \mu(\sigma_i) \tilde{\lambda}_i \rrbracket) \quad (3.4)$$

for tree-level scattering amplitudes in $\mathcal{N} = 4$ SYM. We now have

$$\frac{dt_1}{t_1} du_1 \xrightarrow{z \rightarrow \infty} \frac{1}{z^2} \frac{\hat{t}_1 d\hat{t}_1}{t_n^2} u_n^{\frac{d-1}{d}} \hat{u}_1^{d-1} d\hat{u}_1$$

so that the measure decays as $1/z^2$, while $(1n)$ still behaves as $1/z$ under (3.3) so that the decay is softened to $1/z$ overall. Had we chosen to shift external particles that were not adjacent in the colour ordering, all the $(ii+1)$ brackets would have approached constants as $z \rightarrow \infty$. Representing Yang–Mills amplitudes by $\mathcal{A}_{n,d}$ thus makes it manifest that they behave as $1/z^2$ under BCFW shifts of non-adjacent particles [6, 17]. Once again, this fact is very difficult to see by any other means except the Grassmanian formulation of Yang–Mills amplitudes [3, 25].

⁴We recall that this determinant can be defined to absorb the overall factor of $|1 \dots n|^{-2}$ in (2.6), whereupon the associated Vandermonde determinants are those of the rows and columns that *remain* in Φ .

3.3 Multi-particle factorization

The main ingredient in the proof is multiparticle factorization⁵. Gravitational tree amplitudes have a pole whenever the sum P of any two or more external momenta becomes null, and the residue of this pole is the product of two subamplitudes, summed over the helicities of the particle being exchanged. More specifically, divide the particles into two sets L and R and call

$$P_L := \sum_{i \in L} p_i, \quad P_R := \sum_{j \in R} p_j.$$

Then the amplitude behaves as

$$\mathcal{M}(\Lambda_1, \dots, \Lambda_n) \xrightarrow{P_L^2 \rightarrow 0} \delta^4 \left(\sum_{k=1}^n p_k \right) \int d^8 \eta \widetilde{\mathcal{M}}_L(\{\Lambda_i\}, \Lambda) \frac{1}{P_L^2} \widetilde{\mathcal{M}}_R(-\Lambda, \{\Lambda_j\}) + \dots, \quad (3.5)$$

where $\Lambda_i = \{\lambda_i, \tilde{\lambda}_i, \eta_i\}$ is shorthand for the spinor momenta, and Λ represents the spinor momenta of the internal particle in the strict limit that $P_L^2 = 0$. In this equation, $\widetilde{\mathcal{M}}$ represents an amplitude stripped of its overall (bosonic) momentum δ -function. We can restore these δ -functions by writing

$$\begin{aligned} & \delta^4 \left(\sum_{k=1}^n p_k \right) \int d^8 \eta \widetilde{\mathcal{M}}_L(\{\Lambda_i\}, \Lambda) \frac{1}{P_L^2} \widetilde{\mathcal{M}}_R(-\Lambda, \{\Lambda_j\}) \\ &= \int \frac{d^4 p}{p^2} \delta^4(P_L + p) \widetilde{\mathcal{M}}_L(\{\Lambda_i\}, \Lambda) \delta^4(-p + P_R) \widetilde{\mathcal{M}}_R(-\Lambda, \{\Lambda_j\}) \\ &= \int \langle \lambda d\lambda \rangle d^{2|8} \tilde{\lambda} \frac{ds^2}{s^2} \delta^4(P_L + \lambda \tilde{\lambda} + s^2 q) \widetilde{\mathcal{M}}_L(\{\Lambda_i\}, \Lambda) \delta^4(-\lambda \tilde{\lambda} - s^2 q + P_R) \widetilde{\mathcal{M}}_R(-\Lambda, \{\Lambda_j\}), \end{aligned} \quad (3.6)$$

where in the last line we have parameterized p by

$$p = \lambda \tilde{\lambda} + s^2 q, \quad (3.7)$$

where $\lambda \tilde{\lambda}$ and q are null momenta, with q fixed, and s^2 is a scalar parameter chosen for later convenience. Any 4-momentum may be parametrized this way. Notice also that $p^2 = P_L^2 = s^2 \langle \lambda | q | \tilde{\lambda} \rangle$.

Suppose we approach the factorization channel by taking the limit as $s^2 \rightarrow 0$. If we wish to recover the amplitude (3.5) then the $d^4 p$ integral in the second line of (3.6) should be performed over a copy of real momentum space. However, as the amplitude itself is diverging, it is more sensible to compute the residue of the pole. This may be done by changing the contour in the final line to be an S^1 encircling the pole at $s^2 = 0$, together with an integral over the on-shell phase space of the intermediate particle. One finds

$$\text{Res}_{P_L^2=0} \mathcal{M}(\Lambda_1, \dots, \Lambda_n) = \int \langle \lambda d\lambda \rangle d^{2|8} \tilde{\lambda} \mathcal{M}_L(\{\Lambda_i\}, \Lambda) \mathcal{M}_R(-\Lambda, \{\Lambda_j\}), \quad (3.8)$$

where the δ -functions are now naturally incorporated into the subamplitudes. In particular, this formula shows that the residue itself has no memory of the direction in which the factorization channel was approached.

We must show that $\mathcal{M}_{n,d}$ in (2.6) has the same property. It will actually be convenient first to rewrite the residue on twistor space by transforming the external and internal Λ 's to twistor

⁵Factorization properties of the form (3.4) of Yang–Mills amplitudes were investigated in [21, 31, 34] and the discussions there are closely related to the argument here.

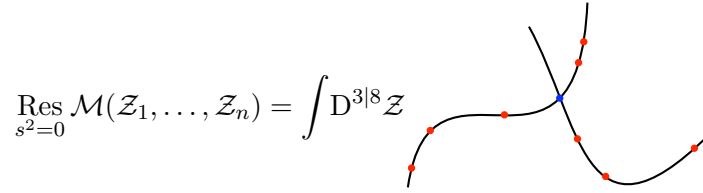


Figure 1. On twistor space, the residue of a factorization channel looks like a nodal curve with the location \mathcal{Z} of the node integrated over.

variables⁶. Doing so, (3.8) becomes [4, 14, 26, 31]⁷

$$\text{Res}_{s^2=0} \mathcal{M}(Z_1, \dots, Z_n) = \int D^{3|8} \mathcal{Z} \wedge \mathcal{M}_L(\{Z_i\}, Z) \wedge \mathcal{M}_R(Z, \{Z_j\}), \quad (3.9)$$

where $\{Z_i\}$ and $\{Z_j\}$ are the sets of twistors associated with external states on the L and R subamplitudes. Notice that on twistor space, $\mathcal{N} = 8$ gravitational amplitudes are homogeneous of degree $+2$ in each of their arguments. Under the assumption (valid at least for 3-particle amplitudes) that these gravitational subamplitudes are associated with curves in twistor space, we see that the residue on a factorization channel corresponds to a nodal curve, with the location \mathcal{Z} of the node integrated over the space (see Fig. 1). Therefore, to prove that $\mathcal{M}_{n,d}$ as given by (2.6) obeys BCFW recursion – and therefore agrees with all tree amplitudes in $\mathcal{N} = 8$ supergravity – we must show both that it has a simple pole on the boundary of the moduli space where the curve degenerates, and further that the residue of this pole is given by (3.9).

A standard way to describe the decomposition of a rational curve into a nodal curve is to introduce a complex parameter s and model the rational curve as the conic⁸

$$\Sigma_s = \{xy = s^2 z^2\} \subset \mathbb{CP}^2,$$

where (x, y, z) are homogeneous coordinates on the complex projective plane. The homogeneous coordinates $\sigma^\alpha = (\sigma^0, \sigma^1)$ intrinsic to the \mathbb{CP}^1 worldsheet are related to these coordinates by

$$(x, y, z) = \left((\sigma^0)^2, (\sigma^1)^2, \frac{\sigma^0 \sigma^1}{s} \right).$$

The degeneration of the curve is controlled by the parameter s^2 , which we will show is the same parameter as appears in (3.7). In the limit we have

$$\lim_{s \rightarrow 0} \Sigma_s = \Sigma_L \cap \Sigma_R,$$

where the \mathbb{CP}^1 's Σ_L and Σ_R are defined by

$$\Sigma_L = \{y = 0\} \subset \mathbb{CP}^2, \quad \Sigma_R = \{x = 0\} \subset \mathbb{CP}^2$$

so that (z, x) form homogeneous coordinates on Σ_L and (z, y) form homogeneous coordinates on Σ_R . The good homogeneous coordinates intrinsic to $\Sigma_{L,R}$ are therefore

$$\sigma_L^\alpha = (z, x) = \sigma^0 (\sigma^1/s, \sigma^0), \quad \sigma_R^\alpha = (z, y) = \sigma^1 (\sigma^0/s, \sigma^1)$$

⁶This is implemented by use of wave functions $h_i = \delta^{3|8}(Z, Z_i) := \int \delta^{4|8}(Z_i - tZ) dt/t^3$ in (2.6) instead of momentum eigenstates; in split signature this is the half-Fourier transform of the momentum-space version, see discussion around (5.3).

⁷We do not distinguish the symbol \mathcal{M} for momentum space amplitudes from that of twistor space ones. Which is meant should be clear from the context.

⁸We emphasize that this is a model for the abstract worldsheet Σ before it is mapped to twistor space.

and the affine coordinate $u = \sigma^\perp/\sigma^0$ on Σ_s is related to the affine coordinates $u_{L,R}$ on $\Sigma_{L,R}$ by

$$u_L = \frac{s}{u}, \quad u_R = su. \quad (3.10)$$

With this choice of coordinates, the node $\Sigma_L \cap \Sigma_R$ is the point $x = y = 0 \in \mathbb{CP}^2$ and is also at the origin in each of the affine coordinates $u_{L,R}$.

As the curve degenerates, the n marked points distribute themselves among the component curves $\Sigma_{L,R}$, with at least two of these points on each curve component. In the degeneration limit, any such distribution defines a boundary divisor in the moduli space $\overline{M}_{0,n}$ of n -pointed rational curves, with the locations of the marked points considered up to $\mathrm{SL}(2; \mathbb{C})$ transformations. The parameter s^2 is then a coordinate transverse to this boundary divisor, which lies at $s^2 = 0$.

Ordinarily, we think of coordinates on $\overline{M}_{0,n}$ as given by a choice of $n - 3$ independent cross-ratios of the marked points. No choice of these cross ratios provides coordinates globally on $\overline{M}_{0,n}$, but we can always make a choice such that a particular boundary divisor arises when one or more cross-ratios approach zero, so that in some conformal frame the marked points in the numerator of these cross ratios are colliding. To relate this description to s^2 , consider the cross-ratios

$$x_k := \frac{(1k)(n-1n)}{(n1)(kn-1)}, \quad (3.11)$$

where without loss of generality we assume that our boundary divisor has $1 \in L$ and $n-1, n \in R$. To study the degeneration, marked points should be described in terms of the coordinates u_L or u_R as appropriate. Using

$$u_i - u_j = \begin{cases} s \frac{u_{jL} - u_{iL}}{u_{iL}u_{jL}}, & i, j \in L, \\ \frac{u_{iR} - u_{jR}}{s}, & i, j \in R, \\ \frac{s^2 - u_{iL}u_{jR}}{su_{jR}}, & i \in L, \quad j \in R \end{cases} \quad (3.12)$$

we see that

$$x_i = s^2 \frac{(u_{1L} - u_{iL})(u_{n-1R} - u_{nR})}{u_{1L}u_{iL}u_{nR}u_{n-1R}} + \mathcal{O}(s^4)$$

when $i \in L$, whereas

$$x_j = \frac{u_j(u_{n-1R} - u_{nR})}{u_{nR}(u_{jR} - u_{n-1R})} + \mathcal{O}(s^2)$$

when $j \in R$. Consequently, as we approach the boundary divisor, any ratio x_i/x_j with $i \in L$ and $j \in R$ will vanish as s^2 , whereas any such ratio with i and j limiting onto the same curve components remains finite, provided we approach a generic point of the boundary divisor (i.e., we only consider a single degeneration). We can now extract s^2 by defining rescaled cross-ratios y_i by

$$x_i =: s^2 y_i \quad \text{for } i \in L,$$

where the y_i are to be considered only up to an overall scaling.

The factor

$$d\mu := \frac{1}{\mathrm{vol}(\mathrm{SL}(2; \mathbb{C}))} \prod_{i=1}^n (\sigma_i d\sigma_i)$$

provides meromorphic top form on the moduli space. This form cannot be written in terms of the cross-ratios alone since it has non-zero homogeneity in each of the σ s. However, fixing the $\text{SL}(2 : \mathbb{C})$ by freezing 1, $n - 1$ and n , at least locally we can write

$$\frac{1}{\text{vol}(\text{SL}(2; \mathbb{C}))} \prod_{i=1}^n (\sigma_i d\sigma_i) = f(u_i) \prod_{i=2}^{n-2} dx_i = f(u_i) \prod_{\substack{i \in L \\ i \neq 1}} dx_i \prod_{\substack{j \in R \\ j \neq n-1, n}} dx_j \quad (3.13)$$

in terms of $n - 3$ of the cross-ratios (3.11) and where the function⁹

$$f(u_i) := \left[\prod_{i \neq n-1} (i n - 1)^2 \right] \left[\frac{(1n)}{(n-1n)(1n-1)} \right]^{n-2}$$

absorbs the homogeneity.

Upon transforming to the new coordinates, for $i \in L$ we wish to replace the x_i cross-ratios by s^2 and y_i , with the y_i treated projectively. To leading order in s^2 , the measure for these L cross-ratios becomes

$$\prod_{\substack{i \in L \\ i \neq 1}} dx_i = s^{2(n_L-2)} ds^2 \epsilon^{i_1 i_2 \dots i_{n_L-1}} y_{i_1} dy_{i_2} \dots dy_{i_{n_L-1}} = s^{2(n_L-2)} ds^2 y_2 \prod_{\substack{i \in L \\ i \neq 1, 2}} dy_i,$$

where in the second line we work on the affine patch $y_2 = \text{const}$. The measure for the R cross-ratios is s -independent and stays as in (3.13). To put this measure in a more familiar form, we now undo the transformation (3.13) separately on the left and on the right, finding

$$\begin{aligned} \prod_{\substack{i \in L \\ i \neq 1, 2}} dy_i &= \left[\frac{(u_{n-1R} - u_{nR})}{u_{n-1R} u_{nR}} \right]^{n_L-2} \prod_{\substack{i \in L \\ i \neq 1, 2}} \frac{du_{iL}}{u_{iL}^2}, \\ \prod_{\substack{j \in R \\ j \neq n-1, n}} dx_j &= \left[(u_{n-1R} - u_{nR}) \frac{u_{n-1R}}{u_{nR}} \right]^{n_R-2} \prod_{\substack{j \in R \\ j \neq n-1, n}} \frac{du_{jR}}{(u_{jR} - u_{n-1R})^2}. \end{aligned}$$

Putting all the pieces together, we have shown that in a neighbourhood of the boundary divisor defined by $s^2 = 0$, the measure for the integration over marked points may be written as

$$\begin{aligned} d\mu &= s^{n_L - n_R - 4} ds^2 \left[\frac{1}{\text{vol}(\text{SL}(2; \mathbb{C}))} \prod_{i \in L \cup \bullet} du_{iL} \right] \left[\frac{1}{\text{vol}(\text{SL}(2; \mathbb{C}))} \prod_{j \in R \cup \bullet} du_{jR} \right] \times \prod_{i \in L} \frac{1}{u_{iL}^2} \\ &:= s^{n_L - n_R - 4} ds^2 d\mu_L d\mu_R \times \prod_{i \in L} \frac{1}{u_{iL}^2}, \end{aligned} \quad (3.14)$$

where we recall that with the coordinates (3.10) assumed a gauge fixing in which the node was at the origin in each curve component. Bearing in mind that the boundary divisor is naturally the product $\overline{M}_{0, n_L+1} \times \overline{M}_{0, n_R+1}$ of the moduli spaces for rational curves with fewer marked points, the factors in square brackets are precisely the expected $(n_{L,R} + 1) - 3$ forms on these spaces. The form ds^2 is normal to this boundary divisor. Thus, to find the residue of our proposed gravity amplitude $\mathcal{M}_{n,d}$ in a factorization channel, we must interpret the contour in (2.6) to

⁹More precisely, $\frac{1}{\text{vol}(\text{SL}(2; \mathbb{C}))} \prod (\sigma_i d\sigma_i)$ is an $n - 3$ form with values in $\otimes_i \mathbb{L}_i^{-1}$, where \mathbb{L}_i is the tautological line bundle on $\overline{M}_{0,n}$ whose fibre is the holomorphic cotangent space of Σ at i . The function $f(u_i)$ is thus really a section of $\otimes_i \mathbb{L}_i^{-1}$.

include an S^1 factor that encircles the boundary divisor $s^2 = 0$, and use this contour to compute the ds^2 integral. Of course, until we study the rest of the integrand in $\mathcal{M}_{n,d}$, it is not clear that we actually have a simple pole there.

Armed with this description of a neighbourhood of the factorization channel, we now investigate the behaviour of the rest of the integrand. Begin by considering the map $\mathcal{Z} : \Sigma_s \rightarrow \mathbb{P}\mathbb{T}$. It is useful to pull out a factor of u^{d_L} and write

$$\begin{aligned} \mathcal{Z}(u, s) &= u^{d_L} \left(\sum_{a=1}^{d_L} \mathcal{Z}'_{d_L-a} u^{-c} + \mathcal{Z}_\bullet + \sum_{b=1}^{d_R} \mathcal{Z}'_{d_L+b} u^b \right) = u^{d_L} \left(\sum_{a=1}^{d_L} \mathcal{Z}_a u_L^a + \mathcal{Z}_\bullet + \sum_{b=1}^{d_R} \mathcal{Z}_b \frac{s^{2b}}{u_L^b} \right) \\ &= u^{d_L} \left(\sum_{a=1}^{d_L} \mathcal{Z}_a \frac{s^{2a}}{u_R^a} + \mathcal{Z}_\bullet + \sum_{b=1}^{d_R} \mathcal{Z}_b u_R^b \right) \end{aligned} \quad (3.15)$$

with the second or third lines the appropriate description for particles limiting onto $\Sigma_{L,R}$, respectively. The coefficients \mathcal{Z}_\bullet , \mathcal{Z}_a and \mathcal{Z}_b are related to the original map coefficients \mathcal{Z}'_c by

$$\mathcal{Z}_a := s^a \mathcal{Z}'_{d_L-a}, \quad \mathcal{Z}_\bullet := \mathcal{Z}'_{d_L}, \quad \mathcal{Z}_b := s^b \mathcal{Z}'_{d_L+b}. \quad (3.16)$$

This shows that as $s^2 \rightarrow 0$, the twistor curve $\mathcal{Z}(\Sigma)$ degenerates into a pair of curves $\mathcal{Z}(\Sigma_L)$ and $\mathcal{Z}(\Sigma_R)$ that are the images of degree $d_{L,R}$ maps, respectively, where¹⁰ $d_L + d_R = d$.

As shown in [34], the δ -functions involving λ_i that are already present in the external wavefunctions combine with those in $\tilde{\lambda}_i$ that are generated by Fourier transforming to momentum space to enforce

$$\sum_{i \in L} \lambda_i \tilde{\lambda}_i - \lambda_\bullet \sum_{i \in L} t_i \tilde{\lambda}_i = s^2 \rho \sum_{i \in L} t_i \tilde{\lambda}_i \frac{1}{u_{iL}} + \mathcal{O}(s^4),$$

where ρ is the λ -component of a map coefficient that limits onto Σ_R . This shows that, as in (3.8), (3.9), a factorization channel in momentum space corresponds to a nodal curve in twistor space, with the same parameter s^2 governing both degenerations.

We can account for the various factors of u^{d_L} and the rescalings in (3.16) as follows. Firstly, unlike in the $\mathcal{N} = 4$ Calabi–Yau case the $\mathcal{N} = 8$ measure is not invariant under the rescaling (3.16) of the map coefficients, but rather behaves as

$$\prod_{c=0}^d d^{4|8} \mathcal{Z}'_c = s^{-2d_L(d_L+1)} s^{-2d_R(d_R+1)} \times d^{4|8} \mathcal{Z}_\bullet \prod_{a=1}^{d_L} d^{4|8} \mathcal{Z}_a \prod_{b=1}^{d_R} d^{4|8} \mathcal{Z}_b. \quad (3.17)$$

Secondly, bearing in mind that the wavefunctions and matrix elements each depend homogeneously on $\mathcal{Z}(u)$, we can treat the map purely as the terms in parentheses in (3.15) provided we also make the replacements

$$h_i(\mathcal{Z}(u_i)) \rightarrow u_i^{2d_L} h_i(\mathcal{Z}(u_i)), \quad \Phi_{ij} \rightarrow (u_i u_j)^{d_L} \Phi_{ij}, \quad \tilde{\Phi}_{ij} \rightarrow \frac{1}{(u_i u_j)^{d_L}} \tilde{\Phi}_{ij}. \quad (3.18)$$

We do this henceforth. In terms of the new coordinates, the product of the replaced wavefunctions becomes

$$\prod_{i=1}^n u_i^{2d_L} h_i(\mathcal{Z}(u_i)) = s^{2d_L(n_L - n_R)} \prod_{i \in L} \frac{h_i(\mathcal{Z}(u_{iL}))}{u_i^{2d_L}} \prod_{j \in R} u_j^{2d_L} h_j(\mathcal{Z}(u_{jR})) \quad (3.19)$$

to leading order in s^2 .

¹⁰By forgetting the data of the map, the moduli space $\overline{M}_{0,n}(\mathbb{P}\mathbb{T}, d)$ of degree d rational maps from an n -pointed curve to $\mathbb{P}\mathbb{T}$ admits a morphism to $\overline{M}_{0,n}$. As we see in the text, a boundary divisor in $\overline{M}_{0,n}(\mathbb{P}\mathbb{T}, d)$ is specified by a boundary divisor in $\overline{M}_{0,n}$, together with choices of degree d_L and degree d_R maps on the two curve components, with $d_L + d_R = d$.

The node itself is mapped to $\mathcal{Z}_\bullet \in \mathbb{P}\mathbb{T}$. It will be convenient to be able to treat the node separately on the two curve components. For this, we introduce a factor

$$1 = \int D^{3|8} \mathcal{Y}_\bullet D^{3|8} \mathcal{Z} \frac{dt}{t} \frac{dr}{r^5} \bar{\delta}^{4|8}(\mathcal{Z} - t\mathcal{Z}_\bullet) \bar{\delta}^{4|8}(\mathcal{Z} - r\mathcal{Y}_\bullet) \quad (3.20)$$

into the integrand of $\mathcal{M}_{n,d}$. To understand this factor, first note that the powers of scaling parameters t and r in the measure are chosen to ensure the whole expression has no weight in any of the three twistors. The integrals can all be performed against the δ -functions, which simply freeze \mathcal{Y}_\bullet to $(t/r)\mathcal{Z}_\bullet$. Now, whenever we describe a particle in R , we write the map as

$$\mathcal{Z}(u_R) = \frac{t}{r} \left(\mathcal{Y}_\bullet + \sum_{b=1}^{d_R} \mathcal{Y}_b u_R^b + \mathcal{O}(s^2) \right),$$

where

$$\mathcal{Y}_b := \frac{r}{t} \mathcal{Z}_b \quad (3.21)$$

are a further rescaling of the d_R map coefficients \mathcal{Z}_b . Note that we do not rescale the d_L coefficients \mathcal{Z}_a . Pulling out this factor of r/t from all the wavefunctions $h_j(\mathcal{Z})$ with $j \in R$, and from all the rows and columns of Φ and $\tilde{\Phi}$ corresponding to particles in R , and also changing the \mathcal{Z}_b measure into that for \mathcal{Y}_b leads to a factor of $(r/t)^2$. In the original formula (2.6), we divided by $\text{vol}(\mathbb{C}^*)$ to account for an overall rescaling of the map coefficients. However, as a consequence of (3.21), the new d_R map coefficients \mathcal{Y}_b are no longer locked to scale like $\{\mathcal{Z}_\bullet, \mathcal{Z}_a\}$ but instead are locked to scale like \mathcal{Y}_\bullet . This factor combines beautifully with the factors in the measure (3.20) to convert those integrals into our standard $\bar{\delta}^{3|8}$'s of homogeneity +2 in each entry. These δ -functions can thus be treated as ‘external data’ for the node. Thus, as in (3.9), as $s^2 \rightarrow 0$ the map degenerates into two independent maps from $(n_{L,R} + 1)$ -pointed curves $\Sigma_{L,R}$, each described by $d_{L,R} + 1$ twistor coefficients, with a point $\bullet_{L,R}$ on each curve mapped to the same point \mathcal{Z} in the target space. The final integral $D^{3|8} \mathcal{Z}$ allows this twistor to be anywhere, just as in the residue calculation (3.9).

Now that we have described the degeneration, we must show that (2.6) has a simple pole there, with the correct residue. Our first aim is to show that to leading order in s^2 , the matrices Φ and $\tilde{\Phi}$ become block diagonal so that their determinants naturally factor into a product of determinants for $\Phi_{L,R}$ and for $\tilde{\Phi}_{L,R}$. Consider first Φ and assume that we choose the $\tilde{d} + 1$ reference points p_r on the diagonal in (2.4) so that $\tilde{d}_{L,R}$ limit onto $\Sigma_{L,R}$, where

$$\tilde{d}_{L,R} := (n_{L,R} + 1) - d_{L,R} - 2.$$

so that

$$\tilde{d} = \tilde{d}_L + \tilde{d}_R.$$

The remaining marked point is chosen to be the node, viewed as being on the *right* when we consider diagonal elements Φ_{ii} with $i \in L$, and on the *left* for diagonal elements Φ_{jj} with $j \in R$. Specifically, we have

$$\begin{aligned} \Phi_{ij} &= \frac{\langle \lambda(u_i) \lambda(u_j) \rangle}{(ij)} (u_i u_j)^{d_L}, \\ \Phi_{ii} &= - \sum_{j \neq i} \Phi_{ij} \prod_{l=1}^{\tilde{d}_L} \frac{(plj)}{(pli)} \prod_{r=1}^{\tilde{d}_R} \frac{(prj)}{(pri)} \frac{(\bullet j)}{(\bullet i)} \frac{\prod_{k \neq i} (ki)}{\prod_{m \neq j} (mj)} \end{aligned}$$

in terms of the original coordinates, where we have accounted for the factors in (3.18).

Using (3.12) to transform to the limit coordinates, we find that Φ can be written as

$$\begin{aligned} \Phi_{ij} &= \frac{\langle \lambda(u_i) \lambda(u_j) \rangle}{(u_i - u_j)} \frac{s^{2d_L-1}}{(u_i u_j)^{d_L-1}} \Big|_L, \\ \Phi_{ii} &= - \sum_{\substack{j \in L \cup \bullet \\ j \neq i}} \left(\Phi_{ij} \left(\frac{u_j}{u_i} \right)^{d_L-1} \prod_{l=1}^{\tilde{d}_L} \frac{(u_l - u_j)(u_\bullet - u_j)}{(u_l - u_i)(u_\bullet - u_i)} \frac{\prod_{\substack{k \in L \cup \bullet \\ k \neq i}} (u_k - u_i)}{\prod_{\substack{m \in L \cup \bullet \\ m \neq j}} (u_m - u_j)} \right) + \mathcal{O}(s^2) \end{aligned} \quad (3.22)$$

when $i, j \in L$, and where the subscript L on means we are using limiting coordinates appropriate for L throughout. Similarly

$$\begin{aligned} \Phi_{ij} &= \frac{\langle \lambda(u_i) \lambda(u_j) \rangle}{u_i - u_j} \frac{(u_i u_j)^{d_L}}{s^{2d_L-1}} \Big|_R, \\ \Phi_{ii} &= - \sum_{\substack{j \in R \cup \bullet \\ j \neq i}} \left(\Phi_{ij} \left(\frac{u_i}{u_j} \right)^{d_L} \prod_{r=1}^{\tilde{d}_R} \frac{(u_r - u_j)(u_\bullet - u_j)}{(u_r - u_i)(u_\bullet - u_i)} \frac{\prod_{\substack{k \in R \cup \bullet \\ k \neq i}} (u_k - u_i)}{\prod_{\substack{m \in R \cup \bullet \\ m \neq j}} (u_m - u_j)} \right) + \mathcal{O}(s^2) \end{aligned} \quad (3.23)$$

when $i, j \in R$ and again we use the R limiting coordinates. Once we extract a power of $1/u_{iL}^{d_L-1}$ from each row and column of (3.22), a power of $u_{iR}^{d_L}$ from each row and column of (3.23) and powers of s from both, these matrices are exactly of the form $\Phi_{L,R}$ for the subamplitudes. Note that in both cases, we have extended the sum on the diagonal term to include the node (located at $u_\bullet = 0$ in our coordinates). This is possible because the choice of the node as a reference point means this term is zero. While $\Phi_{L,R}$ as given here are $n_{L,R} \times n_{L,R}$ matrices (rather than $(n_{L,R} + 1) \times (n_{L,R} + 1)$ matrices), they still have the expected rank $d_{L,R}$, because in each case we were forced to choose one of the reference points to be the node. It is as if the row and column corresponding to the internal particle have ‘already’ been removed.

The off-block-diagonal terms Φ_{ij} with $i \in L, j \in R$ are of the same order in s^2 as the R block diagonal ones in (3.23). Therefore, the leading term in the reduced determinant comes from the block diagonal terms. After also changing variables $u \rightarrow u_{L,R}$ in the Vandermonde determinants, a straightforward but somewhat tedious calculation shows that

$$|\Phi'| = \frac{s^{(2d_L-1)(d_L-d_R)} s^{d_R(d_R-1)}}{s^{-2d_L d_R} s^{d_L(d_L-1)}} |\Phi_L'| |\Phi_R'| + \text{higher order}, \quad (3.24)$$

exactly as required for a product of two subamplitudes, times an overall power of s .

In an exactly parallel computation, transforming $\tilde{\Phi}$ into the L, R coordinates shows that

$$|\tilde{\Phi}'| = \frac{s^{d_R(d_R+1)} s^{2(d_R+1)(d_L+1)}}{s^{d_L(d_L+1)} s^{(2d_L+1)(\tilde{d}_L-\tilde{d}_R)}} |\tilde{\Phi}_L'| |\tilde{\Phi}_R'| \frac{\prod_{i \in L} u_{iL}^{2(d_L+1)}}{\prod_{j \in R} u_{jR}^{2d_L}} \quad (3.25)$$

to leading order in s^2 . Once again the matrices $\tilde{\Phi}_L$ and $\tilde{\Phi}_R$ are precisely as they should be for the left and right subamplitudes, where again we choose the node as one of the reference points.

After these somewhat lengthy calculations, we are finally in position to compute the residue of $\mathcal{M}_{n,d}$ on the boundary of the moduli space corresponding to a factorization channel. First,

collecting powers of s^2 from equations (3.14), (3.17), (3.19), (3.24) and (3.25), a near miraculous cancellation occurs, leaving simply

$$ds^2 \left(\frac{1}{s^2} (\dots) + \mathcal{O}(s^0) \right),$$

showing that the integrand of (2.6) indeed has a simple pole on boundary divisors in the moduli space. Combining all the pieces, the residue of this simple pole is

$$\int D^{3|8} \mathcal{Z} \left[\int d\mu_L \frac{d^{4|8} \mathcal{Z}_\bullet \prod_{a=1}^{d_L} d^{4|8} \mathcal{Z}_a}{\text{vol}(\mathbb{C}^*)} |\Phi_L|' |\tilde{\Phi}_L|' \prod_{i \in L} h_i(\mathcal{Z}(u_{iL})) \bar{\delta}^{3|8}(\mathcal{Z}(u_{\bullet L}), \mathcal{Z}) \right. \\ \left. \times \int d\mu_R \frac{d^{4|8} \mathcal{Y}_\bullet \prod_{b=1}^{d_R} d^{4|8} \mathcal{Y}_b}{\text{vol}(\mathbb{C}^*)} |\Phi_R|' |\tilde{\Phi}_R|' \prod_{j \in R} h_j(\mathcal{Z}(u_{jR})) \bar{\delta}^{3|8}(\mathcal{Z}(u_{\bullet R}), \mathcal{Z}) \right],$$

or in other words exactly the residue

$$\int D^{3|8} \mathcal{Z} \mathcal{M}_L(\{\mathcal{Z}_{i \in L}\}, \mathcal{Z}) \mathcal{M}_R(\mathcal{Z}, \{\mathcal{Z}_{j \in R}\})$$

of the gravitational scattering amplitude.

We have now shown that $\mathcal{M}_{n,d}$ as given by equation (2.6) produces the correct seed amplitudes for BCFW recursion, has the correct $1/z^2$ decay as the BCFW shift parameter $z \rightarrow \infty$ and has a simple pole on any physical factorization channel, with residue correctly given by the product of two subamplitudes on either side of the factorization, integrated over the phase space of the intermediate state.

The only remaining thing to check is that in momentum space, $\mathcal{M}_{n,d}$ has no unwanted *unphysical* poles. This is straightforward. A simple dimension count of integrals versus constraints shows that, as for Yang–Mills [30], when evaluated on momentum eigenstates, $\mathcal{M}_{n,d}$ is inevitably a rational function of the spinor momenta. Thus the only possible singularities are poles. Any unphysical poles in $\mathcal{M}_{n,d}$ which carry some helicity weight would be detected by taking one of the external momenta to become soft. Unphysical “multiparticle” poles, i.e. poles that carry no helicity weight, would also be detected by sequentially taking many particles to become soft. However, the soft limits of $\mathcal{M}_{n,d}$ have recently been checked to agree with those of gravity [13]. We therefore conclude that $\mathcal{M}_{n,d}$ indeed obeys the correct BCFW recursion relation, and have thus demonstrated that it computes all tree amplitudes in $\mathcal{N} = 8$ supergravity.

4 Parity invariance

One of the pleasing features of using (2.6) to describe gravitational scattering amplitudes is that the way these amplitudes break conformal symmetry becomes completely explicit: it arises purely from the infinity twistors $\langle \ , \ \rangle$ and $[\ , \]$ and in Φ and $\tilde{\Phi}$, respectively. On the other hand, parity transformations are not manifest, because parity exchanges twistor space with the dual twistor space. For example, the twistor space $\mathbb{CP}^{3|\mathcal{N}}$ of conformally flat space-time is exchanged with the dual projective space $\mathbb{CP}^{3|\mathcal{N}^*}$. On the original \mathcal{Z} twistor space, $[\ , \]$ is a differential operator while $\langle \ , \ \rangle$ is multiplicative, so the role of these brackets are interchanged under parity. We see this change of roles quite transparently at the level of amplitudes: a parity transformation flips the helicities of all external states, so it exchanges $d \leftrightarrow \tilde{d}$, and one of the key observations

of [15] was that the n -particle N^{d-1} MHV amplitude $\mathcal{M}_{n,d}$ is a monomial of degree d in $\langle \cdot, \cdot \rangle$ and of degree \tilde{d} in $[\cdot, \cdot]$. This strongly suggests that the determinants of Φ and $\tilde{\Phi}$, which hitherto have seemed very different, are naturally parity conjugates of each other. Let's now see this explicitly.

Acting on either momentum or twistor eigenstates, the matrix Φ has elements

$$\Phi_{ij} = \frac{\langle ij \rangle}{(ij)} \frac{1}{t_i t_j}, \quad \Phi_{ii} = - \sum_{j \neq i} \Phi_{ij} \prod_{r=0}^{\tilde{d}} \frac{(jp_r)}{(ip_r)} \frac{\prod_{k \neq i} (ik)}{\prod_{l \neq j} (jl)}.$$

To bring this to the form of $\tilde{\Phi}$, consider making the change of variables $t_i \rightarrow s_i$, defined by

$$t_i s_i := \frac{1}{\prod_{j \neq i} (ij)}. \quad (4.1)$$

This transformation of the scaling parameters played a key role in studying the behaviour of the connected prescription for $\mathcal{N} = 4$ SYM under parity [30, 35]; its relation to a parity transformation will be reviewed below. Under this change of variables, we find

$$\Phi^{(d)}(\langle \cdot, \cdot \rangle, t) = A \circ \tilde{\Phi}^{(\tilde{d})}(\langle \cdot, \cdot \rangle, s) \circ A, \quad (4.2)$$

where $\Phi^{(d)}$ is our usual Φ matrix on a degree d curve¹¹, and $\tilde{\Phi}^{(\tilde{d})}(\langle \cdot, \cdot \rangle, s)$ is the $\tilde{\Phi}$ matrix appropriate for a degree \tilde{d} curve. We also make the replacement $[\cdot, \cdot] \rightarrow \langle \cdot, \cdot \rangle$ in $\tilde{\Phi}^{(\tilde{d})}$. Finally, A is the diagonal matrix whose j^{th} entry is the product $\prod_{k \neq j} (jk)$. Acting with A as in (4.2)

multiplies the rows and columns of $\tilde{\Phi}$ by this product, which accounts for the denominator in (4.1).

In equation (A.3), we saw that $\det'(\Phi)$ and $\det'(\tilde{\Phi})$ behave just as usual determinants under matrix multiplication. In the present case we have

$$\begin{aligned} \det' \left(\Phi^{(d)}(\langle \cdot, \cdot \rangle, t) \right) &= \det' \left(A \circ \tilde{\Phi}^{(\tilde{d})}(\langle \cdot, \cdot \rangle, s) \circ A \right) = (\det A)^2 \det' \left(\tilde{\Phi}^{(\tilde{d})}(\langle \cdot, \cdot \rangle, s) \right) \\ &= |1 \dots n|^4 \det' \left(\tilde{\Phi}^{(\tilde{d})}(\langle \cdot, \cdot \rangle, s) \right). \end{aligned} \quad (4.3)$$

Similarly, if we start from $\tilde{\Phi}^{(d)}([\cdot, \cdot], t)$ and make the same change of variables (4.1), then reading this equation backwards gives

$$\det' \left(\tilde{\Phi}^{(d)}([\cdot, \cdot], t) \right) = |1 \dots n|^{-4} \det' \left(\Phi^{(\tilde{d})}([\cdot, \cdot], s) \right)$$

so that the product is

$$\det' \left(\Phi^{(d)}(\langle \cdot, \cdot \rangle, t) \right) \det' \left(\tilde{\Phi}^{(d)}([\cdot, \cdot], t) \right) = \det' \left(\tilde{\Phi}^{(\tilde{d})}(\langle \cdot, \cdot \rangle, s) \right) \det' \left(\Phi^{(\tilde{d})}([\cdot, \cdot], s) \right),$$

with no extra factors. Note that the roles of Φ and $\tilde{\Phi}$ have been exchanged, along with the exchanges $\langle \cdot, \cdot \rangle \leftrightarrow [\cdot, \cdot]$ and $d \leftrightarrow \tilde{d}$.

As mentioned above, in [30, 35] it was shown that the parity transformation of all the other factors in the $\mathcal{N} = 4$ SYM tree amplitudes $\mathcal{A}_{n,d}$ conspire to produce the transformation (4.1) of scaling parameters. In $\mathcal{N} = 4$ SYM, the measure for the scaling parameters themselves behaves as

$$\frac{dt_i}{t_i} \propto \frac{ds_i}{s_i}$$

¹¹The degree affects the matrices only through the diagonal elements.

under (4.1), with a proportionality factor that is cancelled by the transformation of the fermions. In $\mathcal{N} = 8$ supergravity, the scaling parameters' measure is

$$\frac{dt_i}{t_i^3} \propto ds_i s_i$$

instead, but now the transformation of the $\mathcal{N} = 8$ fermions provides an extra factor of $1/s_i^4$. Exactly the same arguments as given in [30, 35] thus establish the parity symmetry of our formulation of $\mathcal{M}_{n,d}$. Rather than simply repeat those arguments verbatim, we instead make parity manifest by recasting the integral (2.6) in terms of the link variables introduced in [4] for $\mathcal{N} = 4$ SYM.

5 Gravity and the Grassmannian

The aim of this section is to write the tree amplitude $\mathcal{M}_{n,d}$ as an integral over the Grassmannian $G(k, n)$ (with $k = d + 1$) along the lines of [2, 10, 14, 18, 20, 28, 32] for the connected prescription of $\mathcal{N} = 4$ SYM. The most obvious reason to perform this transformation is that as an integral over $G(k, n)$, all δ -function constraints involving external data become linear in the variables and hence trivial to perform. The price for such a simplification is that the number of integrations variables is larger than before. The difference in the number of variables is $(k - 2)(n - k - 2)$ and hence the amplitude becomes a multidimensional contour integral over that many variables.

In $\mathcal{N} = 4$ super Yang–Mills something remarkable happens: repeated applications of the global residue theorem transform the integral into one where all variables can be solved for from linear systems of equations [2, 10, 18, 19, 20, 28, 32]. Computationally this is a major advantage, but it also gives a conceptual advantage because individual residues computed after the application of the global residue theory coincide precisely with BCFW terms and hence, in Yang–Mills, leading singularities of the theory. One can then write down a generating function for all leading singularities [3, 25] that control the behaviour of the theory to all orders in perturbation theory and which has allowed the development of recursion relations for the all loop integrand [1].

These remarkable properties of the Grassmannian formulation of $\mathcal{N} = 4$ SYM should provide sufficient motivation to explore the same avenues in gravity.

It is important to realize that the existence of a Grassmannian formulation *per se* has nothing to do with $\mathcal{N} = 4$ SYM, or Yangian invariance, or even twistors. Rather, it is a completely general consequence of dealing with degree d holomorphic maps from an n -pointed rational curve. To see this [14], recall that we can describe the map $\mathcal{Z} : \mathbb{CP}^1 \rightarrow \mathbb{PT}$ by picking a basis $\{P_0(\sigma), \dots, P_d(\sigma)\}$ of $d + 1$ linearly independent degree d polynomials in the worldsheet coordinates and expanding

$$Z(\sigma) = \sum_{a=0}^d \mathcal{Z}_a P_a(\sigma).$$

The space of such polynomials is $H^0(\mathbb{CP}^1, \mathcal{O}(d)) \cong \mathbb{C}^k$. Given n marked points on the worldsheet, we would like to define a natural embedding of this \mathbb{C}^k into \mathbb{C}^n by ‘evaluating’ each of the $P_a(\sigma)$ at each of the marked points. This can be done once we fix a scale for σ at each marked point – in other words, once we pick a trivialization of $\mathcal{O}(d)$ at each of the σ_i . This is exactly the role of the scaling parameters t_i . Thus, for every choice of n marked points and n scaling parameters, our map defines a k -plane in \mathbb{C}^n , i.e. a point in the Grassmannian $G(k, n)$.

As we integrate over the moduli space of rational maps, we sweep out a $2(n-2)$ -dimensional subvariety of $G(k, n)$. This dimension arises as

$$2(n-2) = (n-3) + (n) + (-1), \quad (5.1)$$

where $(n-3)$ parameters come from the locations of the marked points up to worldsheet $SL(2; \mathbb{C})$ transformations, a further n parameters are the scaling parameters t_i and we lose one parameter from overall rescaling. (Equivalently, we have $2n$ parameters from both components of the worldsheet coordinates σ_i^α , minus four from the quotient by $GL(4; \mathbb{C})$.)

The precise subvariety we obtain may be characterized as follows [2, 14]. The map from the worldsheet to the space of degree d polynomials, considered up to an overall scale, is of course the Veronese map

$$\mathcal{V} : \mathbb{CP}^1 \rightarrow \mathbb{CP}^d. \quad (5.2)$$

The subvariety of the Grassmannian we sweep out is therefore defined by the condition that the n different k -vectors we get by evaluating our polynomials do not simply span a k -plane through the origin in \mathbb{C}^n , or equivalently a $\mathbb{CP}^d \subset \mathbb{CP}^{n-1}$ but rather lie in the image of the Veronese map to that \mathbb{CP}^d . As shown in [2, 10, 18, 19, 20, 28, 32] and obtained again below, this condition amounts to the vanishing of $(d-1)(\tilde{d}-1)$ quartics in the Plücker coordinates of the Grassmannian. Note that

$$\dim G(d+1, n) - (d-1)(\tilde{d}-1) = 2(n-2)$$

giving the dimension expected from (5.1). On transforming to momentum space, the external data specifies $2(n-2)$ divisors in $G(k, n)$ defined by those k -planes in \mathbb{C}^n that contain the 2-plane specified by the λ_i and are orthogonal to the 2-plane specified by the $\tilde{\lambda}_i$. The intersection number of these divisors with the Veronese subvariety is believed to be $\langle \binom{n-3}{k-2} \rangle$, where $\langle \binom{p}{q} \rangle$ is the (p, q) Eulerian number [30].

All the above features of the Grassmannian formulation should thus be common to both $\mathcal{N} = 4$ super Yang–Mills and $\mathcal{N} = 8$ supergravity, purely as a consequence of their having a description in terms of degree d rational maps to twistor space. Of course, the detailed form of the measure on the Grassmannian will be different in the two cases, coming from the external wavefunctions, and from the Parke–Taylor worldsheet denominator in Yang–Mills and from $|\Phi|'$ and $|\tilde{\Phi}|'$ in gravity.

Let us now construct the Grassmannian formulation of tree amplitudes in $\mathcal{N} = 8$ supergravity. We will choose our external wavefunctions to be either twistor or dual twistor eigenstates. More precisely we choose exactly $d+1$ of the wavefunctions to be

$$h_a(\mathcal{Z}(\sigma_a)) = \int \frac{dt_a}{t_a^3} \delta^4 |8(\mathcal{Z}_a - t_a \mathcal{Z}(\sigma_a)) \quad (5.3)$$

that have support only when $\sigma_a \in \Sigma$ is mapped to $\mathcal{Z}_a \in \mathbb{PT}$. The remaining $\tilde{d}+1$ wavefunctions are chosen to be

$$h_r(\mathcal{Z}(\sigma_r)) = \int \frac{dt_r}{t_r^3} \exp(it_r \mathcal{W}_r \cdot \mathcal{Z}(\sigma_r))$$

that have plane-wave dependence on a fixed dual twistor \mathcal{W}_r . We sometimes write components $\mathcal{W}_I = (\tilde{\mu}^\alpha, \tilde{\lambda}_{\dot{\alpha}}, \psi_A)$ dual to the components $\mathcal{Z}^I = (\lambda_\alpha, \mu^\alpha, \chi^A)$ of the original twistors. Notice that both types of wavefunction have homogeneity $+2$ in $\mathcal{Z}(\sigma)$, as required for an $\mathcal{N} = 8$ multiplet on twistor space. To recover the momentum space amplitude from these twistorial

amplitudes, one Fourier transforms¹² $\mu_a \rightarrow \tilde{\lambda}_a$ in twistor variables \mathcal{Z}_a and $\tilde{\mu}_r \rightarrow \lambda_r$ in the \mathcal{W}_r dual twistor variables. Since μ and $\tilde{\mu}$ appear only in the exponentials, this Fourier transform is straightforward.

The main virtue of these external wavefunctions is that they provide exactly enough δ -functions to perform all the integrals over the map \mathcal{Z} . If we pick our basis of polynomials to be

$$\left\{ \prod_{b \neq a} \frac{(\sigma b)}{(ab)} \right\} \quad \text{for } |a| = d + 1, \quad (5.4)$$

we can describe the map by

$$\mathcal{Z}(\sigma) = \sum_{a=1}^{d+1} \mathcal{Y}_a \prod_{b \neq a} \frac{(\sigma b)}{(ab)}. \quad (5.5)$$

Then for $a = 1, \dots, d + 1$, we have simply $\mathcal{Z}(\sigma_a) = \mathcal{Y}_a$, so the $k \times n$ matrix is fixed to be the identity matrix in the k columns corresponding to the a -type particles. In other words, with this choice of basis, our k -plane inside \mathbb{C}^n will be represented by the matrix

$$C_{ai} = \begin{cases} c_{ar} & \text{when } i = r, \\ \delta_{ab} & \text{when } i = b \end{cases} \quad (5.6)$$

for some parameters c_{ra} . These parameters are known as ‘link variables’ [4]. Using a different choice of basis for $H^0(\mathbb{CP}^1, \mathcal{O}(d))$ would lead to a $\text{GL}(k; \mathbb{C})$ transformation of C , but the point it defines in the Grassmannian remains invariant. Note that, with the parametrization given in (5.6), since $c_{ab} = \delta_{ab}$, the link variables can be thought of directly as minors of C_{ai} .

There is a small subtlety in using (5.5) to describe the map, because (5.4) is not quite a basis for $H^0(\mathbb{CP}^1, \mathcal{O}(d))$ since $P_a(\sigma)$ in (5.4) has weight $-d$ in σ_a . We can absorb this by declaring that for each a , \mathcal{Y}_a likewise has weight $+d$ under a rescaling of σ_a , so that \mathcal{Y}_a really takes values in $\mathcal{O}_a(d)$. With the Calabi–Yau $\mathcal{N} = 4$ supertwistor space, this may be done without comment, but with $\mathcal{N} = 8$ supersymmetry we acquire a Jacobian in the measure for the integration over the map, which becomes

$$\prod_{a=0}^d d^{4|8} \mathcal{Z}_a = \prod_{a=0}^d \left[d^{4|8} \mathcal{Y}_a \prod_{b \neq a} (ab)^2 \right] \quad (5.7)$$

in terms of the map coefficients in (5.5). This Jacobian cancels¹³ the scaling of $d^{4|8} \mathcal{Y}_a$. With this subtlety accounted for, the wavefunctions (5.3) enforce $\mathcal{Y}_a = \mathcal{Z}_a/t_a$ allowing us to integrate out the map directly.

We are left with a contribution

$$\prod_a dt_a t_a \prod_r \frac{dt_r}{t_r^3} \exp \left(\sum_{r,a} \mathcal{W}_r \cdot \mathcal{Z}_a \frac{t_r}{t_a} \prod_{b \neq a} \frac{(rb)}{(ab)} \right) \quad (5.8)$$

¹²The Fourier transform applies most directly in (2,2) space-time signature, and should more properly be understood as a contour integral in other signatures.

¹³The Jacobian is in the numerator because of the four extra fermionic components. One can check that it has homogeneity $+4$ in each σ_a .

from the external wavefunctions, where the measure for the t_a 's includes a factor of t_a^4 from solving the δ -functions for the \mathcal{Y}_a 's. The factors

$$\frac{t_r}{t_a} \prod_{b \neq a} \frac{(rb)}{(ab)} = \frac{t_r}{t_a} P_a(\sigma_r)$$

in the exponential are precisely the Grassmannian coordinates c_{ra} that we obtain by the procedure described above. The ratio t_r/t_a in front of $P_a(\sigma_r)$ defines a trivialization of $\mathcal{O}(d)$ at σ_r , and so sets a meaningful scale for this ratio of homogeneous coordinates.

The next step is to manipulate our main formula (2.6) so as to write it purely in terms of the external data $\{\mathcal{W}_r, \mathcal{Z}_a\}$ and the Grassmannian minors c_{ra} (i.e. 'link variables'), treated as independent variables. Many of the required steps follow in close parallel to the computations of [2, 10, 18, 19, 20, 28, 32] in $\mathcal{N} = 4$ SYM. Since we did not find these manipulations to be particularly enlightening, we have postponed them to Appendix B.

The final result is that all tree amplitudes in $\mathcal{N} = 8$ supergravity can be written as the Grassmannian integral

$$\begin{aligned} \mathcal{M}_{n,d}(\{\mathcal{W}_r, \mathcal{Z}_a\}) &= \int \left[\prod_{r,a} \frac{dc_{ra}}{c_{ra}} \right] \frac{D_{12}^{n-1n}}{\prod_a c_{1a} c_{2a} \prod_r c_{rn-1} c_{rn}} \left[\prod_{\substack{r \neq 1,2 \\ a \neq n-1,n}} D_{12}^{n-1n} c_{1a} c_{2a} c_{rn-1} c_{rn} \bar{\delta}(\mathcal{V}_{12r}^{an-1n}) \right] \\ &\times \phi^{(d)} \left(\frac{\langle ab \rangle}{H_{12}^{ab}} \right) \phi^{(\tilde{d})} \left(\frac{[rs]}{H_{rs}^{n-1n}} \right) \exp \left(\sum_{r,a} c_{ra} \mathcal{W}_r \cdot \mathcal{Z}_a \right). \end{aligned} \quad (5.9)$$

Here, D and H are the quadratic polynomials

$$D_{rs}^{ab} := \begin{vmatrix} c_{ra} & c_{rb} \\ c_{sa} & c_{sb} \end{vmatrix} \quad \text{and} \quad H_{rs}^{ab} := \frac{D_{rs}^{ab}}{c_{ra} c_{rb} c_{sa} c_{sb}} \quad (5.10)$$

in the minors c_{ra} , while the $\bar{\delta}$ -functions in the sextic polynomials

$$\mathcal{V}_{12r}^{an-1n} := \begin{vmatrix} c_{1a} c_{1n-1} & c_{1n-1} c_{1n} & c_{1n} c_{1a} \\ c_{2a} c_{2n-1} & c_{2n-1} c_{2n} & c_{2n} c_{2a} \\ c_{ra} c_{rn-1} & c_{rn-1} c_{rn} & c_{rn} c_{ra} \end{vmatrix}$$

restrict the support of the integral to the subvariety of $G(k, n)$ defined by the Veronese map (5.2), as expected. The functions $\phi^{(d)}$ and $\phi^{(\tilde{d})}$ represent the determinants $\det'(\Phi) \det'(\tilde{\Phi})$ from (2.6). $\phi^{(d)}$ is defined to be the determinant of the $d \times d$ symmetric matrix with elements

$$\phi_{ab} = \frac{\langle ab \rangle}{H_{12}^{ab}} \quad \text{for } a \neq b \quad \text{and} \quad a, b \neq n, \quad \phi_{aa} = - \sum_{b \neq a} \frac{\langle ab \rangle}{H_{12}^{ab}} \quad (5.11)$$

running over all of the \mathcal{Z}_a -type particles, except for one which without loss we take to be \mathcal{Z}_n . (Note however that \mathcal{Z}_n does appear in the diagonal entries.) Similarly, $\phi^{(\tilde{d})}$ is the determinant of the $\tilde{d} \times \tilde{d}$ symmetric matrix with elements

$$\phi_{rs} = \frac{[rs]}{H_{rs}^{n-1n}} \quad \text{for } r \neq s \quad \text{and} \quad r, s \neq 1, \quad \phi_{rr} = - \sum_{s \neq r} \frac{[rs]}{H_{12}^{ac}} \quad (5.12)$$

running over all of the \mathcal{W}_r -type particles except for one which we take to be \mathcal{W}_1 (again appearing in the diagonal). Notice that the infinity twistor in the form $[,]$ sees only the \mathcal{W}_r 's

(which contain $\tilde{\lambda}_r$'s), while the infinity twistor \langle , \rangle sees only the \mathcal{Z}_a 's (containing λ_a 's). These are therefore multiplicative operators in both cases. Also notice that, as usual in the link representation, parity invariance is now completely manifest.

Note. As this manuscript was being prepared to be submitted [13] and [22] appeared. The former has overlap with the parity invariance proof given in Section 4 while the latter overlaps with the link representation formula presented in Section 5.

A Some properties of determinants

In this appendix we give a careful definition of the determinants $\det'(\Phi)$ and $\det'(\tilde{\Phi})$ that appear in (2.6), and to gather a few general results about such determinants. All the material in here is standard mathematics.

The symmetric matrix Φ defines an inner product on an n -dimensional vector space that we call V . The m -dimensional kernel of Φ is characterized by an $m \times n$ matrix of relations R . This is summarized in the sequence

$$0 \longrightarrow W \xrightarrow{R} V \xrightarrow{\Phi} V^* \xrightarrow{R^T} W^* \longrightarrow 0,$$

where $\Phi \circ R = 0$ and $R^T \circ \Phi = 0$. The sequence is exact if $\ker \Phi = \text{im} R$ and $\ker R^T = \text{im} \Phi$ so that the matrices otherwise have maximal rank (m for R and $(n - m)$ for Φ). If we are given top exterior forms ϵ and ε on V and W respectively, the determinant $\det'(\Phi)$ may be defined in an invariant way via the equation¹⁴

$$\epsilon^{i_1 \dots i_n} \epsilon^{j_1 \dots j_n} \Phi_{i_{m+1} j_{m+1}} \dots \Phi_{i_n j_n} =: \det'(\Phi) \varepsilon^{r_1 \dots r_m} R_{r_1}^{j_1} \dots R_{r_m}^{j_m} \varepsilon^{s_1 \dots s_m} R_{s_1}^{i_1} \dots R_{s_m}^{i_m}. \quad (\text{A.1})$$

That this identity is true for some $\det'(\Phi)$ follows from the fact that, while the left hand side is non-zero by the assumption on the rank of Φ , it vanishes if contracted with any further copy of Φ . Since the kernel is characterized by the m vectors R , the m upstairs skew indices i_1, \dots, i_m and j_1, \dots, j_m must each be a multiple of the m^{th} exterior power of the R 's. Choosing any values for these free indices, we immediately obtain the standard formula

$$\det'(\Phi) = \frac{|\Phi|_{j_1 \dots j_m}^{i_1 \dots i_m}}{\varepsilon^{r_1 \dots r_m} R_{r_1}^{j_1} \dots R_{r_m}^{j_m} \varepsilon^{s_1 \dots s_m} R_{s_1}^{i_1} \dots R_{s_m}^{i_m}}. \quad (\text{A.2})$$

The above argument shows this expression is independent of the choice of indices.

It will be useful to understand the behaviour of reduced determinants when Φ is multiplied by a non-singular $n \times n$ matrix A . If Φ has kernel R , then ΦA has kernel $A^{-1}R$. We now replace Φ by ΦA in the definition (A.1) and multiplying through by m further A 's. On the left we obtain a factor of the determinant of A , while on the right the multiplication cancels the factors of A^{-1} in the m -fold exterior product of $(A^{-1}R)$. We therefore obtain simply

$$\det'(\Phi A) = \det(A) \det'(\Phi). \quad (\text{A.3})$$

In particular, provided A is non-singular, conjugation $\Phi \rightarrow A^{-1}\Phi A$ does not change the reduced determinant.

In our case, neither the map Φ nor the vector spaces V and W are really fixed, but depend on parameters such as the map to twistor space and the locations of the vertex operators. Because we can rescale these parameters, there are no preferred top exterior forms ϵ or ε . The determinant $\det'(\Phi)$ is not really a number, but a section of the determinant line bundle $(\wedge^n V^*)^2 \otimes (\wedge^m W)^2$ over the space of parameters. We need to check that the determinant line bundles defined

¹⁴Really, $\det'(\Phi)$ is the determinant of the whole sequence.

by Φ and $\tilde{\Phi}$ combine with the rest of the factors in the integrand to form a canonically trivial bundle, so that the whole expression is invariant under local rescalings of the worldsheet and map homogeneous coordinates.

We can keep track of the behaviour under rescalings of the homogeneous coordinates σ and \mathcal{Z} by defining a quantity to take values in $\mathcal{O}_i(1)$ if it has homogeneity 1 under rescaling of σ_i , and values in $\mathcal{O}[1]$ if it has homogeneity 1 under rescaling of \mathcal{Z} . Thus for example the relation $\lambda_i = t_i \lambda(\sigma_i)$ means that¹⁵ $t_i \in \mathcal{O}_i(-d)[-1]$. The weights of Φ_{ij} then identify Φ as a symmetric form on

$$V = \bigoplus_{i=1}^n \mathcal{O}_i(1-d)[-1] \quad (\text{A.4})$$

so that Φ gives a pure number, invariant under rescalings, when evaluated on two elements of V . With the kernel of Φ defined by (2.5), the map R is explicitly

$$R_r^j = \frac{\sigma_j^{\alpha_1} \cdots \sigma_j^{\alpha_{\tilde{d}+1}}}{\prod_{i \neq j} (ij)}. \quad (\text{A.5})$$

Since V is given by (A.4), this identifies W as

$$W = \mathbb{C}^{n-d} \otimes \mathcal{O}[-1] \bigotimes_{i=1}^n \mathcal{O}_i(1),$$

where \mathbb{C}^{n-d} is (dual to) the space $H^0(\Sigma, \mathcal{O}(\tilde{d} + 1))$ of degree $\tilde{d} + 1$ polynomials in σ . The determinant line bundle associated to Φ is thus

$$(\wedge^n V^*)^2 \otimes (\wedge^m W)^2 \cong \mathcal{O}[2d] \bigotimes_i \mathcal{O}_i(2n - 2)$$

so that $\det'(\Phi)$ has homogeneity $2d$ in \mathcal{Z} and $2n - 2$ in each of the n points σ_i .

Considering the exact sequence

$$0 \longrightarrow \tilde{W} \xrightarrow{\tilde{R}} \tilde{V} \xrightarrow{\tilde{\Phi}} \tilde{V}^* \xrightarrow{\tilde{R}^T} \tilde{W}^* \longrightarrow 0$$

for $\tilde{\Phi}$, we can likewise identify

$$\tilde{V} = \bigoplus_{i=1}^n \mathcal{O}_i(d+1)[1] \quad \text{and} \quad \tilde{W} = \mathbb{C}^{d+2} \otimes \mathcal{O}[1],$$

so that $\det'(\tilde{\Phi})$ is a section of the determinant line bundle

$$(\wedge^n \tilde{V}^*)^2 \otimes (\wedge^{d+2} \tilde{W})^2 \cong \mathcal{O}[-2\tilde{d}] \bigotimes_{i=1}^n \mathcal{O}_i(-2d - 2).$$

Combining the two determinants and the explicit Vandermonde factor in (2.6) shows that

$$\frac{\det'(\Phi) \det'(\tilde{\Phi})}{|1 \dots n|^2} \in \mathcal{O}[4(d+1) - 2n] \bigotimes_{i=1}^n \mathcal{O}_i(-2d - 2),$$

¹⁵No other factor in (2.6) has non-trivial behaviour under scalings of the external data, so for the purposes of this discussion, we can keep the external data fixed.

which correctly conspires to cancel the scaling $-4(d+1)$ of the measure $\prod_a d^{4|8} \mathcal{Z}$ and the wavefunctions $\prod_{i=1}^n (\sigma_i d\sigma_i) h_i(\mathcal{Z}(\sigma_i))$. The integrand of (2.6) is thus projectively well-defined.

Finally, let us comment on relation of the definition of $\det'(\Phi)$ given here to that given in [15]. Here, the denominator in (A.2) involves

$$R^{i_1} \wedge R^{i_2} \wedge \dots \wedge R^{i_m} = \frac{|i_1 \dots i_m|}{\prod_{r=1}^m \left(\prod_{k \neq i_r} (i_r k) \right)}, \quad (\text{A.6})$$

where numerator of this expression is the Vandermonde determinant of all the worldsheet coordinates associated with the components of Φ that are absent in (A.1), while the denominator comes from the factors in the denominator of R in (A.5). A little experimentation shows that when $d > 1$ (A.6) can also be written as

$$R^{i_1} \wedge R^{i_2} \wedge \dots \wedge R^{i_m} = \frac{|i_{m+1} \dots i_n|}{|1 \dots n|},$$

where $|i_{m+1} \dots i_n|$ is the Vandermonde determinant corresponding to the components of Φ that are *present* in (A.1). (For $d = 0, 1$ the exterior product of all the R 's gives exactly $|1 \dots n|^{-1}$.) An exactly analogous statement is true for the cokernel defined by R^T . It is amusing to notice that the explicit factor of $|1 \dots n|^2$ in (2.6) can thus be absorbed into $\det'(\Phi)$ if we define this as

$$\det'(\Phi) = \frac{|\Phi|_{j_1 \dots j_m}^{i_1 \dots i_m}}{|i_{m+1} \dots i_n| |j_{m+1} \dots j_n|}$$

instead of by (A.2). This is the definition that was used in [15] and it is often more convenient for explicit calculations.

B Transformation to the link variables

In this appendix we explain how $\mathcal{M}_{n,d}(\{\mathcal{W}_r, \mathcal{Z}_a\})$ can be manipulated so as to be written as an integral over the Grassmannian, gauged fixed to the link representation.

With the aim of simplifying the argument of the exponentials in (5.8) it is useful to replace the scaling parameters (t_a, t_r) by parameters (S_a, T_r) defined as in [18, 32] by

$$S_a := \frac{1}{t_a \prod_{b \neq a} (ab)} \quad \text{and} \quad T_r := t_r \prod_b (rb). \quad (\text{B.1})$$

Notice that

$$S_a = s_a \prod_r (ar), \quad (\text{B.2})$$

where s_a was given in (4.1). In terms of these (S_a, T_r) variables, (5.8) becomes

$$\prod_a \left[\frac{dS_a}{S_a^3} \prod_{b \neq a} \frac{1}{(ab)^2} \right] \prod_r \left[\frac{dT_r}{T_r^3} \prod_b (rb)^2 \right] \exp \left(\sum_{r,a} \mathcal{W}_r \cdot \mathcal{Z}_a \frac{T_r S_a}{(ra)} \right),$$

where the $1/(ra)$ factor appears in the exponential because (ra) is absent in (5.8) but present in the definition of T_r . The factor of $\prod_a \left[\prod_{b \neq a} (ab)^{-1} \right]$ precisely cancels the Jacobian factor in (5.7) associated with our choice of basis polynomials.

An advantage of choosing $d + 1$ wavefunctions to be of the form (5.3) is that we may choose to remove rows and columns from Φ and $\tilde{\Phi}$ in such a way that $\det'(\Phi)$ depends only on the \mathcal{Z}_a while $\det'(\tilde{\Phi})$ depends only on the \mathcal{W}_r . Using the identity (4.3) we have

$$\frac{\det'(\Phi)\det'(\tilde{\Phi})}{|1 \dots n|^2} = |1 \dots n|^2 \det' \left(\tilde{\Phi}^{(\tilde{d})}(\langle \cdot, \cdot \rangle, s) \right) \det' \left(\tilde{\Phi}^{(d)}([\cdot, \cdot], t) \right).$$

We then choose to remove from $\tilde{\Phi}^{(\tilde{d})}(\langle \cdot, \cdot \rangle, s)$ all the $\tilde{d} + 1$ rows and columns corresponding to the \mathcal{W}_r particles, together with the row and column of one of the \mathcal{Z}_a particles. Without loss of generality, we take this to be ‘ n ’. The off-diagonal elements of $\tilde{\Phi}^{(\tilde{d})}(\langle \cdot, \cdot \rangle, s)$ are now independent of the external \mathcal{W}_r ’s. To remove the \mathcal{W}_r ’s from the diagonal elements, we additionally choose the $\tilde{d} + 1$ reference points p_r to be the worldsheet insertion points σ_r of the \mathcal{W}_r wavefunctions. Then the remaining elements of $\tilde{\Phi}^{(\tilde{d})}(\langle \cdot, \cdot \rangle, s)$ are

$$\tilde{\Phi}_{ab}^{(\tilde{d})} = \frac{\langle ab \rangle}{(ab)} S_a S_b \prod_r \frac{1}{(ar)(br)}, \quad \tilde{\Phi}_{aa}^{(\tilde{d})} = - \sum_{b \neq a} \frac{\langle ab \rangle}{(ab)} S_a S_b \prod_r \frac{1}{(ar)^2}, \quad (\text{B.3})$$

where the sum runs only over the \mathcal{Z}_b particles, and we have used (B.2).

Similarly, in $\tilde{\Phi}^{(d)}([\cdot, \cdot], t)$ we choose to remove the $d + 2$ rows and columns corresponding to the \mathcal{Z}_a and to, say, \mathcal{W}_1 . In addition we must choose the $d + 1$ reference points $p_a = \sigma_a$ so as to ensure \mathcal{Z}_a does not arise on the diagonal. The remaining elements of $\tilde{\Phi}$ are then

$$\tilde{\Phi}_{rs} = \frac{[rs]}{(rs)} T_r T_s \prod_b \frac{1}{(rb)(sb)}, \quad \tilde{\Phi}_{rr} = - \sum_{s \neq r} \frac{[rs]}{(rs)} T_r T_s \prod_b \frac{1}{(rb)^2}, \quad (\text{B.4})$$

where again the sum runs only over the \mathcal{W}_s -type particles and where we have used (B.1).

The next step is to examine the reduced determinants. We can remove a factor of $\prod_{a \neq 1} \left[\prod_r (ab)^{-2} \right]$ from the reduced determinant of (B.3) and a factor of $\prod_{r \neq 1} \left[\prod_b (rb)^{-2} \right]$ from the reduced determinant of (B.4). For later convenience, we also multiply every remaining element of (B.3) by $T_1 T_2 / (12)$ and every remaining element of (B.4) by $S_{n-1} S_n / (n-1)n$. Collecting all the factors, the determinants become

$$\begin{aligned} \frac{\det'(\Phi)\det'(\tilde{\Phi})}{|1 \dots n|^2} &= \phi^{(d)} \left(\frac{\langle ab \rangle S_a S_b T_1 T_2}{(12)(ab)} \right) \phi^{(\tilde{d})} \left(\frac{[rs] T_r T_s S_{n-1} S_n}{(rs)(n-1)n} \right) \frac{(12)^d (n-1)n^{\tilde{d}}}{(T_1 T_2)^d (S_{n-1} S_n)^{\tilde{d}}} \\ &\times \frac{|1 \dots n|^2}{|nr_0 \dots r_{\tilde{d}}|^2 |1a_0 \dots a_d|^2} \frac{\prod_r (rn)^2 \prod_a (1a)^2}{\prod_{a,r} (ar)^4}, \end{aligned} \quad (\text{B.5})$$

where $\phi^{(d)}$ is the determinant of the $d \times d$ symmetric matrix defined in (5.11) and $\phi^{(\tilde{d})}$ is the determinant of the $\tilde{d} \times \tilde{d}$ symmetric matrix defined in (5.12). The last line of (B.5) combines with the factor of $\prod_{a,r} (ar)^2$ left over from the change of the measure $t_r \rightarrow T_r$ and then cancels completely.

We now introduce the link variables

$$c_{ra} := \frac{t_r}{t_a} \prod_{b \neq a} \frac{(rb)}{(ab)} = \frac{T_r S_a}{(ra)} \quad (\text{B.6})$$

so that the argument of the exponential in (5.8) becomes simply $\sum_{a,r} c_{ar} \mathcal{W}_r \cdot \mathcal{Z}_a$. We can treat the c_{ra} 's as $(d+1)(\tilde{d}+1)$ independent variables if we enforce the conditions (B.6) by introducing further δ -functions into $\mathcal{M}_{n,d}$ via

$$1 = \prod_{r,a} dc_{ra} \bar{\delta} \left(c_{ra} - \frac{T_r S_a}{(ra)} \right).$$

At this point, almost all of the formula for the amplitude can immediately be written in terms of the c 's and the external data. We find

$$\begin{aligned} \mathcal{M}_{n,d}(\{\mathcal{W}_r, \mathcal{Z}_a\}) &= \int \frac{\prod_{i=1}^n (\sigma_i d\sigma_i)}{\text{vol GL}(2; \mathbb{C})} \prod_r \frac{dT_r}{T_r^3} \prod_a \frac{dS_a}{S_a^3} \left[\prod_{r,a} dc_{ra} \bar{\delta} \left(c_{ra} - \frac{T_r S_a}{(ra)} \right) \right] \frac{(n-1n)^{\tilde{d}-d}}{(S_{n-1} S_n)^{\tilde{d}-d}} (H_{12}^{n-1n})^d \\ &\quad \times \phi^{(d)} \left(\frac{\langle ab \rangle}{H_{12}^{ab}} \right) \phi^{(\tilde{d})} \left(\frac{[rs]}{H_{rs}^{n-1n}} \right) \exp \left(\sum_{r,a} c_{ra} \mathcal{W}_r \cdot \mathcal{Z}_a \right), \end{aligned}$$

where we have defined

$$H_{rs}^{ab} := \frac{1}{c_{ra} c_{sb}} - \frac{1}{c_{rb} c_{sa}} = \frac{c_{rb} c_{sa} - c_{ra} c_{sb}}{c_{ra} c_{rb} c_{sa} c_{sb}}.$$

as in (5.10).

To reach our final form of the Grassmannian representation of gravitational tree amplitudes, depending exclusively on the c_{ra} 's and external data, we must perform the (σ, T, S) integrals. This is a straightforward, if rather lengthy exercise. We choose to fix the $\text{SL}(2; \mathbb{C})$ freedom by freezing σ_1, σ_{n-1} and σ_n to some arbitrary values at the usual expense of a Jacobian $(1n-1)(n-1n)(n1)$, and fix the scaling by freezing $S_n = 1$ (for which the Jacobian is $S_n = 1$). The integrals are then performed using $2n$ of the $(d+1)(\tilde{d}+1)$ δ -functions, and lead directly to (5.9) given in the main text. Note in particular that the Veronese constraints $\bar{\delta}(\mathcal{V}_{12r}^{an-1n})$ that remain in (5.9) arise simply from repeatedly substituting the support of one δ -function into another.

C Conventions

Let us list our conventions. We take \mathbb{PT} to be the $\mathcal{N} = 8$ supertwistor space $\mathbb{CP}^{3|8}$ with a line I removed. We use calligraphic letters to denote supertwistors, lowercase and uppercase Roman indices to denote their four bosonic and \mathcal{N} fermionic components, respectively. We often decompose the bosonic components into two 2-component Weyl spinors with dotted and undotted Greek indices. Thus $\mathcal{Z} = (Z^a, \chi^A) = (\lambda_\alpha, \mu^{\dot{\alpha}}, \chi^A)$. External states are labelled by lowercase Roman indices from the middle of the alphabet $i, j, \dots \in \{1, \dots, n\}$. We use σ^α with $\underline{\alpha}, \underline{\beta}, \dots \in \{1, 2\}$ to denote a homogeneous coordinate on the \mathbb{CP}^1 worldsheet. We often choose italic letters from the beginning of the alphabet to run over the space of degree d polynomials in the worldsheet coordinates, so $a, b, \dots \in \{0, \dots, d\}$. It is also useful to separately allow $r, s, \dots \in \{0, \dots, \tilde{d}\}$. We use $[\ , \]$ to denote dotted spinor contractions, $\langle \ , \ \rangle$ for undotted contractions, and $(\ , \)$ for contractions of the homogeneous coordinates σ on the worldsheet. When affine coordinates are more convenient, we will choose them so that $\sigma = (1, u)$. We shall denote the data of external spinor supermomenta by $\Lambda = (\lambda_\alpha, \tilde{\lambda}_{\dot{\alpha}}, \eta_A)$.

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