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Levi-Civita's Theorem for Noncommutative Tori*

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Abstract. We show how to define Riemannian metrics and connections on a noncommutative torus in such a way that an analogue of Levi-Civita's theorem on the existence and uniqueness of a Riemannian connection holds. The major novelty is that we need to use two different notions of noncommutative vector field. Levi-Civita's theorem makes it possible to define Riemannian curvature using the usual formulas.

Key words: noncommutative torus; noncommutative vector field; Riemannian metric; Levi-Civita connection; Riemannian curvature; Gauss–Bonnet theorem

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To Marc Rieffel, with admiration and appreciation

Introduction

In his lecture series at the Focus Program on Noncommutative Geometry and Quantum Groups at the Fields Institute in June, 2013, Masoud Khalkhali gave a very beautiful description of recent work by Connes and Moscovici [3] (building on earlier work of Connes and Tretkoff [4]) and by Fathizadeh and Khalkhali [7, 8, 9] on a calculation of what one can call "scalar curvature" for metrics on noncommutative tori obtained by (noncommutative) conformal deformation of a flat metric. At the same time, Khalkhali explained that defining curvature in terms of the spectral geometry of the Laplacian is basically forced on us by a lack in the noncommutative setting of the standard machinery of Riemannian geometry, whereby one would define curvature using derivatives of the Levi-Civita connection. In the same Focus Program, Marc Rieffel in his lecture gave a definition of Riemannian metric in the noncommutative setting, albeit only for finite-dimensional algebras (which, roughly speaking, correspond to zero-dimensional manifolds). The purpose of this paper is to show that one can give a very natural and quite general definition of Riemannian metrics on a noncommutative torus, without assuming *a priori* that the metric is a conformal deformation of a flat metric, and that for this definition one can prove an analogue of Levi-Civita's theorem ([10]; for a modern formulation and proof see for example [5, Chapter 2, § 3]) on the existence and uniqueness of a torsion-free connection compatible with the metric. For our notion of Riemannian metric we also obtain a notion of Riemannian curvature, which we compute explicitly in the two-dimensional case.

Admittedly our definition of Riemannian metric still has certain drawbacks. It would be nice to be able to prove a uniformization theorem for two-dimensional noncommutative tori, stating that in some sense all Riemannian metrics are equivalent to conformal deformations of a standard flat metric, and are thus of the form studied by Connes–Moscovici and by Fathizadeh–Khalkhali. The problem, however, is that generic smooth two-dimensional noncommutative tori

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are quite rigid – their diffeomorphism groups are not much bigger than the group of smooth inner automorphisms [6] – and this makes it hard to see how a uniformization theorem could be true without using a very different definition of Riemannian metric.

1 Riemannian metrics and connections

While it would be desirable to have a theory of Riemannian metrics and connections on arbitrary "noncommutative manifolds," there is a problem in general in understanding what a tangent vector or vector field should be. However, tori are parallelizable, so on a torus, a vector field is simply a linear combination (with the coefficients being arbitrary functions) of the (commuting) coordinate vector fields ∂_j. This definition carries over without difficulty to a noncommutative torus, though as we will see, there are also other ways of defining vector fields. We begin with basic definitions and notation.

Definition 1.1. Let Θ be a skew-symmetric $n \times n$ matrix with entries in $\mathbb{R}$. (When $n = 2$, we write $\Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$ for $\theta \in \mathbb{R}$.) The *noncommutative torus* of dimension n with noncommutativity parameter Θ is the universal C^*-algebra A_Θ on n unitaries $U_1, \dots, U_n$ with commutation relations $U_j U_k = e^{2\pi i \Theta_{jk}} U_k U_j$. (When $n = 2$, there are only two generators U_1 and U_2 and there is only one relation, $U_1 U_2 = e^{2\pi i \theta} U_2 U_1$, and we write A_θ for A_Θ.) We will sometimes think of A_Θ as the algebra of "functions" on a "noncommutative manifold" $\mathbb{T}^n_\Theta$. The algebra A_Θ carries an ergodic action of the n-torus $\mathbb{T}^n$ via $t \cdot U_j = t_j U_j$, $t \in \mathbb{T}^n$. The infinitesimal generators of this action are the (unbounded) $*$-derivations ∂_j with $\partial_j(U_k) = \delta_{jk} 2\pi i U_k$. The C^∞ vectors for this action constitute the algebra A^∞_Θ called the *smooth noncommutative torus.* This algebra can be identified with the (noncommutative) rapidly decreasing Fourier series

$$\left\{ \sum_{m_1,\dots,m_n} c_{m_1,\dots,m_n} U_1^{m_1} \cdots U_n^{m_n} \,\middle|\, \{c_{m_1,\dots,m_n}\} \text{ rapidly decreasing} \right\}.$$

When $\Theta = 0$ (the commutative case), this is $C^\infty(\mathbb{T}^n)$. In general this is always indistinguishable from $C^\infty(\mathbb{T}^n)$ as a Fréchet space, even though the algebra structures are different.

Now we get to the point where the noncommutative theory diverges from the usual geometry of manifolds. On a smooth (ordinary) closed manifold, there are three equivalent ways of defining vector fields: as infinitesimal generators of one-parameter groups of diffeomorphisms (i.e., flows), or as smooth sections of the tangent bundle, or as homogeneous linear first-order differential operators annihilating the constants. The problem is that in the noncommutative case, these definitions do not all agree, and so we need more than one notion.

Definition 1.2. The space $\mathcal{X}_\Theta$ of *vector fields* on the smooth noncommutative torus is defined to be the free rank-n left A^∞_Θ-module with basis $\partial_1, \dots, \partial_n$. In other words, a vector field is just a formal linear combination of "partial derivatives" with "function" coefficients in A^∞_Θ. Vector fields operate on A^∞_Θ in the obvious way as linear "first-order differential operators" annihilating the "constant functions" $\lambda \cdot 1$, $\lambda \in \mathbb{C}$.

The problem with this definition is that, unlike the standard commutative situation, an element of $\mathcal{X}_\Theta$ is *not* usually a derivation, and commutator of such vector fields is not necessarily a vector field, since

$$[b\partial_j, c\partial_k]a = b\partial_j(c\partial_k a) - c\partial_k(b\partial_j a) = b(\partial_j c)(\partial_k a) + bc(\partial_j\partial_k a) - c(\partial_k b)(\partial_j a) - cb(\partial_k\partial_j a),$$

so

$$[b\partial_j, c\partial_k] = b(\partial_j c)\partial_k - c(\partial_k b)\partial_j + [b, c]\partial_j\partial_k$$

and the second order term $[b, c]\partial_j\partial_k$ does not necessarily cancel out.

Definition 1.3. Accordingly, we introduce another linear space $\mathcal{D}_\Theta$, consisting of the $*$-derivations $\delta\colon A_\Theta^\infty \to A_\Theta^\infty$. By [1, Corollary 5.3, C2], any such δ is automatically continuous in the Fréchet topology. It is clear that $\mathcal{D}_\Theta$ is a Lie algebra under the commutator bracket (since the bracket of derivations is a derivation), so this remedies one of the defects of Definition 1.2. We can view $\mathcal{D}_\Theta$ as the Lie algebra of the infinite dimensional group $\mathrm{Diff}(\mathbb{T}_\Theta^n) = \mathrm{Aut}(A_\Theta^\infty)$ of $*$-automorphisms of A_Θ^∞. Furthermore, by [1, Corollary 5.3, D2], any $\delta \in \mathcal{D}_\Theta$ has a unique decomposition as $a_1\partial_1 + \cdots + a_n\partial_n + \delta_0$, where δ_0 is approximately inner and $a_1, \ldots, a_n$ lie in the center of A_Θ^∞.

Now we need the following result:

Theorem 1.4 (Bratteli, Elliott, and Jorgensen)**.** *If Θ is "generically transcendental" (in a rather complicated sense made precise in [1], but satisfied for almost all skew-adjoint matrices), then any $\delta \in \mathcal{D}_\Theta$ has a unique decomposition as $a_1\partial_1 + \cdots + a_n\partial_n + \delta_0$, where δ_0 is **inner**, hence bounded in the C^*-algebra norm, and $a_1, \ldots, a_n \in \mathbb{C}$.*

Proof. See [1, Remark 4.3]. It also follows that δ is a pregenerator of a one-parameter subgroup of $\mathrm{Aut}(A_\Theta^\infty)$ (in fact sometimes this is even true without genericity of Θ, see [1, Theorem 5.4]). ■

Definition 1.5. A *Riemannian metric* $g = \langle\cdot,\cdot\rangle$ on A_Θ^∞ is defined to be a (positive) A_Θ^∞-valued inner product on vector fields, or in other words, a sesquilinear map $\langle\cdot,\cdot\rangle\colon \mathcal{X}_\Theta \times \mathcal{X}_\Theta \to A_\Theta^\infty$ satisfying the axioms of a (pre-)Hilbert module:

1. $\langle X + X', Y\rangle = \langle X, Y\rangle + \langle X', Y\rangle$ and $\langle aX, Y\rangle = a\langle X, Y\rangle$ for $X, X', Y \in \mathcal{X}_\Theta$, $a \in A_\Theta^\infty$ (so g is A_Θ^∞-linear in the first variable);
2. $\langle X, Y\rangle^* = \langle Y, X\rangle$ (hermitian symmetry) – together with (1), this implies $\langle X, aY\rangle = \langle X, Y\rangle a^*$;
3. $\langle X, X\rangle \geq 0$ in the sense of the C^*-algebra A_Θ, with equality only if $X = 0$.

Note that the metric is uniquely determined by the matrix $(g_{jk}) = (\langle\partial_j, \partial_k\rangle)$ in $M_n(A_\Theta^\infty)$. This matrix must be a positive element of $M_n(A_\Theta)$ since

$$0 \leq \left\langle \sum_j a_j\partial_j, \sum_k a_k\partial_k \right\rangle = \sum_{j,k} a_j g_{jk} a_k^* \qquad \text{for any} \quad a_j \in A_\Theta^\infty.$$

Actually, the axioms so far only correspond to a hermitian metric on the complexified tangent bundle. Recall that a Riemannian metric must assign a real-valued (i.e., self-adjoint) inner product to two real vector fields. Since the "real vector fields" are generated by the ∂_j, we need to add one additional condition:

4. For each j, k, $\langle X_j, X_k\rangle$ is self-adjoint, and thus $\langle X_j, X_k\rangle = \langle X_k, X_j\rangle$.

Definition 1.6. A *connection* on A_Θ^∞ is a way of defining covariant derivatives for vector fields satisfying the Leibniz rule

$$\nabla_X(aY) = (X \cdot a)Y + a\nabla_X Y. \tag{1.1}$$

But here's the tricky aspect of this. For (1.1) even to make sense, we need to be able to multiply Y on the left by an element of the algebra, so we want $Y \in \mathcal{X}_\Theta$. On the other hand, applying (1.1) to $\nabla_X\big((ab)Y\big)$ and comparing with the expansion of $\nabla_X\big(a(bY)\big)$, we obtain

$$(X \cdot (ab))Y = \big((X \cdot a)b + a(X \cdot b)\big)Y,$$

and since Y is arbitrary, this forces X to be a derivation. So we need $X \in \mathcal{D}_\Theta$ rather than $X \in \mathcal{X}_\Theta$.

In other words, a connection is a map $\nabla\colon \mathcal{D}_\Theta \times \mathcal{X}_\Theta \to \mathcal{X}_\Theta$, written $(X, Y) \mapsto \nabla_X Y$, satisfying the following axioms:

1. ∇ is linear in the first variable, so that $\nabla_{\lambda X} Y = \lambda \nabla_X Y$ and $\nabla_{X+X'} Y = \nabla_X Y + \nabla_{X'} Y$, for $X, X' \in \mathcal{D}_\Theta$, $\lambda \in \mathbb{C}$;
2. ∇ is $\mathbb{C}$-linear in the second variable, so $\nabla_X(Y + Y') = \nabla_X Y + \nabla_X Y'$, $\nabla_X(\lambda Y) = \lambda \nabla_X Y$, for $X, Y, Y' \in \mathcal{X}_\Theta$, $\lambda \in \mathbb{C}$;
3. For $X, Y \in \mathcal{X}_\Theta$, $a \in A_\Theta^\infty$, $\nabla_X(aY) = (X \cdot a)Y + a\, \nabla_X Y$.

Normally (i.e., in the classical case $\Theta = 0$) the axioms for a connection require that ∇ be A_Θ^∞-linear in the first variable, but this does not make sense in our context since $\mathcal{D}_\Theta$ is not a left A_Θ^∞-module. However, let's assume that Θ is generic in the sense of Theorem 1.4, so that any element of $\mathcal{D}_\Theta$ differs from a linear combination of $\partial_1, \ldots, \partial_n$ by an inner derivation. We need an extra axiom to pin down the values of $\nabla_{\operatorname{ad} a}$, $a \in A_\Theta^\infty$. We have no classical precedent for this since there are no inner derivations in the commutative case, but from (1.1) we obtain

$$\nabla_{\operatorname{ad} a}(bY) = [a, b]Y + b\nabla_{\operatorname{ad} a} Y, \qquad \text{or} \qquad [\nabla_{\operatorname{ad} a}, b] = [a, b].$$

The easiest way to satisfy this is to take $\nabla_{\operatorname{ad} a}$ = left multiplication by a. However $\operatorname{ad} a$ only determines a up to addition of a constant, so we use the canonical trace τ on A_Θ to normalize things. Given the derivation $\operatorname{ad} a$, a is unique subject to the condition that $\tau(a) = 0$, and we add as another axiom:

4. For any $a \in A_\Theta^\infty$ with $\tau(a) = 0$, $\nabla_{\operatorname{ad} a}$ = left multiplication by a.

For simplicity we write ∇_j for ∇_{∂_j}. The operators $\nabla_j \colon \mathcal{X}_\Theta \to \mathcal{X}_\Theta$ determine the connection, because of condition (4), Theorem 1.4, and the linearity axiom, condition (1). Once again, the axioms so far correspond to a connection on the complexified tangent bundle, so it's natural to require the covariant derivative of a "real" vector field in a "real" direction to be "real-valued". In the presence of a Riemannian metric satisfying Definition 1.5(1)–(4), this corresponds to the additional axiom

5. For any j, k, and ℓ, $\langle \nabla_j \partial_k, \partial_\ell \rangle$ is *self-adjoint*.

We call the connection *torsion-free* if for all $j, k \le n$, $\nabla_j \partial_k = \nabla_k \partial_j$. This is the exact analogue of the corresponding condition in the commutative case (since ∂_k and ∂_j commute).

We say the connection is *compatible with a Riemannian metric* $g = \langle \cdot, \cdot \rangle$ (in the sense of Definition 1.5) if for all $X, Y \in \mathcal{X}_\Theta$, $Z \in \mathcal{D}_\Theta$,

$$Z \cdot \langle X, Y \rangle = \langle \nabla_Z X, Y \rangle + \langle X, \nabla_Z Y \rangle.$$

Remark 1.7. Note that a definition of connections on "vector bundles" over noncommutative tori (and other noncommutative spaces also equipped with a Lie group of symmetries) was given about 30 years ago by Connes in his classic paper [2]. This definition is also based on (1.1) (except with left modules replaced by right modules), but with the vector field X restricted to have "constant coefficients," or in our situation, to be a $\mathbb{C}$-linear combination of the ∂_j. In this same paper Connes gives the definition of compatibility with a metric, and it is the same as ours. However, he does not address the notion of torsion for a connection, nor does he attempt to prove a version of Levi-Civita's theorem.

2 Levi-Civita's theorem

Now we can state and prove the analogue of Levi-Civita's theorem.

Theorem 2.1. *Let Θ be a generic skew-symmetric $n \times n$ matrix in the sense of Theorem 1.4 and let $g = \langle \cdot, \cdot \rangle$ be a Riemannian metric on A_Θ^∞ in the sense of Definition 1.5 (including condition 1.5(4)). Then there is one and only one connection on $\mathcal{X}_\Theta$ in the sense of Definition 1.6 (including conditions 1.6(4) and 1.6(5)) that is torsion-free and compatible with the metric. This connection, called the* ***Levi-Civita connection****, is determined by the formula*

$$\langle \nabla_j \partial_k, \partial_\ell \rangle = \frac{1}{2}\big[\partial_j \langle \partial_k, \partial_\ell \rangle + \partial_k \langle \partial_j, \partial_\ell \rangle - \partial_\ell \langle \partial_j, \partial_k \rangle\big]. \tag{2.1}$$

Proof. First we prove uniqueness. Suppose we have a torsion-free connection ∇ compatible with the metric. We have (because of compatibility with the metric)

$$\begin{aligned} \partial_j \langle \partial_k, \partial_\ell \rangle &= \langle \nabla_j \partial_k, \partial_\ell \rangle + \langle \partial_k, \nabla_j \partial_\ell \rangle, \\ \partial_k \langle \partial_\ell, \partial_j \rangle &= \langle \nabla_k \partial_\ell, \partial_j \rangle + \langle \partial_\ell, \nabla_k \partial_j \rangle, \\ \partial_\ell \langle \partial_j, \partial_k \rangle &= \langle \nabla_\ell \partial_j, \partial_k \rangle + \langle \partial_j, \nabla_\ell \partial_k \rangle. \end{aligned} \tag{2.2}$$

Via conditions 1.5(2), 1.5(4) and 1.6(5), together with the torsion-free condition, we can rewrite (2.2) as

$$\langle \nabla_j \partial_k, \partial_\ell \rangle = \partial_j \langle \partial_k, \partial_\ell \rangle - \langle \nabla_\ell \partial_j, \partial_k \rangle, \tag{2.3}$$

$$\langle \nabla_j \partial_k, \partial_\ell \rangle = \partial_k \langle \partial_j, \partial_\ell \rangle - \langle \nabla_\ell \partial_k, \partial_j \rangle, \qquad \text{and} \tag{2.4}$$

$$0 = -\partial_\ell \langle \partial_j, \partial_k \rangle + \langle \nabla_\ell \partial_j, \partial_k \rangle + \langle \nabla_\ell \partial_k, \partial_j \rangle. \tag{2.5}$$

Adding (2.3), (2.4), and (2.5) gives (2.1).

Next we prove existence, by showing that (2.1) determines a unique connection which is compatible with the metric and torsion-free. To begin with, given j and k, knowing $\langle \nabla_j \partial_k, \partial_\ell \rangle$ for all ℓ determines $\nabla_j \partial_k$, since the metric is nondegenerate. We get a unique extension to a definition of $\nabla_X \partial_k$ for all vector fields $X \in \mathcal{D}_\Theta$ by making $\nabla_X \partial_k$ linear in X and requiring that $\nabla_{\operatorname{ad} a} \partial_k = a \partial_k$ when $\tau(a) = 0$. (We are using genericity of Θ in order to appeal to Theorem 1.4.) Then since the ∂_k are a free A_Θ^∞-basis for $\mathcal{X}_\Theta$, knowing $\nabla_X \partial_k \in \mathcal{X}_\Theta$ for each k uniquely determines $\nabla_X Y \in \mathcal{X}_\Theta$ for each $Y \in \mathcal{X}_\Theta$, since we have a unique expression $Y = \sum_k a_k \partial_k$ and can set

$$\nabla_X \left(\sum_k a_k \partial_k \right) = \sum_k (X \cdot a_k) \partial_k + a_k \nabla_X \partial_k.$$

This gives us a definition of ∇ satisfying the axioms of Definition 1.6(1), (2) and 1.6(4). Condition 1.6(5) holds because of condition 1.5(4) and the fact that the ∂_j are $*$-preserving. We need to show that 1.6(3) is also satisfied, which means we need to check that

$$\nabla_X \big(ab\partial_k\big) = (X \cdot a)\big(b\partial_k\big) + a\nabla_X \big(b\partial_k\big). \tag{2.6}$$

The left-hand side of (2.6) is defined to be

$$\big(X \cdot (ab)\big)\partial_k + ab\nabla_X \partial_k.$$

Since $X \in \mathcal{D}_\Theta$ and is thus a derivation, this becomes

$$(X \cdot a)\big(b\partial_k\big) + a(X \cdot b)\partial_k + ab\nabla_X \partial_k = (X \cdot a)\big(b\partial_k\big) + a\nabla_X \big(b\partial_k\big),$$

which agrees with the right-hand side of (2.6), as required.

The right-hand side of (2.1) is clearly symmetric under interchange of j and k, because of the fact that Definition 1.5(4) ensures that $\langle \partial_j, \partial_k\rangle = \langle \partial_k, \partial_j\rangle$, so ∇ is torsion-free. We have just one more thing to check, which is compatibility with the metric. From (2.1), we have

$$\begin{aligned}\langle \nabla_j \partial_k, \partial_\ell\rangle + \langle \partial_k, \nabla_j \partial_\ell\rangle = \frac{1}{2}\Big[&\partial_j\langle \partial_k, \partial_\ell\rangle + \partial_k\langle \partial_j, \partial_\ell\rangle - \partial_\ell\langle \partial_j, \partial_k\rangle \\ &+ \partial_j\langle \partial_\ell, \partial_k\rangle + \partial_\ell\langle \partial_j, \partial_k\rangle - \partial_k\langle \partial_j, \partial_\ell\rangle\Big] = \partial_j\langle \partial_k, \partial_\ell\rangle,\end{aligned}$$

which is what is required. This completes the proof. ■

3 Riemannian curvature

With Theorem 2.1 in place, it now makes sense to define curvature for a Riemannian metric using derivatives of the Levi-Civita connection. There are different sign conventions used by different authors; here we are following [5].

Definition 3.1. Let Θ be a generic skew-symmetric $n \times n$ matrix in the sense of Theorem 1.4 and let $g = \langle \cdot, \cdot\rangle$ be a Riemannian metric on A_Θ^∞ in the sense of Definition 1.5 (including condition 1.5(4)). Let ∇ be the associated Levi-Civita connection from Theorem 2.1. Define the associated *Riemann curvature operator* by

$$R(X,Y) = \nabla_Y \nabla_X - \nabla_X \nabla_Y + \nabla_{[X,Y]} \colon \mathcal{X}_\Theta \to \mathcal{X}_\Theta, \qquad X, Y \in \mathcal{D}_\Theta.$$

We also define the *Riemannian curvature* by

$$R_{j,k,\ell,m} = \langle R(\partial_j, \partial_k)\partial_\ell, \partial_m\rangle .$$

Remark 3.2. Recall by Theorem 1.4 that $\mathcal{D}_\Theta$ splits as the direct sum of the n-dimensional vector space spanned by the ∂_j and the set of $\operatorname{ad} a$, $a \in A_\Theta^\infty$. The second summand actually has no effect on the curvature the way we've normalized the connection, for if $a \in A_\Theta^\infty$, $\tau(a) = 0$, and if $X \in \mathcal{D}_\Theta$, then first of all $\tau(Xa) = 0$ (this is obvious for X inner, so we only need to check it for $X = \partial_j$, where it follows from the fact that the gauge action of $\mathbb{T}^n$ on A_Θ preserves τ — see also [11, Lemma 2.1]). So for $b \in A_\Theta^\infty$,

$$\begin{aligned}[\operatorname{ad}(a), X]b &= [a, X\cdot b] - X\cdot([a,b]) = a(X\cdot b) - (X\cdot b)a - X\cdot(ab - ba) \\ &= a(X\cdot b) - (X\cdot b)a - (X\cdot a)b - a(X\cdot b) + (X\cdot b)a - b(X\cdot a) \\ &= -[X\cdot a, b] = \operatorname{ad}(-X\cdot a)b,\end{aligned}$$

and we have $[\operatorname{ad} a, X] = \operatorname{ad}(-X\cdot a)$. So

$$\begin{aligned}R(\operatorname{ad} a, X)Z &= \left(\nabla_X \nabla_{\operatorname{ad} a} - \nabla_{\operatorname{ad} a}\nabla_X + \nabla_{[\operatorname{ad} a, X]}\right) Z \\ &= \nabla_X(aZ) - a(\nabla_X Z) + \nabla_{\operatorname{ad}(-X\cdot a)} Z \\ &= (X\cdot a)Z + a(\nabla_X Z) - a(\nabla_X Z) + (-X\cdot a)Z = 0.\end{aligned}$$

Since $R(X,Y)$ is bilinear and antisymmetric in X and Y, it follows that $R(X,Y)$ only depends on the projections of X and Y into the $\mathbb{C}$-span of $\partial_1, \dots, \partial_n$.

Proposition 3.3. *In the context of Definition 3.1, if $X, Y \in \mathcal{D}_\Theta$, then $R(X,Y)$ is A_Θ^∞-linear, i.e., "is a tensor".*

Proof. The classical proof works without change. Just expand

$$R(X,Y)(aZ) = \left(\nabla_Y\nabla_X - \nabla_X\nabla_Y + \nabla_{[X,Y]}\right)(aZ)$$

using the Leibniz rule and observe that all the terms involving $X\cdot a$ and $Y\cdot a$ cancel out. ■

Proposition 3.4. *In the context of Definition 3.1, the Riemannian curvature satisfies the following symmetry properties for all j, k, ℓ, m:*

1. $R_{j,k,\ell,m} + R_{k,\ell,j,m} + R_{\ell,j,k,m} = 0$ (***Bianchi identity***);
2. $R_{j,k,\ell,m} = -R_{k,j,\ell,m}$;
3. $R_{j,k,\ell,m} = -R_{j,k,m,\ell}$;
4. $R_{j,k,\ell,m} = R_{\ell,m,j,k}$.

Proof. The easiest is (2), which is immediate from the fact that the definition of $R(X,Y)$ is antisymmetric in X and Y. To prove (3), it's enough to show that for $X, Y \in \mathcal{D}_\Theta$ and $Z \in \mathcal{X}_\Theta$, $\langle R(X,Y)Z, Z\rangle = 0$. Using compatibility of ∇ with the metric and symmetry of the metric, we find that

$$\begin{aligned}\langle R(X,Y)Z, Z\rangle &= \big\langle \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z, Z\big\rangle \\ &= Y\langle \nabla_X Z, Z\rangle - X\langle \nabla_Y Z, Z\rangle + \frac{1}{2}[X,Y]\langle Z, Z\rangle \\ &= \frac{1}{2}\Big[YX\langle Z,Z\rangle - XY\langle Z,Z\rangle + [X,Y]\langle Z,Z\rangle\Big] = 0.\end{aligned}$$

Next we prove (1). We expand $R(\partial_j, \partial_k)\partial_\ell$, etc., using the fact that ∇ is torsion-free, and obtain

$$\begin{aligned}&R(\partial_j,\partial_k)\partial_\ell + R(\partial_k,\partial_\ell)\partial_j + R(\partial_\ell,\partial_j)\partial_k \\ &\quad = \nabla_k\nabla_j\partial_\ell - \nabla_j\nabla_k\partial_\ell + \nabla_\ell\nabla_k\partial_j - \nabla_k\nabla_\ell\partial_j + \nabla_j\nabla_\ell\partial_k - \nabla_\ell\nabla_j\partial_k \\ &\quad = (\nabla_k\nabla_\ell\partial_j - \nabla_k\nabla_\ell\partial_j) + (\nabla_\ell\nabla_j\partial_k - \nabla_\ell\nabla_j\partial_k) + (\nabla_j\nabla_k\partial_\ell - \nabla_j\nabla_k\partial_\ell) = 0,\end{aligned}$$

proving the Bianchi identity.

Finally, we prove (4). From (1), we have

$$\begin{aligned}&R_{j,k,\ell,m} + R_{k,\ell,j,m} + R_{\ell,j,k,m} = 0, \\ &R_{m,j,k,\ell} + R_{j,k,m,\ell} + R_{k,m,j,\ell} = 0, \\ &R_{\ell,m,j,k} + R_{m,j,\ell,k} + R_{j,\ell,m,k} = 0, \\ &R_{k,\ell,m,j} + R_{\ell,m,k,j} + R_{m,k,\ell,j} = 0.\end{aligned}$$

Sum these and use (2) and (3). We obtain

$$2R_{\ell,j,k,m} + 2R_{k,m,j,\ell} = 0,$$

which shows that $R_{k,m,j,\ell} = -R_{\ell,j,k,m} = R_{j,\ell,k,m}$, which is equivalent to (4). ■

As in classical Riemannian geometry, the symmetry properties of the curvature (Theorem 3.4) greatly cut down the number of independent components of the curvature, especially in low dimension. For example, for the irrational rotation algebra A_θ (the case $n = 2$), the curvature is entirely determined by $R_{1,2,1,2}$.

4 An example

We illustrate our theory in the case of the irrational rotation algebra A_θ (for generic θ), and a Riemannian metric that is a conformal deformation of the flat metric associated to the complex elliptic curve $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$. For this flat metric, we have $\langle \partial_j, \partial_k\rangle = \delta_{jk}$, so we choose a conformal factor $h = h^* \in A_\theta^\infty$ and suppose $\langle \partial_j, \partial_k \rangle = e^h \delta_{jk}$. Formula (2.1) then determines the Levi-Civita connection; we have

$$\langle \nabla_j \partial_k, \partial_l \rangle = \frac{1}{2}\big[\delta_{kl}\partial_j\big(e^h\big) + \delta_{jl}\partial_k\big(e^h\big) - \delta_{jk}\partial_\ell\big(e^h\big)\big]. \tag{4.1}$$

For example, from (4.1),

$$\langle \nabla_1 \partial_1, \partial_1 \rangle = \frac{1}{2}\partial_1\big(e^h\big), \qquad \langle \nabla_2 \partial_2, \partial_2 \rangle = \frac{1}{2}\partial_2\big(e^h\big).$$

We also get

$$\langle \nabla_1 \partial_1, \partial_2 \rangle = -\frac{1}{2}\partial_2\big(e^h\big), \qquad \langle \nabla_2 \partial_2, \partial_1 \rangle = -\frac{1}{2}\partial_1\big(e^h\big).$$

So these imply that

$$\nabla_1\partial_1 = -\nabla_2\partial_2 = \frac{1}{2}\big(\partial_1\big(e^h\big)e^{-h}\partial_1 - \partial_2\big(e^h\big)e^{-h}\partial_2\big) = \frac{1}{2}\big(k_1\partial_1 - k_2\partial_2\big),$$

where we have written $k_j = \partial_j(e^h)e^{-h}$. Similarly

$$\nabla_2\partial_1 = \nabla_1\partial_2 = \frac{1}{2}\big(\partial_2\big(e^h\big)e^{-h}\partial_1 + \partial_1\big(e^h\big)e^{-h}\partial_2\big) = \frac{1}{2}\big(k_2\partial_1 + k_1\partial_2\big).$$

This makes it possible to compute the curvature. We obtain

$$\begin{aligned} R_{1,2,1,2} &= \langle R(\partial_1,\partial_2)\partial_1, \partial_2\rangle = \langle \nabla_2\nabla_1\partial_1 - \nabla_1\nabla_2\partial_1, \partial_2\rangle \\ &= \frac{1}{2}\Big\langle \nabla_2\big(k_1\partial_1 - k_2\partial_2\big) - \nabla_1\big(k_2\partial_1 + k_1\partial_2\big), \partial_2\Big\rangle \\ &= \frac{1}{2}\Big\langle \partial_2(k_1)\partial_1 + k_1\nabla_2\partial_1 - \partial_2(k_2)\partial_2 - k_2\nabla_2\partial_2 \\ &\qquad - \partial_1(k_2)\partial_1 - k_2\nabla_1\partial_1 - \partial_1(k_1)\partial_2 - k_1\nabla_1\partial_2, \partial_2\Big\rangle. \end{aligned}$$

The four terms without a ∇_j in them contribute

$$-\frac{1}{2}\big\langle\big(\partial_2(k_2) + \partial_1(k_1)\big)\partial_2, \partial_2\big\rangle = -\frac{1}{2}\big(\partial_2(k_2) + \partial_1(k_1)\big)e^h.$$

The remaining four terms contribute

$$\begin{aligned} &\frac{1}{2}\Big[k_1\langle\nabla_2\partial_1,\partial_2\rangle - k_2\langle\nabla_2\partial_2,\partial_2\rangle - k_2\langle\nabla_1\partial_1,\partial_2\rangle - k_1\langle\nabla_1\partial_2,\partial_2\rangle\Big] \\ &\quad = \frac{1}{4}\Big[k_1\partial_1\big(e^h\big) - k_2\partial_2\big(e^h\big) + k_2\partial_2\big(e^h\big) - k_1\partial_1\big(e^h\big)\Big] = 0. \end{aligned}$$

So we conclude that

$$R_{1,2,1,2} = -\frac{1}{2}\big(\partial_2(k_2) + \partial_1(k_1)\big)e^h \tag{4.2}$$

and expanding using the definitions of k_1 and k_2:

$$\begin{aligned} &= -\frac{1}{2}\big(\partial_2^2\big(e^h\big)e^{-h} - \partial_2\big(e^h\big)e^{-h}\partial_2\big(e^h\big)e^{-h} + \partial_1^2\big(e^h\big)e^{-h} - \partial_1\big(e^h\big)e^{-h}\partial_1\big(e^h\big)e^{-h}\big)e^h \\ &= -\frac{1}{2}\big(\Delta\big(e^h\big) - \partial_1\big(e^h\big)e^{-h}\partial_1\big(e^h\big) - \partial_2\big(e^h\big)e^{-h}\partial_2\big(e^h\big)\big), \end{aligned} \tag{4.3}$$

where Δ is the Laplacian $\partial_1^2+\partial_2^2$. If h and its derivatives all commute, then k_j would just be $\partial_j(h)$ and this would reduce to $-\frac{1}{2}e^h\Delta h$. On the other hand, in the commutative case, we would really want the *Gaussian curvature* K, which would be $e^{-2h}R_{1,2,1,2}$ (since the vector fields ∂_1 and ∂_2 are orthogonal but not normalized), and we'd get the classical formula $K=-\frac{1}{2}e^{-h}\Delta h$. Our calculation is clearly related to, but vastly simpler, than the calculations in [3] and [7]. Reconciling these very different approaches to curvature in noncommutative geometry is an important problem for the future. However, we note that we do have an analogue of the Gauss–Bonnet theorem in our context, which can be formulated as follows:

Proposition 4.1 (Gauss–Bonnet theorem)**.** *Let A_θ^∞ be a smooth irrational rotation algebra, with generic θ, equipped with a Riemannian metric $\langle\partial_j,\partial_k\rangle=e^h\delta_{jk}$, $h=h^*\in A_\theta^\infty$ as above. Then if τ is the canonical trace on A_θ, we have $\tau(R_{1,2,1,2}e^{-h})=0$.*

Proof. By formula (4.2), $R_{1,2,1,2}\,e^{-h}$ is, up to a factor of $-\frac{1}{2}$, just $\partial_1(k_1)+\partial_2(k_2)$. But $\tau(\partial_j(a))=0$ for any a, since τ is invariant under the gauge action of $\mathbb{T}^2$ (see also [11, Lemma 2.1]). ■

Remark 4.2. We should explain why Proposition 4.1, in the commutative case of $\mathbb{T}^2$ ($\theta=0$), really is the Gauss–Bonnet theorem. In that case, τ is integration against Haar measure, and so $\tau(\,\cdot\,e^h)$ is integration against the Riemannian volume form, which differs from the standard volume form by $\sqrt{\det(g)}=e^h$. Since $K=e^{-2h}R_{1,2,1,2}$, the integral of K against the Riemannian volume form is thus $\tau(R_{1,2,1,2}\,e^{-2h}e^h)=\tau(R_{1,2,1,2}\,e^{-h})$.

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