

$\mathfrak{spo}(2|2)$ -Equivariant Quantizations on the Supercircle $S^{1|2}$

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Abstract. We consider the space of differential operators $\mathcal{D}_{\lambda\mu}$ acting between λ - and μ -densities defined on $S^{1|2}$ endowed with its standard contact structure. This contact structure allows one to define a filtration on $\mathcal{D}_{\lambda\mu}$ which is finer than the classical one, obtained by writing a differential operator in terms of the partial derivatives with respect to the different coordinates. The space $\mathcal{D}_{\lambda\mu}$ and the associated graded space of symbols \mathcal{S}_δ ($\delta = \mu - \lambda$) can be considered as $\mathfrak{spo}(2|2)$ -modules, where $\mathfrak{spo}(2|2)$ is the Lie superalgebra of contact projective vector fields on $S^{1|2}$. We show in this paper that there is a unique isomorphism of $\mathfrak{spo}(2|2)$ -modules between \mathcal{S}_δ and $\mathcal{D}_{\lambda\mu}$ that preserves the principal symbol (i.e. an $\mathfrak{spo}(2|2)$ -equivariant quantization) for some values of δ called non-critical values. Moreover, we give an explicit formula for this isomorphism, extending in this way the results of [Mellouli N., *SIGMA* **5** (2009), 111, 11 pages] which were established for second-order differential operators. The method used here to build the $\mathfrak{spo}(2|2)$ -equivariant quantization is the same as the one used in [Mathonet P., Radoux F., *Lett. Math. Phys.* **98** (2011), 311–331] to prove the existence of a $\mathfrak{pgl}(p+1|q)$ -equivariant quantization on $\mathbb{R}^{p|q}$.

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1 Introduction

The concept of equivariant quantization over \mathbb{R}^n was introduced by P. Lecomte and V. Ovsienko in [14]. An equivariant quantization is a linear bijection between a space of differential operators and its corresponding space of symbols that commutes with the action of a Lie subalgebra of vector fields over \mathbb{R}^n and preserves the principal symbol.

In their seminal work [14], P. Lecomte and V. Ovsienko considered spaces of differential operators acting between densities and the Lie algebra of projective vector fields over \mathbb{R}^n , $\mathfrak{sl}(n+1)$. In this situation, they showed the existence and uniqueness of an equivariant quantization.

The results of [14] were generalized in many references: in [2, 3, 6, 12] the authors considered other spaces of differential operators or other Lie subalgebras of vector fields over the Euclidean space. In [13], P. Lecomte globalized the problem of equivariant quantization by defining the problem of natural invariant quantization on arbitrary manifolds. Finally in [4, 5, 7, 9, 10, 18, 19, 20, 21], the authors proved the existence of such quantizations by using different methods in more and more general contexts.

Recently, several papers dealt with the problem of equivariant quantizations in the context of supergeometry: the thesis [24] dealt with *conformally equivariant quantizations* over super-

cotangent bundles, the papers [22] and [16] exposed and solved respectively the problems of the $\mathfrak{pgl}(p+1|q)$ -equivariant quantization over $\mathbb{R}^{p|q}$ and of the $\mathfrak{osp}(p+1, q+1|2r)$ -equivariant quantization over $\mathbb{R}^{p+q|2r}$, whereas in [17], the authors define the problem of the natural and projectively invariant quantization on arbitrary supermanifolds and show the existence of such a map.

In [8, 23] the problem of equivariant quantizations over the supercircles $S^{1|1}$ and $S^{1|2}$ endowed with canonical contact structures was considered. These quantizations are equivariant with respect to Lie superalgebras of contact projective vector fields. These Lie superalgebras are the intersections of the Lie superalgebras of contact vector fields on $S^{1|1}$ and $S^{1|2}$ and of the projective Lie superalgebras $\mathfrak{pgl}(2|1)$ and $\mathfrak{pgl}(2|2)$. In these works, the spaces of differential operators are endowed with filtrations which are defined thanks to the contact structures and which are finer than the classical ones. The spaces of symbols are then the graded spaces corresponding to these finer filtrations. In [8], the authors show the existence of equivariant quantizations at an arbitrary order whereas in [23], N. Mellouli proved the existence of equivariant quantizations up to order two.

In this paper, we aim to build an $\mathfrak{spo}(2|2)$ -equivariant quantization at an arbitrary order on $S^{1|2}$, where $\mathfrak{spo}(2|2)$ stands for the Lie superalgebra of contact projective vector fields on $S^{1|2}$. Moreover, we derive an explicit formula for this quantization, extending in this way the results of [23]. The method used here to build the quantization is the same as the one linked to the Casimir operators used in [22] to build the $\mathfrak{pgl}(p+1|q)$ -equivariant quantization on $\mathbb{R}^{p|q}$.

The paper is organized as follows. In Section 2, we recall the definitions of the objects that occur in the problem of quantization such as densities, differential operators and symbols. In Section 3, we expose the tools that we are going to use to build the quantization. These tools have already been defined in [22]. The main task performed in Section 3 is the computation and the comparison of the second-order Casimir operators of $\mathfrak{spo}(2|2)$ acting on the space of differential operators and on the space of symbols on $S^{1|2}$. Section 4 is devoted to the explicit construction of the $\mathfrak{spo}(2|2)$ -equivariant quantization map, built from the techniques developed in Section 3. The method used in Section 4 allows one to find in Section 5 explicit formulae for the $\mathfrak{spo}(2|2)$ -equivariant quantization at an arbitrary order.

2 Notation and problem setting

In this section, we recall some tools pertaining to the problem of equivariant quantization such as tensor (or weighted) densities, differential operators, symbols, contact projective vector fields on $S^{1|2}$. These objects were already exposed in [23].

The only point that we will deepen concerns the Lie superalgebra of contact projective vector fields, $\mathfrak{spo}(2|2)$. Actually, we will realize this Lie superalgebra as a Lie subsuperalgebra of $\mathfrak{gl}(2|2)$, allowing in this way to define easily Casimir operators associated with representations of $\mathfrak{spo}(2|2)$. This point will be crucial in the sequel.

2.1 Functions and vector fields on $S^{1|2}$

We define the supercircle $S^{1|2}$ by describing its graded commutative algebra of functions which we denote by $C^\infty(S^{1|2})$ and which is constituted by the elements

$$f(x, \theta_1, \theta_2) = f_0(x) + \theta_1 f_1(x) + \theta_2 f_2(x) + \theta_1 \theta_2 f_{12}(x),$$

where x is the coordinate corresponding to one of the two affine coordinates system on $\mathbb{R}P^1$, θ_1 and θ_2 are odd Grassmann coordinates and where $f_0, f_{12}, f_1, f_2 \in C^\infty(S^1)$ are functions with complex values. We define the parity function $\tilde{\cdot}$ by setting $\tilde{x} = 0$ and $\tilde{\theta}_1 = \tilde{\theta}_2 = 1$.

A *vector field* on $S^{1|2}$ is a derivation of the graded commutative algebra $C^\infty(S^{1|2})$. It can be expressed as

$$X = f\partial_x + g_1\partial_{\theta_1} + g_2\partial_{\theta_2},$$

where $f, g_1, g_2 \in C^\infty(S^{1|2})$, $\partial_x = \frac{\partial}{\partial x}$ and $\partial_{\theta_i} = \frac{\partial}{\partial \theta_i}$, for $i = 1, 2$. The space of vector fields on $S^{1|2}$ is a Lie superalgebra which we shall denote by $\text{Vect}(S^{1|2})$.

2.2 The Lie superalgebra of contact vector fields

The *standard contact* structure on $S^{1|2}$ is defined by the data of a linear distribution $\langle \bar{D}_1, \bar{D}_2 \rangle$ on $S^{1|2}$ generated by the odd vector fields

$$\bar{D}_1 = \partial_{\theta_1} - \theta_1\partial_x, \quad \bar{D}_2 = \partial_{\theta_2} - \theta_2\partial_x.$$

A vector field X on $S^{1|2}$ is called a *contact vector field* if it preserves the contact distribution, that is, satisfies the condition:

$$[X, \bar{D}_1] = \psi_{1X}\bar{D}_1 + \psi_{2X}\bar{D}_2, \quad [X, \bar{D}_2] = \phi_{1X}\bar{D}_1 + \phi_{2X}\bar{D}_2,$$

where $\psi_{1X}, \psi_{2X}, \phi_{1X}, \phi_{2X} \in C^\infty(S^{1|2})$ are functions depending on X . The space of contact vector fields is a Lie superalgebra which we shall denote by $\mathcal{K}(2)$.

It is well-known that every contact vector field can be expressed, for some function $f \in C^\infty(S^{1|2})$, by

$$X_f = f\partial_x - (-1)^{\tilde{f}} \frac{1}{2} (\bar{D}_1(f)\bar{D}_1 + \bar{D}_2(f)\bar{D}_2).$$

The function f is said to be a *contact Hamiltonian* of the field X_f . The space $C^\infty(S^{1|2})$ is therefore identified with the Lie superalgebra $\mathcal{K}(2)$ and is equipped with a structure of Lie superalgebra thanks to the following contact bracket:

$$\{f, g\} = fg' - f'g - (-1)^{\tilde{f}} \frac{1}{2} (\bar{D}_1(f)\bar{D}_1(g) + \bar{D}_2(f)\bar{D}_2(g)),$$

where $f' = \partial_x(f)$.

2.3 The Lie superalgebra $\mathfrak{spo}(2|2)$

The Lie superalgebra $\mathfrak{spo}(2|2)$ is the intersection of the Lie superalgebra $\mathcal{K}(2)$ and the Lie superalgebra of projective vector fields $\mathfrak{pgl}(2|2)$ exposed in [22]. The Lie superalgebra $\mathfrak{spo}(2|2)$ is thus a 4|4-dimensional Lie superalgebra spanned by the contact vector fields associated with the following contact Hamiltonians:

$$\{1, x, \theta_1, \theta_2, \theta_1\theta_2, x^2, x\theta_1, x\theta_2\}.$$

The Lie subsuperalgebra $\mathfrak{Aff}(2|2)$ of $\mathfrak{spo}(2|2)$ spanned by the contact vector fields associated with the contact Hamiltonians $\{1, x, \theta_1, \theta_2, \theta_1\theta_2\}$ will be called the *affine* Lie superalgebra.

Actually, the Lie superalgebra $\mathfrak{spo}(2|2)$ can be realized as the embedding of a Lie superalgebra consisted of matrices belonging to $\mathfrak{gl}(2|2)$ into $\text{Vect}(S^{1|2})$.

We shall also denote this matrix realization by $\mathfrak{spo}(2|2)$ (remark that this matrix realization has not to be confused with the special Poisson Lie superalgebra). The matrix realization of $\mathfrak{spo}(2|2)$ is e.g. exposed in [15, p. 419]. In this reference, this Lie superalgebra is denoted

by $\mathfrak{osp}^{sk}(2|2)$; this is the Lie subsuperalgebra of $\mathfrak{gl}(2|2)$ made of the matrices A that preserve a particular superskewsymmetric even bilinear form ω defined on $\mathbb{R}^{2|2}$, in the sense that

$$\omega(AU, V) + (-1)^{\tilde{A}\tilde{U}} \omega(U, AV) = 0 \quad \text{for all } U, V \in \mathbb{R}^{2|2}.$$

This particular form ω is defined on $\mathbb{R}^{2|2}$ by $\omega(U, V) = V^t G U$, where

$$G = \begin{pmatrix} J & 0 \\ 0 & S \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The Lie superalgebra $\mathfrak{spo}(2|2)$ is then constituted by the matrices A in $\mathfrak{gl}(2|2)$ such that

$$A^{st}G + GA = 0,$$

where the supertranspose of the matrix A , A^{st} , is defined by

$$\begin{pmatrix} A_1^t & -A_3^t \\ A_2^t & A_4^t \end{pmatrix}$$

if the matrix A is equal to

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}.$$

The blocks constituting the matrix A have thus to verify the following properties:

- $A_1^t J + J A_1 = 0$, i.e. $A_1 \in \mathfrak{sp}(2)$;
- $A_4^t + A_4 = 0$, i.e. $A_4 \in \mathfrak{o}(2)$;
- $A_3 = -A_2^t J$.

The Lie superalgebra $\mathfrak{spo}(2|2)$ is then 4|4-dimensional and one of its bases is constituted by the following matrices:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

The upper-left blocks of the first three matrices provide a basis of $\mathfrak{sp}(2)$, whereas the lower-right block of the fourth provides a basis of $\mathfrak{o}(2)$.

With the above considerations, it is obvious that $\mathfrak{spo}(2|2)$ can be embedded into the projective Lie superalgebra $\mathfrak{pgl}(2|2)$ via the map ι defined in the following way:

$$\iota : \mathfrak{spo}(2|2) \rightarrow \mathfrak{pgl}(2|2) : A \mapsto [A].$$

Now, $\mathfrak{pgl}(2|2)$ can be embedded into the Lie superalgebra of vector fields on $S^{1|2}$ thanks to the projective embedding defined in [22] in the following way:

$$\left[\begin{pmatrix} 0 & \xi \\ v & B \end{pmatrix} \right] \mapsto - \sum_{i=1}^3 v^i \partial_{y^i} - \sum_{i,j=1}^3 (-1)^{\tilde{j}(\tilde{i}+\tilde{j})} B_j^i y^j \partial_{y^i} + \sum_{j=1}^3 (-1)^{\tilde{j}} \xi_j y^j \partial_{y^i},$$

where $v \in \mathbb{R}^{1|2}$, $\xi \in \mathbb{R}^{1|2*}$, $B \in \mathfrak{gl}(1|2)$ and where the coordinates y^1, y^2, y^3 correspond respectively to the coordinates x, θ_1, θ_2 .

Composing ι with the projective embedding, we can embed $\mathfrak{spo}(2|2)$ into the Lie superalgebra of vector fields on $S^{1|2}$. If we compute this embedding on the generators of $\mathfrak{spo}(2|2)$ written above, we obtain respectively $2X_x, X_{x^2}, -X_1, 2X_{\theta_1\theta_2}, -2X_{x\theta_1}, -2X_{x\theta_2}, 2X_{\theta_1}$ and $2X_{\theta_2}$.

2.4 Modules of weighted densities

For any contact vector field X_f , we define a family of differential operators of order one on $C^\infty(S^{1|2})$, denoted by $L_{X_f}^\lambda$, in the following way:

$$L_{X_f}^\lambda = X_f + \lambda f',$$

where the parameter λ is an arbitrary (complex) number and where f' denotes the left multiplication by f' . The map $X_f \mapsto L_{X_f}^\lambda$ is a homomorphism of Lie superalgebras. We thus obtain a family of $\mathcal{K}(2)$ -modules on $C^\infty(S^{1|2})$ which we shall denote by \mathcal{F}_λ and which we shall call spaces of *weighted densities* of weight λ .

2.5 Differential operators and symbols

In the sequel, we will call “natural number” a non-negative integer.

For every (half)-natural number k , we denote by $\mathcal{D}_{\lambda\mu}^k$ the space of differential operators acting between λ - and μ -densities that are of the form

$$\sum_{l+\frac{m}{2}+\frac{n}{2}\leq k} a_{l,m,n} (\partial_x)^l \bar{D}_1^m \bar{D}_2^n,$$

where $a_{l,m,n} \in C^\infty(S^{1|2})$ for all l, m, n . Furthermore, since $\partial_x = -\bar{D}_1^2 = -\bar{D}_2^2$, we can assume $m, n \leq 1$. The space $\mathcal{D}_{\lambda\mu}^k$ will be called the space of differential operators of order k .

We define then $\mathcal{D}_{\lambda\mu}$, the space of differential operators acting between λ - and μ -densities as the union of the spaces $\mathcal{D}_{\lambda\mu}^k$:

$$\mathcal{D}_{\lambda\mu} = \bigcup_{k \in \frac{\mathbb{N}}{2}} \mathcal{D}_{\lambda\mu}^k,$$

where $\frac{\mathbb{N}}{2}$ denotes the set consisting of natural and half-natural numbers. Since $\mathcal{D}_{\lambda\mu}^k \subset \mathcal{D}_{\lambda\mu}^{k+\frac{1}{2}}$, the space $\mathcal{D}_{\lambda\mu}$ is a filtered space. This space has a structure of a $\mathcal{K}(2)$ -module defined in the following way: if $D \in \mathcal{D}_{\lambda\mu}$, then the Lie derivative of D in the direction of X_f , denoted by $\mathcal{L}_{X_f} D$, is given by the differential operator

$$L_{X_f}^\mu \circ D - (-1)^{\tilde{f}\tilde{D}} D \circ L_{X_f}^\lambda.$$

Moreover, it turns out that the action of $\mathcal{K}(2)$ preserves the order of D .

The graded space associated with the filtered space $\mathcal{D}_{\lambda\mu}$ is called the space of symbols and is denoted by \mathcal{S}_δ , where $\delta = \mu - \lambda$:

$$\mathcal{S}_\delta = \bigoplus_{k \in \frac{\mathbb{N}}{2}} \mathcal{S}_\delta^k,$$

where $\mathcal{S}_\delta^k = \mathcal{D}_{\lambda\mu}^k / \mathcal{D}_{\lambda\mu}^{k-\frac{1}{2}}$ for every (half)-natural number k .

The principal symbol map, denoted by σ , is the map defined on $\mathcal{D}_{\lambda\mu}$ whose the restriction to $\mathcal{D}_{\lambda\mu}^k$, denoted by σ_k , is simply defined by:

$$\sigma_k : \mathcal{D}_{\lambda\mu}^k \rightarrow \mathcal{S}_\delta^k : D \mapsto [D].$$

If k is a natural number, the space of symbols of degree k , \mathcal{S}_δ^k , is isomorphic as vector space to $\mathcal{F}_{\delta-k} \oplus \mathcal{F}_{\delta-k}$ through the following identification:

$$[F_1 \partial_x^k + F_2 \partial_x^{k-1} \bar{D}_1 \bar{D}_2] \longleftrightarrow (F_1, F_2).$$

Thanks to the fact that the action of $\mathcal{K}(2)$ on $\mathcal{D}_{\lambda\mu}$ preserves the filtration of this space, the $\mathcal{K}(2)$ -module structure on $\mathcal{D}_{\lambda\mu}^k$ induces a $\mathcal{K}(2)$ -module structure on \mathcal{S}_δ^k . If we denote by L_{X_f} the Lie derivative of a symbol in the direction of X_f , we have

$$L_{X_f}[D] := [\mathcal{L}_{X_f}D].$$

If a symbol of degree k is represented by a pair of densities (F_1, F_2) , it is easy to see that $L_{X_f}(F_1, F_2)$ corresponds to the pair

$$(L_{X_f}^{\delta-k} F_1, L_{X_f}^{\delta-k} F_2).$$

If k is half of a natural number, the space of symbols of degree k , \mathcal{S}_δ^k , is also isomorphic as vector space to $\mathcal{F}_{\delta-k} \oplus \mathcal{F}_{\delta-k}$ through the following identification:

$$\left[F_1 \partial_x^{k-\frac{1}{2}} \bar{D}_1 + F_2 \partial_x^{k-\frac{1}{2}} \bar{D}_2 \right] \longleftrightarrow (F_1, F_2).$$

As in the case where k is a natural number, the $\mathcal{K}(2)$ -module structure on $\mathcal{D}_{\lambda\mu}^k$ induces a $\mathcal{K}(2)$ -module structure on \mathcal{S}_δ^k . If a symbol of degree k is represented by a pair of densities (F_1, F_2) , it is easy to see that $L_{X_f}(F_1, F_2)$ corresponds to the pair

$$\left(L_{X_f}^{\delta-k} F_1 - \frac{1}{2} \bar{D}_1 \bar{D}_2(f) F_2, L_{X_f}^{\delta-k} F_2 + \frac{1}{2} \bar{D}_1 \bar{D}_2(f) F_1 \right).$$

We are now in position to define the notion of $\mathfrak{spo}(2|2)$ -equivariant quantization:

Definition 1. An $\mathfrak{spo}(2|2)$ -equivariant quantization is a linear bijection Q between \mathcal{S}_δ and $\mathcal{D}_{\lambda\mu}$ that preserves the principal symbol in the sense that $\sigma \circ Q = \text{Id}$ and that commutes with the action of $\mathfrak{spo}(2|2)$, in the sense that

$$\mathcal{L}_{X_f} \circ Q = Q \circ L_{X_f}$$

for all $X_f \in \mathfrak{spo}(2|2)$.

3 Tools used to build the quantization

In order to tackle the problem of $\mathfrak{spo}(2|2)$ -equivariant quantization, we will need to adapt the tools used in [22] for the case where $\mathfrak{g} = \mathfrak{pgl}(p+1|q)$. The main ingredients are the affine quantization map and the difference between the representations $(\mathcal{S}_\delta, \mathcal{L})$ and (\mathcal{S}_δ, L) of $\mathfrak{spo}(2|2)$ measured by the map γ (see Section 3.2). In the sequel, $\frac{1}{2} + \mathbb{N}$ will denote the set of half-natural numbers.

3.1 The affine quantization map

The *affine quantization map* Q_{Aff} is the linear bijection between \mathcal{S}_δ and $\mathcal{D}_{\lambda\mu}$ defined in the following way:

$$Q_{\text{Aff}}|_{\mathcal{S}_\delta^k}(F_1, F_2) = \begin{cases} F_1 \partial_x^k + F_2 \partial_x^{k-1} \bar{D}_1 \bar{D}_2, & \text{if } k \in \mathbb{N}, \\ F_1 \partial_x^{k-\frac{1}{2}} \bar{D}_1 + F_2 \partial_x^{k-\frac{1}{2}} \bar{D}_2, & \text{if } k \in \frac{1}{2} + \mathbb{N}. \end{cases}$$

Thanks to the map Q_{Aff} , we can carry the $\text{Vect}(S^{1|2})$ -module structure of $\mathcal{D}_{\lambda,\mu}$ to \mathcal{S}_δ by defining a Lie derivative \mathcal{L} on \mathcal{S}_δ in the following way:

$$\mathcal{L}_X = Q_{\text{Aff}}^{-1} \circ \mathcal{L}_X \circ Q_{\text{Aff}}$$

for all X in $\text{Vect}(S^{1|2})$.

The existence of an $\mathfrak{spo}(2|2)$ -equivariant quantization is then equivalent to the existence, for all (half)-natural number k , of an $\mathfrak{spo}(2|2)$ -equivariant map

$$Q : (\mathcal{S}_\delta^k, L) \rightarrow (\mathcal{S}_\delta, \mathcal{L})$$

such that the homogeneous part of highest degree of $Q(S)$ is equal to S for all $S \in \mathcal{S}_\delta^k$. Indeed, Q is a map of this type if and only if $Q_{\text{Aff}} \circ Q$ is an $\mathfrak{spo}(2|2)$ -equivariant quantization in the sense of Definition 1.

3.2 The map γ

The difference between the representations $(\mathcal{S}_\delta, \mathcal{L})$ and (\mathcal{S}_δ, L) of $\mathfrak{spo}(2|2)$ is measured by the map

$$\gamma : \mathfrak{spo}(2|2) \rightarrow \mathfrak{gl}(\mathcal{S}_\delta, \mathcal{S}_\delta) : X_f \mapsto \gamma(X_f) = \mathcal{L}_{X_f} - L_{X_f}.$$

In order to compute this map, the following lemma, which gives some commutators, will be useful.

Lemma 1. *If $k \in \mathbb{N}$ and $X_f \in \mathfrak{spo}(2|2)$, then, if we consider that the operators X_f , ∂_x , \bar{D}_1 and \bar{D}_2 act on \mathcal{F}_λ (where $\lambda \in \mathbb{R}$), we obtain the following relations:*

$$\begin{aligned} [X_f, \partial_x^k] &= -kf' \partial_x^k + k \frac{(-1)^{\bar{f}}}{2} (\bar{D}_1(f') \partial_x^{k-1} \bar{D}_1 + \bar{D}_2(f') \partial_x^{k-1} \bar{D}_2) - \frac{k(k-1)}{2} f'' \partial_x^{k-1}, \\ [X_f, \bar{D}_1] &= -\frac{1}{2} f' \bar{D}_1 + \frac{1}{2} \bar{D}_1 \bar{D}_2(f) \bar{D}_2, \quad [X_f, \bar{D}_2] = -\frac{1}{2} f' \bar{D}_2 - \frac{1}{2} \bar{D}_1 \bar{D}_2(f) \bar{D}_1 \end{aligned}$$

and

$$[\partial_x^k, f] = kf' \partial_x^{k-1} + \frac{k(k-1)}{2} f'' \partial_x^{k-2},$$

if f denotes the left multiplication by f .

Proof. The first formula can be proved simply by induction. If $k = 1$, the relation is obvious. If the formula is true for k then

$$\begin{aligned} [X_f, \partial_x^{k+1}] &= [X_f, \partial_x \partial_x^k] = \left(-f' \partial_x + \frac{(-1)^{\bar{f}}}{2} (\bar{D}_1(f') \bar{D}_1 + \bar{D}_2(f') \bar{D}_2) \right) \partial_x^k \\ &\quad + \partial_x \left(-kf' \partial_x^k + k \frac{(-1)^{\bar{f}}}{2} (\bar{D}_1(f') \partial_x^{k-1} \bar{D}_1 + \bar{D}_2(f') \partial_x^{k-1} \bar{D}_2) - \frac{k(k-1)}{2} f'' \partial_x^{k-1} \right). \end{aligned}$$

It is then easy to see that the formula is true also for $k + 1$.

The formulae for $[X_f, \bar{D}_1]$ and $[X_f, \bar{D}_2]$ can be proved simply by a straightforward computation while the last formula can be proved e.g. by an easy induction. \blacksquare

We are now in position to compute the map γ . We shall say that a symbol (F_1, F_2) is even (resp. odd) if $\tilde{F}_1 = \tilde{F}_2 = 0$ (resp. $\tilde{F}_1 = \tilde{F}_2 = 1$). If (F_1, F_2) is a homogeneous symbol, we shall denote by \tilde{F} the number $\tilde{F}_1 = \tilde{F}_2$.

Proposition 1. *The map γ vanishes on the affine Lie subsuperalgebra $\mathfrak{Aff}(2|2)$.*

Also, for any $f \in \{x\theta_1, x\theta_2, x^2\}$, $\gamma(f)$ maps \mathcal{S}_δ^k to $\mathcal{S}_\delta^{k-\frac{1}{2}} \oplus \mathcal{S}_\delta^{k-1}$. More precisely, if $(F_1, F_2) \in \mathcal{S}_\delta^k$ is a homogeneous symbol, we have

$$(Q_{\text{Aff}} \circ \gamma(X_f))(F_1, F_2) = (-1)^{\bar{f}} \left((-1)^{\tilde{F}} \frac{k}{2} \bar{D}_1(f') F_1 + (-1)^{\tilde{F}} \left(\frac{k}{2} + \lambda \right) \bar{D}_2(f') F_2 \right) \partial_x^{k-1} \bar{D}_1$$

$$\begin{aligned}
& + (-1)^{\tilde{f}} \left((-1)^{\tilde{F}} \frac{k}{2} \bar{D}_2(f') F_1 - (-1)^{\tilde{F}} \left(\frac{k}{2} + \lambda \right) \bar{D}_1(f') F_2 \right) \partial_x^{k-1} \bar{D}_2 \\
& - k \left(\frac{k-1}{2} + \lambda \right) f'' F_1 \partial_x^{k-1} - (k-1) \left(\frac{k}{2} + \lambda \right) f'' F_2 \partial_x^{k-2} \bar{D}_1 \bar{D}_2,
\end{aligned}$$

for $k \in \mathbb{N}$ and

$$\begin{aligned}
(Q_{\text{Aff}} \circ \gamma(X_f))(F_1, F_2) & = -(-1)^{\tilde{F}+\tilde{f}} \left(\frac{k-\frac{1}{2}}{2} + \lambda \right) (\bar{D}_1(f') F_1 + \bar{D}_2(f') F_2) \partial_x^{k-\frac{1}{2}} \\
& + (-1)^{\tilde{f}} \frac{k-\frac{1}{2}}{2} \left((-1)^{\tilde{F}} \bar{D}_1(f') F_2 - (-1)^{\tilde{F}} \bar{D}_2(f') F_1 \right) \partial_x^{k-\frac{3}{2}} \bar{D}_1 \bar{D}_2 \\
& + \left(k - \frac{1}{2} \right) \left(\left(\frac{-k+\frac{1}{2}}{2} - \lambda \right) f'' F_1 \right) \partial_x^{k-\frac{3}{2}} \bar{D}_1 + \left(k - \frac{1}{2} \right) \left(\left(\frac{-k+\frac{1}{2}}{2} - \lambda \right) f'' F_2 \right) \partial_x^{k-\frac{3}{2}} \bar{D}_2
\end{aligned}$$

for $k \in \frac{1}{2} + \mathbb{N}$.

Proof. If $k \in \mathbb{N}$, $Q_{\text{Aff}}(F_1, F_2)$ can be expressed formally in the following matrix form:

$$Q_{\text{Aff}}(F_1, F_2) = (F_1, F_2) \begin{pmatrix} \partial_x^k \\ \partial_x^{k-1} \bar{D}_1 \bar{D}_2 \end{pmatrix}.$$

If we denote by D the column vector $\begin{pmatrix} \partial_x^k \\ \partial_x^{k-1} \bar{D}_1 \bar{D}_2 \end{pmatrix}$, $\mathcal{L}_{X_f}(Q_{\text{Aff}}(F_1, F_2))$ is then equal to

$$(X_f + \mu f')((F_1, F_2)D) - (-1)^{\tilde{F}\tilde{f}}(F_1, F_2)D(X_f + \lambda f'),$$

which is equal to

$$\begin{aligned}
& (X_f \cdot F_1, X_f \cdot F_2)D + (-1)^{\tilde{F}\tilde{f}}(F_1, F_2)[X_f, D] + (\mu - \lambda)f'(F_1, F_2)D \\
& - (-1)^{\tilde{F}\tilde{f}}\lambda(F_1, F_2)[D, f'].
\end{aligned}$$

Thanks to the relations established in Lemma 1, we can see that $[X_f, D]$ is equal to

$$\begin{aligned}
& \begin{pmatrix} -k f' \partial_x^k + \frac{k}{2} (-1)^{\tilde{f}} (\bar{D}_1(f') \partial_x^{k-1} \bar{D}_1 + \bar{D}_2(f') \partial_x^{k-1} \bar{D}_2) \\ -k f' \partial_x^{k-1} \bar{D}_1 \bar{D}_2 + \frac{k}{2} (-1)^{\tilde{f}} (\bar{D}_2(f') \partial_x^{k-1} \bar{D}_1 - \bar{D}_1(f') \partial_x^{k-1} \bar{D}_2) \end{pmatrix} \\
& + \begin{pmatrix} -\frac{k(k-1)}{2} f'' \partial_x^{k-1} \\ -\frac{k(k-1)}{2} f'' \partial_x^{k-2} \bar{D}_1 \bar{D}_2 \end{pmatrix},
\end{aligned}$$

while $[D, f']$ is equal to

$$\begin{pmatrix} k f'' \partial_x^{k-1} \\ (-1)^{\tilde{f}+1} (\bar{D}_2 f') \partial_x^{k-1} \bar{D}_1 + (-1)^{\tilde{f}} (\bar{D}_1 f') \partial_x^{k-1} \bar{D}_2 + (k-1) f'' \partial_x^{k-2} \bar{D}_1 \bar{D}_2 \end{pmatrix}.$$

The terms $(X_f \cdot F_1, X_f \cdot F_2)D$, $(\mu - \lambda)f'(F_1, F_2)D$ and

$$(-1)^{\tilde{F}\tilde{f}}(F_1, F_2) \begin{pmatrix} -k f' \partial_x^k \\ -k f' \partial_x^{k-1} \bar{D}_1 \bar{D}_2 \end{pmatrix}$$

are used to reconstruct $Q_{\text{Aff}}(L_{X_f}(F_1, F_2))$. It is easy to see that the sum of the other terms is equal to $Q_{\text{Aff}}(\gamma(X_f)(F_1, F_2))$.

If $k \in \frac{1}{2} + \mathbb{N}$, $Q_{\text{Aff}}(F_1, F_2)$ can be expressed formally in the following matrix form:

$$Q_{\text{Aff}}(F_1, F_2) = (F_1, F_2) \begin{pmatrix} \partial_x^{k-\frac{1}{2}} \bar{D}_1 \\ \partial_x^{k-\frac{1}{2}} \bar{D}_2 \end{pmatrix}.$$

If we denote by D the column vector $\begin{pmatrix} \partial_x^{k-\frac{1}{2}} \bar{D}_1 \\ \partial_x^{k-\frac{1}{2}} \bar{D}_2 \end{pmatrix}$, $\mathcal{L}_{X_f}(Q_{\text{Aff}}(F_1, F_2))$ is then equal to

$$(X_f + \mu f')((F_1, F_2)D) - (-1)^{\tilde{f}(\tilde{F}+1)}(F_1, F_2)D(X_f + \lambda f'),$$

which is equal to

$$\begin{aligned} & (X_f \cdot F_1, X_f \cdot F_2)D + (-1)^{\tilde{f}\tilde{F}}(F_1, F_2)[X_f, D] + (\mu - \lambda)f'(F_1, F_2)D \\ & - (-1)^{\tilde{f}(\tilde{F}+1)}\lambda(F_1, F_2)[D, f']. \end{aligned}$$

Thanks to the relations established in the Lemma 1, we can see that $[X_f, D]$ is equal to

$$\begin{aligned} & \begin{pmatrix} -kf' \partial_x^{k-\frac{1}{2}} \bar{D}_1 + \frac{1}{2} \bar{D}_1 \bar{D}_2(f) \partial_x^{k-\frac{1}{2}} \bar{D}_2 \\ -kf' \partial_x^{k-\frac{1}{2}} \bar{D}_2 - \frac{1}{2} \bar{D}_1 \bar{D}_2(f) \partial_x^{k-\frac{1}{2}} \bar{D}_1 \end{pmatrix} \\ & - \begin{pmatrix} \frac{k-\frac{1}{2}}{2} (-1)^{\tilde{f}} \left(\bar{D}_1(f') \partial_x^{k-\frac{1}{2}} + \bar{D}_2(f') \partial_x^{k-\frac{3}{2}} \bar{D}_1 \bar{D}_2 \right) + \frac{(k-\frac{1}{2})^2}{2} f'' \partial_x^{k-\frac{3}{2}} \bar{D}_1 \\ -\frac{k-\frac{1}{2}}{2} (-1)^{\tilde{f}} \left(\bar{D}_1(f') \partial_x^{k-\frac{3}{2}} \bar{D}_1 \bar{D}_2 - \bar{D}_2(f') \partial_x^{k-\frac{1}{2}} \right) + \frac{(k-\frac{1}{2})^2}{2} f'' \partial_x^{k-\frac{3}{2}} \bar{D}_2 \end{pmatrix}, \end{aligned}$$

while $[D, f']$ is equal to

$$\begin{pmatrix} (-1)^{\tilde{f}} \left(k - \frac{1}{2} \right) f'' \partial_x^{k-\frac{3}{2}} \bar{D}_1 + \bar{D}_1(f') \partial_x^{k-\frac{1}{2}} \\ (-1)^{\tilde{f}} \left(k - \frac{1}{2} \right) f'' \partial_x^{k-\frac{3}{2}} \bar{D}_2 + \bar{D}_2(f') \partial_x^{k-\frac{1}{2}} \end{pmatrix}.$$

The terms $(X_f \cdot F_1, X_f \cdot F_2)D$, $(\mu - \lambda)f'(F_1, F_2)D$ and

$$(-1)^{\tilde{f}\tilde{F}}(F_1, F_2) \begin{pmatrix} -kf' \partial_x^{k-\frac{1}{2}} \bar{D}_1 + \frac{1}{2} \bar{D}_1 \bar{D}_2(f) \partial_x^{k-\frac{1}{2}} \bar{D}_2 \\ -kf' \partial_x^{k-\frac{1}{2}} \bar{D}_2 - \frac{1}{2} \bar{D}_1 \bar{D}_2(f) \partial_x^{k-\frac{1}{2}} \bar{D}_1 \end{pmatrix}$$

are used to reconstruct $Q_{\text{Aff}}(L_{X_f}(F_1, F_2))$. It is easy to see that the sum of the other terms is equal to $Q_{\text{Aff}}(\gamma(X_f)(F_1, F_2))$. ■

If $k \in \mathbb{N}$, then the component of $\gamma(X_f)|_{\mathcal{S}_\delta^k}$ with respect to $\mathcal{S}_\delta^{k-\frac{1}{2}}$ can be put in the following matrix form:

$$(-1)^{\tilde{f}+\tilde{F}} \begin{pmatrix} \frac{k}{2} \bar{D}_1(f') & \left(\frac{k}{2} + \lambda\right) \bar{D}_2(f') \\ \frac{k}{2} \bar{D}_2(f') & -\left(\frac{k}{2} + \lambda\right) \bar{D}_1(f') \end{pmatrix}$$

while the component with respect to \mathcal{S}_δ^{k-1} can be written in the following way:

$$-f'' \begin{pmatrix} k \left(\frac{k-1}{2} + \lambda\right) & 0 \\ 0 & (k-1) \left(\frac{k}{2} + \lambda\right) \end{pmatrix}.$$

If $k \in \frac{1}{2} + \mathbb{N}$, then the component of $\gamma(X_f)|_{\mathcal{S}_\delta^k}$ with respect to $\mathcal{S}_\delta^{k-\frac{1}{2}}$ can be put in the following matrix form:

$$-(-1)^{\tilde{f}+\tilde{F}} \begin{pmatrix} \left(\frac{k-\frac{1}{2}}{2} + \lambda\right) \bar{D}_1(f') & \left(\frac{k-\frac{1}{2}}{2} + \lambda\right) \bar{D}_2(f') \\ \frac{k-\frac{1}{2}}{2} \bar{D}_2(f') & -\left(\frac{k-\frac{1}{2}}{2}\right) \bar{D}_1(f') \end{pmatrix}$$

while the component with respect to \mathcal{S}_δ^{k-1} can be written in the following way:

$$\left(k - \frac{1}{2}\right) \left(\frac{-k + \frac{1}{2}}{2} - \lambda\right) f'' \text{Id.}$$

3.3 Casimir operators

The method that we are going to use here to build the $\mathfrak{spo}(2|2)$ -equivariant quantization is linked to the Casimir operators, as in [22]. Actually, the method is based on the comparison of the Casimir operators C and \mathcal{C} of $\mathfrak{spo}(2|2)$ associated with the representations L and \mathcal{L} on \mathcal{S}_δ . In this section, we are going to show that the Casimir operator C is diagonalizable and that there is a simple relation between C and \mathcal{C} .

First of all, let us recall the definition of the Casimir operator associated with a representation of a Lie superalgebra (see e.g. [1, 11, 25, 26, 27]).

Definition 2. We consider a Lie superalgebra \mathfrak{l} endowed with an even non-degenerate supersymmetric bilinear form K and a representation (V, β) of \mathfrak{l} . The Casimir operator C_β of \mathfrak{l} associated with (V, β) is defined by

$$C_\beta = \sum_i \beta(u_i^*) \beta(u_i),$$

where u_i and u_i^* are K -dual bases of \mathfrak{l} , in the sense that $K(u_i, u_j^*) = \delta_{i,j}$ for all i, j .

In the sequel, the bilinear form that we will use to define the Casimir operators of $\mathfrak{spo}(2|2)$ will be the form K defined in this way:

$$K(A, B) = 2 \text{str}(AB) \quad \forall A, B \in \mathfrak{spo}(2|2).$$

The following lemma gives the bases that will be used to define the Casimir operators of $\mathfrak{spo}(2|2)$.

Lemma 2. *The K -dual basis corresponding to the basis*

$$\{X_1, X_{\theta_1}, X_{\theta_2}, X_x, X_{\theta_1\theta_2}, X_{x\theta_1}, X_{x\theta_2}, X_{x^2}\}$$

of $\mathfrak{spo}(2|2)$ is given by the basis

$$\left\{ -\frac{1}{2} X_{x^2}, -X_{x\theta_1}, -X_{x\theta_2}, X_x, X_{\theta_1\theta_2}, X_{\theta_1}, X_{\theta_2}, -\frac{1}{2} X_1 \right\}.$$

Proof. The result is easily proved using the correspondence between vector fields of $\mathfrak{spo}(2|2)$ and matrices established in Section 2.3. \blacksquare

We are now in position to compute the Casimir operator associated with the representation $(\mathcal{S}_\delta^k, L)$. Actually, this operator is simply a multiple of the identity.

Proposition 2. *The Casimir operator associated with the representation (S_δ^k, L) of $\mathfrak{spo}(2|2)$ is given by $C|_{S_\delta^k} = \alpha_k \text{Id}$, where*

$$\alpha_k = \begin{cases} (-k + \delta)^2, & \text{if } k \in \mathbb{N}, \\ (-k + \delta)^2 - \frac{1}{4}, & \text{if } k \in \frac{1}{2} + \mathbb{N}. \end{cases}$$

Proof. First, using Definition 2 and Lemma 2, the Casimir operator $C|_{S_\delta^k}$ is equal to

$$\begin{aligned} & -\frac{1}{2}L_{X_{x_2}}L_{X_1} - L_{X_{x\theta_1}}L_{X_{\theta_1}} - L_{X_{x\theta_2}}L_{X_{\theta_2}} \\ & + (L_{X_x})^2 + (L_{X_{\theta_1\theta_2}})^2 + L_{X_{\theta_1}}L_{X_{x\theta_1}} + L_{X_{\theta_2}}L_{X_{x\theta_2}} - \frac{1}{2}L_{X_1}L_{X_{x_2}}. \end{aligned}$$

If $k \in \mathbb{N}$, thanks to the form of the hamiltonian vector fields, it is easy to see that this Casimir operator is a differential operator that can be written in the following way:

$$f_0 + f_1\partial_x + f_2\bar{D}_1 + f_3\bar{D}_2 + f_4\partial_x^2 + f_5\partial_x\bar{D}_1 + f_6\partial_x\bar{D}_2 + f_7\bar{D}_1\bar{D}_2,$$

where $f_i \in C^\infty(S^{1|2})$ for all i . Since the Casimir operator commutes with the vector field $X_1 = \partial_x$ and since ∂_x commutes with \bar{D}_1 and \bar{D}_2 , the coefficients f_i can not depend on the coordinate x .

Since the Casimir operator commutes with the action of the vector field X_{θ_1} , which is equal to $\frac{1}{2}(\partial_{\theta_1} + \theta_1\partial_x)$, and the action of the vector field X_{θ_2} , which is equal to $\frac{1}{2}(\partial_{\theta_2} + \theta_2\partial_x)$, and since these vector fields commute with ∂_x , \bar{D}_1 and \bar{D}_2 , the coefficients f_i are invariant under the actions of X_{θ_1} and X_{θ_2} , thus these coefficients can not depend on the coordinates θ_1 and θ_2 .

To summarize, $C|_{S_\delta^k}$ has constant coefficients.

Because of the expressions of the Lie derivative and of the vector fields X_{x_2} , $X_{x\theta_1}$, $X_{x\theta_2}$ and $X_{\theta_1\theta_2}$, the terms $-\frac{1}{2}L_{X_{x_2}}L_{X_1}$, $L_{X_{x\theta_1}}L_{X_{\theta_1}}$, $L_{X_{x\theta_2}}L_{X_{\theta_2}}$ and $(L_{X_{\theta_1\theta_2}})^2$ of $C|_{S_\delta^k}$ give then no contribution.

The term $-\frac{1}{2}L_{X_1}L_{X_{x_2}}$ gives the same contribution as the term

$$-\frac{1}{2}L_{[X_1, X_{x_2}]} = -\frac{1}{2}L_{X_{\{1, x^2\}}} = -L_{X_x}.$$

The contribution of this last term is equal to $-(\delta - k)\text{Id}$.

The term $(L_{X_x})^2$ gives as for it a contribution equal to $(\delta - k)^2\text{Id}$.

Eventually, each of the two terms $L_{X_{\theta_1}}L_{X_{x\theta_1}}$ and $L_{X_{\theta_2}}L_{X_{x\theta_2}}$ gives a contribution equal to $\frac{1}{2}(\delta - k)\text{Id}$.

To conclude, if $k \in \mathbb{N}$, $C|_{S_\delta^k} = (\delta - k)^2\text{Id}$.

If $k \in \frac{1}{2} + \mathbb{N}$, it is easy to see that the component number 1 (resp. 2) of $C|_{S_\delta^k}(F_1, F_2)$ can be written in the following way:

$$\begin{aligned} & (f_0 + f_1\partial_x + f_2\bar{D}_1 + f_3\bar{D}_2 + f_4\partial_x^2 + f_5\partial_x\bar{D}_1 + f_6\partial_x\bar{D}_2 + f_7\bar{D}_1\bar{D}_2)F_1 \\ & + (g_0 + g_1\partial_x + g_2\bar{D}_1 + g_3\bar{D}_2)F_2 \\ & (\text{resp. } (f_0 + f_1\partial_x + f_2\bar{D}_1 + f_3\bar{D}_2 + f_4\partial_x^2 + f_5\partial_x\bar{D}_1 + f_6\partial_x\bar{D}_2 + f_7\bar{D}_1\bar{D}_2)F_2 \\ & + (g_0 + g_1\partial_x + g_2\bar{D}_1 + g_3\bar{D}_2)F_1), \end{aligned}$$

where $f_i, g_i \in C^\infty(S^{1|2})$ for all i . As above, thanks to the fact that $C|_{S_\delta^k}$ commutes with the vector fields X_1 , X_{θ_1} and X_{θ_2} , it turns out that the coefficients f_i and g_i are constant.

It is then easily seen that the only terms that did not occur in the case $k \in \mathbb{N}$ and that give contributions in the computation of $C|_{S_\delta^k}$ come from the term $(L_{X_{\theta_1\theta_2}})^2$ and give a contribution equal to $-\frac{1}{4}\text{Id}$. ■

As in [22], we define an operator which measures the difference between the Casimir operators C and \mathcal{C} . This map will be called the operator N , as in the previous reference.

Definition 3. The operator N is defined by

$$N : \mathcal{S}_\delta \rightarrow \mathcal{S}_\delta : S \mapsto \mathcal{C}(S) - C(S).$$

From the expression of the map γ , it is easy to deduce an explicit formula for N .

Proposition 3. The operator N maps \mathcal{S}_δ^k into $\mathcal{S}_\delta^{k-\frac{1}{2}} \oplus \mathcal{S}_\delta^{k-1}$.

If $k \in \mathbb{N}$, the component with respect to $\mathcal{S}_\delta^{k-\frac{1}{2}}$ of $N|_{\mathcal{S}_\delta^k}$ can be written in a matrix form in the following way:

$$-(-1)^{\tilde{F}} \begin{pmatrix} \frac{k}{2} \bar{D}_1 & (\frac{k}{2} + \lambda) \bar{D}_2 \\ \frac{k}{2} \bar{D}_2 & -(\frac{k}{2} + \lambda) \bar{D}_1 \end{pmatrix},$$

while the component with respect to \mathcal{S}_δ^{k-1} of this operator is equal to

$$2 \begin{pmatrix} k \left(\frac{k-1}{2} + \lambda\right) \partial_x & 0 \\ 0 & (k-1) \left(\frac{k}{2} + \lambda\right) \partial_x \end{pmatrix}.$$

If $k \in \frac{1}{2} + \mathbb{N}$, the component with respect to $\mathcal{S}_\delta^{k-\frac{1}{2}}$ of $N|_{\mathcal{S}_\delta^k}$ is equal to

$$(-1)^{\tilde{F}} \begin{pmatrix} \left(\frac{k-\frac{1}{2}}{2} + \lambda\right) \bar{D}_1 & \left(\frac{k-\frac{1}{2}}{2} + \lambda\right) \bar{D}_2 \\ \frac{k-\frac{1}{2}}{2} \bar{D}_2 & -\frac{k-\frac{1}{2}}{2} \bar{D}_1 \end{pmatrix},$$

while the component with respect to \mathcal{S}_δ^{k-1} of this operator can be written in the following way:

$$2 \left(k - \frac{1}{2}\right) \begin{pmatrix} \frac{k-\frac{1}{2}}{2} + \lambda \\ 0 \end{pmatrix} \begin{pmatrix} \partial_x & 0 \\ 0 & \partial_x \end{pmatrix}.$$

Proof. Using the expressions of \mathcal{C} and C and the definition of γ , we obtain immediately

$$\begin{aligned} N &= -\frac{1}{2} \gamma(X_{x^2}) L_{X_1} - \gamma(X_{x\theta_1}) L_{X_{\theta_1}} - \gamma(X_{x\theta_2}) L_{X_{\theta_2}} + L_{X_{\theta_1}} \gamma(X_{x\theta_1}) \\ &\quad + L_{X_{\theta_2}} \gamma(X_{x\theta_2}) - \frac{1}{2} L_{X_1} \gamma(X_{x^2}). \end{aligned}$$

Suppose first that $k \in \mathbb{N}$. If we write the Lie derivatives in a matrix way, we obtain then that the component with respect to $\mathcal{S}_\delta^{k-\frac{1}{2}}$ of $N|_{\mathcal{S}_\delta^k}$ is equal to

$$\begin{aligned} &2(-1)^{\tilde{F}} \begin{pmatrix} \frac{k}{2} \theta_1 & (\frac{k}{2} + \lambda) \theta_2 \\ \frac{k}{2} \theta_2 & -(\frac{k}{2} + \lambda) \theta_1 \end{pmatrix} \begin{pmatrix} \partial_x & 0 \\ 0 & \partial_x \end{pmatrix} - (-1)^{\tilde{F}} \begin{pmatrix} \frac{k}{2} & 0 \\ 0 & -(\frac{k}{2} + \lambda) \end{pmatrix} \begin{pmatrix} \partial_{\theta_1} + \theta_1 \partial_x & 0 \\ 0 & \partial_{\theta_1} + \theta_1 \partial_x \end{pmatrix} \\ &- (-1)^{\tilde{F}} \begin{pmatrix} 0 & \frac{k}{2} + \lambda \\ \frac{k}{2} & 0 \end{pmatrix} \begin{pmatrix} \partial_{\theta_2} + \theta_2 \partial_x & 0 \\ 0 & \partial_{\theta_2} + \theta_2 \partial_x \end{pmatrix}, \end{aligned}$$

i.e. to

$$= -(-1)^{\tilde{F}} \begin{pmatrix} \frac{k}{2} \bar{D}_1 & (\frac{k}{2} + \lambda) \bar{D}_2 \\ \frac{k}{2} \bar{D}_2 & -(\frac{k}{2} + \lambda) \bar{D}_1 \end{pmatrix}.$$

It turns out that the component with respect to \mathcal{S}_δ^{k-1} of $N|_{\mathcal{S}_\delta^k}$ is equal to

$$2 \begin{pmatrix} k \left(\frac{k-1}{2} + \lambda\right) & 0 \\ 0 & (k-1) \left(\frac{k}{2} + \lambda\right) \end{pmatrix} \begin{pmatrix} \partial_x & 0 \\ 0 & \partial_x \end{pmatrix}.$$

Similar computations allow to show that the formulae for the case $k \in \mathbb{N} + \frac{1}{2}$ are also correct. \blacksquare

4 Construction of the quantization

We begin this section with the definition of the critical values of δ . As in [22], the set of critical values of δ is a set outside of which the $\mathfrak{spo}(2|2)$ -equivariant quantization exists and is unique.

4.1 Critical values

Definition 4. A value of δ is *critical* if there exist $k, l \in \frac{\mathbb{N}}{2}$ with $l < k$ such that $\alpha_k = \alpha_l$.

Proposition 4. *The set of critical values of δ is given by*

$$\left\{ \frac{k+l}{2} : k, l \in \mathbb{N}, l < k \right\} \cup \left\{ \frac{k^2 - l^2 + \frac{1}{4}}{2(k-l)} : k \in \mathbb{N}, l \in \mathbb{N} + \frac{1}{2}, l < k \right\} \\ \cup \left\{ \frac{k^2 - l^2 - \frac{1}{4}}{2(k-l)} : k \in \mathbb{N} + \frac{1}{2}, l \in \mathbb{N}, l < k \right\}.$$

Proof. The result can be easily proved from a straightforward computation using Proposition 2. ■

4.2 The construction

The main result is given by the following theorem.

Theorem 1. *If δ is not critical, there exists a unique $\mathfrak{spo}(2|2)$ -equivariant quantization from \mathcal{S}_δ to $\mathcal{D}_{\lambda, \mu}$.*

Proof. The proof is similar to the one of [22]. First, remark that for every $S \in \mathcal{S}_\delta^k$, there exists a unique eigenvector \hat{S} of \mathcal{C} with eigenvalue α_k such that

$$\begin{cases} \hat{S} = S_k + S_{k-\frac{1}{2}} + S_{k-1} + \cdots + S_0, & S_k = S, \\ S_l \in \mathcal{S}_\delta^l & \text{for all } l \leq k - \frac{1}{2}. \end{cases}$$

Indeed, the fact that \hat{S} has to be an eigenvector of \mathcal{C} with eigenvalue α_k implies that

$$\begin{cases} (C - \alpha_k \text{Id})S_{k-\frac{1}{2}} = -\text{pr}_{k-\frac{1}{2}}(N(S_k)), \\ (C - \alpha_k \text{Id})S_{k-l} = -(\text{pr}_{k-l}(N(S_{k-l+\frac{1}{2}})) + \text{pr}_{k-l}(N(S_{k-l+1}))) \end{cases} \quad (1)$$

for $l = 1, \frac{3}{2}, \dots, k$, where pr_i stands for the natural projection $\mathcal{S}_\delta \rightarrow \mathcal{S}_\delta^i$.

As δ is not critical, the differences $\alpha_k - \alpha_l$ are different from 0, the operators $(C - \alpha_k \text{Id})|_{\mathcal{S}_\delta^{k-l}}$ are thus all invertible and therefore this system of equations has a unique solution.

Now, define the quantization Q by $Q|_{\mathcal{S}_\delta^k}(S) = \hat{S}$. It is clearly a bijection and it also fulfills

$$Q \circ L_{X_f} = \mathcal{L}_{X_f} \circ Q \quad \text{for all } X_f \in \mathfrak{spo}(2|2).$$

Indeed, for all $S \in \mathcal{S}_\delta^k$, the symbols $Q(L_{X_f}S)$ and $\mathcal{L}_{X_f}(Q(S))$ share the following properties:

- they are eigenvectors of \mathcal{C} of eigenvalue α_k because, on the one hand, \mathcal{C} commutes with \mathcal{L}_{X_f} for all $X_f \in \mathfrak{spo}(2|2)$ and, on the other hand, C commutes with L_{X_f} for all $X_f \in \mathfrak{spo}(2|2)$.
- their term of degree k is exactly $L_{X_f}S$.

The first part of the proof shows that they have to coincide. An $\mathfrak{spo}(2|2)$ -equivariant quantization is thus given by $Q_{\text{Aff}} \circ Q$.

This quantization is the unique $\mathfrak{spo}(2|2)$ -equivariant quantization. Indeed, if S is an eigenvector of C of eigenvalue α_k , then, if Q is $\mathfrak{spo}(2|2)$ -equivariant, $Q(S)$ has to be an eigenvector of \mathcal{C} of the same eigenvalue because \mathcal{C} and C are built from the Lie derivatives \mathcal{L} and L . The uniqueness of the solution of the system (1) implies then the uniqueness of the $\mathfrak{spo}(2|2)$ -equivariant quantization. \blacksquare

5 Explicit formulae for the $\mathfrak{spo}(2|2)$ -equivariant quantization

In this section, we give in the non-critical situations the explicit formula for the $\mathfrak{spo}(2|2)$ -equivariant quantization in the cases where $k \in \mathbb{N}$ and where $k \in \mathbb{N} + \frac{1}{2}$.

5.1 Case $k \in \mathbb{N}$

Proposition 5. *If δ is non-critical and if l is a natural number, the symbol S_{k-l} in the proof of Theorem 1 is given by:*

$$C_l \begin{pmatrix} A_k B_{k-l} \partial_x^l & 0 \\ 0 & A_{k-l} B_k \partial_x^l \end{pmatrix} S_k + D_l \begin{pmatrix} A_k B_{k-l} \partial_x^l & -B_k B_{k-l} \bar{D}_1 \bar{D}_2 \partial_x^{l-1} \\ A_k A_{k-l} \bar{D}_1 \bar{D}_2 \partial_x^{l-1} & A_{k-l} B_k \partial_x^l \end{pmatrix} S_k, \quad (2)$$

where the coefficients A_k , B_k , C_l and D_l are given in the following way:

$$A_k = -k, \quad B_k = -(k + 2\lambda),$$

$$C_l = \frac{\prod_{i=1}^{l-1} A_{k-i} B_{k-i}}{\prod_{i=1}^l (\alpha_k - \alpha_{k-i})}, \quad D_l = \frac{-l \prod_{i=1}^{l-1} A_{k-i} B_{k-i}}{2(\alpha_k - \alpha_{k-\frac{1}{2}}) \prod_{i=1}^l (\alpha_k - \alpha_{k-i})}.$$

If $l \in \mathbb{N} + \frac{1}{2}$, we have:

$$S_{k-l} = (-1)^{\bar{S}_k+1} E_l \left[\begin{pmatrix} A_k \bar{D}_1 & B_k \bar{D}_2 \\ A_k \bar{D}_2 & -B_k \bar{D}_1 \end{pmatrix} \partial_x^{[l]} \right] S_k, \quad (3)$$

where $[l]$ is the integer part of l and where E_l is given in the following way:

$$E_l = \frac{\prod_{i=1}^{l-\frac{1}{2}} A_{k-i} B_{k-i}}{2(\alpha_k - \alpha_{k-\frac{1}{2}}) \prod_{i=1}^{l-\frac{1}{2}} (\alpha_k - \alpha_{k-i})}.$$

The first coefficients C_l , D_l and E_l are given by the following formulae:

$$C_1 = \frac{1}{\alpha_k - \alpha_{k-1}}, \quad D_1 = -\frac{1}{2(\alpha_k - \alpha_{k-\frac{1}{2}})(\alpha_k - \alpha_{k-1})}, \quad E_{\frac{1}{2}} = \frac{1}{2(\alpha_k - \alpha_{k-\frac{1}{2}})}.$$

Proof. The symbols S_{k-l} are built by induction using the formula (1). When $l = \frac{1}{2}$ and when $l = 1$, the formulae (2) and (3) can be easily proved using the relation (1), Propositions 2 and 3.

Suppose that the formulae for the symbols $S_{k-l'}$ are true if $l' \leq l$ and prove that the formula for $S_{k-l-\frac{1}{2}}$ is also true.

Using (1), we can first write

$$S_{k-l-\frac{1}{2}} = \frac{1}{\alpha_k - \alpha_{k-l-\frac{1}{2}}} \left[N_{k-l-\frac{1}{2}}^{k-l} (S_{k-l}) + N_{k-l-\frac{1}{2}}^{k-l+\frac{1}{2}} \left(S_{k-(l-\frac{1}{2})} \right) \right], \quad (4)$$

where the component with respect to $\mathcal{S}_\delta^{k-l-i}$ of $N|_{\mathcal{S}_\delta^{k-l}}$ is denoted by N_{k-l-i}^{k-l} .

Two situations may occur:

If l is a natural number, then $(k-l)$ is a natural number and $(k-l+\frac{1}{2}) \in \mathbb{N} + \frac{1}{2}$. In this case, the relation (4) shows, using Propositions 2 and 3, the relation (2) and the relation (3), that $S_{k-l-\frac{1}{2}}$ has the same form as the right hand side of (3) and that

$$E_{l+\frac{1}{2}} = \frac{A_{k-l}B_{k-l}}{2(\alpha_k - \alpha_{k-(l+\frac{1}{2})})} (C_l + 2D_l + 2E_{l-\frac{1}{2}}).$$

Using the induction hypothesis on the coefficients C_l , D_l and $E_{l-\frac{1}{2}}$, it is easy to see that $E_{l+\frac{1}{2}}$ is given by the right formula.

If $l \in \mathbb{N} + \frac{1}{2}$, then $(k-l) \in \mathbb{N} + \frac{1}{2}$ and $(k-l+\frac{1}{2})$ is a natural number. Computations similar to the computations above show that $S_{k-l-\frac{1}{2}}$ has the same form as the right hand side of (2) and that the coefficients $C_{l+\frac{1}{2}}$ and $D_{l+\frac{1}{2}}$ are given by the following formulae:

$$C_{l+\frac{1}{2}} = \frac{A_{k-l+\frac{1}{2}}B_{k-l+\frac{1}{2}}}{(\alpha_k - \alpha_{k-(l+\frac{1}{2})})} C_{l-\frac{1}{2}},$$

$$D_{l+\frac{1}{2}} = \frac{1}{(\alpha_k - \alpha_{k-(l+\frac{1}{2})})} (-E_l + A_{k-l+\frac{1}{2}}B_{k-l+\frac{1}{2}}D_{l-\frac{1}{2}}).$$

Using the induction hypothesis on the coefficients $C_{l-\frac{1}{2}}$, E_l and $D_{l-\frac{1}{2}}$, it is easy to see that $C_{l+\frac{1}{2}}$ and $D_{l+\frac{1}{2}}$ are given by the right formulae. ■

5.2 Case $k \in \frac{1}{2} + \mathbb{N}$

Proposition 6. *If δ is non-critical and if l is a natural number, the symbol S_{k-l} given in the proof of Theorem 1 is given by:*

$$\left[C'_l \begin{pmatrix} \partial_x^l & 0 \\ 0 & \partial_x^l \end{pmatrix} + D'_l \begin{pmatrix} \partial_x^l & -\bar{D}_1 \bar{D}_2 \partial_x^{l-1} \\ \bar{D}_1 \bar{D}_2 \partial_x^{l-1} & \partial_x^l \end{pmatrix} \right] S_k,$$

where the coefficients C'_l and D'_l are given in the following way:

$$C'_l = \frac{\prod_{i=0}^{l-1} (A_{k-\frac{1}{2}-i} B_{k-\frac{1}{2}-i})}{\prod_{i=1}^l (\alpha_k - \alpha_{k-i})}, \quad D'_l = \frac{-l \prod_{i=0}^{l-1} (A_{k-\frac{1}{2}-i} B_{k-\frac{1}{2}-i})}{2(\alpha_k - \alpha_{k-\frac{1}{2}}) \prod_{i=1}^l (\alpha_k - \alpha_{k-i})}.$$

If $l \in \mathbb{N} + \frac{1}{2}$, we have:

$$S_{k-l} = (-1)^{\bar{S}_{k+1}} E'_l \left[\begin{pmatrix} B_{k-l} \bar{D}_1 & B_{k-l} \bar{D}_2 \\ A_{k-l} \bar{D}_2 & -A_{k-l} \bar{D}_1 \end{pmatrix} \partial_x^{[l]} \right] S_k,$$

where $[l]$ is the integer part of l and where the coefficients E'_l are given by the following formulae:

$$E'_l = \frac{\prod_{i=0}^{l-\frac{3}{2}} \left(A_{k-\frac{1}{2}-i} B_{k-\frac{1}{2}-i} \right)}{2 \left(\alpha_k - \alpha_{k-\frac{1}{2}} \right) \prod_{i=1}^{l-\frac{1}{2}} \left(\alpha_k - \alpha_{k-i} \right)}.$$

The first coefficient $E'_{\frac{1}{2}}$ is defined in the following way:

$$E'_{\frac{1}{2}} = \frac{1}{2 \left(\alpha_k - \alpha_{k-\frac{1}{2}} \right)}.$$

Proof. The proof is completely similar to the previous one. ■

Remark 1. If $k \in \mathbb{N}$ (resp. $k \in \mathbb{N} + \frac{1}{2}$), if $l \in \mathbb{N} + \frac{1}{2}$ is between $\frac{3}{2}$ and $k - \frac{1}{2}$ (resp. k) and if δ is such that the difference $\alpha_k - \alpha_{k-l}$ vanishes, the explicit formula given in Proposition 5 (resp. Proposition 6) is still well-defined. Since these formulae give $\mathfrak{spo}(2|2)$ -equivariant quantizations and are continuous in neighborhoods of these values of δ , these formulae give also $\mathfrak{spo}(2|2)$ -equivariant quantizations for these values of δ , by continuity.

If $k \in \mathbb{N}$ (resp. $k \in \mathbb{N} + \frac{1}{2}$), an $\mathfrak{spo}(2|2)$ -equivariant quantization for symbols of degree k exists thus also for the values of δ equal to

$$\frac{k^2 - l^2 + \frac{1}{4}}{2(k-l)} \quad \left(\text{resp.} \quad \frac{k^2 - l^2 - \frac{1}{4}}{2(k-l)} \right),$$

when the half-natural number l (resp. the natural number l) varies from $\frac{1}{2}$ (resp. 0) to $k - \frac{3}{2}$.

Remark 2. If $k \in \mathbb{N}$ and if $\alpha_k - \alpha_{k-\frac{1}{2}} = 0$ (i.e. if $\delta = k$) or if $\alpha_k - \alpha_{k-1} = 0$ (i.e. if $\delta = k - \frac{1}{2}$), then there is no $\mathfrak{spo}(2|2)$ -equivariant quantization. Indeed, in these situations, the equations whose the unknowns are $S_{k-\frac{1}{2}}$ and S_{k-1} in the system (1) admit no solution because the left hand side of these equations vanishes whereas their right hand side is not equal to zero. Now, the existence of an $\mathfrak{spo}(2|2)$ -equivariant quantization implies the existence of a solution of the system (1), as explained in the proof of Theorem 1. For the same reason, if $2 \leq l \leq k$ and if $\alpha_k - \alpha_{k-l} = 0$ (i.e. if $\delta = k - \frac{l}{2}$), then there is no $\mathfrak{spo}(2|2)$ -equivariant quantization if the numbers $B_{k-1}, \dots, B_{k-l+1}$ are different from zero (i.e. if λ is different from the numbers $-\frac{k-1}{2}, \dots, -\frac{k-l+1}{2}$). By cons, if one of these numbers vanishes, the system (1) has infinitely many solutions. In these situations, there are certainly infinitely many $\mathfrak{spo}(2|2)$ -equivariant quantizations, but our method linked to the Casimir operators does not allow to conclude.

Exactly in the same way, if $k \in \mathbb{N} + \frac{1}{2}$ and if $\alpha_k - \alpha_{k-\frac{1}{2}} = 0$ (i.e. if $\delta = k - \frac{1}{2}$), there is no $\mathfrak{spo}(2|2)$ -equivariant quantization. If $1 \leq l \leq k$ and if $\alpha_k - \alpha_{k-l} = 0$ (i.e. if $\delta = k - \frac{l}{2}$), then there is no $\mathfrak{spo}(2|2)$ -equivariant quantization if the numbers $B_{k-\frac{1}{2}}, \dots, B_{k-l+\frac{1}{2}}$ are different from zero (i.e. if λ is different from the numbers $-\frac{k-\frac{1}{2}}{2}, \dots, -\frac{k-l+\frac{1}{2}}{2}$). As in the case $k \in \mathbb{N}$, if one of these numbers vanishes, there are surely infinitely many $\mathfrak{spo}(2|2)$ -equivariant quantizations, but the proof of their existence seems *a priori* difficult.

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References

- [1] Berezin F.A., Introduction to superanalysis, *Mathematical Physics and Applied Mathematics*, Vol. 9, D. Reidel Publishing Co., Dordrecht, 1987.
- [2] Boniver F., Hansoul S., Mathonet P., Poncin N., Equivariant symbol calculus for differential operators acting on forms, *Lett. Math. Phys.* **62** (2002), 219–232, [math.RT/0206213](#).
- [3] Boniver F., Mathonet P., IFFT-equivariant quantizations, *J. Geom. Phys.* **56** (2006), 712–730, [math.RT/0109032](#).
- [4] Bouarroudj S., Projectively equivariant quantization map, *Lett. Math. Phys.* **51** (2000), 265–274, [math.DG/0003054](#).
- [5] Čap A., Šilhan J., Equivariant quantizations for AHS-structures, *Adv. Math.* **224** (2010), 1717–1734, [arXiv:0904.3278](#).
- [6] Duval C., Lecomte P., Ovsienko V., Conformally equivariant quantization: existence and uniqueness, *Ann. Inst. Fourier (Grenoble)* **49** (1999), 1999–2029, [math.DG/9902032](#).
- [7] Fox D.J.F., Projectively invariant star products, *Int. Math. Res. Pap.* (2005), 461–510, [math.DG/0504596](#).
- [8] Gargoubi H., Mellouli N., Ovsienko V., Differential operators on supercircle: conformally equivariant quantization and symbol calculus, *Lett. Math. Phys.* **79** (2007), 51–65, [math-ph/0610059](#).
- [9] Hansoul S., Projectively equivariant quantization for differential operators acting on forms, *Lett. Math. Phys.* **70** (2004), 141–153.
- [10] Hansoul S., Existence of natural and projectively equivariant quantizations, *Adv. Math.* **214** (2007), 832–864, [math.DG/0601518](#).
- [11] Kac V.G., Lie superalgebras, *Adv. Math.* **26** (1977), 8–96.
- [12] Lecomte P.B.A., Classification projective des espaces d’opérateurs différentiels agissant sur les densités, *C. R. Acad. Sci. Paris Sér. I Math.* **328** (1999), 287–290.
- [13] Lecomte P.B.A., Towards projectively equivariant quantization, *Progr. Theoret. Phys. Suppl.* (2001), no. 144, 125–132.
- [14] Lecomte P.B.A., Ovsienko V.Yu., Projectively equivariant symbol calculus, *Lett. Math. Phys.* **49** (1999), 173–196, [math.DG/9809061](#).
- [15] Leites D., Poletaeva E., Serganova V., On Einstein equations on manifolds and supermanifolds, *J. Nonlinear Math. Phys.* **9** (2002), 394–425, [math.DG/0306209](#).
- [16] Leuther T., Mathonet P., Radoux F., One $\mathfrak{osp}(p+1, q+1|2r)$ -equivariant quantizations, *J. Geom. Phys.* **62** (2012), 87–99, [arXiv:1107.1387](#).
- [17] Leuther T., Radoux F., Natural and projectively invariant quantizations on supermanifolds, *SIGMA* **7** (2011), 034, 12 pages, [arXiv:1010.0516](#).
- [18] Mathonet P., Radoux F., Natural and projectively equivariant quantizations by means of Cartan connections, *Lett. Math. Phys.* **72** (2005), 183–196, [math.DG/0606554](#).
- [19] Mathonet P., Radoux F., Cartan connections and natural and projectively equivariant quantizations, *J. Lond. Math. Soc. (2)* **76** (2007), 87–104, [math.DG/0606556](#).
- [20] Mathonet P., Radoux F., On natural and conformally equivariant quantizations, *J. Lond. Math. Soc. (2)* **80** (2009), 256–272, [arXiv:0707.1412](#).
- [21] Mathonet P., Radoux F., Existence of natural and conformally invariant quantizations of arbitrary symbols, *J. Nonlinear Math. Phys.* **17** (2010), 539–556, [arXiv:0811.3710](#).
- [22] Mathonet P., Radoux F., Projectively equivariant quantizations over the superspace $\mathbb{R}^{p|q}$, *Lett. Math. Phys.* **98** (2011), 311–331, [arXiv:1003.3320](#).
- [23] Mellouli N., Second-order conformally equivariant quantization in dimension $1|2$, *SIGMA* **5** (2009), 111, 11 pages, [arXiv:0912.5190](#).
- [24] Michel J.-P., Quantification conformément équivariante des fibrés supercotangents, Ph.D. thesis, Université de la Méditerranée - Aix-Marseille II, 2009, available at <http://tel.archives-ouvertes.fr/tel-00425576>.
- [25] Musson I.M., On the center of the enveloping algebra of a classical simple Lie superalgebra, *J. Algebra* **193** (1997), 75–101.
- [26] Pinczon G., The enveloping algebra of the Lie superalgebra $\mathfrak{osp}(1, 2)$, *J. Algebra* **132** (1990), 219–242.
- [27] Sergeev A., The invariant polynomials on simple Lie superalgebras, *Represent. Theory* **3** (1999), 250–280, [math.RT/9810111](#).