Twisted Cyclic Cohomology and Modular Fredholm Modules

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Abstract. Connes and Cuntz showed in [*Comm. Math. Phys.* **114** (1988), 515–526] that suitable cyclic cocycles can be represented as Chern characters of finitely summable semifinite Fredholm modules. We show an analogous result in twisted cyclic cohomology using Chern characters of modular Fredholm modules. We present examples of modular Fredholm modules arising from Podleś spheres and from $\mathrm{SU}_q(2)$.

Key words: twisted cyclic cohomology; spectral triple; modular theory; KMS weight

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1 Introduction

Let $\mathcal{A}$ be an associative algebra over the field of complex numbers $\mathbb{C}$, $\mathcal{A} *_{\mathbb{C}} \mathcal{A}$ the free product, and[1] $q\mathcal{A}$ the ideal generated by $\iota(a) - \bar{\iota}(a)$, $a \in \mathcal{A}$, where ι, $\bar{\iota}$ are the two canonical inclusions of $\mathcal{A}$ in $\mathcal{A} *_{\mathbb{C}} \mathcal{A}$. In [5], it was shown that those cyclic cocycles for $\mathcal{A}$ which arise from positive traces on $(qA)^n$ are Chern characters of finitely summable semifinite Fredholm modules.

In this note we show that those twisted cyclic cocycles arising from KMS weights on $(qA)^n$ are Chern characters of finitely summable modular Fredholm modules, a twisted version of the usual notion of Fredholm modules. While this is not in any way a practical method of obtaining such representing Fredholm modules, it shows that in general one must consider the semifinite and modular settings.

The examples treated in the last two sections, the Podleś spheres $S^2_{q,s}$ and the $\mathrm{SU}_q(2)$ quantum group, show that we can construct non-trivial twisted cyclic cocycles from naturally arising modular Fredholm modules. Moreover these cocycles encode the correct classical dimension, in the sense that the Hochschild class of these cocycles is non-vanishing at the classical dimension. This was first observed in [15] for the standard Podleś sphere. Thus using twisted cohomology avoids the 'dimension drop' phenomena, at least in these examples.

We observe that the objects and phenomena studied here seem to have little to do with [6] and related papers such as [8]. The use of twisted commutators in these papers leads to a need for twisted traces, but ultimately these produce actual (not twisted) cyclic cocycles.

[1]There is a clash in the standard notations: in this section q is used for $q\mathcal{A}$ and $q(a)$, while later in the paper q is used as a deformation parameter. The different usages will be clear from context.

2 The algebraic background

We begin with a short recollection of the twisted cyclic cohomology of an algebra $\mathcal{A}$.

Definition 1. Let $\mathcal{A}$ be an algebra and σ be an automorphism of $\mathcal{A}$. We say that $\phi\colon \mathcal{A}^{\otimes(n+1)} \to \mathbb{C}$ is a σ-twisted cyclic n-cocycle if,

- ϕ is σ-invariant:

$$\phi(a_0, a_1, \dots, a_n) = \phi(\sigma(a_0), \sigma(a_1), \dots, \sigma(a_n));$$

- ϕ is σ-cyclic:

$$\phi(a_0, a_1, \dots, a_n) = (-1)^n \phi(\sigma(a_n), a_0, a_1, \dots, a_{n-1});$$

- ϕ is a σ-twisted Hochschild cocycle

$$\begin{aligned}(b^\sigma \phi)(a_0, a_1, \dots, a_n, a_{n+1}) = \sum_{k=0}^{n} &(-1)^k \phi(a_0, \dots, a_k a_{k+1}, \dots, a_{n+1}) \\ &+ (-1)^{n+1} \phi(\sigma(a_{n+1}) a_0, a_1, \dots, a_n) = 0,\end{aligned}$$

where $a_0, \dots, a_{n+1} \in \mathcal{A}$.

In the examples we consider, one can use the algebraic tensor product, as we work with polynomial subalgebras. Alternatively, one could complete $\mathcal{A}$ in a suitable Fréchet topology and use the projective tensor product: see [1] and [4, III, Appendix B] for more information. As all our algebras will have a natural C^*-completion, we define all K-theories in terms of such completions. It will transpire for our examples that generators of the relevant K-theory groups will belong to the polynomial subalgebras we will work with.

Now, let us present a simple generalisation of a result of Connes and Cuntz [5] to the twisted cyclic theory.

Let $\mathcal{A}$ be a unital algebra and $q\mathcal{A}$ be an algebra generated by the elements $q(a)$, $q(a)b$, and $aq(b)$ for $a, b \in \mathcal{A}$, subject to the relation $q(\lambda a + b + \mu) = \lambda q(a) + q(b)$ and

$$q(ab) = q(a)b + aq(b) - q(a)q(b), \qquad a, b \in \mathcal{A}. \tag{2.1}$$

Equivalently, one may identify $q\mathcal{A}$ with the ideal within the unital free product algebra $\mathcal{A} *_{\mathbb{C}} \mathcal{A}$ generated by the elements $q(a) := \iota(a) - \bar{\iota}(a)$ for $a \in \mathcal{A}$. If $\mathcal{A}$ is an involutive algebra, then so is $q\mathcal{A}$ with the involution defined by $q(a)^* = q(a^*)$ for $a \in \mathcal{A}$.

Setting $\mathcal{J} := q\mathcal{A} \subset \mathcal{A} *_{\mathbb{C}} \mathcal{A}$, we can define $\mathcal{J}^n$ to be the ideal of $\mathcal{A} *_{\mathbb{C}} \mathcal{A}$ generated by the products $a_0 q(a_1) \cdots q(a_m)$ and $q(a_1) \cdots q(a_m)$ for $m \geq n$. If σ is an automorphism of $\mathcal{A}$, then we can extend σ to an automorphism of $\mathcal{J}$ and $\mathcal{J}^n$ by setting $\sigma(q(a)) := q(\sigma(a))$.

Proposition 1 (see [5, Proposition 3])**.** *Let $\mathcal{A}$ be a unital algebra, σ an automorphism of $\mathcal{A}$, and let $\mathcal{J}$ be the ideal $q\mathcal{A}$ of $\mathcal{A} *_{\mathbb{C}} \mathcal{A}$ described above. Suppose that T is a σ-twisted trace on $\mathcal{J}^n$ for some even integer n. That is, T is a linear functional such that*

$$T(xy) = T(\sigma(y)x), \qquad \forall\, x \in J^k, \quad y \in J^l, \quad k + l = n \tag{2.2}$$

with the convention that $J^0 = \mathcal{A} *_{\mathbb{C}} \mathcal{A}$. Then the formula

$$\tau(a_0, a_1, \dots, a_n) := T(q(a_0) q(a_1) \cdots q(a_n)), \qquad a_0, a_1, \dots, a_n \in \mathcal{A}$$

defines a σ-twisted cyclic n-cocycle τ on $\mathcal{A}$.

Proof. Setting $x = 1$ in (2.2), we obtain that T is σ-invariant. The σ-cyclicity follows by setting $x = q(a_0)\cdots q(a_{n-1})$ and $y = q(a_n)$. It remains to verify the σ-twisted Hochschild cocycle condition. If $a_0,\dots,a_n$ are elements of $\mathcal{A}$, (2.1) implies

$$\sum_{k=0}^{n}(-1)^k T\left(q(a_0)\cdots q(a_k a_{k+1})\cdots q(a_{n+1})\right)$$
$$= T(a_0 q(a_1)\cdots q(a_{n+1})) + T(q(a_0)\cdots q(a_n)a_{n+1}) - T(q(a_0)\cdots q(a_{n+1})).$$

Then, using (2.2), one sees that this is equal to

$$T((q(\sigma(a_{n+1}))a_0 + \sigma(a_{n+1})q(a_0) - q(\sigma(a_{n+1}))q(a_0))q(a_1)\cdots q(a_n)).$$

Again by (2.1), we obtain

$$\sum_{k=0}^{n}(-1)^k T\left(q(a_0)\cdots q(a_k a_{k+1})\cdots q(a_{n+1})\right) = T\left(q(\sigma(a_{n+1})a_0)q(a_1)\cdots q(a_n)\right),$$

which is equivalent to the desired equality $b^\sigma\tau = 0$. ∎

For the analogous statement for odd cocycles, we need to extend the automorphism σ to $\mathcal{J}^n$ in a different way, cf. [5, Lemma 4]. We define $\tilde\sigma$ via the formula

$$\tilde\sigma\left(a_0 q(a_1)\cdots q(a_m)\right) = (-1)^m(\sigma(a_0) - q(\sigma(a_0)))q(\sigma(a_1))\cdots q(\sigma(a_m)),$$
$$\tilde\sigma\left(q(a_1)\cdots q(a_m)\right) = (-1)^m q(\sigma(a_1))\cdots q(\sigma(a_m)).$$

Then it is easy to check that $\tilde\sigma$ is indeed an automorphism of $q\mathcal{A}$ and, just as above, we have

Proposition 2. *If T is a $\tilde\sigma$-twisted trace on $\mathcal{J}^n$, for n an odd integer, then the formula*

$$\tau(a_0,a_1,\dots,a_n) := T(q(a_0)q(a_1)\cdots q(a_n)), \qquad a_0,a_1,\dots,a_n \in \mathcal{A},$$

defines a σ-twisted n-cyclic cocycle on $\mathcal{A}$.

3 The analytic picture

In this section we look at a version of [5, Theorem 15] in twisted cyclic cohomology. In brief, [5] shows that positive traces on certain ideals in the free product $\mathcal{A} *_{\mathbb{C}} \mathcal{A}$ give rise to cyclic cocycles on $\mathcal{A}$. These cyclic cocycles can be represented as the Chern characters of semifinite Fredholm modules. By replacing traces with KMS functionals, we arrive at an analogue of this result in twisted cyclic theory. There are also some analytic differences in our starting assumptions, which we discuss at the end of this section.

We let A be a unital C^*-algebra and consider the unital full free product C^*-algebra $\mathsf{A} *_{\mathbb{C}} \mathsf{A}$. We denote by ι, $\bar\iota$ the two canonical inclusions of A in $\mathsf{A} *_{\mathbb{C}} \mathsf{A}$, and by $q\mathsf{A}$ the ideal generated by elements of the form $q(a) := \iota(a) - \bar\iota(a)$ for $a \in \mathsf{A}$.

Similarly, if $\mathcal{A} \subset \mathsf{A}$ is a dense subalgebra, then we let $q\mathcal{A}$ be the analogously defined ideal in $\mathcal{A} *_{\mathbb{C}} \mathcal{A}$. Introduce the shorthand $J^k := (q\mathsf{A})^k$ and $\mathcal{J}^k := (q\mathcal{A})^k$ for $k \in \mathbb{N}$.

Our starting point is a lower semicontinuous and semifinite weight ϕ on the C^*-algebra J^{2p} [19, Chapter VI] which satisfies the KMS$_\beta$ condition for a strongly continuous one parameter group $\sigma_\bullet\colon \mathbb{R} \to \mathrm{Aut}(J^{2p})$. We will assume that $\mathcal{J}^{2p} \subset \mathrm{dom}(\phi)$ and that $\mathcal{J}^{2p}$ consists of analytic vectors for $\sigma_\bullet$, and that

$$\phi(xx^*) = 0 \;\Leftrightarrow\; \phi(x^*x) = 0, \qquad x \in \mathsf{A}. \tag{3.1}$$

The weight ϕ gives, via the GNS construction, a Hilbert space $\mathcal{H}_\phi$ with a nondegenerate representation $\pi_\phi\colon J^{2p} \to \mathcal{B}(\mathcal{H}_\phi)$, and a linear map $\Lambda\colon \operatorname{dom}^{1/2}(\phi) \subset J^{2p} \to \mathcal{H}_\phi$. The condition (3.1) implies that $\{x \in \mathsf{A} \mid \phi(xx^*) = 0\}$ is the kernel of this representation, and that $\pi_\phi\big(J^{2p}\big)$ admits a faithful weight which induces ϕ. Since $\sigma_\bullet$ leaves ϕ invariant, it descends to $\pi_\phi(J^{2p})$.

There is a canonical faithful normal semifinite extension Φ of ϕ to $\big(\pi_\phi\big(J^{2p}\big)\big)''$ satisfying $\phi = \Phi \circ \pi_\phi$ and $\sigma_t^\Phi \circ \pi_\phi = \pi_\phi \circ \sigma_{-\beta t}$. See [19, Proposition 1.5, Chapter VIII] for a proof.

The KMS property implies that for a, $b \in J^{2p}$ we have

$$\phi(ab) = \phi(\sigma(b)a),$$

where we define the (non-$*$) automorphism σ to be the value of the extension of the one-parameter group $\sigma_\bullet$ to the complex value $t = i\beta$.

We observe that the representation of J^{2p} on $\mathcal{H}_\phi$ extends naturally to a representation of $\mathsf{A} *_\mathbb{C} \mathsf{A}$ on $\mathcal{H}_\phi$, denoted λ, such that $\lambda(\mathsf{A} *_\mathbb{C} \mathsf{A}) \subset \big(\pi_\phi\big(J^{2p}\big)\big)''$. This is the usual extension, defined on the dense subspace $\pi_\phi\big(J^{2p}\big)\mathcal{H}_\phi$ by $\lambda(\alpha)(j\xi) := (\alpha j)\xi$ for $\alpha \in \mathsf{A} *_\mathbb{C} \mathsf{A}$, $j \in J^{2p}$ and $\xi \in \mathcal{H}_\phi$. If T is in the commutant of $\pi_\phi(J^{2p})$ then

$$T(\lambda(\alpha)(j\xi)) = T((\alpha j)\xi) = (\alpha j)(T\xi) = \lambda(\alpha)(j(T\xi)) = \lambda(\alpha)(T(j\xi)),$$

showing that $\lambda(\alpha)$ is indeed in $\pi_\phi(J^{2p})''$ for all $\alpha \in \mathsf{A} *_\mathbb{C} \mathsf{A}$.

By [19, Theorem 2.6, Chapter VII], the (image under Λ of) $\operatorname{dom}^{1/2}\Phi \cap \big(\operatorname{dom}^{1/2}\Phi\big)^*$ is a full left Hilbert algebra, which we denote by U. Moreover, the left von Neumann algebra of U is precisely $\big(\pi_\phi\big(J^{2p}\big)\big)''$. We record the following Lemma, whose proof follows immediately from the definitions.

Lemma 1. *Let N be the left von Neumann algebra of the left Hilbert algebra U and Φ the corresponding faithful normal semifinite weight. Then for all $\alpha \in J^p \cap \operatorname{dom}^{1/2}(\phi)$ we have $\lambda(\alpha) \in \operatorname{dom}^{1/2}(\Phi)$, and $\Phi(\lambda(\alpha)^*\lambda(\alpha)) = \phi(\alpha^*\alpha)$.*

Definition 2. Let $\mathcal{N}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, and Φ be a faithful normal semifinite weight on $\mathcal{N}$. Then we say that $(\mathcal{A}, \mathcal{H}, F)$ is an n-summable *unital modular Fredholm module* with respect to $(\mathcal{N}, \Phi)$ if

- o) $\mathcal{A}$ is a separable unital $*$-subalgebra of $\mathcal{N}$;
- i) $\mathcal{A}$ is globally invariant under the group σ^Φ, and consists of analytic vectors for it;
- ii) F is a self-adjoint operator in the fixed point algebra $\mathcal{M} := \mathcal{N}^{\sigma^\Phi}$ with $F^2 = 1_\mathcal{N}$;
- iii) $[F, a]^n \in \operatorname{dom}(\Phi)$ for all $a \in \mathcal{A}$.

If there exists a self adjoint element γ of $\mathcal{M}$ satisfying $\gamma^2 = 1$, $\gamma a = a\gamma$ for all $a \in \mathcal{A}$ and $\gamma F + F\gamma = 0$, the quadruple $(\mathcal{A}, \mathcal{H}, F, \gamma)$ is said to be an even module. In contrast, a Fredholm module without the grading γ is said to be odd.

The Chern character of a modular Fredholm module is the class of the $\sigma := \sigma_i^\Phi$ twisted cyclic n-cocycle defined by the formula

$$\operatorname{Ch}_n(a_0, a_1, \dots, a_n) = \lambda_n \frac{1}{2}\Phi(\gamma F[F, a_0][F, a_1]\cdots[F, a_n]), \qquad a_0, a_1, \dots, a_n \in \mathcal{A}.$$

Here we set $\gamma = 1$ if the module is odd. The constants λ_n are given by

$$\lambda_n = \begin{cases} (-1)^{n(n-1)/2}\Gamma\left(\dfrac{n}{2}+1\right), & n \text{ even}, \\ \sqrt{2i}(-1)^{n(n-1)/2}\Gamma\left(\dfrac{n}{2}+1\right), & n \text{ odd}. \end{cases}$$

Theorem 1. *Suppose that $\mathcal{A}$ is a $*$-subalgebra of a C^*-algebra A, and ϕ is a weight on J^{2p} which is lower semicontinuous, semifinite, and satisfies (3.1). We further assume that it satisfies the KMS$_\beta$ condition for a one parameter group σ such that $\mathcal{J}^{2p} := (q\mathcal{A})^{2p}$ consists of analytic vectors in the domain of ϕ. Then there exists a $2p$-summable modular Fredholm module for $\mathcal{A}$. The modular Fredholm module has Chern character*

$$\mathrm{Ch}_{2p}(a_0, a_1, \dots, a_{2p}) = \lambda_{2p}(-1)^p \phi(q(a_0)\, q(a_1) \cdots q(a_{2p})).$$

Proof. The universal property of $\mathsf{A} *_{\mathbb{C}} \mathsf{A}$ gives two $*$-homomorphisms π_ϕ and $\bar{\pi}_\phi$ from A to $\mathcal{B}\big(L^2\big(J^{p+1}, \phi\big)\big)$, whose images lie in $N = \pi_\phi\big(J^{2p}\big)''$. The modular Fredholm module is given by the data:

- the Hilbert space $\mathcal{H} := L^2\big(J^{p+1}, \phi\big) \oplus L^2\big(J^{p+1}, \phi\big)$;
- the representation $\pi_2 \colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$, $\pi_2(a) = \pi_\phi(a) \oplus \bar{\pi}_\phi(a)$;
- the operator $F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$;
- the von Neumann algebra $M_2(N)$;
- the weight $\Phi \circ \mathrm{Tr}_2$.

Observe that

$$[F, \pi_2(a)] = \begin{pmatrix} 0 & \bar{\pi}_\phi(a) - \pi_\phi(a) \\ \pi_\phi(a) - \bar{\pi}_\phi(a) & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\pi_\phi(q(a)) \\ \pi_\phi(q(a)) & 0 \end{pmatrix}.$$

Since $J^{2p+1} \subset J^{2p}$, the Chern character is well-defined. The computation of the Chern character is straightforward. ■

The odd case is similar using the $\tilde{\sigma}$-automorphism of $\mathcal{J}^n$.

Remark 1. When σ is trivial, the above construction reduces to a particular case of the one in [5, Section V]. Since we started from a positive (unbounded) functional on the enveloping C^*-algebra rather than a one on the algebraic object $\mathcal{J}^{2p}$, the analytic argument leading to the existence of the underlying left Hilbert algebra is greatly simplified. If we were to assume only a 'positive twisted trace' on $\mathcal{J}^{2p}$ in some sense, we would not necessarily have enough control to prove this pre-closedness.

4 The modular index and pairing with K-theory

We recall here the construction of the modular index and its computation through the pairing between the equivariant K-theory and twisted cyclic cohomology. This section adapts [16] to the notation and notions used here.

Let $\mathcal{N}$ be a von Neumann algebra endowed with a faithful normal semifinite weight Φ, and $(\mathcal{A}, \mathcal{H}, F, \gamma)$ be a $2n$-summable even modular Fredholm module with respect to $(\mathcal{N}, \Phi)$, as defined in the previous section.

Furthermore, let us assume that there exists a densely defined operator Ξ in $\mathcal{H}$ which implements the modular automorphism. Thus, Ξ is an unbounded self adjoint operator satisfying

$$[F, \Xi] = 0, \qquad [\Xi, \gamma] = 0, \qquad \Xi^{-1} a \Xi = \sigma(a), \qquad \forall\, a \in \mathcal{A},$$

where we identify a with the operator in the representation $\pi(\mathcal{A})$.

With this set up we can make the following definition, and we assume in what follows that $(\mathcal{N}, \Phi) = \big(\mathcal{B}(\mathcal{H}), \mathrm{Tr}\,\big(\Xi^{1/2} \cdot \Xi^{1/2}\big)\big)$, as this is the context we shall be working with in the examples. Extending the definition to the more general situation is straightforward using the theory of Breuer–Fredholm operators as in [3].

Definition 3. Let F be a Fredholm operator, which commutes with Ξ. We define the *modular index* of F to be

$$\text{q-Ind}(F) = \text{Tr}(\Xi|_{\ker F}) - \text{Tr}(\Xi|_{\text{coker} F}).$$

This definition is well defined, since both kernel and cokernel are finite-dimensional and at the same time invariant subspaces of Ξ, so in fact both traces are finite expressions. We omit the proof of the next standard construction.

Proposition-Definition 1. *Suppose that $(\mathcal{A}, \mathcal{H}, F, \gamma)$ is an even modular Fredholm module, and let $p \in \mathsf{A}$ be a projection which is fixed by the modular automorphism group $\sigma_\bullet$. Replacing $\mathcal{H}$ by $p\mathcal{H}$, $\mathcal{N}$ by $\mathcal{N}_p = p\mathcal{N}p$, Φ by $\Phi_p = \Phi|_{\mathcal{N}_p}$, F by pF_+p in Definition 3, we obtain a Fredholm operator for $(\mathcal{N}_p, \Phi_p)$. We define* $\text{q-Ind}^F(p)$ *to be its modular index* $\text{q-Ind}(pF_+p)$.

More generally, we can extend the above index pairing on the classes in the equivariant K_0-group as follows. An element of the equivariant K_0-group $K_0^{\mathbb{R}}(\mathcal{A})$ is given by a formal difference of invariant projections in the $\mathbb{R}$-algebras of the form $\mathcal{A} \otimes \text{End}(X)$, where $U\colon \mathbb{R} \to \text{End}(X)$ is an arbitrary finite-dimensional representation of $\mathbb{R}$ [16, Theorem 3.1]. Assume that $p \in \mathcal{A} \otimes \text{End}(X)$ is such a projection. Then we extend the modular Fredholm module to $(\mathcal{A} \otimes \text{End}(X), \mathcal{H} \otimes X, F \otimes \text{Id}, \gamma \otimes \text{Id})$ with respect to $(\mathcal{N} \otimes \text{End}(X), \Phi \otimes G_X)$, where $G_X(T) = \text{Tr}(U_{-i}T)$ for $T \in \text{End}(X)$. The above consideration gives the number $\text{q-Ind}^F(p)$, which only depends on the $K_0^{\mathbb{R}}$-class of p, [17, Lemma 3.15]. This way, we obtain the map

$$\text{q-Ind}^F\colon K_0^{\mathbb{R}}(\mathcal{A}) \to \mathbb{R}.$$

With this definition the following two propositions follow as in [16].

Proposition 3. *Let $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where $\gamma_{\mathcal{H}_\pm} = \pm\,\text{Id}$ and let us denote F_+ the restriction of F to $\mathcal{H}_+$. Let $U\colon \mathbb{R} \to \text{End}(X)$ be a unitary representation of $\mathbb{R}$ on a finite-dimensional Hilbert space X. Let D be the generator of U, and put $\Delta_X = e^{-D}$. For any projection $p \in \mathcal{A} \otimes \text{End}(X)$ invariant under the action $(\sigma_t \otimes \text{Ad}_{\Delta_X^{it}})_{t\in\mathbb{R}}$, the modular index of $p(F_+ \otimes \text{Id}_X)p$ is given by*

$$\text{q-Ind}^F(p) = (-1)^n\, \text{Tr}_{\mathcal{H}} \otimes \text{Tr}_X((\Xi \otimes \Delta_X)(\gamma \otimes \text{Id}_X)p[F \otimes \text{Id}_X, p]^{2n}). \tag{4.1}$$

Outline of the proof. The proof follows standard lines, see [4, pp. 296–297] and [16, pp. 370–371]. The broad plan is to observe that

$$p - p(F \otimes \text{Id}_X)p(F \otimes \text{Id}_X)p = -p[(F \otimes \text{Id}_X), p][(F \otimes \text{Id}_X), p]p.$$

Then we observe that $\tau := \text{Tr}_{\mathcal{H}} \otimes \text{Tr}_X((\Xi \otimes \Delta_X)\cdot)$ is a trace on the fixed point algebra $(\mathcal{B}(\mathcal{H}) \otimes \text{End}(X))^{\sigma\otimes\text{Ad}\Delta_X}$. Since $(p - p(F \otimes \text{Id}_X)p(F \otimes \text{Id}_X)p)^n$ is trace class with respect to τ, we find that the τ-index of $p(F_+ \otimes \text{Id}_X)p$ is given by $\tau(\gamma(p - p(F \otimes \text{Id}_X)p(F \otimes \text{Id}_X)p)^n)$ (see [9, Proposition 4.2]). By definition of τ, this is just $\text{q-Ind}^F(p)$. ■

There is the notion of modular index pairing for odd modular Fredholm modules and invariant unitaries. Let $(\mathcal{A}, \mathcal{H}, F)$ be an odd modular Fredholm module with respect to $(\mathcal{N}, \Phi)$, and set $E = \frac{1}{2}(1 + F) \otimes \text{Id}_X$.

Proposition-Definition 2. *Let X be a finite-dimensional unitary representation of $\mathbb{R}$, and suppose that v is an unitary element in $\mathcal{A} \otimes \text{End}(X)$ which is invariant under $(\sigma_t \otimes \text{Ad}_{\Delta_X^{it}})_{t\in\mathbb{R}}$. Then, $EvEv^*$, as an operator from $vE(\mathcal{H} \otimes X)$ to $E(\mathcal{H} \otimes X)$, becomes a modular Fredholm operator. We let* $\text{q-Ind}^F(v)$ *denote its modular index. This number only depends on the equivariant K_1-class of v.*

Proposition 4. *Under the above setting, the modular index of $EvEv^*$ is given by*

$$\text{q-Ind}^F(v) = \frac{(-1)^n}{2^{2n}} \operatorname{Tr}_{\mathcal{H}} \otimes \operatorname{Tr}_X \left(\Xi \otimes \Delta_X \left([F,v][F,v^*] \right)^n \right). \tag{4.2}$$

Outline of the proof. As in the even case, we observe that $(E - vEv^*)^2 = -\frac{1}{4}E[F,v][F,v^*]$, and so by [2, Theorem 3.1] and the definition of τ above, the τ-index of $EvEv^*$ is given by $\tau((E - vEv^*)^{2n} - (E - v^*Ev)^{2n})$, and after standard algebraic manipulations (see [1, pp. 51–52]), we find the result of the proposition. ■

5 The Podleś spheres

5.1 The algebra

Given parameters $0 \le q < 1$ and $0 \le s \le 1$, the Podleś quantum sphere $\mathcal{A}\big(S^2_{q,s}\big)$ is defined as the universal $*$-algebra with generators $A = A^*$, B, and B^* subject to the relations

$$B^*B + (A-1)\big(A+s^2\big) = 0, \qquad BB^* + \big(q^2A - 1\big)\big(q^2A + s^2\big) = 0, \qquad AB = q^{-2}BA.$$

When $0 < s$, the algebra $\mathcal{A}\big(S^2_{q,s}\big)$ has two inequivalent irreducible representations π_+ and π_- on $\ell^2(\mathbb{N})$. In terms of is the standard orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$, they are given by the formulae

$$\begin{aligned} \pi_+(B)e_k &= \sqrt{1-q^{2k}}\sqrt{s^2+q^{2k}}\,e_{k-1}, \qquad & \pi_+(A)e_k &= q^{2k}e_k, \\ \pi_-(B)e_k &= s\sqrt{1-q^{2k}}\sqrt{1+s^2q^{2k}}\,e_{k-1}, \qquad & \pi_-(A)e_k &= -s^2q^{2k}e_k. \end{aligned} \tag{5.1}$$

The algebra $\mathcal{A}\big(S^2_{q,s}\big)$ can be completed to a C^*-algebra $C\big(S^2_{q,s}\big)$ by means of the operator norm induced by the representation $\pi_+ \oplus \pi_-$. The modular group $(\sigma_t)_{t\in\mathbb{R}}$ is periodic, and it extends by continuity to an action of $\mathrm{U}(1)$ on $C\big(S^2_{q,s}\big)$. These $\mathrm{U}(1)$-C^*-algebras are known to be isomorphic to the fibre product of the two copies of the Toeplitz algebra $\mathcal{T}$ with respect to the symbol map $\mathcal{T} \to C(S^1)$ [18]. Here, we consider the gauge action of $\mathrm{U}(1)$ on each copy of $\mathcal{T}$ and the translation action on $C(S^1)$.

Definition 4. We construct an even Fredholm module $\big(\mathcal{A}\big(S^2_{q,s}\big), F, \mathcal{H}\big)$ by taking $\mathcal{H} = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$, endowed with the representation of $\mathcal{A}\big(S^2_{q,s}\big)$ defined by the formula

$$\pi(a) = \begin{pmatrix} \pi_+(a) & 0 \\ 0 & \pi_-(a) \end{pmatrix}, \qquad a \in \mathcal{A}\big(S^2_{q,s}\big),$$

along with the grading operator and Fredholm operator F given by

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In addition, we let K be the diagonal modular operator on $\mathcal{H}$ defined by

$$Ke_{k,\pm} = q^{-2k}e_{k,\pm}, \qquad k \in \mathbb{N}.$$

Here, $e_{k,+}$ is a basis in the first direct summand (supporting π_+) $\ell^2(\mathbb{N})$ of $\mathcal{H}$, and $e_{k,-}$ is in the second direct summand (supporting π_-).

Lemma 2. *For $0 < s \le 1$ and $0 < q < 1$, the Fredholm module $\big(\mathcal{A}\big(S^2_{q,s}\big), F, \mathcal{H}\big)$ can be regarded as a 2-summable modular Fredholm module with the following data. The von Neumann algebra $\mathcal{N}$ is $\mathcal{B}(\mathcal{H})$, with the weight Φ defined by*

$$\Phi(S) = \operatorname{Tr}\big(K^{1/2}SK^{1/2}\big), \qquad 0 \le S \in \mathcal{B}(\mathcal{H}).$$

The modular automorphism $\sigma(T) = K^{-1}TK$ leaves $\mathcal{A}(S^2_{q,s})$ invariant, and the restriction can be expressed as

$$\sigma(A) = A, \qquad \sigma(B) = q^{-2}B, \qquad \sigma(B^*) = q^2B^*.$$

Proof. By [19, Theorem 2.11], Φ is a faithful normal semifinite weight on $\mathcal{B}(\mathcal{H})$, with modular group given by $T \mapsto K^{it}TK^{-it}$.

As a next step, we show that for any $a \in \mathcal{A}(S^2_{q,s})$ the operator $K[F,\pi(a)]$ is bounded and $[F,\pi(a)]$ is of trace class. Applying the definitions of the representation π and the operator F yields

$$K[F,\pi(A)]e_{k,\pm} = \pm\big(1+s^2\big)e_{k,\mp},$$
$$K[F,\pi(B)]e_{k,\pm} = \pm q^{-2k}\sqrt{1-q^{2k}}\left(\sqrt{s^2+q^{2k}} - s\sqrt{1+s^2q^{2k}}\right)e_{k-1,\mp}.$$

To make an estimate for the last expression we observe that

$$\sqrt{s^2+q^{2k}} - s\sqrt{1+s^2q^{2k}} = \frac{\big(s^2+q^{2k}\big) - s^2\big(1+s^2q^{2k}\big)}{\sqrt{s^2+q^{2k}} + s\sqrt{1+s^2q^{2k}}} = \frac{\big(1-s^4\big)q^{2k}}{\sqrt{s^2+q^{2k}} + s\sqrt{1+s^2q^{2k}}}.$$

Since the denominator is greater than or equal to $2s$, we find

$$\left|\sqrt{s^2+q^{2k}} - s\sqrt{1+s^2q^{2k}}\right| = \frac{\big(1-s^4\big)q^{2k}}{\sqrt{s^2+q^{2k}} + s\sqrt{1+s^2q^{2k}}} \le \frac{1-s^4}{2s}q^{2k}.$$

Now, since $[F,\pi(a)] = K^{-1}(K[F,\pi(a)])$ and K^{-1} is a trace class operator, it follows directly that $[F,\pi(a)]$ is of trace class for any $a \in \mathcal{A}(S^2_{q,s})$.

Therefore, in the end we obtain that for any $a_0, a_1 \in \mathcal{A}(S^2_{q,s})$, the operator $K[F,a_0][F,a_1]$ is of trace class, and so $[F,a_0][F,a_1]$ is in the domain of Φ. ■

Corollary 1. *The 3-linear functional ϕ defined by the formula*

$$\phi(a_0,a_1,a_2) = \Phi(\gamma F[F,a_0][F,a_1][F,a_2]), \qquad a_0,a_1,a_2 \in \mathcal{A}(S^2_{q,s}),$$

determines a σ-twisted cyclic cocycle over $\mathcal{A}(S^2_{q,s})$.

To see that the cyclic cocycle we obtained is non-trivial, we explicitly compute its pairing with the twisted cyclic cycle ω_2, found by Hadfield [10]. In our notation the twisted cyclic 2-cycle ω_2 is given by

$$\begin{aligned}\omega_2 = 2\,\big(&A\otimes B\otimes B^* - A\otimes B^*\otimes B + 2B\otimes B^*\otimes A - 2q^{-2}B\otimes A\otimes B^* \\ &+ \big(q^4-1\big)A\otimes A\otimes A\big) + \big(1-q^{-2}\big)s^2\big(1-s^2\big)1\otimes 1\otimes 1 \\ &+ \big(1-s^2\big)\left(1\otimes B^*\otimes B - q^{-2}1\otimes B\otimes B^* + \big(1-q^2\big)1\otimes A\otimes A\right).\end{aligned}$$

The pairing of ω_2 with ϕ (we skip the straightforward computations) gives

$$(\phi,\omega_2) = \big(1+s^2\big)^3.$$

Since ω_2 comes from $\mathrm{HH}_2^\sigma\big(\mathcal{A}(S^2_{q,s})\big)$, this also shows that the Hochschild class of ϕ is nontrivial.

5.2 The index pairing, local picture

Consider the projection $P \in M_2\big(\mathcal{A}(S^2_{q,s})\big)$ defined by

$$P = \frac{1}{1+s^2}\begin{pmatrix} 1-q^2A & B \\ B^* & A+s^2 \end{pmatrix}.$$

This projection becomes $\mathbb{R}$-invariant with respect to the representation $(\sigma_t \otimes \Delta^{it})_{t\in\mathbb{R}}$ of $\mathbb{R}$ on $\mathcal{A}(S^2_{q,s}) \otimes \mathbb{C}^2$, where $\Delta \in M_2(\mathbb{C})$ is given by

$$\Delta = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}. \tag{5.2}$$

The classes of 1 and P generate the U(1)-equivariant K_0 group of $C\big(S^2_{q,s}\big)$ [21]. We explicitly compute the index pairing of P with the twisted cocycle given by the Chern character of the modular Fredholm module constructed above.

Proposition 5. *The pairing of the Chern character of F and $[P]$ is equal to q.*

Proof. Expanding the relevant definitions, one has

$$\begin{aligned}\mathrm{Ch}_2(P,P,P) &= -\frac{1}{2}\frac{2q}{(1+s^2)^2} \\ &\times \sum_{k=0}^{\infty}\Big(2s^2\big(q^4-1\big)q^{4k} + 2s(a_{k+1}-a_k) + \big(q^2-1\big)\big(1-s^2\big)^2 q^{2k} + 2s(b_k - b_{k+1})\Big),\end{aligned}$$

where

$$a_k = \sqrt{s^2+q^{2k}}\sqrt{1+s^2q^{2k}}\,q^{2k}, \qquad b_k = \sqrt{s^2+q^{2k}}\sqrt{1+s^2q^{2k}}.$$

We compute the sum explicitly. First, observe that

$$\sum_{k=0}^{\infty}(a_{k+1}-a_k) = -\big(1+s^2\big), \qquad \sum_{k=0}^{\infty}(b_k - b_{k+1}) = \big(1+s^2\big) - s,$$

which then allows the rest of the sum to be computed to yield

$$\mathrm{Ch}_2(P,P,P) = -\frac{1}{2}\frac{2q}{\big(1+s^2\big)^2}\Big(-2s^2 - 2s\big(1+s^2\big) - \big(1-s^2\big)^2 + 2s\big(1+s^2-s\big)\Big) = q.$$

This proves the assertion. ■

Note that the index pairing is independent of the parameter s, since the K-theoretic data is invariant under continuous deformation as we shall see in detail in the next section.

5.3 The index pairing, global picture

In this section we give an alternative global picture of the index pairing. For this purpose it is convenient to use a different set of generators of the equivariant K-theory group.

Since $A \in \mathcal{A}\big(S^2_{q,s}\big)$ is σ-invariant, the spectral projections of the selfadjoint operator A give elements of the σ-equivariant K_0 group of $C\big(S^2_{q,s}\big)$. For each $k \in \mathbb{N}$, let us denote the projection onto the span of $e_{k,+}$ (resp. $e_{k,-}$) by $p_k^{(+)}$ (resp. by $p_k^{(-)}$).

By (5.1), the spectral projections of A for the positive (resp. negative) eigenvalues are given by the $p_k^{(+)} \oplus 0$ (resp. $0 \oplus p_k^{(-)}$) for $k \in \mathbb{N}$. In what follows we abbreviate these projections as $p_k^{(+)}$ and $p_k^{(-)}$.

From the description of K in Lemma 2, Proposition 3 implies

$$\mathrm{Ch}_2\big(p_k^{(\pm)}, p_k^{(\pm)}, p_k^{(\pm)}\big) = \pm q^{-2k}, \qquad k \in \mathbb{N}.$$

Let us relate this computation to the calculation of Proposition 5. One may think of P as a family of projections on $\mathcal{H} \otimes \mathbb{C}^2$ parametrized by $0 \leq q < 1$ and $0 < s \leq 1$. Moreover, the C^*-algebras $C(S^2_{q,s})$ can be identified with each other because they have the same image under the representation π. From its presentation, P is operator norm continuous in the parameters $q \in [0,1)$ and $s \in (0,1]$. Hence the class of P in $K_0^{\mathrm{U}(1)} C\big(S^2_{q,s}\big)$ is independent of q and s.

Now, the projection $(\pi_+ \oplus \pi_-)(P)$ at $q = 0$ and $s = 1$ can be written as

$$\frac{1}{2}\begin{pmatrix} 1 & S^* \\ S & 1 + p_0^{(+)} \end{pmatrix} \oplus \frac{1}{2}\begin{pmatrix} 1 & S^* \\ S & 1 - p_0^{(-)} \end{pmatrix},$$

where S is the isometry $e_k \mapsto e_{k+1}$ on $\ell^2(\mathbb{N})$. This projection and

$$\begin{pmatrix} 1 & 0 \\ 0 & p_0^{(+)} \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

are connected by a continuous path of U(1)-invariant projections $f_t^{(+)} \oplus f_t^{(-)}$ ($t \in [0,1]$) defined by

$$f_t^{(+)} = \frac{1}{2}\begin{pmatrix} \sqrt{1-t^2} + 1 & tS^* \\ tS & (1 - \sqrt{1-t^2})(1 - p_0^{(+)}) + 2p_0^{(+)} \end{pmatrix}$$

and

$$f_t^{(-)} = \frac{1}{2}\begin{pmatrix} \sqrt{1-t^2} + 1 & tS^* \\ tS & (1 - \sqrt{1-t^2})(1 - p_0^{(-)}) \end{pmatrix}.$$

Using the representation of K in equation (5.2), we obtain

$$\mathrm{Ch}_2(P,P,P) = \mathrm{Ch}_2(1,1,1)q^{-1} + \mathrm{Ch}_2\big(p_0^{(+)}, p_0^{(+)}, p_0^{(+)}\big)q = q,$$

which gives a 'global' picture of the index pairing.

6 The modular Fredholm modules over $\mathcal{A}(\mathrm{SU}_q(2))$

As an example of odd-dimensional case, let us present the quantum group $\mathrm{SU}_q(2)$. In this section the parameter q takes value in $(0,1)$. The $*$-algebra $\mathcal{A}(\mathrm{SU}_q(2))$ is universally generated by a and b satisfying the relations

$$ba = qab, \qquad bb^* = b^*b, \qquad b^*a = qab^*, \qquad aa^* + bb^* = 1, \qquad a^*a + q^2bb^* = 1.$$

In this section we shall demonstrate that the fundamental Fredholm module presented first in [14] and the Fredholm module arising from the spectral triple constructed in [7] both give rise to non-trivial twisted cyclic cocycles. We explicitly compute the pairing of these cocycles with an element from the equivariant K_1 group, and show that the two pairings are both non-zero.

6.1 The basic Fredholm module

We briefly review the construction of the module Fredholm module. The Hilbert space is $\mathcal{H} = \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{Z})$, with an representation π_0 of the $\mathcal{A}(\mathrm{SU}_q(2))$ defined by

$$\pi_0(a)e_{k,l} = \sqrt{1-q^{2k+2}}e_{k+1,l}, \qquad \pi_0(b)e_{k,l} = q^k e_{k,l+1},$$

in terms of the standard basis for $k \geq 0$, and $l \in \mathbb{Z}$. The Fredholm operator F is chosen to be $Fe_{k,l} = \mathrm{sign}(l)e_{k,l}$, where we put $\mathrm{sign}(0) = 1$.

Lemma 3. *The triple $(F, \pi_0, \mathcal{H})$ is a 3-summable modular Fredholm module with respect to the von Neumann algebra $\mathcal{B}(\mathcal{H})$ and weight Φ defined as follows. Define the modular operator by*

$$Ke_{k,l} = q^{-2k}e_{k,l}.$$

Then the weight Φ is given by $\Phi(T) := \mathrm{Tr}\left(K^{1/2}TK^{1/2}\right)$, for $T \geq 0$.

Proof. From the way π_0 is defined, one obtains the commutation relation

$$[F, \pi_0(a)] = 0, \qquad [F, \pi_0(b)]e_{k,l} = 2q^k\delta_{l,-1}e_{k,l+1}. \tag{6.1}$$

It follows that for any $x \in \mathcal{A}(\mathrm{SU}_q(2))$, the matrix coefficient of $[F, \pi_0(x)]$ decays by the order of q^k with respect to the index $k \in \mathbb{N}$. Therefore, for any elements x, y in $\mathcal{A}(\mathrm{SU}_q(2))$, the operator $K[F, \pi_0(x)][F, \pi_0(y)]$ is bounded, and for any three elements x, y, and z, the operator $K[F, \pi_0(x)][F, \pi_0(y)][F, \pi_0(z)]$ is of trace class. This means that $[F, \pi_0(\mathcal{A}(\mathrm{SU}_q(2)))]^3$ is in the domain of Φ. ∎

The modular automorphism $\sigma(T) = K^{-1}TK$ is given on the generators by

$$\sigma(b) = b, \qquad \sigma(b^*) = b^*, \qquad \sigma(a) = q^{-2}a, \qquad \sigma(a^*) = q^2a^*.$$

Since the Fredholm module is odd, we can use it to construct a twisted 3-cyclic cocycle. Let us investigate the pairing of this cocycle with the equivariant K_1-group. Consider the unitary V in $\mathcal{A}(\mathrm{SU}_q(2)) \otimes M_2(\mathbb{C})$ given by

$$V = \begin{pmatrix} -qb^* & a \\ a^* & b \end{pmatrix}. \tag{6.2}$$

This gives a generator of the $U(1)$-equivariant K_1-group of $C(\mathrm{SU}_q(2))$ with respect to σ_t. If we extend the action of the modular operator to $\mathcal{H} \otimes \mathbb{C}^2$ by the generator

$$\tilde{K} = K \otimes \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix},$$

then we see that V is invariant under $\sigma = \mathrm{Ad}_{\tilde{K}}$. We can now compute the modular index pairing using (4.2).

Proposition 6. *The equivariant pairing between the class of Ch_3 and the class of V is nontrivial and is equal to q.*

Proof. We compute the pairing explicitly:

$$\begin{aligned}\langle[\mathrm{Ch}_3], [V]\rangle &= \frac{(-1)^2}{2^4}\,\mathrm{Tr}\left(\tilde{K}F[F,V][F,V^*][F,V][F,V^*]\right) \\ &= \frac{1}{16}\left(\sum_{k=0}^{\infty}\sum_{l\in\mathbb{Z}} 16q^{-2k}\,\mathrm{sign}(l)\left(q\delta_{l,0}q^{4k} + q^3\delta_{l,-1}q^{4k}\right)\right) \\ &= \sum_{k=0}^{\infty} q\left(1-q^2\right)q^{2k} = q.\end{aligned}$$

∎

It follows from results of Hadfield and Krähmer (see [11, Lemma 4.6]) that the map

$$I\colon\ \mathrm{HH}_3^\sigma(\mathcal{A}(\mathrm{SU}_q(2))) \to \mathrm{HC}_3^\sigma(\mathcal{A}(\mathrm{SU}_q(2)))$$

is surjective. Therefore, the Hochschild cohomology class of the Chern character is nontrivial in $\mathrm{HH}^3_\sigma(\mathcal{A}(\mathrm{SU}_q(2)))$. Similar comments apply to the Chern character in the next section.

6.2 The modular Fredholm module of $\mathrm{SU}_q(2)$ from its spectral triple

The Fredholm module for $\mathcal{A}(\mathrm{SU}_q(2))$ presented above gives (up to sign) the same (ordinary) K-homology class as the Fredholm module arising from the spectral triple over $\mathcal{A}(\mathrm{SU}_q(2))$ discovered in [7]. It is therefore not surprising that the modular index pairings and the twisted cyclic three-cocycles obtained from these two examples are both nontrivial.

Let us briefly recall the construction of the equivariant spectral triple over $\mathcal{A}(\mathrm{SU}_q(2))$ due to [7]. We will use the notation from that work, and refer there for more details.

When $j \in \frac{1}{2}\mathbb{N}$ is a half integer, put $j^\pm = j \pm 1/2$ when $j \in \frac{1}{2}\mathbb{N}$. For each $j \in \frac{1}{2}\mathbb{N}$, consider finite-dimensional Hilbert spaces

$$W_j^\uparrow = \mathrm{span}\big\{\, |j\mu n\uparrow\rangle \mid \mu = -j, -j+1, \dots, j,\ \text{and } n = -j^+, \dots, j^+\big\},$$
$$W_j^\downarrow = \mathrm{span}\big\{\, |j\mu n\downarrow\rangle \mid \mu = -j, -j+1, \dots, j,\ \text{and } n = -j^-, \dots, j^-\big\},$$

where the elements of the respective sets form orthonormal bases. The spectral triple is realized on the completion $\mathcal{H}$ of the pre-Hilbert space $\oplus_{2j=0}^\infty W_j^\uparrow \oplus W_j^\downarrow$. The action of $\mathcal{A}(\mathrm{SU}_q(2))$ on $\mathcal{H}$ is given ([7], Proposition 4.4) as follows. First, the action of a is given by

$$\pi'(a)\,|j\mu n\uparrow\rangle = \alpha^+_{j\mu n\uparrow\uparrow}\,\big|j^+\mu^+n^+\uparrow\big\rangle + \alpha^+_{j\mu n\downarrow\uparrow}\,\big|j^+\mu^+n^+\downarrow\big\rangle + \alpha^-_{j\mu n\uparrow\uparrow}\,\big|j^-\mu^+n^+\uparrow\big\rangle,$$
$$\pi'(a)\,|j\mu n\downarrow\rangle = \alpha^+_{j\mu n\downarrow\downarrow}\,\big|j^+\mu^+n^+\downarrow\big\rangle + \alpha^-_{j\mu n\downarrow\downarrow}\,\big|j^-\mu^+n^+\downarrow\big\rangle + \alpha^-_{j\mu n\uparrow\downarrow}\,\big|j^-\mu^+n^+\uparrow\big\rangle,$$

where the coefficients $\alpha^\pm_{j\mu n}$ are given by (writing $[k] = (q^{-k} - q^k)(q^{-1} - q)^{-1}$)

$$\begin{pmatrix} \alpha^+_{j\mu n\uparrow\uparrow} & \alpha^+_{j\mu n\uparrow\downarrow} \\ \alpha^+_{j\mu n\downarrow\uparrow} & \alpha^+_{j\mu n\downarrow\downarrow} \end{pmatrix} = q^{\left(\mu+n-\frac{1}{2}\right)/2}\,[j+\mu+1]^{\frac{1}{2}} \begin{pmatrix} q^{-j-\frac{1}{2}}\frac{[j+n+\frac{3}{2}]^{\frac{1}{2}}}{[2j+2]} & 0 \\ q^{\frac{1}{2}}\frac{[j-n+\frac{1}{2}]^{\frac{1}{2}}}{[2j+1][2j+2]} & q^{-j}\frac{[j+n+\frac{1}{2}]^{\frac{1}{2}}}{[2j+1]} \end{pmatrix}$$

and

$$\begin{pmatrix} \alpha^-_{j\mu n\uparrow\uparrow} & \alpha^-_{j\mu n\uparrow\downarrow} \\ \alpha^-_{j\mu n\downarrow\uparrow} & \alpha^-_{j\mu n\downarrow\downarrow} \end{pmatrix} = q^{\left(\mu+n-\frac{1}{2}\right)/2}\,[j-\mu]^{\frac{1}{2}} \begin{pmatrix} q^{j+1}\frac{[j-n+\frac{1}{2}]^{\frac{1}{2}}}{[2j+1]} & -q^{\frac{1}{2}}\frac{[j+n+\frac{1}{2}]^{\frac{1}{2}}}{[2j][2j+1]} \\ 0 & q^{j+\frac{1}{2}}\frac{[j-n-\frac{1}{2}]^{\frac{1}{2}}}{[2j]} \end{pmatrix}.$$

Similarly, the action of b can be expressed as

$$\pi'(b)\,|j\mu n\uparrow\rangle = \beta^+_{j\mu n\uparrow\uparrow}\,\big|j^+\mu^+n^-\uparrow\big\rangle + \beta^+_{j\mu n\downarrow\uparrow}\,\big|j^+\mu^+n^-\downarrow\big\rangle + \beta^-_{j\mu n\uparrow\uparrow}\,\big|j^-\mu^+n^-\uparrow\big\rangle,$$
$$\pi'(b)\,|j\mu n\downarrow\rangle = \beta^+_{j\mu n\downarrow\downarrow}\,\big|j^+\mu^+n^-\downarrow\big\rangle + \beta^-_{j\mu n\downarrow\downarrow}\,\big|j^-\mu^+n^-\downarrow\big\rangle + \beta^-_{j\mu n\uparrow\downarrow}\,\big|j^-\mu^+n^-\uparrow\big\rangle,$$

where the coefficients are given by

$$\begin{pmatrix} \beta^+_{j\mu n\uparrow\uparrow} & \beta^+_{j\mu n\uparrow\downarrow} \\ \beta^+_{j\mu n\downarrow\uparrow} & \beta^+_{j\mu n\downarrow\downarrow} \end{pmatrix} = q^{\left(\mu+n-\frac{1}{2}\right)/2}\,[j+\mu+1]^{\frac{1}{2}} \begin{pmatrix} \frac{[j-n+\frac{3}{2}]^{\frac{1}{2}}}{[2j+2]} & 0 \\ -q^{-j-1}\frac{[j+n+\frac{1}{2}]^{\frac{1}{2}}}{[2j+1][2j+2]} & q^{-\frac{1}{2}}\frac{[j-n+\frac{1}{2}]^{\frac{1}{2}}}{[2j+1]} \end{pmatrix}$$

and

$$\begin{pmatrix} \beta^{-}_{j\mu n\uparrow\uparrow} & \beta^{-}_{j\mu n\uparrow\downarrow} \\ \beta^{-}_{j\mu n\downarrow\uparrow} & \beta^{-}_{j\mu n\downarrow\downarrow} \end{pmatrix} = q^{\left(\mu+n-\frac{1}{2}\right)/2} \left[j-\mu\right]^{\frac{1}{2}} \begin{pmatrix} -q^{-\frac{1}{2}} \frac{\left[j+n+\frac{1}{2}\right]^{\frac{1}{2}}}{[2j+1]} & -q^{j} \frac{\left[j-n+\frac{1}{2}\right]^{\frac{1}{2}}}{[2j][2j+1]} \\ 0 & -\frac{\left[j+n-\frac{1}{2}\right]^{\frac{1}{2}}}{[2j]} \end{pmatrix}.$$

The Dirac operator D acts as the scalar j on $W_j^{\uparrow}$ and as $-j$ on $W_j^{\downarrow}$. The phase F of D is therefore given by the factor 1 on $W_j^{\uparrow}$ and by -1 on $W_j^{\downarrow}$.

In this basis, the modular element K is represented by

$$K\left|j\mu n\uparrow\downarrow\right\rangle = q^{-2(\mu+n)}\left|j\mu n\uparrow\downarrow\right\rangle.$$

We take the von Neumann algebra $\mathcal{B}(\mathcal{H})$, and the weight $\Phi(T) := \operatorname{Tr}\left(K^{1/2}TK^{1/2}\right)$ for $0 \leq T \in \mathcal{B}(\mathcal{H})$.

Proposition 7. *The triple $(\mathcal{A}(\mathrm{SU}_q(2)), H, F)$ is an odd 3-summable modular Fredholm module with respect to $(\mathcal{B}(\mathcal{H}), \Phi)$.*

Proof. Since $x \mapsto [F, x]$ is a derivation, we only need to verify the summability condition for the generators $x = a, b$. Let $P^{\uparrow}$ (resp. $P^{\downarrow}$) denote the projection onto $\oplus_j W_j^{\uparrow}$ (resp. $\oplus_j W_j^{\downarrow}$). Then the commutator $[F, x]$ can be expressed as $P^{\uparrow}xP^{\downarrow} - P^{\downarrow}xP^{\uparrow}$. Thus, for example,

$$\begin{aligned} [F,a]\left|j\mu n\uparrow\right\rangle &\mapsto q^{\left(\mu+n-\frac{1}{2}\right)/2}\left[j+\mu+1\right]^{\frac{1}{2}} q^{\frac{1}{2}} \frac{\left[j-n+\frac{1}{2}\right]^{\frac{1}{2}}}{[2j+1]\,[2j+2]} \left|j^{+}\mu^{+}n^{+}\downarrow\right\rangle, \\ [F,a]\left|j\mu n\downarrow\right\rangle &\mapsto q^{\left(\mu+n-\frac{1}{2}\right)/2}\left[j-\mu\right]^{\frac{1}{2}} q^{\frac{1}{2}} \frac{\left[j+n+\frac{1}{2}\right]^{\frac{1}{2}}}{[2j]\,[2j+1]} \left|j^{-}\mu^{+}n^{+}\uparrow\right\rangle. \end{aligned} \tag{6.3}$$

Therefore, we need to establish that the coefficients in the above expressions are summable with respect to the modular weight. The asymptotics of $[k]$ is the same as that of q^{-k} as k tends to infinity. Hence the asymptotics of the first component of $[F,a]K^{\frac{1}{3}}$ is bounded from above by

$$\frac{q^{-\left(j+\frac{2}{3}\mu+\frac{1}{3}n\right)}}{q^{-4j}}.$$

Similarly, from (6.3), the second component of $[F,a]K^{\frac{1}{3}}$ is asymptotically bounded from above by

$$\frac{q^{-\left(j+\frac{1}{3}\mu-\frac{2}{3}n\right)}}{q^{-4j}},$$

and one can see that it is a trace class operator. Analogously for $x = b$, using the expression of the matrices $\beta^{\pm}_{j\mu n}$, the 'matrix coefficients' of $[F,b]K^{\frac{1}{3}}$ are asymptotically bounded from above by

$$\max\left(\frac{q^{-\left(2j+\frac{2}{3}\mu+\frac{2}{3}n\right)}}{q^{-4j}}, \frac{q^{-\left(j+\frac{1}{3}\mu-\frac{1}{3}n\right)}}{q^{-4j}}\right),$$

and similar analysis shows that $[F,b]K^{\frac{1}{3}} \in L^1(H) \subset L^3(H)$. This proves the assertion. Observe that the Fredholm module $(\mathcal{A}(\mathrm{SU}_q(2)), H, F)$ is not 2-summable, as could be easily seen by computing the asymptotics of $[F,b]K^{\frac{1}{2}}$, which shows that this is only bounded but not compact. ∎

Since the product of at least three commutators with F is in the domain of the modular weight $\Phi(\cdot) = \mathrm{Tr}\left(K^{1/2} \cdot K^{1/2}\right)$ we can define the twisted Chern character of the modular Fredholm module as before:

$$\mathrm{Ch}_3(x_0, x_1, x_2, x_3) = \lambda_3 \frac{1}{2} \mathrm{Tr}(F[F, x_0][F, x_1][F, x_2][F, x_3]K), \qquad x_i \in \mathcal{A}(\mathrm{SU}_q(2)).$$

We now compute the pairing of Ch_3 with the equivariant odd K-group. Taking V as in (6.2) and with the same extension of K to $\mathcal{H} \otimes \mathbb{C}^2$, we obtain the following:

Proposition 8. *The modular index of V relative to the above Fredholm module is equal to* 1*.*

Proof. First, observe that V can be written as $V = SU$, where S and U are given by

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} a & b \\ -qb^* & a^* \end{pmatrix}.$$

Recall that by [20], the operator PUP, where $P = \frac{1}{2}(1 + F) \otimes \mathrm{Id}$, has a trivial cokernel, whereas its kernel is one-dimensional and spanned by

$$\xi_0 = \begin{pmatrix} \left|0, 0, -\frac{1}{2} \uparrow\right\rangle \\ -q^{-1} \left|0, 0, \frac{1}{2} \uparrow\right\rangle \end{pmatrix}.$$

Since the matrix S commutes with the projection P, ξ_0 also spans the kernel of PVP. It is then easy to check that the eigenvalue of the modular operator $\tilde{K}$ acting on ξ_0 is 1. ■

Remark 2. The above computations show that, the two modular Fredholm modules of Sections 6.1 and 6.2 are related by multiplying a nontrivial 1-dimensional character of U(1) if one considers the associated U(1)-equivariant K-homology classes.

7 Conclusions

The significance of the results in [5] is that we can represent those cyclic cocycles arising from traces on $\mathcal{J}^n$ as Chern characters of n-summable semifinite Fredholm modules. Theorem 1 shows that we can represent those twisted cyclic cocycles arising from KMS weights on $\mathcal{J}^n$ as Chern characters of modular Fredholm modules.

This Fredholm module approach to twisted traces works well, and in the examples avoids the dimension drop phenomena which plague q-deformations. An unbounded approach, in the spirit of spectral triples, is still a work in progress, but see [12, 13, 17].

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