

# Irrationality of the Roots of the Yablonskii–Vorob’ev Polynomials and Relations between Them

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**Abstract.** We study the Yablonskii–Vorob’ev polynomials, which are special polynomials used to represent rational solutions of the second Painlevé equation. Divisibility properties of the coefficients of these polynomials, concerning powers of 4, are obtained and we prove that the nonzero roots of the Yablonskii–Vorob’ev polynomials are irrational. Furthermore, relations between the roots of these polynomials for consecutive degree are found by considering power series expansions of rational solutions of the second Painlevé equation.

*Key words:* second Painlevé equation; rational solutions; power series expansion; irrational roots; Yablonskii–Vorob’ev polynomials

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## 1 Introduction

In this paper we study the Yablonskii–Vorob’ev polynomials  $Q_n$ , with a special interest in their roots. These polynomials were derived by Yablonskii and Vorob’ev, while examining the hierarchy of rational solutions of the second Painlevé equation. The Yablonskii–Vorob’ev polynomials are defined by the differential-difference equation

$$Q_{n+1}Q_{n-1} = zQ_n^2 - 4(Q_nQ_n'' - (Q_n')^2), \quad (1)$$

with  $Q_0 = 1$  and  $Q_1 = z$ . From the recurrence relation, it is clear that the functions  $Q_n$  are rational, though it is far from obvious that they are polynomials, since in every iteration one divides by  $Q_{n-1}$ . The Yablonskii–Vorob’ev polynomials  $Q_n$  are monic polynomials of degree  $\frac{1}{2}n(n+1)$ , with integer coefficients. The first few are given in Table 1.

Yablonskii [1] and Vorob’ev [2] expressed the rational solutions of the second Painlevé equation,

$$P_{II}(\alpha) : w''(z) = 2w(z)^3 + zw(z) + \alpha,$$

with complex parameter  $\alpha$ , in terms of the Yablonskii–Vorob’ev polynomials, as summarized in the following theorem:

**Theorem 1.**  $P_{II}(\alpha)$  has a rational solution iff  $\alpha = n \in \mathbb{Z}$ . For  $n \in \mathbb{Z}$  the rational solution is unique and if  $n \geq 1$ , then it is equal to

$$w_n = \frac{Q'_{n-1}}{Q_{n-1}} - \frac{Q'_n}{Q_n}.$$

The other rational solutions are given by  $w_0 = 0$  and for  $n \geq 1$ ,  $w_{-n} = -w_n$ .

**Table 1.**


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Yablonskii–Vorob’ev polynomials

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$$Q_2 = 4 + z^3$$

$$Q_3 = -80 + 20z^3 + z^6$$

$$Q_4 = z(11200 + 60z^6 + z^9)$$

$$Q_5 = -6272000 - 3136000z^3 + 78400z^6 + 2800z^9 + 140z^{12} + z^{15}$$

$$Q_6 = -38635520000 + 19317760000z^3 + 1448832000z^6 - 17248000z^9 + 627200z^{12} \\ + 18480z^{15} + 280z^{18} + z^{21}$$

$$Q_7 = z(-3093932441600000 - 49723914240000z^6 - 828731904000z^9 + 13039488000z^{12} \\ + 62092800z^{15} + 5174400z^{18} + 75600z^{21} + 504z^{24} + z^{27})$$

$$Q_8 = -991048439693312000000 - 743286329769984000000z^3 \\ + 37164316488499200000z^6 + 1769729356595200000z^9 + 126696533483520000z^{12} \\ + 407736096768000z^{15} - 6629855232000z^{18} + 124309785600z^{21} + 2018016000z^{24} \\ + 32771200z^{27} + 240240z^{30} + 840z^{33} + z^{36}$$


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The rational solutions of  $P_{\text{II}}$  can also be determined, using the Bäcklund transformations, first given by Gambier [3], of the second Painlevé equation, by

$$w_{n+1} = -w_n - \frac{2n+1}{2w_n^2 + 2w_n' + z}, \quad w_{-n} = -w_n,$$

with “seed solution”  $w_0 = 0$ ; see also Lukashevich [4] and Noumi [5].

We note that the Yablonskii–Vorob’ev polynomials find many applications in physics. For instance, solutions of the Korteweg–de Vries equation (Airault, McKean and Moser [6]) and the Boussinesq equation (Clarkson [7]) can be expressed in terms of these polynomials. Clarkson and Mansfield [8] studied the structure of the roots of the Yablonskii–Vorob’ev polynomials  $Q_n$  and observed that the roots, of each of these polynomials, form a highly regular triangular-like pattern, for  $n \leq 7$ , suggesting that they have interesting properties. This further motivates studying the zeros of the Yablonskii–Vorob’ev polynomials.

In Section 2 the divisibility of the coefficients of the Yablonskii–Vorob’ev polynomials by powers of 4 is examined. From the divisibility properties found, we conclude that nonzero roots of the Yablonskii–Vorob’ev polynomials are irrational. In Section 3 we study power series expansions of (functions related to) the rational solution  $w_n$  of  $P_{\text{II}}(n)$ , around poles of  $w_n$ . This leads to relations between the roots of  $Q_{n-1}$  and  $Q_n$ . These relations suggest deeper connections between the zeros of  $Q_{n-1}$  and  $Q_n$ . Similarly, we look at power series expansions of (functions related to) the rational solution  $w_n$  of  $P_{\text{II}}(n)$  around 0, in Section 4. We obtain polynomial expressions in  $n$ , with rational coefficients, for sums of fixed negative powers of the nonzero roots of  $Q_n$ .

## 2 Nonzero roots are irrational

The Yablonskii–Vorob’ev polynomials  $Q_n$  are monic polynomials of degree  $\frac{1}{2}n(n+1)$ , and Taneda [9] proved:

- if  $n \equiv 1 \pmod{3}$ , then  $\frac{Q_n}{z} \in \mathbb{Z}[z^3]$ ;
- if  $n \not\equiv 1 \pmod{3}$ , then  $Q_n \in \mathbb{Z}[z^3]$ .

Therefore, we have

$$Q_n = z^{\frac{1}{2}n(n+1)} + a_1^n z^{\frac{1}{2}n(n+1)-3} + a_2^n z^{\frac{1}{2}n(n+1)-6} + \dots + a_{\lfloor \frac{1}{6}n(n+1) \rfloor}^n z^{\frac{1}{2}n(n+1)-3\lfloor \frac{1}{6}n(n+1) \rfloor}, \quad (2)$$

for certain  $a_s^n \in \mathbb{Z}$ , with convention  $a_0^n = 1$ , where  $\lfloor \cdot \rfloor$  denotes the floor function.

**Lemma 1.** *For every  $0 \leq m \leq \lceil \frac{1}{6}n(n+1) \rceil$ , we have  $4^m \mid a_m^n$ .*

**Proof.** We proceed by proving the following statement, by induction, for all  $M \in \mathbb{N}$ :

For every  $1 \leq m \leq M$ , for all  $n \in \mathbb{N}$ , whenever  $m \leq \lceil \frac{1}{6}n(n+1) \rceil$ , we have  $4^m \mid a_m^n$ , and

$$4^M \mid a_{M+1}^n, \quad 4^M \mid a_{M+2}^n, \quad \dots, \quad 4^M \mid a_{\lceil \frac{1}{6}n(n+1) \rceil}^n.$$

Observe that the case  $M = 0$  is trivial. Now suppose the statement is true for  $M \in \mathbb{N}$ . Then there are  $b_s^n \in \mathbb{Z}$ , such that for every  $n \in \mathbb{N}$ ,

$$Q_n = z^{\frac{1}{2}n(n+1)} + 4b_1^n z^{\frac{1}{2}n(n+1)-3} + 4^2 b_2^n z^{\frac{1}{2}n(n+1)-6} + \dots + 4^M b_M^n z^{\frac{1}{2}n(n+1)-3M} + 4^M P_n,$$

where  $P_n \in \mathbb{Z}[z]$  is zero or has degree less or equal to  $\frac{1}{2}n(n+1)-3(M+1)$ , and if  $m > \lceil \frac{1}{6}n(n+1) \rceil$ , then  $b_m^n = 0$ .

To complete the induction, we need to show that for every  $n \in \mathbb{N}$ ,  $4 \mid P_n$ . We prove this by induction with respect to  $n$ . Observe that  $P_0 = 0$  and  $P_1 = 0$ , therefore, indeed  $4 \mid P_0$  and  $4 \mid P_1$ . Assume  $4 \mid P_{n-1}$  and  $4 \mid P_n$ . Then  $4^M P_n \equiv 0 \pmod{4^{M+1}}$ , therefore, modulo  $4^{M+1}$ , we have:

$$\begin{aligned} z^{\max(0, n(n+1)-3M+1)} \mid zQ_n^2, \quad z^{\max(0, n(n+1)-3M+1)} \mid 4Q_n Q_n'', \\ z^{\max(0, n(n+1)-3M+1)} \mid 4(Q_n')^2. \end{aligned}$$

By the definition of  $Q_{n+1}$  (1),

$$Q_{n+1}Q_{n-1} = zQ_n^2 - 4(Q_n Q_n'' - (Q_n')^2),$$

so

$$z^{\max(0, n(n+1)-3M+1)} \mid Q_{n+1}Q_{n-1} \pmod{4^{M+1}}. \quad (3)$$

Let us consider  $Q_{n+1}Q_{n-1}$ . Since  $4 \mid P_{n-1}$ , we have

$$4^M P_{n-1} \equiv 0 \pmod{4^{M+1}},$$

therefore, modulo  $4^{M+1}$ ,

$$\begin{aligned} Q_{n+1}Q_{n-1} \equiv Q_{n+1}z^{\frac{1}{2}n(n-1)} + Q_{n+1}(4b_1^{n-1}z^{\frac{1}{2}n(n-1)-3} \\ + 4^2 b_2^{n-1}z^{\frac{1}{2}n(n-1)-6} + \dots + 4^M b_M^{n-1}z^{\frac{1}{2}n(n-1)-3M}). \end{aligned} \quad (4)$$

Since

$$\begin{aligned} Q_{n+1} = z^{\frac{1}{2}(n+1)(n+2)} + 4b_1^{n+1}z^{\frac{1}{2}(n+1)(n+2)-3} \\ + 4^2 b_2^{n+1}z^{\frac{1}{2}(n+1)(n+2)-6} + \dots + 4^M b_M^{n+1}z^{\frac{1}{2}(n+1)(n+2)-3M} + 4^M P_{n+1}, \end{aligned}$$

we have, modulo  $4^{M+1}$ ,

$$\begin{aligned} z^{\max(0, n(n+1)-3M+1)} \mid Q_{n+1}(4b_1^{n-1}z^{\frac{1}{2}n(n-1)-3} + 4^2 b_2^{n-1}z^{\frac{1}{2}n(n-1)-6} \\ + \dots + 4^M b_M^{n-1}z^{\frac{1}{2}n(n-1)-3M}). \end{aligned}$$

Hence, by (3) and (4),

$$z^{\max(0, n(n+1)-3M+1)} \mid Q_{n+1}z^{\frac{1}{2}n(n-1)} \pmod{4^{M+1}},$$

which implies

$$z^{\max(0, \frac{1}{2}(n+1)(n+2)-3M)} \mid Q_{n+1} \pmod{4^{M+1}}.$$

Since

$$\begin{aligned} Q_{n+1} &= z^{\frac{1}{2}(n+1)(n+2)} + 4b_1^{n+1} z^{\frac{1}{2}(n+1)(n+2)-3} \\ &\quad + 4^2 b_2^{n+1} z^{\frac{1}{2}(n+1)(n+2)-6} + \dots + 4^M b_M^{n+1} z^{\frac{1}{2}(n+1)(n+2)-3M} + 4^M P_{n+1}, \end{aligned}$$

we have, therefore,  $4 \mid P_{n+1}$ . Hence, by induction, for all  $n \in \mathbb{N}$ ,  $4 \mid P_n$ .

The lemma follows by induction on  $M$ . ■

Let us denote the coefficient of the lowest degree term in  $Q_n$  by

$$x_n := a_{\lfloor \frac{1}{6}n(n+1) \rfloor}^n,$$

i.e.  $x_n$  is the constant coefficient in  $Q_n$  if  $n \not\equiv 1 \pmod{3}$ , and  $x_n$  is the coefficient of  $z$  in  $Q_n$  if  $n \equiv 1 \pmod{3}$ . Fukutani, Okamoto, and Umemura [10] proved that the roots of the Yablonskii–Vorob’ev polynomials are simple, hence  $x_n$  is nonzero. Let  $p_n$  be the multiplicity of 2 in the prime factorization of  $x_n$ . As a consequence of Lemma 1, we obtain that  $p_n \geq 2 \lfloor \frac{1}{6}n(n+1) \rfloor$ . We prove

$$p_n = \left\lfloor \frac{1}{3}n(n+1) \right\rfloor.$$

Observe that  $x_n = Q_n(0)$  if  $n \not\equiv 1 \pmod{3}$ , and  $x_n = Q_n'(0)$  if  $n \equiv 1 \pmod{3}$ . Fukutani, Okamoto, and Umemura [10] derived the following identity for the Yablonskii–Vorob’ev polynomials:

$$Q_{n+1}'Q_{n-1} - Q_{n+1}Q_{n-1}' = (2n+1)Q_n^2.$$

Using this identity at 0, we obtain

$$x_{n+1}x_{n-1} = \begin{cases} (2n+1)x_n^2 & \text{if } n \equiv 0 \pmod{3}, \\ -(2n+1)x_n^2 & \text{if } n \equiv 2 \pmod{3}. \end{cases}$$

By evaluating equation (1) at 0,

$$x_{n+1}x_{n-1} = 4x_n^2, \quad \text{if } n \equiv 1 \pmod{3}.$$

Therefore, we have the following recursion for  $(x_n)_n$ :

$$\begin{aligned} x_0 &= 1, & x_1 &= 1 & \text{and} \\ x_{n+1}x_{n-1} &= \begin{cases} (2n+1)x_n^2 & \text{if } n \equiv 0 \pmod{3}, \\ 4x_n^2 & \text{if } n \equiv 1 \pmod{3}, \\ -(2n+1)x_n^2 & \text{if } n \equiv 2 \pmod{3}. \end{cases} \end{aligned}$$

So, we obtain the following recursion for  $(p_n)_n$ :

$$\begin{aligned} p_0 &= 0, & p_1 &= 0 & \text{and} \\ p_{n+1} &= \begin{cases} 2p_n - p_{n-1} & \text{if } n \not\equiv 1 \pmod{3}, \\ 2 + 2p_n - p_{n-1} & \text{if } n \equiv 1 \pmod{3}. \end{cases} \end{aligned}$$

Using this recursion, the formula  $p_n = \lfloor \frac{1}{3}n(n+1) \rfloor$ , can be proven directly, by induction.

**Remark 1.** Kaneko and Ochiai [11] found an explicit expression for the coefficients  $x_n$ . But deriving the formula  $p_n = \lceil \frac{1}{3}n(n+1) \rceil$  directly from this expression seems to be a difficult task.

**Theorem 2.** *The nonzero roots of the Yablonskii–Vorob’ev polynomials are irrational.*

**Proof.** Let  $n \not\equiv 1 \pmod{3}$ . Suppose  $x$  is a rational root of  $Q_n$ . Since  $Q_n \in \mathbb{Z}[z]$  is monic, by Gauss’s lemma,  $x \in \mathbb{Z}$ . By Lemma 1,

$$Q_n \equiv z^{\frac{1}{2}n(n+1)} \pmod{4},$$

so  $x$  is even. Let  $y := \frac{x}{2}$ , then, by equation (2),

$$0 = (2y)^{\frac{1}{2}n(n+1)} + a_1^n (2y)^{\frac{1}{2}n(n+1)-3} + a_2^n (2y)^{\frac{1}{2}n(n+1)-6} + \cdots + a_{\frac{1}{6}n(n+1)-1}^n (2y)^3 + a_{\frac{1}{6}n(n+1)}^n.$$

By Lemma 1, for every  $m \leq \frac{1}{6}n(n+1)$ , we have  $4^m \mid a_m^n$ . Hence

$$\begin{aligned} 2^{\frac{1}{2}n(n+1)} \mid (2y)^{\frac{1}{2}n(n+1)}, & \quad 2^{\frac{1}{2}n(n+1)-1} \mid a_1^n (2y)^{\frac{1}{2}n(n+1)-3}, \\ 2^{\frac{1}{2}n(n+1)-2} \mid a_2^n (2y)^{\frac{1}{2}n(n+1)-6}, & \quad \dots, \quad 2^{\frac{1}{2}n(n+1)-\frac{1}{6}n(n+1)+1} \mid a_{\frac{1}{6}n(n+1)-1}^n (2y)^3. \end{aligned}$$

So

$$2^{\frac{1}{3}n(n+1)+1} \mid a_{\frac{1}{6}n(n+1)}^n = x_n,$$

which implies

$$p_n \geq \frac{1}{3}n(n+1) + 1.$$

But  $p_n = \frac{1}{3}n(n+1)$ , a contradiction, hence roots of  $Q_n$  are irrational.

If  $n \equiv 1 \pmod{3}$ , we can apply the same reasoning to  $\frac{Q_n}{z}$ , and show that roots of  $\frac{Q_n}{z}$  are irrational. Therefore, nonzero roots of  $Q_n$  are irrational.  $\blacksquare$

This result raises the question whether the Yablonskii–Vorob’ev polynomials, excluding the trivial factor  $z$  in case  $n \equiv 1 \pmod{3}$ , are irreducible in  $\mathbb{Q}[z]$ . Kametaka [12] showed that for  $n \leq 23$ , the Yablonskii–Vorob’ev polynomials  $Q_n$  are indeed irreducible.

### 3 Relations between roots of the Yablonskii–Vorob’ev polynomials

By Theorem 1, for  $n \geq 1$ , the unique rational solution of  $P_{\text{II}}(n)$  is given by

$$w_n = \frac{Q'_{n-1}}{Q_{n-1}} - \frac{Q'_n}{Q_n}.$$

Fukutani, Okamoto, and Umemura [10] proved that the roots of the Yablonskii–Vorob’ev polynomials are simple, hence

$$w_n = \sum_{k=1}^{\frac{1}{2}n(n-1)} \frac{1}{z - z_{n-1,k}} - \sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{1}{z - z_{n,k}}, \quad (5)$$

where the  $z_{m,k}$  are the roots of  $Q_m$ . From equation (5) and the fact that  $w_n$  is the rational solution of  $P_{\text{II}}(n)$ , we obtain relations between the zeros of  $Q_{n-1}$  and  $Q_n$ .

**Theorem 3.** For  $1 \leq j \leq \frac{1}{2}n(n-1)$ :

$$\begin{aligned} & \sum_{k=1, k \neq j}^{\frac{1}{2}n(n-1)} \frac{1}{z_{n-1,j} - z_{n-1,k}} - \sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n-1,j} - z_{n,k}} = 0, \\ & \sum_{k=1, k \neq j}^{\frac{1}{2}n(n-1)} \frac{1}{(z_{n-1,j} - z_{n-1,k})^2} - \sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{1}{(z_{n-1,j} - z_{n,k})^2} = \frac{z_{n-1,j}}{6}, \\ & \sum_{k=1, k \neq j}^{\frac{1}{2}n(n-1)} \frac{1}{(z_{n-1,j} - z_{n-1,k})^3} - \sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{1}{(z_{n-1,j} - z_{n,k})^3} = -\frac{n+1}{4}, \\ & \sum_{k=1, k \neq j}^{\frac{1}{2}n(n-1)} \frac{1}{(z_{n-1,j} - z_{n-1,k})^5} - \sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{1}{(z_{n-1,j} - z_{n,k})^5} = z_{n-1,j} \left( \frac{n+1}{24} - \frac{1}{36} \right). \end{aligned}$$

For  $1 \leq j \leq \frac{1}{2}n(n+1)$ :

$$\begin{aligned} & \sum_{k=1}^{\frac{1}{2}n(n-1)} \frac{1}{z_{n,j} - z_{n-1,k}} - \sum_{k=1, k \neq j}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,j} - z_{n,k}} = 0, \\ & \sum_{k=1}^{\frac{1}{2}n(n-1)} \frac{1}{(z_{n,j} - z_{n-1,k})^2} - \sum_{k=1, k \neq j}^{\frac{1}{2}n(n+1)} \frac{1}{(z_{n,j} - z_{n,k})^2} = -\frac{z_{n,j}}{6}, \\ & \sum_{k=1}^{\frac{1}{2}n(n-1)} \frac{1}{(z_{n,j} - z_{n-1,k})^3} - \sum_{k=1, k \neq j}^{\frac{1}{2}n(n+1)} \frac{1}{(z_{n,j} - z_{n,k})^3} = -\frac{n-1}{4}, \\ & \sum_{k=1}^{\frac{1}{2}n(n-1)} \frac{1}{(z_{n,j} - z_{n-1,k})^5} - \sum_{k=1, k \neq j}^{\frac{1}{2}n(n+1)} \frac{1}{(z_{n,j} - z_{n,k})^5} = z_{n,j} \left( \frac{n-1}{24} + \frac{1}{36} \right). \end{aligned}$$

**Proof.** Let  $1 \leq j \leq \frac{1}{2}n(n-1)$  and define  $\omega := z_{n-1,j}$  and  $u := w_n - \frac{1}{z-\omega}$ . Since  $\gcd(Q_{n-1}, Q_n) = 1$ , see Fukutani, Okamoto, and Umemura [10], equation (5) shows that  $u$  is holomorphic in a neighbourhood of  $\omega$ . Hence  $u$  has a power series expansion, say

$$\sum_{m=0}^{\infty} a_m (z - \omega)^m,$$

which converges in an open disc centered at  $\omega$ .

Since  $w_n$  is a solution of  $P_{\text{II}}(n)$ ,  $u$  satisfies

$$\begin{aligned} (z - \omega)^2 u'' &= 6u + 6(z - \omega)u^2 + 2(z - \omega)^2 u^3 + (n+1)(z - \omega)^2 + \omega(z - \omega) \\ &\quad + (z - \omega)^3 u + \omega(z - \omega)^2 u. \end{aligned}$$

Hence we have the following identity in an open disc centered at  $\omega$ :

$$\begin{aligned} \sum_{m=2}^{\infty} (m-1) m a_m (z - \omega)^m &= 6 \sum_{m=0}^{\infty} a_m (z - \omega)^m + 6(z - \omega) \left( \sum_{m=0}^{\infty} a_m (z - \omega)^m \right)^2 \\ &\quad + 2(z - \omega)^2 \left( \sum_{m=0}^{\infty} a_m (z - \omega)^m \right)^3 + (n+1)(z - \omega)^2 + \omega(z - \omega) \end{aligned}$$

$$+ (z - \omega)^3 \sum_{m=0}^{\infty} a_m (z - \omega)^m + \omega (z - \omega)^2 \sum_{m=0}^{\infty} a_m (z - \omega)^m.$$

By considering coefficients of  $(z - \omega)^n$ ,  $n = 0, 1, 2, 4$ , it is easy to deduce that  $a_0 = 0$ ,  $a_1 = -\frac{\omega}{6}$ ,  $a_2 = -\frac{n+1}{4}$  and  $a_4 = \omega \left( \frac{n+1}{24} - \frac{1}{36} \right)$ . Note that  $a_3$  does not follow from considering coefficients of  $(z - \omega)^3$ .

By Taylor’s theorem and equation (5),

$$a_m = \frac{u^{(m)}(z_{n-1,j})}{m!} = (-1)^m \left( \sum_{k=1, k \neq j}^{\frac{1}{2}n(n-1)} \frac{1}{(z_{n-1,j} - z_{n-1,k})^{m+1}} - \sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{1}{(z_{n-1,j} - z_{n,k})^{m+1}} \right).$$

The first half of the theorem follows, the second half is proved analogously.  $\blacksquare$

Note that countably many nontrivial relations can be found between the  $a_m$  in the above proof, by considering the coefficient of  $(z - \omega)^n$ , for  $n \in \mathbb{N}$ .

In Kudryashov and Demina [13] similar relations for the roots of  $Q_n$  are obtained using the Korteweg–de Vries equation. In particular, the following results are presented in [13] for  $1 \leq j \leq \frac{1}{2}n(n+1)$ :

$$\begin{aligned} \sum_{k=1, k \neq j}^{\frac{1}{2}n(n+1)} \frac{1}{(z_{n,j} - z_{n,k})^2} &= -\frac{z_{n,j}}{12}, & \sum_{k=1, k \neq j}^{\frac{1}{2}n(n+1)} \frac{1}{(z_{n,j} - z_{n,k})^3} &= 0, \\ \sum_{k=1, k \neq j}^{\frac{1}{2}n(n+1)} \frac{1}{(z_{n,j} - z_{n,k})^5} &= -\frac{z_{n,j}}{144}. \end{aligned}$$

From these relations and Theorem 3, we obtain the following corollary:

**Corollary 1.** For  $1 \leq j \leq \frac{1}{2}n(n-1)$ :

$$\begin{aligned} \sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{1}{(z_{n-1,j} - z_{n,k})^2} &= -\frac{z_{n-1,j}}{4}, & \sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{1}{(z_{n-1,j} - z_{n,k})^3} &= \frac{n+1}{4}, \\ \sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{1}{(z_{n-1,j} - z_{n,k})^5} &= -z_{n-1,j} \left( \frac{n+1}{24} - \frac{1}{48} \right). \end{aligned}$$

For  $1 \leq j \leq \frac{1}{2}n(n+1)$ :

$$\begin{aligned} \sum_{k=1}^{\frac{1}{2}n(n-1)} \frac{1}{(z_{n,j} - z_{n-1,k})^2} &= -\frac{z_{n,j}}{4}, & \sum_{k=1}^{\frac{1}{2}n(n-1)} \frac{1}{(z_{n,j} - z_{n-1,k})^3} &= -\frac{n-1}{4}, \\ \sum_{k=1}^{\frac{1}{2}n(n-1)} \frac{1}{(z_{n,j} - z_{n-1,k})^5} &= z_{n,j} \left( \frac{n-1}{24} + \frac{1}{48} \right). \end{aligned}$$

In Theorem 3, we have obtained 4 times  $\frac{1}{2}n(n-1)$  plus 4 times  $\frac{1}{2}n(n+1)$  equations satisfied by the  $\frac{1}{2}n(n+1)$  roots of  $Q_n$ , suggesting that these equations can be used to determine the roots of the polynomials  $Q_n$  recursively. If so, then these equations may be of use to derive properties of the roots of the Yablonskii–Vorob’ev polynomials. We shall not pursue this issue further here.

## 4 Sums of negative powers of roots

In Section 2, the rational solutions  $w_n$  of  $P_{\text{II}}(n)$  were studied around roots of the Yablonskii–Vorob’ev polynomials. In this section, we consider  $w_n$  at 0.

Let  $n \equiv 0 \pmod{3}$ , then 0 is not a root of  $Q_{n-1}$  or  $Q_n$ . Therefore, by equation (5),  $w_n$  is holomorphic in a neighbourhood of 0. So  $w_n$  has a power series expansion, say

$$\sum_{m=0}^{\infty} a_m z^m,$$

which converges on an open disc centered at 0.

By Taylor’s theorem and equation (5), we have

$$a_m = - \left( \sum_{k=1}^{\frac{1}{2}n(n-1)} \frac{1}{z_{n-1,k}^{m+1}} - \sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,k}^{m+1}} \right).$$

Let  $\omega := e^{\frac{2\pi i}{3}}$ . Since  $n \equiv 0 \pmod{3}$ ,  $Q_n \in \mathbb{Z}[z^3]$ . Therefore, the roots of  $Q_n$  are invariant under multiplication by  $\omega$ . Hence

$$\sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,k}^{m+1}} = \sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{1}{(\omega z_{n,k})^{m+1}} = \frac{1}{\omega^{m+1}} \sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,k}^{m+1}},$$

therefore, if  $m \not\equiv 2 \pmod{3}$ ,

$$\sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,k}^{m+1}} = 0. \tag{6}$$

By the same reason, if  $m \not\equiv 2 \pmod{3}$ ,

$$\sum_{k=1}^{\frac{1}{2}n(n-1)} \frac{1}{z_{n-1,k}^{m+1}} = 0.$$

So  $a_m = 0$ , if  $m \not\equiv 2 \pmod{3}$ , and in an open disc centered at 0,

$$w_n(z) = \sum_{m=0}^{\infty} a_{3m+2} z^{3m+2}.$$

Since  $w_n$  is a solution of  $P_{\text{II}}(n)$ , we have the following identity in an open disc centered at 0:

$$\sum_{m=0}^{\infty} (3m+1)(3m+2) a_{3m+2} z^{3m} = 2 \left( \sum_{m=0}^{\infty} a_{3m+2} z^{3m+2} \right)^3 + \sum_{m=0}^{\infty} a_{3m+2} z^{3m+3} + n.$$

Comparing coefficients gives  $a_2 = \frac{1}{2}n$ ,  $a_5 = \frac{1}{40}n$  and  $a_8 = \frac{1}{2240}n + \frac{1}{224}n^3$ . We have obtained the following relations for  $n \equiv 0 \pmod{3}$ :

$$\sum_{k=1}^{\frac{1}{2}n(n-1)} \frac{1}{z_{n-1,k}^3} - \sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,k}^3} = -\frac{n}{2}, \quad \sum_{k=1}^{\frac{1}{2}n(n-1)} \frac{1}{z_{n-1,k}^6} - \sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,k}^6} = -\frac{n}{40},$$



$$\sum_{k=1}^{\frac{1}{2}n(n-1)} \frac{1}{z_{n-1,k}^9} - \sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,k}^9} = -\frac{1}{2240}n - \frac{1}{224}n^3.$$

If  $n \equiv 1 \pmod{3}$ , then  $u := w_n + \frac{1}{z}$  is holomorphic at 0 and satisfies

$$z^2 u'' = 6u - 6zu^2 + 2z^2 u^3 + z^3 u + (n-1)z^2.$$

By considering the power series expansion of  $u = w_n + \frac{1}{z}$  around 0, the following relations are found:

$$\begin{aligned} \sum_{k=1}^{\frac{1}{2}n(n-1)} \frac{1}{z_{n-1,k}^3} - \sum_{k=1, z_{n,k} \neq 0}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,k}^3} &= \frac{1}{4}(n-1), \\ \sum_{k=1}^{\frac{1}{2}n(n-1)} \frac{1}{z_{n-1,k}^6} - \sum_{k=1, z_{n,k} \neq 0}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,k}^6} &= \frac{1}{56}(n-1) + \frac{3}{112}(n-1)^2, \\ \sum_{k=1}^{\frac{1}{2}n(n-1)} \frac{1}{z_{n-1,k}^9} - \sum_{k=1, z_{n,k} \neq 0}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,k}^9} &= \frac{1}{2800}(n-1) + \frac{9}{5600}(n-1)^2 + \frac{1}{448}(n-1)^3. \end{aligned}$$

If  $n \equiv 2 \pmod{3}$ , then  $u := w_n - \frac{1}{z}$  is holomorphic at 0 and satisfies

$$z^2 u'' = 6u - 6zu^2 + 2z^2 u^3 + z^3 u + (n+1)z^2.$$

By considering the power series expansion of  $u = w_n - \frac{1}{z}$  around 0, the following relations are found:

$$\begin{aligned} \sum_{k=1, z_{n-1,k} \neq 0}^{\frac{1}{2}n(n-1)} \frac{1}{z_{n-1,k}^3} - \sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,k}^3} &= \frac{1}{4}(n+1), \\ \sum_{k=1, z_{n-1,k} \neq 0}^{\frac{1}{2}n(n-1)} \frac{1}{z_{n-1,k}^6} - \sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,k}^6} &= \frac{1}{56}(n+1) - \frac{3}{112}(n+1)^2, \\ \sum_{k=1, z_{n-1,k} \neq 0}^{\frac{1}{2}n(n-1)} \frac{1}{z_{n-1,k}^9} - \sum_{k=1}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,k}^9} &= \frac{1}{2800}(n+1) - \frac{9}{5600}(n+1)^2 + \frac{1}{448}(n+1)^3. \end{aligned}$$

**Remark 2.** Considering higher order coefficients, we see that for every threefold  $m \geq 3$ , polynomial expressions in  $n$ , with rational coefficients, depending on  $n \pmod{3}$ , exist for

$$\sum_{k=1, z_{n-1,k} \neq 0}^{\frac{1}{2}n(n-1)} \frac{1}{z_{n-1,k}^m} - \sum_{k=1, z_{n,k} \neq 0}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,k}^m}.$$

As a corollary of these relations, by induction, we obtain:

$$\sum_{k=1, z_{n,k} \neq 0}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,k}^3} = \begin{cases} \frac{n}{4} & \text{if } n \equiv 0 \pmod{3}, \\ 0 & \text{if } n \equiv 1 \pmod{3}, \\ -\frac{n+1}{4} & \text{if } n \equiv 2 \pmod{3}, \end{cases}$$

$$\sum_{k=1, z_{n,k} \neq 0}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,k}^6} = \begin{cases} \frac{1}{40}n^2 + \frac{1}{80}n & \text{if } n \equiv 0 \pmod{3}, \\ -\frac{1}{560}n^2 - \frac{1}{560}n + \frac{1}{280} & \text{if } n \equiv 1 \pmod{3}, \\ \frac{1}{40}n^2 + \frac{3}{80}n + \frac{1}{80} & \text{if } n \equiv 2 \pmod{3}, \end{cases}$$

$$\sum_{k=1, z_{n,k} \neq 0}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,k}^9} = \begin{cases} \frac{n + 7n^2 + 10n^3}{4480} & \text{if } n \equiv 0 \pmod{3}, \\ \frac{2 - n - n^2}{22400} & \text{if } n \equiv 1 \pmod{3}, \\ \frac{-20 - 85n - 115n^2 - 50n^3}{22400} & \text{if } n \equiv 2 \pmod{3}. \end{cases}$$

By Remark 2, for every threefold  $m \geq 3$ , polynomial expressions in  $n$ , with rational coefficients, depending on  $n \pmod{3}$ , exist for

$$\sum_{k=1, z_{n,k} \neq 0}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,k}^m}.$$

If  $m \not\equiv 0 \pmod{3}$ , see equation (6), then

$$\sum_{k=1, z_{n,k} \neq 0}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,k}^m} = 0.$$

So, for all  $n, m \in \mathbb{N}$ ,

$$\sum_{k=1, z_{n,k} \neq 0}^{\frac{1}{2}n(n+1)} \frac{1}{z_{n,k}^m} \in \mathbb{Q},$$

even though the nonzero roots of the Yablonskii–Vorob’ev polynomials are irrational.

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