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Induced Modules for Affine Lie Algebras*

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Abstract. We study induced modules of nonzero central charge with arbitrary multiplicities over affine Lie algebras. For a given pseudo parabolic subalgebra $\mathcal{P}$ of an affine Lie algebra $\mathfrak{G}$, our main result establishes the equivalence between a certain category of $\mathcal{P}$-induced $\mathfrak{G}$-modules and the category of weight $\mathcal{P}$-modules with injective action of the central element of $\mathfrak{G}$. In particular, the induction functor preserves irreducible modules. If $\mathcal{P}$ is a parabolic subalgebra with a finite-dimensional Levi factor then it defines a unique pseudo parabolic subalgebra $\mathcal{P}^{ps}$, $\mathcal{P} \subset \mathcal{P}^{ps}$. The structure of $\mathcal{P}$-induced modules in this case is fully determined by the structure of $\mathcal{P}^{ps}$-induced modules. These results generalize similar reductions in particular cases previously considered by V. Futorny, S. König, V. Mazorchuk [*Forum Math.* **13** (2001), 641–661], B. Cox [*Pacific J. Math.* **165** (1994), 269–294] and I. Dimitrov, V. Futorny, I. Penkov [*Comm. Math. Phys.* **250** (2004), 47–63].

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1 Introduction

It is difficult to over-estimate the importance of Kac–Moody algebras for modern mathematics and physics. These algebras were introduced in 1968 by V. Kac and R. Moody as a generalization of simple finite-dimensional Lie algebras, by relaxing the condition of Cartan matrix to be positive definite. We address to [20] for the basics of the Kac–Moody theory.

Affine Lie algebras are the most studied among infinite-dimensional Kac–Moody algebras, and have very wide applications. They correspond to the case of positive semidefinite matrix ($\det(a_{ij}) = 0$, with positive principal minors).

All results in the paper hold for both *untwisted* and *twisted* affine Lie algebras of rank greater than 1. Let $\mathfrak{G}$ be an affine Kac–Moody algebra with a 1-dimensional center $Z = \mathbb{C}c$.

A natural way to construct representations of affine Lie algebras is via induction from parabolic subalgebras. Induced modules play an important role in the classification problem of irreducible modules. For example, in the finite-dimensional setting any irreducible weight module is a quotient of the module induced from an irreducible module over a parabolic subalgebra, and this module is *dense* (that is, it has the largest possible set of weights) as a module over the Levi subalgebra of the parabolic [9, 10, 7]. In particular, dense irreducible module is always torsion free if all weight spaces are finite-dimensional. In the affine case,

*This paper is a contribution to the Special Issue on Kac–Moody Algebras and Applications. The full collection is available at http://www.emis.de/journals/SIGMA/Kac-Moody_algebras.html a similar conjecture [11, Conjecture 8.1] singles out induced modules as construction devices for irreducible weight modules. This conjecture has been shown for $A_1^{(1)}$ [12, Proposition 6.3] and for all affine Lie algebras in the case of modules with finite-dimensional weight spaces and nonzero central charge [18]. In the latter case, a phenomenon of reduction to modules over a proper subalgebra (finite-dimensional reductive) provides a classification of irreducible modules. Moreover, recent results of I. Dimitrov and D. Grantcharov [6] show the validity of the conjecture also for modules with finite-dimensional weight spaces and zero central charge.

For highest weight modules (with respect to nonstandard Borel subalgebras) with nonzero central charge such reduction was shown in [4, 17]. The case of induced modules from a parabolic subalgebra with a finite-dimensional Levi factor was considered in [15]. In particular, it was shown that such categories of modules are related to projectively stratified algebras [16]. A more general setting of toroidal Lie algebras was considered in [5] for induced modules from general Borel subalgebras.

The main purpose of the present paper is to show, that in the affine setting all known cases of the reduction are just particular cases of a general reduction phenomenon for modules with nonzero central charge.

We assign to each *parabolic subset* P of the root system Δ of $\mathfrak{G}$, the *parabolic subalgebra* $\mathfrak{G}_P = \mathfrak{G}_P^0 \oplus \mathfrak{G}_P^+$ of $\mathfrak{G}$ with the Levi subalgebra $\mathfrak{G}_P^0$ (cf. Section 3.1). The subalgebra $\mathfrak{G}_P^0$ is infinite-dimensional if and only if $P \cap -P$ contains imaginary roots. If, in the same time, $P \cap -P$ contains some real roots then we define our key subalgebra $\mathfrak{G}_P^{ps} \subset \mathfrak{G}_P$, which will be called *pseudo parabolic subalgebra*.

Denote by $U(\mathfrak{G})$ the universal enveloping algebra of $\mathfrak{G}$.

Let $\mathcal{P} = \mathcal{P}_0 \oplus \mathfrak{N}$ be a (pseudo) parabolic subalgebra of $\mathfrak{G}$ with the Levi subalgebra $\mathcal{P}_0$. If N is a weight $\mathcal{P}_0$-module then it can be viewed as a $\mathcal{P}$-module with a trivial action of $\mathfrak{N}$. Then one can construct the induced $\mathfrak{G}$-module $\mathrm{ind}(\mathcal{P}, \mathfrak{G}; N) = U(\mathfrak{G}) \otimes_{\mathcal{P}} N$. Hence

$$\mathrm{ind}(\mathcal{P}, \mathfrak{G}) : \ N \longmapsto \mathrm{ind}(\mathcal{P}, \mathfrak{G}; N)$$

defines a functor from the category of weight $\mathcal{P}_0$-modules to the category of weight $\mathfrak{G}$-modules.

Denote by $W(\mathcal{P}_0)$ the full subcategory of weight $\mathcal{P}_0$-modules on which the central element c acts injectively. Our main result is the following *reduction* theorem.

Theorem 1.1. *Let $\mathfrak{G}$ be affine Lie algebra of rank greater than 1, $\mathcal{P} = \mathcal{P}_0 \oplus \mathfrak{N}$ a pseudo parabolic subalgebra of $\mathfrak{G}$, $\mathcal{P}_0$ infinite-dimensional and $\mathrm{ind}_0(\mathcal{P}, \mathfrak{G})$ the restriction of the induction functor $\mathrm{ind}(\mathcal{P}, \mathfrak{G})$ onto $W(\mathcal{P}_0)$. Then the functor $\mathrm{ind}(\mathcal{P}, \mathfrak{G})$ preserves the irreducibles.*

This result allows to construct new irreducible modules for affine algebras using parabolic induction from affine subalgebras.

Theorem 1.1 follows from a more general result (see Theorems 3.1 and 4.1) which establishes an equivalence of certain categories of modules.

In the case when the Levi factor $\mathfrak{G}_P^0$ is finite-dimensional, we define a certain subalgebra $\mathfrak{m}_P \subset \mathfrak{G}$, which leads to a parabolic decomposition $\mathfrak{G} = \mathfrak{N}_P^- \oplus \mathfrak{m}_P \oplus \mathfrak{N}_P^+$ with $\mathfrak{N}_P^+ \subset \mathfrak{G}_P^+$ and $\mathfrak{G}_P^0 \subset \mathfrak{m}_P$ (cf. Section 2). If $\mathfrak{m}_P$ is finite-dimensional then $\mathfrak{G}_P^0 = \mathfrak{m}_P$ and $\mathfrak{G}_P^+ = \mathfrak{N}_P^+$. In this case there is no reduction. If $\mathfrak{m}_P$ is infinite-dimensional then $\mathfrak{m}_P \oplus \mathfrak{N}_P^+$ is pseudo parabolic subalgebra and the reduction theorem above implies Theorem 8 in [15].

The structure of the paper is as follows. In Section 2 we recall the classification of Borel subalgebras and parabolic subalgebras. Section 3 is devoted to the study of parabolic and pseudo parabolic induction. In particular, we prove Theorem 3.1 that describes the structure of induced modules. In the last section we introduce certain categories of $\mathfrak{G}$-modules and of $\mathcal{P}$-modules and establish their equivalence (Theorem 4.1).

2 Preliminaries on Borel subalgebras and parabolic subalgebras

Let $\mathfrak{H}$ be a Cartan subalgebra of $\mathfrak{G}$ with the following decomposition

$$\mathfrak{G} = \mathfrak{H} \oplus (\oplus_{\alpha\in\mathfrak{H}^*\setminus\{0\}}\mathfrak{G}_\alpha),$$

where $\mathfrak{G}_\alpha := \{x \in \mathfrak{G} \,|\, [h, x] = \alpha(h)x$ for every $h \in \mathfrak{H}\}$. Denote by $\Delta = \{\alpha \in \mathfrak{H}^*\setminus\{0\} \,|\, \mathfrak{G}_\alpha \neq 0\}$ the root system of $\mathfrak{G}$. Let π be a basis of the root system Δ. For a subset $S \subset \pi$ denote by $\Delta_+(S)$ the subset of Δ, consisting of all linear combinations of elements of S with nonnegative coefficients. Thus, $\Delta_+(\pi)$ is the set of positive roots with respect to π. Let $\delta \in \Delta_+(\pi)$ be the indivisible imaginary root. Then the set of all imaginary roots is $\Delta^{im} = \{k\delta|k \in \mathbb{Z}\setminus\{0\}\}$. Denote by G a Heisenberg subalgebra of $\mathfrak{G}$ generated by the root spaces $\mathfrak{G}_{k\delta}$, $k \in \mathbb{Z}\setminus\{0\}$.

Denote by S_π a root subsystem generated by S and δ. Let $S_\pi^+ = S_\pi \cap \Delta_+(\pi)$. For a subset $T \subset \Delta$ denote by $\mathfrak{G}(T)$ the subalgebra of $\mathfrak{G}$ generated by the root subspaces $\mathfrak{G}_\alpha$, $\alpha \in T$, and let $\mathfrak{H}(T) = \mathfrak{H} \cap \mathfrak{G}(T)$. The subalgebra $\mathfrak{G}(-T)$ will be called the *opposite subalgebra* to $\mathfrak{G}(T)$.

Let $S \subset \pi$, $S = \cup_i S_i$ where all S_i's are connected and $S_i \cap S_j = \varnothing$ if $i \neq j$.

Proposition 2.1 ([15, Proposition 2]). *$\mathfrak{G}(S_\pi) = \mathfrak{G}^S + G(S) + \mathfrak{H}$, where $\mathfrak{G}^S = \sum_i \mathfrak{G}^i$, $[\mathfrak{G}^i, \mathfrak{G}^j] = 0$, $i \neq j$, $\mathfrak{G}^i$ is the derived algebra of an affine Lie algebra of rank $|S_i| + 1$, $[\mathfrak{G}^S, G(S)] = 0$, $G(S) \subset G$, $G(S) + (G \cap \mathfrak{G}^S) = G$, $\mathfrak{G}^S \cap G(S) = \cap_i \mathfrak{G}^i = Z$.*

Let V be a weight $\mathfrak{G}$-module, that is $V = \oplus_{\mu\in\mathfrak{H}^*} V_\mu$, $V_\mu = \{v \in V \,|\, hv = \mu(h)v, \forall\, h \in \mathfrak{H}\}$. If V is irreducible then c acts as a scalar on V, it is called the *central charge* of V. A classification of irreducible modules in the category of all weight modules is still an open question even in the finite-dimensional case. In the affine case, a classification is only known in the subcategory of modules with finite-dimensional weight spaces ([18] for nonzero charge and [6] for zero charge), and in certain subcategories of induced modules with some infinite-dimensional weight spaces [11, 15]. If V is a weight module (with respect to a fixed Cartan subalgebra) then we denote by $w(V)$ the set of weight, that is $w(V) = \{\lambda \in \mathfrak{H}^*|V_\lambda \neq 0\}$.

The affine Lie algebra $\mathfrak{G}$ has the associated simple finite-dimensional Lie algebra $\mathfrak{g}$ (for details see [20]). Of course, in the untwisted case $\mathfrak{G}$ is just the affinization of $\mathfrak{g}$:

$$\mathfrak{G} = \mathfrak{g} \otimes \mathbb{C}\big[t, t^{-1}\big] \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

where d is the degree derivation: $d(x \otimes t^n) = n(x \otimes t^n)$, $d(c) = 0$, for all $x \in \mathfrak{G}$, $n \in \mathbb{Z}$.

A closed subset $P \subset \Delta$ is called a *partition* if $P\cap(-P) = \varnothing$ and $P\cup(-P) = \Delta$. In the case of finite-dimensional simple Lie algebras, every partition corresponds to a choice of positive roots in Δ, and all partitions are conjugate by the Weyl group. The situation is different in the infinite-dimensional case. In the case of affine Lie algebras the partitions are divided into a finite number of Weyl group orbits (cf. [19, 11]).

Given a partition P of Δ, we define a *Borel* subalgebra $\mathfrak{B}_P \subset \mathfrak{G}$ generated by $\mathfrak{H}$ and the root spaces $\mathfrak{G}_\alpha$ with $\alpha \in P$. Hence, in the affine case not all of the Borel subalgebras are conjugate but there exists a finite number of conjugacy classes.

A *parabolic* subalgebra of $\mathfrak{G}$ corresponds to a *parabolic* subset $P \subset \Delta$, which is a closed subset in Δ such that $P \cup (-P) = \Delta$. Given such a parabolic subset P, the corresponding parabolic subalgebra $\mathfrak{G}_P$ of $\mathfrak{G}$ is generated by $\mathfrak{H}$ and all the root spaces $\mathfrak{G}_\alpha$, $\alpha \in P$.

The conjugacy classes of Borel subalgebras of $\mathfrak{G}$ are parameterized by the parabolic subalgebras of the associated finite-dimensional simple Lie algebra $\mathfrak{g}$. We just recall the construction in the untwisted case. Parabolic subalgebras of $\mathfrak{g}$ are defined as above, they correspond to parabolic subsets of the roots system of $\mathfrak{g}$. Let $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_+$ be a parabolic subalgebra of $\mathfrak{g}$ containing a fixed Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$. Define

$$\mathfrak{B}(\mathfrak{p}) = \mathfrak{p}_+ \otimes \mathbb{C}[t, t^{-1}] \oplus \mathfrak{p}_0 \otimes t\mathbb{C}[t] \oplus \mathfrak{b} \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

For any Borel subalgebra $\mathfrak{B}$ of $\mathfrak{G}$ there exists a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ such that $\mathfrak{B}$ is conjugate to $\mathfrak{B}(\mathfrak{p})$ [19, 11].

Any Borel subalgebra conjugated to $\mathfrak{B}(\mathfrak{g})$ is called *standard*. It is determined by a choice of positive roots in $\mathfrak{G}$. Another extreme case $\mathfrak{p}_0 = \mathfrak{H}$ corresponds to the *natural* Borel subalgebra of $\mathfrak{G}$.

We will also use the geometric description of Borel subalgebras in $\mathfrak{G}$ following [8]. Let $W = \mathrm{Span}_{\mathbb{R}} \Delta$, $n = \dim W$. Let

$$F = \{\{0\} = F_n \subset F_{n-1} \subset \cdots \subset F_1 \subset F_0 = W\}$$

be a flag of maximal length in W. The flag F of maximal length is called oriented if, for each i, one of the connected components of $F_i \backslash F_{i+1}$ is labelled by $+$ and the other one is labelled by $-$. Such an oriented flag F determines the partition

$$P := \Delta \cap \left(\cup_i \left(F_i \backslash F_{i+1}\right)^+ \right)$$

of Δ. Denote $P_i^0 = \Delta \cap F_i$. The subsets P_i^0 are important invariants of the partition P.

Next statement follows immediately from the description of partitions of root systems [13].

Proposition 2.2. *Let $P \subset \Delta$ be a partition. There exists an oriented flag of maximal length $F = \{\{0\} = F_n \subset F_{n-1} \subset \cdots \subset F_1 \subset F_0 = W\}$ which determines the partition P.*

Given a partition P of Δ we will denote by $F(P)$ the corresponding oriented flag of maximal length.

Let $\lambda : B_P \to \mathbb{C}$ be a 1-dimensional representation of B_P. Then one defines an induced *Verma type* $\mathfrak{G}$-module

$$M_P(\lambda) = U(\mathfrak{G}) \otimes_{U(B_P)} \mathbb{C}.$$

The module $M_{\Delta_+(\pi)}(\lambda)$ is a classical Verma module with the highest weight λ [20]. In the case of natural Borel subalgebra we obtain *imaginary* Verma modules studied in [14].

Let $\mathfrak{B}_P = \mathfrak{H} \oplus \mathfrak{N}_P$, where $\mathfrak{N}_P$ is generated by $\mathfrak{G}_\alpha$, $\alpha \in P$. Note that the module $M_P(\lambda)$ is $U(\mathfrak{N}_P^-)$-free, where $\mathfrak{N}_P^-$ is the opposite subalgebra to $\mathfrak{N}_P$. The theory of Verma type modules was developed in [11]. It follows immediately from the definition that Verma type module with highest weight λ has a unique maximal submodule. Also, unless it is a classical Verma module, it has both finite and infinite-dimensional weight spaces and it can be obtained using the parabolic induction from a classical Verma module M with highest weight λ over a certain infinite-dimensional Lie subalgebra. Moreover, if the central charge of such Verma type module is nonzero, then the structure of this module is completely determined by the structure of module M, which is well-known [11, 4, 17].

Let $P \subset \Delta$ be a parabolic subset, $P \cup (-P) = \Delta$, $\mathfrak{G}_P$ the corresponding parabolic subalgebra of $\mathfrak{G}$. Set $P^0 = P \cap -P$. Then

$$\mathfrak{G} = \mathfrak{G}_P^- \oplus \mathfrak{G}_P^0 \oplus \mathfrak{G}_P^+,$$

where $\mathfrak{G}_P^{\pm} = \sum_{\alpha \in P \setminus (-P)} \mathfrak{G}_{\pm\alpha}$ and $\mathfrak{G}_{\mathcal{P}}^0$ is generated by $\mathfrak{H}$ and the subspaces $\mathfrak{G}_\alpha$ with $\alpha \in P^0$. The subalgebra $\mathfrak{G}_P^0$ is the *Levi factor* of $\mathfrak{G}$. A classification of parabolic subsets of Δ and parabolic subalgebras of $\mathfrak{G}$ was obtained in [11, 13].

Every parabolic subalgebra $\mathcal{P} \subset \mathfrak{G}$ containing a Borel subalgebra $\mathfrak{B}$ has a Levi decomposition $\mathcal{P} = \mathcal{P}_0 \oplus \mathfrak{N}$, with the Levi factor $\mathcal{P}_0$ and $\mathfrak{N} \subset \mathfrak{B}$. Following [11] we say that $\mathcal{P}$ has *type I* if $\mathcal{P}_0$ is finite-dimensional reductive Lie algebra, and $\mathcal{P}$ has *type II* if $\mathcal{P}_0$ contains the Heisenberg subalgebra G, generated by the imaginary root spaces of $\mathfrak{G}$. In the latter case, $\mathcal{P}_0$ is an extension of a sum of some affine Lie subalgebras by a central subalgebra and by a certain subalgebra of G [10]. Note that the radical $\mathfrak{N}$ is solvable only for the type II parabolic subalgebras. Type I parabolic subalgebras are divided also into two essentially different types depending on whether $\mathfrak{N}$ belongs to some standard Borel subalgebra (*type Ia*) or not (*type Ib*).

It is easy to see that there are parabolic subalgebras $\mathcal{P}$ which do not correspond to any triangular decomposition of $\mathfrak{G}$ [11]. In fact, if $\mathcal{P}$ contains $\mathfrak{G}_{\alpha+k\delta}$ for some α and infinitely many both positive and negative integers k then this parabolic subalgebra does not correspond to any triangular decomposition of $\mathfrak{G}$. In particular, this is always the case for parabolic subalgebras of type II.

If $\mathcal{P} = \mathcal{P}_0 \oplus \mathfrak{N}$ is a parabolic subalgebra of type II then as soon as $\mathfrak{N}$ contains $\mathfrak{G}_{\alpha+k\delta}$ for some real root α and some $k \in \mathbb{Z}$, it also contains $\mathfrak{G}_{\alpha+r\delta}$ for all $r \in \mathbb{Z}$.

Geometrically parabolic subsets correspond to partial oriented flags of maximal length. Let P be a parabolic subset which contains a partition $\tilde{P}$ and $F(\tilde{P})$ the corresponding full oriented flag: $F(\tilde{P}) = \{\{0\} = F_n \subset F_{n-1} \subset \cdots \subset F_1 \subset F_0\}$. If $P^0 \subset F_k$ for some k and P^0 is not in F_{k+1} then P is completely determined by the partial oriented flag

$$F(P) = \{\{0\} = F_k \subset \cdots \subset F_1 \subset F_0 = W\}.$$

Here $P^0 = F_k \cap \Delta$, $\mathcal{P}_0 = \mathfrak{G}_P^0$. The corresponding parabolic subalgebra $\mathcal{P}$ has type II if $\Delta^{\mathrm{Im}} \subset F_k$ and it has type Ib if $\Delta^{\mathrm{Im}} \subset F_s$ for some s, $1 \le s < k$.

Let $\mathcal{P} = \mathcal{P}_0 \oplus \mathfrak{N}$ be a non-solvable parabolic subalgebra of type II, $\mathcal{P}_0 = [\mathcal{P}_0, \mathcal{P}_0] \oplus G(\mathcal{P}) + \mathfrak{H}$, where $G(\mathcal{P}) \subset G$ is the orthogonal completion (with respect to the Killing form) of the Heisenberg subalgebra of $[\mathcal{P}_0, \mathcal{P}_0]$, that is $G(\mathcal{P}) + ([\mathcal{P}_0, \mathcal{P}_0] \cap G) = G$ and $G(\mathcal{P}) \cap [\mathcal{P}_0, \mathcal{P}_0] = \mathbb{C}c$. Note that by Proposition 2.1, $[\mathcal{P}_0, \mathcal{P}_0]$ is a sum of affine subalgebras of $\mathfrak{G}$. Let

$$G(\mathcal{P}) = G(\mathcal{P})_- \oplus \mathbb{C}c \oplus G(\mathcal{P})_+$$

be a triangular decomposition of $G(\mathcal{P})$. Define a pseudo parabolic subalgebra $\mathcal{P}^{ps} = \mathcal{P}_0^{ps} \oplus \mathfrak{N}^{ps}$, where $\mathcal{P}_0^{ps}$ is generated by the root spaces $\mathfrak{G}_\alpha$, $\alpha \in P \cap -P \cap \Delta^{\mathrm{re}}$ and $\mathfrak{H}$, while $\mathfrak{N}^{ps}$ is generated by the root spaces $\mathfrak{G}_\alpha$, $\alpha \in P \setminus (-P)$ and $G(\mathcal{P})_+$. Then $\mathcal{P}^{ps}$ is a proper subalgebra of $\mathcal{P}$.

Suppose $\mathcal{P} = \mathcal{P}_0 \oplus \mathfrak{N}$ is a parabolic subalgebra of type Ib, P the corresponding parabolic subset and $F(P)$ the partial flag. Then $\mathcal{P}_0$ is a finite-dimensional reductive Lie algebra. Assume $\mathfrak{G}_\delta \subset \mathfrak{N}$ and $\Delta^{\mathrm{Im}} \subset F_s$, with the largest such s, $1 \le s < k$. Note that for any $\alpha \in P^0$,

$P \setminus P^0$ contains the roots of the form $\alpha + k\delta$ and $-\alpha + k\delta$ for all $k > 0$. Denote by $\mathfrak{m}_P$ a subalgebra of $\mathfrak{G}$ generated by $F_s \cap \Delta$ and H. This is an infinite-dimensional Lie algebra which contains $\mathcal{P}_0$ and $\tilde{\mathfrak{m}}_P = \mathfrak{m}_P \cap \mathcal{P}$ is a parabolic subalgebra of $\mathfrak{m}_P$ of type Ia.

Let N_P be the span of all root subspaces $\mathfrak{G}_\beta$, $\beta \in P$ which are not in $\mathfrak{m}_P$, $N_P \subset \mathfrak{N}$. Then $\mathcal{P} = \tilde{\mathfrak{m}}_P \oplus N_P$. It follows immediately that

Proposition 2.3 ([15, Proposition 3]). *$\tilde{\mathfrak{m}}_P = \mathfrak{m}_P \oplus N_P$ is a parabolic subalgebra of $\mathfrak{G}$ of type II and $\mathfrak{G} = N_P^- \oplus \mathfrak{m}_P \oplus N_P$, where N_P^- is the opposite algebra to N_P.*

Hence, any parabolic subalgebra of type Ib can be extended canonically to the parabolic subalgebra of type II. Moreover, it can be extended canonically to the pseudo parabolic subalgebra $\tilde{\mathfrak{m}}_P^{ps}$:

$$\mathcal{P} \subset \tilde{\mathfrak{m}}_P^{ps} \subset \tilde{\mathfrak{m}}_P.$$

3 Parabolic induction

Let P be a parabolic subset of Δ. Let N be a weight (with respect to $\mathfrak{H}$) module over the parabolic subalgebra $\mathcal{P} = \mathfrak{G}_P$ (respectively pseudo parabolic subalgebra $\mathcal{P}^{ps}$), with a trivial action of $\mathfrak{G}_P^+$ (respectively $(\mathfrak{G}_P^{ps})^+$), and let

$$M_P(N) = \mathrm{ind}(\mathfrak{G}_P, \mathfrak{G}; N), \qquad M_P^{ps}(N) = \mathrm{ind}(\mathfrak{G}_P^{ps}, \mathfrak{G}; N)$$

be the induced $\mathfrak{G}$-modules. If N is irreducible then $M_P(N)$ (respectively $M_P^{ps}(N)$) has a unique irreducible quotient $L_P(N)$ (respectively $L_P^{ps}(N)$). If N is irreducible $\mathcal{P}$-module such that $G(\mathcal{P})_+$ acts trivially on N then $M_P(N) \simeq M_P^{ps}(N)$ and $L_P(N) \simeq L_P^{ps}(N)$.

If $\mathfrak{G}_P^0 \neq \mathfrak{G}$ then $L_P(N)$ is said to be *parabolically induced*. Following [7] we will say that irreducible $\mathfrak{G}$-module V is *cuspidal* if it is not of type $L_P(N)$ for any proper parabolic subset $P \subset \Delta$ and any N.

We see right away that a classification of irreducible $\mathfrak{G}$-modules reduces to the classification of all irreducible cuspidal modules over Levi subalgebras of $\mathfrak{G}$. Namely we have

Proposition 3.1. *Let V be an irreducible weight $\mathfrak{G}$-module. Then there exists a parabolic subalgebra $\mathcal{P} = \mathcal{P}_0 \oplus \mathfrak{N}$ of $\mathfrak{G}$ (possibly equal $\mathfrak{G}$) and an irreducible weight cuspidal $\mathcal{P}_0$-module N such that $V \simeq L_P(N)$.*

Example 3.1.

- Let V be an irreducible weight cuspidal $\mathfrak{g}$-module then $V \otimes \mathbb{C}[t, t^{-1}]$ is an irreducible cuspidal $\mathfrak{G}$-module with zero central charge.
- Modules obtained by the parabolic induction from cuspidal modules over the Heisenberg subalgebra are called *loop* modules [3].
- Pointed (that is, all weight spaces are 1-dimensional) cuspidal modules were studied in [22].

A Levi subalgebra of $\mathfrak{G}$ is *cuspidal* if it admits a weight cuspidal module. All cuspidal Levi subalgebras of type Ia and Ib parabolics were classified in [9]. They are the subalgebras with simple components of type A and C. All cuspidal Levi factors of type II parabolic subalgebras were described in [13, 11]. For any affine Lie algebra the simplest Levi subalgebra of type II is a Heisenberg subalgebra. Below we provide a list of all other Levi subalgebras of type II with the connected root system.

$\mathfrak{G}$	$\mathcal{P}_0$
$A_n^{(1)}$	$A_k^{(1)}$, $1 \le k \le n-1$
$B_n^{(1)}$	$A_k^{(1)}$, $1 \le k \le n-1$, $C_2^{(1)}$, $B_k^{(1)}$, $3 \le k \le n-1$
$C_n^{(1)}$	$A_k^{(1)}$, $1 \le k \le n-1$, $C_k^{(1)}$, $2 \le k \le n-1$
$D_n^{(1)}$	$A_k^{(1)}$, $1 \le k \le n-1$, $D_k^{(1)}$, $4 \le k \le n-1$
$G_2^{(1)}$, $D_4^{(3)}$	$A_1^{(1)}$
$F_4^{(1)}$	$A_1^{(1)}$, $A_2^{(1)}$, $C_2^{(1)}$, $C_3^{(1)}$, $B_3^{(1)}$
$E_l^{(1)}$, $l = 6, 7, 8$	$A_k^{(1)}$, $1 \le k \le l-1$, $D_k^{(1)}$, $4 \le k \le l-1$, $E_k^{(1)}$, $6 \le k \le l-1$
$A_{2n}^{(2)}$	$A_k^{(1)}$, $1 \le k \le n-1$, $A_{2k}^{(2)}$, $1 \le k \le n-1$, $E_k^{(1)}$, $6 \le k \le l-1$
$D_n^{(2)}$	$A_k^{(1)}$, $1 \le k \le n-2$, $D_k^{(2)}$, $3 \le k \le n-1$, $E_k^{(1)}$, $6 \le k \le l-1$
$A_{2n-1}^{(2)}$	$A_k^{(1)}$, $1 \le k \le n-2$, $A_{2k-1}^{(2)}$, $3 \le k \le n-1$, $D_3^{(2)}$
$E_6^{(2)}$	$A_1^{(1)}$, $A_2^{(1)}$, $D_3^{(2)}$, $D_4^{(2)}$, $D_5^{(2)}$

A nonzero element v of a $\mathfrak{G}$-module V is called *$\mathcal{P}$-primitive* if $\mathfrak{G}_P^+ v = 0$. Let Q_P be the free Abelian group generated by P^0. The following statement is standard.

Proposition 3.2. *Let V be an irreducible weight $\mathfrak{G}$-module with a $\mathcal{P}$-primitive element of weight λ, $P^0 \neq \Delta$, $N = \sum_{\nu \in Q_P} V_{\lambda+\nu}$. Then N is an irreducible $\mathfrak{G}_P$-module and V is isomorphic to $L_P(N)$.*

If V is generated by a $\mathfrak{B}$-primitive element $v \in V_\lambda$ then V is a highest weight module with highest weight λ. If $\mathcal{P}$ is of type Ia then $M_P(N)$ is a *generalized Verma module* [15, Section 2]. A classification of all irreducible N with finite-dimensional weight spaces (and hence of $L_P(N)$ if $\mathcal{P}$ is of type I) is known due to Proposition 3.2, [21] and [9]. Also a classification is known when $\mathcal{P}$ is of type II, N has finite-dimensional weight spaces and a nonzero charge [18]. In this case, N is the irreducible quotient of $\mathrm{ind}(\mathcal{P}_0, \mathcal{P}'; N')$, where $\mathcal{P}' = \mathcal{P}'_0 \oplus \mathfrak{N}'$ is a parabolic subalgebra of $\mathcal{P}_0$ of type Ia and $\mathfrak{N}'$ is an irreducible $\mathcal{P}'$-module with a trivial action of $\mathfrak{N}'$.

3.1 Reduction theorem for type II

Let π be a basis of the root system Δ, $\alpha_0 \in \pi$ such that $-\alpha + \delta \in \sum_{\beta \in \pi \setminus \{\alpha_0\}} \mathbb{Z}\beta$ and either $-\alpha + \delta \in \Delta$ or $1/2(-\alpha + \delta) \in \Delta$. Let $\dot{\pi} = \pi \setminus \{\alpha_0\}$ and $\dot{\Delta}_+$ the free semigroup generated by $\dot{\pi}$. Choose a proper subset $S \in \dot{\pi}$ and the root subsystem S_π generated by S and δ. Set

$$P_+ = \{\alpha + n\delta | \alpha \in \dot{\Delta}_+ \setminus S_\pi, n \in \mathbb{Z}\} \cap \Delta.$$

Then $P_S = S_\pi \cup P_+$ is a parabolic subset with $S_\pi = P_S \cap -P_S$. Let $\mathfrak{G}_{P_S}$ be the corresponding parabolic subalgebra. Then it is of type II with

$$\mathfrak{G}^0_{P_S} = \mathfrak{G}(S_\pi) = \sum_{\alpha \in S_\pi} \mathfrak{G}_\alpha \oplus \mathfrak{H}.$$

Proposition 3.3 ([11])**.** *If $\mathcal{P}$ is a parabolic subalgebra of $\mathfrak{G}$ of type II then there exist π, α_0 and S as above such that $\mathcal{P}$ is conjugate to $\mathfrak{G}_{P_S}$.*

Hence, it suffices to consider the parabolic subalgebras of type II in the form $\mathfrak{G}_{P_S}$.

Let $S_\pi = \cup_i S_i$ be the decomposition of S_π into connected components and let $\mathfrak{G}^0_{P_S} = \sum_i \mathfrak{G}_i \oplus G(P_S)$ be the corresponding decomposition of $\mathfrak{G}^0_{P_S}$ (see Proposition 2.1).

Theorem 3.1. *Let $\mathfrak{G}$ be of rank > 1, P a parabolic subset of Δ such that $P \cap -P$ contains real and imaginary roots simultaneously. Consider a weight $\mathfrak{G}^{ps}_P$-module V which is annihilated by $(\mathfrak{G}^{ps}_P)^+$ and on which the central element c acts injectively (object of $W(\mathfrak{G}^{ps}_P)$). Then for any submodule U of $\mathrm{ind}(\mathfrak{G}^{ps}_P, \mathfrak{G}; V)$ there exists a submodule V_U of V such that*

$$U \simeq \mathrm{ind}(\mathfrak{G}^{ps}_P, \mathfrak{G}; V_U).$$

In particular, $\mathrm{ind}(\mathfrak{G}^{ps}_P, \mathfrak{G}; V)$ is irreducible if and only V is irreducible.

Proof. The proof follows general lines of the proof of Lemma 5.4 in [11]. Denote $M^{ps}(V) = \mathrm{ind}(\mathfrak{G}^{ps}_P, \mathfrak{G}; V)$ and $\hat{M}^{ps}(V) = \sum_{\nu \in Q_P, \lambda \in w(V)} M^{ps}(V)_{\lambda+\nu} = 1 \otimes V$. Then $\hat{M}^{ps}(V)$ is a $\mathfrak{G}_P$-submodule of $M^{ps}(V)$ isomorphic to V, which consists of $\mathcal{P}$-primitive elements.

Let $\mathfrak{G}^{ps}_P = \mathcal{P}_0 \oplus \mathfrak{N}$, $\mathfrak{N}_-$ is the opposite subalgebra to $\mathfrak{N}$.

Let U be a nonzero submodule of $M^{ps}(V)$ and $v \in U$ a nonzero homogeneous element. Then

$$v = \sum_{i \in I} u_i v_i,$$

where $u_i \in U(\mathfrak{N}_-)$ are linearly independent homogeneous, $v_i \in M^{ps}(V)$.

Given a root $\varphi \in \Delta$ denote by $\mathrm{ht}(\varphi)$ the number of simple roots of $\mathfrak{N}_-$ in the decomposition of φ and by $\mathrm{ht}_1(\varphi)$ the number of all simple roots in the decomposition of φ. Suppose $u_i \in U(\mathfrak{N}_-)_{-\varphi_i}$. We can assume that all φ_i's have the same ht. Let $\mathrm{ht}(\varphi_i) = 1$ for all i. Choose i_0 such that $\mathrm{ht}_1(\varphi_{i_0})$ is the least possible. Then there exists a nonzero $x \in \mathfrak{N}$ such that $0 \neq xv \in M^{ps}(V)$ and $[x, u_{i_0}] \in U(G \cap \mathfrak{N}_-)$ since $[\mathfrak{N}, \mathfrak{N}_-] \cap G = G(\mathfrak{G}^{ps}_P)$. But $U(G \cap \mathfrak{N}_-)$ is irreducible $G(\mathfrak{G}^{ps}_P)$-module. Hence, there exists $y \in U(\mathfrak{N})$ such that $y[x, u_{i_0}]v_{i_0} = v_{i_0}$. In the same time $yxu_iv_i = 0$ if $i \neq i_0$. Thus, we obtain $v_{i_0} \in U$ and $v - u_{i_0}v_{i_0} \in U$. Applying the induction on $|I|$ we conclude that $v_i \in U$ for all $i \in I$. This completes the proof in the case $\mathrm{ht}(\varphi) = 1$. The induction step is considered similarly. Hence, U is generated by $U \cap M^{ps}(V)$ which implies the statements. ■

Corollary 3.1. *For each i, let V_i be an irreducible G_i-module with a nonzero action of the central element. Then $\mathrm{ind}(\mathfrak{G}^{ps}_P, \mathfrak{G}; \otimes_i V_i)$ is irreducible.*

Proof. Since $\otimes_i V_i$ is irreducible $\mathfrak{G}^{ps}_P$-module, the statement follows immediately from Theorem 3.1. ■

Corollary 3.2. *Let V be an irreducible weight non-cuspidal $\mathfrak{G}$-module with an injective action of the central element c. Then there exists a parabolic subalgebra $\mathcal{P} = \mathcal{P}_0 \oplus \mathfrak{N}$ of $\mathfrak{G}$ and an irreducible weight cuspidal $\mathcal{P}_0$-module N such that $V \simeq L_P(N)$, where P is the corresponding parabolic subset of Δ. Moreover, $V \simeq M_P(N)$ if $\mathcal{P}_0$ is infinite-dimensional and $\mathfrak{N}^{ps}N = 0$.*

Proof. First statement is obvious. If $\mathfrak{N}^{ps}N = 0$ then $M_P(N) \simeq M^{ps}_P(N)$ which is irreducible by Theorem 3.1. ■

If $\mathcal{P}_0 = \mathfrak{H}$ then Corollary 3.2 implies reduction theorem for Verma type modules [4, 17].

Note that in general $\mathrm{ind}(\mathfrak{G}_P, \mathfrak{G}; N)$ need not be irreducible if N is irreducible. On the other hand we have

Corollary 3.3. *For each i, let V_i be an irreducible G_i-module with the action of the central element by a nonzero scalar a, V an irreducible highest weight $G(\mathfrak{G}_P)$-module with highest weight a. Then*

$$\operatorname{ind}(\mathfrak{G}_P, \mathfrak{G}; \otimes_i V_i \otimes V)$$

is irreducible.

Proof. Note that V is isomorphic to the Verma module with highest weight a. Then

$$\operatorname{ind}(\mathfrak{G}_P, \mathfrak{G}; \otimes_i V_i \otimes V) \simeq \operatorname{ind}(\mathfrak{G}_P^{ps}, \mathfrak{G}; \otimes_i V_i)$$

which is irreducible by Corollary 3.1. ■

Corollary 3.4. *Let $\lambda \in \mathfrak{H}^*$, $\lambda(c) \neq 0$, $\mathfrak{B}$ a non-standard Borel subalgebra of $\mathfrak{G}$, P corresponding partition of Δ and*

$$F = \{\{0\} = F_n \subset \cdots \subset F_1 \subset F_0 = W\}$$

the corresponding full flag of maximal length. Suppose $\delta \in F_{s-1} \setminus F_s$ for some s, $1 \leq s < n$. Denote $\mathfrak{m}_s$ the Lie subalgebra of $\mathfrak{G}$ generated by the root subspaces with roots in $\Delta \cap F_{s-1}$ and by $\mathfrak{H}$. Then $\mathfrak{m}_s$ is infinite-dimensional and $M_P(\lambda)^s = U(\mathfrak{m}_s)v_\lambda$ is a highest weight module over $\mathfrak{m}_s$. Moreover, $M_P(\lambda)$ is irreducible if and only if $M_P(\lambda)^s$ is irreducible.

Proof. Indeed, the flag F defines a parabolic subalgebra $\mathcal{P}$ of $\mathfrak{G}$ whose Levi subalgebra is $\mathfrak{m}_s$. Hence,

$$M_P(\lambda) \simeq \operatorname{ind}(\mathcal{P}, \mathfrak{G}; M_P(\lambda)^s).$$

The statement follows immediately from Corollary 3.2. ■

3.2 Reduction theorem for type Ib

Let $\mathcal{P} = \mathcal{P}_0 \oplus \mathfrak{N}$ be a parabolic subalgebra of $\mathfrak{G}$ of type Ib, $\dim \mathcal{P}_0 < \infty$, P corresponding parabolic subset of Δ and

$$F = \{\{0\} = F_k \subset \cdots \subset F_1 \subset F_0 = W\}$$

the corresponding partial flag of maximal length. Then $\mathcal{P}_0$ has the root system $F_k \cap \Delta$. Since $\mathcal{P}$ is of type Ib, there exists s, $1 \leq s < k$, such that $\delta \in F_s$ and $\delta \notin F_{s+1}$. Then, a Lie subalgebra $\mathfrak{m}_P$ of $\mathfrak{G}$ generated by $\mathfrak{H}$ and the root spaces corresponding to the roots from $F_s \cap \Delta$, is infinite-dimensional. Obviously, it can be extended to a parabolic subalgebra $\mathfrak{m}_P \oplus \mathfrak{N}_P$ of $\mathfrak{G}$ of type II, where $\mathfrak{m}_P$ is the Levi subalgebra.

Corollary 3.5. *Let P be a parabolic subset of Δ such that $\mathfrak{G}_P^0$ is finite-dimensional and $\mathfrak{m}_P$ is infinite-dimensional. Consider a weight $\mathfrak{m}_P$-module V which is annihilated by $\mathfrak{N}_P^{ps}$ and on which the central element c acts injectively. Then for any submodule U of $\operatorname{ind}(\mathfrak{m}_P \oplus \mathfrak{N}_P, \mathfrak{G}; V)$ there exists a submodule V_U of V such that*

$$U \simeq \operatorname{ind}(\mathfrak{m}_P \oplus \mathfrak{N}_P, \mathfrak{G}; V_U).$$

In particular, $\operatorname{ind}(\mathfrak{m}_P \oplus \mathfrak{N}_P, \mathfrak{G}; V)$ is irreducible if and only if V is irreducible.

Proof. Consider the pseudo parabolic subalgebra $\mathfrak{m}_P^{ps} \oplus \mathfrak{N}_P^{ps} \subset \mathfrak{m}_P \oplus \mathfrak{N}_P$, $\mathfrak{G}_P \subset \mathfrak{m}_P^{ps} \oplus \mathfrak{N}_P^{ps}$. Then

$$\mathrm{ind}(\mathfrak{m}_P \oplus \mathfrak{N}_P, \mathfrak{G}; U(\mathfrak{m}_P)V) \simeq \mathrm{ind}(\mathfrak{m}_P^{ps} \oplus \mathfrak{N}_P^{ps}, \mathfrak{G}; U(\mathfrak{m}_P^{ps})V),$$

since $U(\mathfrak{m}_P^{ps})V \simeq \mathrm{ind}(\mathfrak{G}_P^0, \mathfrak{m}_P^{ps}; V)$ and $U(\mathfrak{m}_P)V \simeq U(\mathfrak{m}_P^{ps})V \otimes M$, where M is the highest weight $G(\mathfrak{G}_P)$-module of highest weight a. Hence, the statement follows from Theorem 3.1. ■

Consider now a weight $\mathcal{P}$-module V such that $\mathfrak{N}$ (and hence $\mathfrak{N}^{ps}$) acts trivially on V and c acts by a multiplication by a nonzero scalar. Then

$$M_P(V) = \mathrm{ind}(\mathcal{P}, \mathfrak{G}; V) \simeq \mathrm{ind}\big(\mathfrak{m}_P^{ps} \oplus \mathfrak{N}_P^{ps}, \mathfrak{G}; U(\mathfrak{m}_P^{ps})V\big)$$

and we obtain immediately

Corollary 3.6. *Let P be a parabolic subset of Δ such that $\mathfrak{G}_P^0$ is finite-dimensional and $\mathfrak{m}_P$ is infinite-dimensional. Consider an irreducible weight $\mathfrak{G}_P^0$-module V which is annihilated by $\mathfrak{G}_P^+$ and on which the central element c acts injectively. Then for any submodule U of $M_P(V)$ there exists a submodule V_U of $U(\mathfrak{m}_P)V$ such that*

$$U \simeq \mathrm{ind}(\mathfrak{m}_P \oplus \mathfrak{N}, \mathfrak{G}; V_U).$$

Moreover, $M_P(V)$ is irreducible if and only if $U(\mathfrak{m}_P)V$ is is irreducible.

Corollary 3.6 is essentially Theorem 8 of [15]. In particular it reduces the case of parabolic subalgebras of type Ib to the case of parabolic subalgebras of type II.

Example 3.2.

- Examples of irreducible dense modules with non-zero central charge were constructed in [2] as tensor products oh highest and lowest weight modules. Applying functor of pseudo parabolic induction to these modules one obtains new examples of irreducible modules over $\mathfrak{G}$ with infinite-dimensional weight spaces.
- Series of irreducible cuspidal modules over the Heisenberg subalgebra with a non-zero central charge were constructed in [1]. These modules have infinite-dimensional weight spaces. We can not apply functor of pseudo parabolic induction to these modules since the action of the Heisenberg subalgebra is torsion free. On the other hand, in the case of $A_1^{(1)}$, the functor of parabolic induction applied to such modules gives again new irreducible modules.

Parabolic induction can be easily generalized to the non-weight case as follows (cf. [15]). Let $\mathcal{P} = \mathcal{P}_0 \oplus \mathfrak{N}$ be a parabolic subalgebra of type II, P corresponding parabolic subset of Δ, $P^0 = P \cap -P$. Let $\mathfrak{H}'$ be the linear span of $[\mathfrak{G}_\alpha, \mathfrak{G}_{-\alpha}]$, $\alpha \in P^0$ and $\mathfrak{H}_P$ a complement of $\mathfrak{H}'$ in $\mathfrak{H}$ such that $[\mathcal{P}_0, \mathfrak{H}_P] = 0$.

Let Λ be an arbitrary Abelian category of $\mathcal{P}_0$-modules (note that Λ may have a different Abelian structure than the category of modules over $\mathcal{P}_0$). Given $V \in \Lambda$ and $\lambda \in \mathfrak{H}_P^*$ one makes V into a $\mathcal{P}$-module with $h|_V = \lambda(h)Id$ for any $h \in \mathfrak{H}_P$ and $\mathfrak{N}V = 0$. Then one can construct a $\mathfrak{G}$-module $M_P(V, \lambda)$ by parabolic induction. It follows from the construction that $M_P(V, \lambda)$ is $\mathfrak{H}_P$-diagonalizable. Analogously to Theorem 3.1 one can show

Theorem 3.2. *If the central element c acts injectively on $V \in \Lambda$ then for any submodule U of $M_P(V, \lambda)$ there exists a submodule V_U of V such that*

$$U \simeq M_P(V_U, \lambda).$$

In particular, $M_P(V, \lambda)$ is irreducible if and only V is irreducible.

4 Categories of induced modules

Let $\mathfrak{G}$ be an affine Lie algebra of rank > 1, $\mathcal{P} = \mathcal{P}_0 \oplus \mathfrak{N}$ a *pseudo* parabolic subalgebra of $\mathfrak{G}$ of type II, P corresponding parabolic subset of Δ, $W(\mathcal{P}_0)$ the category of weight (with respect to $\mathfrak{H}$) $\mathcal{P}_0$-modules V with an injective action of c.

Denote by $\mathcal{O}(\mathfrak{G}, \mathcal{P})$ the category of weight $\mathfrak{G}$-modules M such that the action of the central element c on M is injective and M contains a nonzero $\mathcal{P}$-primitive element. Modules $M_P(V)$ and $L_P(V)$ are typical objects of $\mathcal{O}(\mathfrak{G}, \mathcal{P})$.

For $M \in \mathcal{O}(\mathfrak{G}, \mathcal{P})$ we denote by $M^{\mathfrak{N}}$ the subspace of M consisting of $\mathcal{P}$-primitive elements, that is the subspace of $\mathfrak{N}$-invariants. Clearly, $M^{\mathfrak{N}}$ is a $\mathcal{P}_0$-module and hence $M^{\mathfrak{N}} \in W(\mathcal{P}_0)$. Let $\tilde{\mathcal{O}}(\mathfrak{G}, \mathcal{P})$ be the full subcategory of $\mathcal{O}(\mathfrak{G}, \mathcal{P})$ whose objects M are generated by $M^{\mathfrak{N}}$. Again $M_P(V)$ and $L_P(V)$ are objects of $\tilde{\mathcal{O}}(\mathfrak{G}, \mathcal{P})$.

Both categories $\mathcal{O}(\mathfrak{G}, \mathcal{P})$ and $\tilde{\mathcal{O}}(\mathfrak{G}, \mathcal{P})$ are closed under the operations of taking submodules, quotients and countable direct sums.

The parabolic induction provides a functor

$$I: \; W(\mathcal{P}_0) \to \mathcal{O}(\mathfrak{G}, \mathcal{P}), \qquad V \mapsto M_P(V) = \mathrm{ind}(\mathcal{P}, \mathfrak{G}; V).$$

Note that $M_P(N) \simeq M_P^{ps}(N)$. The canonical image of V in $M_P(V)$ is annihilated by $\mathfrak{N}$ and, hence, $I(V)$ is generated by its $\mathcal{P}$-primitive elements. Thus $I(V) \in \tilde{\mathcal{O}}(\mathfrak{G}, \mathcal{P})$.

In the opposite direction we have a well defined functor

$$R: \; \mathcal{O}(\mathfrak{G}, \mathcal{P}) \to W(\mathcal{P}_0), \qquad M \mapsto M^{\mathfrak{N}}.$$

Denote by $\tilde{R}$ the restriction of R onto the category $\tilde{\mathcal{O}}(\mathfrak{G}, \mathcal{P})$.

We need the following lemma.

Lemma 4.1.

- *Let $V \in W(\mathcal{P}_0)$, M a subquotient of $M_P(V)$. Then $w(M^{\mathfrak{N}}) \subset w(V)$.*
- *$M_P(V \oplus V') \simeq M_P(V) \oplus M_P(V')$.*
- *Let $M \in \tilde{\mathcal{O}}(\mathfrak{G}, \mathcal{P})$ be generated by $\mathcal{P}$-primitive elements then M is a direct sum of modules of type $M_P(V)$.*

Proof. Without loss of generality we will assume that M is a quotient of $M_P(V)$. Suppose $M = M_P(V)/M'$. Then $M^{\mathfrak{N}} \simeq V/(M' \cap V)$ (identifying V with $1 \otimes V$) by Theorem 3.1, and the first statement follows. Second statement follows immediately from the definition. Let $M \in \tilde{\mathcal{O}}(\mathfrak{G}, \mathcal{P})$ and $M^{\mathfrak{N}} = M' \oplus M''$ is a direct sum of $\mathcal{P}_0$-modules. If M is generated by $M^{\mathfrak{N}}$ then $M \simeq M_P(M^{\mathfrak{N}})$ by Theorem 3.1, and hence $M \simeq M_P(M') \oplus M_P(M'')$. ∎

Theorem 4.1.

- *The functor* $I : W(\mathcal{P}_0) \to \mathcal{O}(\mathfrak{G}, \mathcal{P})$ *is a left adjoint to the functor* $R : \mathcal{O}(\mathfrak{G}, \mathcal{P}) \to W(\mathcal{P}_0)$*, that is* $R \circ I$ *is naturally isomorphic to the identity functor on* $W(\mathcal{P}_0)$*.*
- *The functors* $\tilde{R}$ *and* I *are mutually inverse equivalences of* $W(\mathcal{P}_0)$ *and the subcategory* $\tilde{\mathcal{O}}(\mathfrak{G}, \mathcal{P})$*.*

Proof. Let $V \in W(\mathcal{P}_0)$. Clearly, V is naturally embedded into $M_P(V)^{\mathfrak{N}}$. On the other hand, $w(M_P(V)^{\mathfrak{N}}) \subset w(V)$ by Lemma 4.1. Since $U(\mathcal{P}_0)V \simeq V$ then $M_P(V)^{\mathfrak{N}} \simeq V$. If $M = M_P(V)$ then $R(M) \simeq V$ and $(I \circ R)(M_P(V)) \simeq M_P(V)$ by Theorem 3.1. If M is an arbitrary object in $\mathcal{O}(\mathfrak{G}, \mathcal{P})$ then M is generated by $M^{\mathfrak{N}} = R(M)$ and $M^{\mathfrak{N}} = \oplus_i M_i$, where M_i are $\mathcal{P}_0$-modules and $w(M_i) \cap w(M_j) = \varnothing$ if $i \neq j$. Then $M \simeq \oplus_i M_P(M_i)$ by Theorem 3.1. On the other hand, $I(M^(\mathfrak{N})) \simeq \oplus_i M_P(M_i)$ by Lemma 4.1. Hence, $(I \circ R)(M) \simeq M$, implying the statement. ∎

Theorem 1.1 is an immediate consequence of Theorem 3.1 or Theorem 4.1.

Let $F(P)$ be a partial flag of the parabolic subset P of Δ:

$$F(P) = \{\{0\} = F_k \subset \cdots \subset F_1 \subset F_0 = W\},$$

$P^0 = F_k \cap \Delta$, $\mathcal{P}_0 = \mathfrak{G}_P^0$, $\Delta^{\mathrm{Im}} \subset F_k$. Then $\mathcal{P}_0$ is a subalgebra of $\mathfrak{G}$ generated by $\mathfrak{G}_\alpha$, $\alpha \in F_k \cap \Delta$ and $\mathfrak{H}$.

Fix s, $k \leq s < n$. Then $\delta \in F_s$ and the Lie subalgebra $\mathfrak{m}_P^s \subset \mathfrak{G}$, generated by $\mathfrak{H}$ and the root spaces corresponding to the roots from $F_s \cap \Delta$, is infinite-dimensional. Obviously, it can be extended to a parabolic subalgebra of $\mathfrak{G}$ of type II, where $\mathfrak{m}_P^s$ is the Levi subalgebra: $\mathcal{P}_s = \mathfrak{m}_P^s \oplus \mathfrak{N}_s$, $\mathfrak{N}_s \subset \mathfrak{N}$. If V is a $\mathcal{P}$-module with $\mathfrak{N}N = 0$ then $V' = U(\mathfrak{m}_P^s)V$ is a $\mathcal{P}_s$-module, $\mathfrak{N}_s V' = 0$. If V' is a $\mathcal{P}_s$-module with a trivial action of $\mathfrak{N}_s$ and injective action of the central element c then the structure of the induced module $\mathrm{ind}(\mathcal{P}_s, \mathfrak{G}; V')$ is completely determined by the structure of V' by Theorem 3.1. In particular, this module is irreducible if and only if V' is irreducible. Moreover, since

$$M_P(V) \simeq \mathrm{ind}(\mathcal{P}_s, \mathfrak{G}; V'),$$

we have the following interesting observation.

Corollary 4.1. *If* $V \in W(\mathcal{P}_0)$*,* s *is such that* $k \leq s < n$ *and* $V' = U(\mathfrak{m}_P^s)V$ *then the submodule structure of the induced module* $M = \mathrm{ind}(\mathcal{P}_s, \mathfrak{G}; V')$ *is determined by the submodule structure of* V *(as in Theorem 3.1). In particular,* M *is irreducible if and only if* V *is irreducible.*

Denote by $W(\mathfrak{m}_P^s)$ the category of weight $\mathfrak{m}_P^s$-modules with injective action of the central element c and by $\mathcal{O}(\mathfrak{G}, \mathcal{P}_s)$ the category of weight $\mathfrak{G}$-modules M such that the action of c on M is injective and M contains a nonzero $\mathcal{P}_s$-primitive element. Modules $M_{P_s}(V)$ and $L_{P_s}(V)$ are the objects of $\mathcal{O}(\mathfrak{G}, \mathcal{P}_s)$, $V \in W(\mathcal{P}_s)$. For $M \in \mathcal{O}(\mathfrak{G}, \mathcal{P}_s)$ we denote by $M^{\mathfrak{N}_s}$ the subspace of M consisting of $\mathcal{P}_s$-primitive elements. Denote by $\tilde{\mathcal{O}}(\mathfrak{G}, \mathcal{P}_s)$ the full subcategory of $\mathcal{O}(\mathfrak{G}, \mathcal{P}_s)$ whose objects M are generated by $M^{\mathfrak{N}_s}$.

Then we have the following functors

$$I_s : \; W(\mathfrak{m}_P^s) \to \mathcal{O}(\mathfrak{G}, \mathcal{P}_s), \qquad V \mapsto M_{P_s}(V) = \mathrm{ind}(\mathcal{P}_s, \mathfrak{G}; V),$$
$$R_s : \; \mathcal{O}(\mathfrak{G}, \mathcal{P}_s) \to W(\mathfrak{m}_P^s), \qquad M \mapsto M^{\mathfrak{N}_s}$$

and $\tilde{R}_s$ the restriction of R_s onto $\tilde{\mathcal{O}}(\mathfrak{G}, \mathcal{P}_s)$.

Theorem 4.2. *For any s, $k \leq s < n$,*

- *the functor I_s is a left adjoint to R_s;*
- *the functors $\tilde{R}_s$ and I_s are mutually inverse equivalences of $W(\mathfrak{m}_P^s)$ and the subcategory $\tilde{\mathcal{O}}(\mathfrak{G}, \mathcal{P}_s)$.*

Note that for $k \leq s < r$ we have

$$\begin{array}{ccc} W(\mathfrak{m}_P^s) & \underset{\tilde{R}_s}{\overset{I_s}{\rightleftarrows}} & \tilde{\mathcal{O}}(\mathfrak{G}, \mathcal{P}_s) \\ \downarrow & & \downarrow \\ W(\mathfrak{m}_P^r) & \underset{\tilde{R}_r}{\overset{I_r}{\rightleftarrows}} & \tilde{\mathcal{O}}(\mathfrak{G}, \mathcal{P}_r) \end{array}$$

Hence, I_s is just the restriction of I_r onto $W(\mathfrak{m}_P^s)$, while $\tilde{R}_s$ is the restriction of $\tilde{R}_r$ onto $\tilde{\mathcal{O}}(\mathfrak{G}, \mathcal{P}_s)$.

Remark 4.1. One can establish similar category equivalences for non-weight modules (cf. Section 5 in [15]).

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